Matching with Frictions and Entry with Poisson Distributed Buyers and Sellers.*

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Abstract

I consider a simple directed search model with a finite number of buyers and sellers. The main innovation is that buyers and sellers are randomly drawn from independent Poisson distributions. This provides a simple justification for the usual equilibrium selection where only symmetric randomizations by buyers are considered, because any equilibrium is payoff equivalent to such symmetric equilibrium. The Poisson assumption also makes the model more tractable. A simple proof of uniqueness of equilibrium prices uses the fact that prices are strategic complements. It also becomes tractable to handle entry with a finite number of players, which is difficult with a fixed finite set of sellers.

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1 Introduction

This paper considers a simple version of a directed search model, where a finite set of buyers and sellers are drawn from independent Poisson distributions. The most important consequence of this modelling strategy is that there is no longer any loss of generality in focusing on equilibria where all buyers follow a symmetric mixed strategy.

In standard finite directed search models, such as Peters (2000) and Burdett et al (2001) there are equilibria where identities are used in order to improve coordination. To understand this, consider the simplest case with two sellers and two buyers. It is then easy to see that there is a multiplicity of continuation equilibria following any pair of prices that are not too far apart. One possibility is that buyer 1 visits seller 1 and seller 2 visits seller 2. Another possibility is that the buyers randomize in a way so that both buyers are indifferent. All players, including the seller, are strictly worse off in the mixed strategy continuation equilibrium. Hence buyers can use randomized continuation equilibria as credible threats to support a continuum of equilibrium prices.[1]

In order to obtain predictive power the directed search literature has focused on equilibria where the buyers (or workers in labor market applications) follow the same mixed strategy after any history of play. This equilibrium selection is usually justified by an informal argument based on the idea that using identities to coordinate is difficult in a market with many buyers and sellers (or workers and firms). A more formal argument justifying the standard equilibrium selection can be found in Bland and Loertscher (2012).

In contrast, there is no need to make an equilibrium selection in the framework considered in this paper. Any equilibrium is payoff equivalent to an equilibrium in which all buyers use a symmetric mixed strategy after any history of play. There are typically equilibria where, say, green buyers are more likely to visit firm 1 and red buyers are more likely to visit firm 2, but we may as well ignore them as both the buyers and the sellers earn the same payoffs in a symmetric equilibrium. This payoff equivalence makes it impossible to build equilibria built on punishing and rewarding sellers by selecting different types of continuation equilibria depending on which prices are posted. Hence, the uniqueness of equilibrium prices that will be discussed further below is not contingent on selecting the symmetric mixed strategy in every continuation game as it is in Galenianos and Kircher, (2012) and Kim and Camera (2014).

While justifying symmetric mixed strategies is arguably the most important contribution of the paper, the setup with Poisson distributed players also makes the analysis considerably more tractable than standard finite models. There are two main sources for the improved tractability. The first reason is that equilibrium arrivals of buyers are distributed in accordance with a Poisson distribution which makes the seller profit maximization problem somewhat cleaner. However, a more important change is that the Poisson assumption creates a model with what Myerson (2000) refers to as *environmental equivalence*: "being born" (or being of any particular type) does not affect the beliefs over the number or the types of what other players are present. This property is particularly important in that it facilitates the construction of equilibrium entry strategies, which

[1] See Burdett et al (2001) for details.

otherwise is a rather difficult task with finite numbers.

While differing in terms of specifics, most papers in the applied directed search literature consider what I henceforth refer to as *market utility models*. These models assume that all agents are small, so that the choice of any particular seller, firm or mechanism designer, cannot affect the utility of the agents that search. This simplifies the analysis in many ways, but is also conceptually awkward as one cannot ask what happens if a single actor deviates from the equilibrium path. To build in sequential rationality it is therefore necessary to use various tricks, such as asking what will happen if a small measure of firms would deviate and then take the limit as this measure approaches zero.

Naturally, some papers have asked whether there are game theoretic foundations for the market utility approach. In particular, Peters (2000) and Burdett et al (2001) consider simple finite models that are very similar to the one studied in this paper and establish that sequences of subgame perfect equilibria converge to the market utility outcome. [2] More recently, Galenianos and Kircher (2012) provide a substantial generalization of Peters (2000) that allows for more flexible production and matching functions.

Market utility models simplify the analysis for two reasons. Firstly, the absence of strategic effects creates a much simpler fixed point problem than in the finite version. Secondly, if there is an entry decision, equilibrium entry boils down to a simple indifference condition. The literature has focused on the first of these issues, but largely ignored the second.[3]

Naively, the problem of handling entry with a finite set of agents may appear rather uninteresting. One may think that indifference can be replaced by a condition that says that the entry cost must be in between the equilibrium seller profits and the profits in case an extra seller enters. This is not the case. The reason is that the profit that a firm makes when m other firms enter is higher when the firm enters the market unexpectedly compared to the case when the other m firms expect the firm to enter. For this reason, equilibria with pure entry strategies will generally fail to exist.

In this paper I consider two variants of the simplest possible directed search model with homogenous buyers and sellers. In both cases, buyers are drawn from a Poisson distribution, but I have results for both the case with a fixed set of sellers and with Poisson distributed sellers.

The analysis with a fixed set of sellers contains no surprising results. Using arguments that are more or less identical with those in Galenianos and Kircher (2012) one shows strict concavity of the profit function in the relevant range. Also, again reproducing a known result for the case with a fixed set of buyers and sellers, I show that there is a unique equilibrium.[4] While this is not a new result in itself, the proof is different and, arguably, more instructive. The argument relies on two properties, symmetry and prices being strategic complements. That is, I first show the best response is strictly increasing in the price posted by any other seller that attracts buyers with

[2] See also Peters (1984), Julien et al (2000), and Lester (2011).

[3] The one exception I am aware of is Geromichalos (2012).

[4] See Galenianos and Kircher (2012) and Kim and Camera (2014).

positive probability. This strategic complementarity together with symmetry immediately rules out any asymmetric equilibrium as the best response of the seller posting the higher price is lower, which is a contradiction. It follows from a direct calculation that there exists a unique symmetric equilibrium, and combined with the non-existence of asymmetric equilibria, this is indeed the unique equilibrium of the model.

Adding entry to the model I first demonstrate that pure strategy equilibria generally fail to exist in the model with a given set of potential buyers. Presumably mixed strategy equilibria exist, but these are not easy to characterize. In contrast, when sellers are distributed in accordance with a Poisson distribution an essentially unique equilibrium is remarkably simple to characterize. In equilibrium, all sellers that enter the market post the same price. The price is strictly decreasing in the entry probability, and the equilibrium is essentially unique in that the average entry probability is determined uniquely by a zero profit condition. In the limit as the number of buyers and sellers go to infinity, the equilibrium approaches the market utility benchmark.

2 The Model

The model is a variant of Burdett et al (2001) where buyers and (potential) sellers are drawn from a Poisson distribution, and where sellers face a non-trivial entry decision. As moves are sequential it is not strictly speaking a Poisson game in the sense of Myerson (2000), but the analysis is very similar.[5]

There are s sellers drawn from a Poisson distribution with expected value m and b buyers drawn from a Poisson distribution with expected value n. All buyers value the object at 1 and all sellers value the object at some $K \geq 0$ which is common knowledge. An alternative interpretation is that K is the cost of entering the marketplace.

The extensive form is as follows:

1. First, nature draws s potential sellers and b buyers. To allow for coordination we also assume that each seller and buyer is equipped with an observable characteristic. Formally, we let $t \in T = \{1, ..., T\}$ be the type of a seller and $c \in C = \{1, ..., C\}$. Type t sellers are drawn from a Poisson with expected value m_t and type c buyers are drawn from a Poisson with expected value n_c where $m = \sum_{t \in T} m_t$ and $n = \sum_{c \in C} n_c$ are the expected number of buyers and sellers. Types are payoff irrelevant in the baseline model as all sellers of any type attach value $K \geq 0$ to it and all buyers of any type value the good at 1.

2. Each potential seller decides whether or not to enter the market. Conditional on entry sellers post prices and sellers don't observe how many other sellers or which types are realized when taking their actions. A seller strategy is thus a pair (e, p) where $e : T \to [0, 1]$ maps the type of player to a probability of entry and $p : T \to R_+$ is the pricing strategy.

[5] In directed search models, buyers react to the posted prices. Hence, buyers must be able to ex post distinguish between different sellers.

3. Buyers observe how many sellers of each type are realized and the posted and decide which seller to visit. That is, a history is a vector $((m_1, p_1), .., (m_T, p_T)) \in (N \times R_+)^T$ where N is the set of non-negative integers. Hence, a strategy is a map $\boldsymbol{\theta} : C \times (N \times R_+)^T \to \Delta^{m_1+m_2+.....+m_T+1}$, where we note that the assumption is that buyers can distinguish between different sellers of the same type. The reason that it is the $m_1+m_2+.....+m_T+1$ dimensional simplex and not $m_1+m_2+.....+m_T$ is that we allow the buyer to visit nobody. If the buyers ignore the type of the player (or if there is a fixed known set of players, which is a case that will be considered below) a strategy can be written $\boldsymbol{\theta} : [0,1]^m \to \Delta^{m+1}$.

4. If at least one buyer visits a seller one of the visiting buyers is chosen to receive the good at the posted price.

As buyers don't know how many other buyers there are when deciding which seller to visit the second stage is not a proper subgame, so subgame perfection is formally not applicable. However, the solution algoritm is still just like how one would solve for a subgame perfect equilibrium. After any set of posted prices the buyers must optimize and, while moves of nature are involved, the relevant beliefs over the number of buyers are given by the prior. Clearly, it is possible for a firm to post a price high enough so that no buyer shows up in equilibrium, implying that the arrival of a buyer is off the equilibrium. However, this is irrelevant as there are no moves left for the firms, implying that an equilibrium (say weak perfect Bayesian) will be characterized by backwards induction.

3 The Market Utility Benchmark

Consider a "competitive" version of the model where we imagine a continuum of buyers and sellers. Then, in equilibrium, any buyer must be indifferent between all sellers, so we let U be the utility for a buyer that visits any active seller. Also assume that the ratio of buyers to potential sellers is given by ρ and that a proportion α of the potential sellers that are active. The ratio of buyers to active sellers is then $\frac{\rho}{\alpha}$. Also, assume that a single seller posts price p and let $\lambda(p, U)$ be the Poisson arrival at this firm as a function of it's posted price and the utility of visiting any other active seller.

Note that if the Poisson arrival probability of buyers at a seller is λ and that the seller randomizes with equal probabilities when picking a buyer , then the probability that a buyer is successful when visiting this seller is $\frac{(1-e^{-\lambda})}{\lambda}$.[6] It follows that the condition that buyers are indifferent between visiting the firm posting price p and any other firm is

$$(1-p)\frac{\left(1-e^{-\lambda(p,U)}\right)}{\lambda(p,U)} = U. \tag{1}$$

[6] This is a well-known property of the Poisson distribution, but a direct calculation is provided below in (5) in the context of the finite model in order to make the paper self contained.

A representative seller will therefore solve the problem $\max_p p\left(1-e^{-\lambda(p,U)}\right)$, where $\lambda(p,U)$ is defined in (1). However, noting that $\frac{1-e^{-x}}{x}$ is strictly decreasing in x we may change variable from p to λ. That is, (suppressing arguments and) solving (1) we have that $p = 1 - \frac{\lambda U}{(1-e^{-\lambda})}$ implying that the problem for an individual seller is

$$\max_{\lambda}\left(1-\frac{\lambda U}{(1-e^{-\lambda})}\right)\left(1-e^{-\lambda}\right) = \max_{\lambda} 1-e^{-\lambda}-\lambda U. \tag{2}$$

The problem in (2) is a strictly concave problem and by taking the first order condition and substituting the condition $U = (1-p)\frac{1-e^{-\lambda}}{\lambda}$ which must hold in a symmetric equilibrium we have that the price must be

$$p(\lambda) = \frac{1-e^{-\lambda}(1+\lambda)}{1-e^{-\lambda}} = 1-\frac{\lambda}{e^{\lambda}-1}. \tag{3}$$

Substituting the price in (3) back into the objective function one finds that the maximized equilibrium profit is $1-e^{-\lambda}(1+\lambda)$, which is strictly increasing in the Poisson arrival parameter λ. To make the entry decision non-trivial, I assume that $1-e^{-\rho}(1+\rho) < K < 1$. This means that the cost of entry is somewhere in between the profit sellers make if all sellers are active and the limiting profit for the case when sellers match with a buyer for sure. Equilibrium entry must then be such that all sellers are indifferent between staying outside the market and being active. That is, the probability of entry, α^*, must solve

$$1-e^{-\frac{\rho}{\alpha^*}}\left(1+\frac{\rho}{\alpha^*}\right) = K, \tag{4}$$

which has a unique solution. We conclude that $\lambda = \frac{\rho}{\alpha^*}$ in order to make active and passive sellers indifferent, and the associated equilibrium price is given by (3) evaluated at $\frac{\rho}{\alpha^*}$

The procedure described in this Section is a simple example of a market utility model with entry. Clearly, there are several somewhat ad hoc steps in the analysis, and the rest of the paper is concerned with the question of whether this analysis can be viewed as a limit of a model with finite numbers of sellers and buyers.

4 The Case with a Deterministic Set of Sellers

As a first step in the analysis I consider the case with m sellers who assign a zero reservation value of the product. The reason for this is twofold. Firstly, it allows me to identify why handling entry is difficult with a known finite set of firms. Secondly, I show that all equilibria are payoff equivalent with an equilibrium in which all buyers follow the same mixed strategy, which is sometimes referred to as a "directed search equilibrium". As a by-product, I also show that prices are strategic complements which allows me to construct a proof of uniqueness of equilibria which is considerable simpler than existing arguments.

4.1 Buyer Payoffs with Symmetric Buyer Strategies

If the number of buyers is a Poisson with expected value n and all buyers go to seller i with probability θ_i then the number of buyers at seller i is a Poisson with expected value $n\theta_i$. This is a well known property of the Poisson distribution, but for the convenience of the reader I demonstrate this in the appendix.

Next, a Poisson setup exhibits what Myerson (2000) labels *environmental equivalence*.[7] That is, if the unconditional probability distribution over the number of buyers at seller i is a Poisson with parameter $n\theta_i$, then, from the point of view of an individual buyer, the probability distribution of *the number of other buyers at seller* i is also a Poisson with parameter $n\theta_i$. Hence, "being born" contains no information over how many other agents are realized. It follows that the probability that a buyer is successful and acquires the good when visiting seller i is

$$\begin{aligned}\Pr\left[\text{buying from seller } i\right] &= \sum_{v=0}^{\infty}\frac{1}{v+1}\underbrace{\frac{e^{-n\theta_i}}{v!}\left(n\theta_i\right)^v}_{\text{Prob } v \text{ other buyers}} = \frac{1}{n\theta_i}\sum_{v=0}^{\infty}\frac{e^{-n\theta_i}}{(v+1)!}\left[n\theta_i\right]^{v+1} \\ &= \frac{1}{n\theta_i}\left[\sum_{v=0}^{\infty}\frac{e^{-n\theta_i}}{v!}\left[n\theta_i\right]^v - \frac{e^{-n\theta_i}}{0!}\right] = \frac{\left(1-e^{-n\theta_i}\right)}{n\theta_i}.\end{aligned} \tag{5}$$

I write $\theta_i\left(\mathbf{p}\right)$ for the probability to visit firm i as a function of the price vector $\mathbf{p},$ which so far is restricted to be the same for all buyers. To simplify notation slightly, I define $\lambda_i\left(\mathbf{p}\right) = n\theta_i\left(\mathbf{p}\right)$ for each seller i. As choosing $\boldsymbol{\lambda}\left(\mathbf{p}\right) = \left(\lambda_1\left(\mathbf{p}\right), ..., \lambda_n\left(\mathbf{p}\right)\right)$ such that $\sum_i \lambda_i\left(\mathbf{p}\right) = n$ is a just change in variables from choosing a mixed strategy it is irrelevant whether we view $\boldsymbol{\lambda}$ or $\boldsymbol{\theta}$ as the strategic variable. I will use this change of variables for the remainder of the paper.

When considering an arbitrary $\mathbf{p}$ and seller i there are two possibilities. Either no buyer visits seller i in which case getting the good at price p_i is weakly worse than visiting any other seller. The other possibility is that the arrival rate is positive in which case the buyer must be indifferent between firm i and any other active seller, that is

$$\frac{\left(1-p_i\right)\left(1-e^{-\lambda_i(\mathbf{p})}\right)}{\lambda_i\left(\mathbf{p}\right)} = \frac{\left(1-p_j\right)\left(1-e^{-\lambda_j(\mathbf{p})}\right)}{\lambda_j\left(\mathbf{p}\right)}, \tag{6}$$

must hold for each (i, j) such that $\lambda_i\left(\mathbf{p}\right) > 0$ and $\lambda_j\left(\mathbf{p}\right) > 0$.

4.2 The Poisson Assumption Justifies Lack of Coordination

In directed search models with a finite set of agents there is a plethora of equilibria that can be supported by buyers coordinating on qualitatively different continuation equilibria after different price vector posted by the sellers. Almost all of the literature, however, focus on the case where buyers play a symmetric mixed strategy after any history of prices posted. This approach is typically justified by arguing that coordination is hard in large anonymous markets.[8] In contrast, the setup with Poisson distributed buyers is an environment in which there is lack of coordination an any equilibrium.

[7] This property is also plays a role in Lester et.al. (2014) where it is called independence.

[8] See Bland and Loertscher (2012) for a more formal justification.

Proposition 1 *Suppose that* $\widehat{\boldsymbol{\theta}} : C \times R_+^m \to \Delta^m$ *is an equilibrium continuation strategy for the buyers.*[9] *Then there exists a symmetric equilibrium continuation strategy* $\boldsymbol{\theta} : R_+^m \to \Delta^m$ *such that all buyers and sellers earn the same payoffs as when buyers are following* $\widehat{\boldsymbol{\theta}}$.

Proof. Let $\widehat{\theta}_{i1}$ be the probability that type 1 visits i and $\widehat{\theta}_{i2}$ be the probability that type 2 visits seller i. Since buyers of the two types are drawn independently the probability that v buyers visit seller i is

$$\begin{aligned}\sum_{k=0}^{v} \frac{e^{-\left(\widehat{\theta}_{i1}n_1\right)}\left(\widehat{\theta}_{i1}n_1\right)^k}{k!}\frac{e^{-\left(\widehat{\theta}_{i2}n_2\right)}\left(\widehat{\theta}_{i2}n_2\right)^{v-k}}{(v-k)!} &= \frac{e^{-\left(\widehat{\theta}_{i1}n_1+\widehat{\theta}_{i2}n_2\right)}}{v!}\sum_{k=0}^{v}\frac{v!}{k!\,(v-k)!}\left(\widehat{\theta}_{ic}n_1\right)^k\left(\widehat{\theta}_{i2}n_2\right)^{v-k} \\ &= \frac{e^{-\left(\widehat{\theta}_{i1}n_1+\widehat{\theta}_{i2}n_2\right)}}{v!}\left(\widehat{\theta}_{ic}n_1+\widehat{\theta}_{i2}n_2\right)^v, \end{aligned} \tag{7}$$

by use of the binomial theorem. We conclude that the number of sellers that are either type 1 or 2 is a Poisson distribution with parameter $\widehat{\theta}_{ic}n_1+\widehat{\theta}_{i2}n_2$. By induction it follows that the probability that v buyers from C visit i is a Poisson with expected queue length $\sum_{c\in C}\widehat{\theta}_{ic}n_c$. For every price vector $\mathbf{p}$ and every seller i let $\theta_i(\mathbf{p})$ be defined as

$$\theta_i(\mathbf{p}) = \frac{\sum_{c\in C}\widehat{\theta}_{ic}(\mathbf{p})\,n_c}{\sum_{c\in C}n_c}. \tag{8}$$

If each buyer of any type follows $\theta_i(\mathbf{p})$ the expected queue length is $\theta_i(\mathbf{p})\sum_{c\in C}n_c = \sum_{c\in C}\widehat{\theta}_{ic}(\mathbf{p})\,n_c$. Hence, the payoffs for all buyers and sellers are unchanged, implying that the symmetric randomization is also an equilibrium. ■

Hence, neither buyers or sellers can do any better or worse by conditioning on payoff irrelevant buyer characteristics. I will therefore assume that all buyers follow a symmetric mixed strategy in the remainder of the paper.

4.3 Seller Payoffs

Proposition 1 establishes that for any price vector $\mathbf{p}$ we may restrict attention to a symmetric continuation equilibrium in which the arrival rate at firm i is some $\lambda_i(\mathbf{p})$. A seller earns p_i provided that at least one buyer contacts the seller. The probability that no buyer visits is $e^{-\lambda_i(\mathbf{p})}$, so a sale is consummated with probability $1-e^{-\lambda_i(\mathbf{p})}$. The payoff function for a seller, taking sequentially rational continuation strategies by the buyers into consideration, is thus on the form

$$\pi_i(\mathbf{p}) = p_i\left(1-e^{-\lambda_i(\mathbf{p})}\right), \tag{9}$$

exactly like the market utility benchmark.

The difference with the market utility model is that the equilibrium buyer utility will be affected by prices set by individual sellers. However, the form of (9) is nevertheless making the model more tractable than the standard setting with a given finite set of buyers.

[9] We have removed seller types.The argument trivially generalizes to the case with random sellers with different observables. The reason is that we assume that buyers can distinguish different sellers of the same type, which implies that the type is redundant at this stage.

4.4 Equilibrium

While there are formally no proper subgames we can define an equilibrium in analogy with subgame perfection:

Definition 1 *The pair* $(\boldsymbol{\lambda}^*, \boldsymbol{p}^*)$ *with* $\boldsymbol{\lambda}^* : [0,1]^m \to \left\{x \in R_+^n \middle| \sum_{i=1}^m x_i = n\right\}$ *and* $\boldsymbol{p}^* \in [0,1]^m$ *is an equilibrium if:*

1. *The indifference condition (6) holds each* $\mathbf{p}$ *and each* (i,j) *such that* $\lambda_i^*(\mathbf{p}) > 0$ *and* $\lambda_j^*(\mathbf{p}) > 0$*. Moreover* $\lambda_i^*(\mathbf{p}) = 0$ *holds if there exists* $j \neq i$ *such that*
$$1 - p_i \leq \frac{(1-p_j)\left(1 - e^{-\lambda_j(\mathbf{p})}\right)}{\lambda_j(\mathbf{p})}.$$

2. $p_i^* \in \arg\max_{p_i} p_i \left(1 - e^{-\lambda_i^*(p_i, \mathbf{p}_{-i}^*)}\right)$.

Condition 1 says that all buyers must follow a (common) sequentially rational rule to select which seller to visit after every conceivable price vector. This generates a sequentially rational expected arrival function, and the second condition says that each seller must maximize profits, taking prices by other sellers and the sequentially rational demand as given.

4.5 Equilibrium Characterization

4.5.1 Concavity of Seller's Profit Function

The typical approach to characterize equilibria in directed search models is to focus on symmetric equilibria. This allows for a change in variable from price to the Poisson arrival rate when solving the seller profit maximization problem, following exactly the same logic as the derivation of the payoff function (2) in the market utility benchmark. As I allow for asymmetric pricing strategies this approach does no longer work. Prices are therefore kept as the strategic variables for the firms.

To simplify notation I define
$$g(x) = \frac{1 - e^{-x}}{x}, \tag{10}$$
and rewrtie the indifference conditions (6) as
$$(1 - p_i)\, g(\lambda_i(\mathbf{p})) - (1 - p_j)\, g(\lambda_j(\mathbf{p})) = 0. \tag{11}$$
One should note that $\lambda_i(\mathbf{p}) = 0$ for p_i that are too close to 1. Specifically, if $p_i \geq \overline{p}_i(\mathbf{p}_{-i})$ defined as the unique solution to
$$(1 - p_i) - (1 - p_j)\, g(\lambda_j(\mathbf{p}_{-i})) = 0, \tag{12}$$
then $\lambda_i(\mathbf{p}) = 0$. This creates a non-concavity of the seller profit function that is discussed in further detail by Galenianos and Kircher (2012). Because of the simple structure of payoffs in this paper,

the non-concavity does not create any analytical difficulties as it occurs in a range where no firm likes to be.

For each firm i there are $m-1$ indifference conditions and, as all remaining indifference conditions are satisfied provided that a buyer is indifferent between i and any other seller we may simply ignore all other conditions. It is easy to use the implicit function theorem to check that $\lambda_i(\cdot)$ is continuously twice differentiable in all arguments (explicit expression for the derivatives are given in (15) and (16)) so we may differentiate the profit function in (9) with respect to p_i to obtain the first and second derivative

$$\begin{aligned}\frac{\partial \pi_i(\mathbf{p})}{\partial p_i} &= \left(1-e^{-\lambda_i(\mathbf{p})}\right)+p_i e^{-\lambda_i(\mathbf{p})}\frac{\partial \lambda_i(\mathbf{p})}{\partial p_i} \\ \frac{\partial^2 \pi_i(\mathbf{p})}{\partial p_i^2} &= 2e^{-\lambda_i(\mathbf{p})}\frac{\partial \lambda_i(\mathbf{p})}{\partial p_i}-p_i e^{-\lambda_i(\mathbf{p})}\left[\frac{\partial \lambda_i(\mathbf{p})}{\partial p_i}\right]^2+p_i e^{-\lambda_i(\mathbf{p})}\frac{\partial^2 \lambda_i(\mathbf{p})}{\partial p_i^2}.\end{aligned} \tag{13}$$

Given that all sellers are active (which is the case in equilibrium) we have that the derivatives of $(\lambda_1(\mathbf{p}),....,\lambda_m(\mathbf{p}))$ are implicitly defined by the $m-1$ indifference conditions in (11). Differentiating the identities in (11) with respect to p_i we get $m-1$ distinct equalities

$$-g(\lambda_i(\mathbf{p}))+(1-p_i)g'(\lambda_i(\mathbf{p}))\frac{\partial \lambda_i(\mathbf{p})}{\partial p_i}=(1-p_j)g'(\lambda_j(\mathbf{p}))\frac{\partial \lambda_j(\mathbf{p})}{\partial p_i}. \tag{14}$$

Rearranging and summing over the the conditions in (14) we express the partial derivative of the expected number of arriving buyers with respect to p_i as

$$\frac{\partial \lambda_i(\mathbf{p})}{\partial p_i}=\frac{g(\lambda_i(\mathbf{p}))}{(1-p_i)g'(\lambda_i(\mathbf{p}))}\frac{\sum_{j\neq i}\frac{g(\lambda_j(\mathbf{p}))}{g'(\lambda_j(\mathbf{p}))}}{\sum_{j=1}^m \frac{g(\lambda_i(\mathbf{p}))}{g'(\lambda_j(\mathbf{p}))}}. \tag{15}$$

and that the cross derivatives are given by

$$\begin{aligned}\frac{\partial \lambda_j(\mathbf{p})}{\partial p_i} &= -\frac{g(\lambda_i(\mathbf{p}))}{(1-p_i)g'(\lambda_i(\mathbf{p}))}\frac{\frac{g(\lambda_j(\mathbf{p}))}{g'(\lambda_j(\mathbf{p}))}}{\sum_{k=1}^m \frac{g(\lambda_k(\mathbf{p}))}{g'(\lambda_k(\mathbf{p}))}} \\ &= -\frac{\partial \lambda_i(\mathbf{p})}{\partial p_i}\frac{\frac{g(\lambda_j(\mathbf{p}))}{g'(\lambda_j(\mathbf{p}))}}{\sum_{k\neq i}\frac{g(\lambda_k(\mathbf{p}))}{g'(\lambda_k(\mathbf{p}))}}\end{aligned} \tag{16}$$

It is easy to check that $\frac{\partial \lambda_i(\mathbf{p})}{\partial p_i}<0$ and $\frac{\partial \lambda_j(\mathbf{p})}{\partial p_i}>0$ for every $j\neq i$. This is also easy to understand, as an increase in the price charged by seller i reduces the expected payoff for a buyer who visits seller i. To restore indifference the probability that a buyer is served must increase, and to keep indifference across all other sellers the probability that a buyer is served must all move in the same direction. Taken together, the only possibility is that $\lambda_i(\mathbf{p})$ decreases and $\lambda_j(\mathbf{p})$ increases for every $j\neq i$ as p_i is increased and all other prices are held fix.

It involves more work to evaluate the second derivative of the profit function. To do so it turns out that the following fact is very helpful.

Lemma 1 *For any $x>0$ we have that $g'(x)^2<g(x)g''(x)<2[g'(x)]^2$.*

For completeness, a direct calculation that establishes Lemma 1 is in the appendix, but it is easy to check the result by simply plotting the functions.

Using the two inequalities in Lemma 1 it is easy to show that $\lambda_i(\mathbf{p})$ is strictly concave in p_i, which in turn implies that the payoff function is globally concave in p_i in the range where $\lambda_i(\mathbf{p}) = 0$. This should be of no surprise given Lemma 3 in Galenianos and Kircher (2012), but since the matching function is not exactly the standard urn-ball matching, a proof is required.

Proposition 2 $\frac{\partial^2 \lambda_i(\mathbf{p})}{\partial p_i^2} < 0$ *for every seller* i *and every* $\mathbf{p} \in [0,1]^m$ *such that all firms are visited with positive probability. Hence, the profit function for firm* i *is globally strictly concave in* p_i *on* $[0, \overline{p}_i(\mathbf{p}_{-i})]$.

Obviously, the non-concavity is not relevant if all sellers use pure strategies. In that case we can simply rule out $p_i > \overline{p}_i(\mathbf{p}_{-i})$ as attracting no buyers is the worst possible outcome. Indeed, using the symmetry of the model we can show that any equilibrium must be in pure strategies, implying that the non-concavity at the upper end is irrelevant for the analysis:

Proposition 3 *There can be no mixed strategy equilibrium*

Proof. For each player i let $\underline{p}_1$ be the lower bound on the support of the mixed strategy used by player i in equilibrium. Also, relabel the players so that $\underline{p}_1 \leq \underline{p}_2 \leq .... \leq \underline{p}_m$. First consider seller 1 and note that

$$\underline{p}_1 \in \arg\max \int_{\mathbf{p}_{-1}} \pi_i(p_1, \mathbf{p}_{-1}) \prod_{j=2}^{m} dF(p_j)$$

where the cumulative distribution $F(p_j)$ is the (possibly) mixed strategy played by j. Obviously $\underline{p}_1 < \overline{p}_1(\mathbf{p}_{-1})$ for any $\mathbf{p}_{-1}$ in the support of the (possibly) mixed strategy equilibrium. As 1) sums of strictly concave functions are strictly concave; 2) each $\pi_1(p_1, \mathbf{p}_{-1})$ is strictly concave on $[0, \inf \overline{p}_1(\mathbf{p}_{-1})] > \underline{p}_1$, and 3) $\pi_1\left(\underline{p}_1, \mathbf{p}_{-1}\right) > \pi_1(p_1, \mathbf{p}_{-1}) = 0$ if $p_1 \geq \overline{p}_1(\mathbf{p}_{-1})$. Taken together this implies that $\int_{\mathbf{p}_{-1}} \pi_i\left(\underline{p}_1, \mathbf{p}_{-1}\right) \prod_{j=2}^{m} dF(p_j) > \int_{\mathbf{p}_{-1}} \pi_i(p_1, \mathbf{p}_{-1}) \prod_{j=2}^{m} dF(p_j)$ for any $p_1 \neq \underline{p}_1$ so player 1 must play a pure strategy. For induction, assume that players $1, ..., m$ play pure strategies. Then, $\underline{p}_i < \overline{p}_i(\mathbf{p}_{-i})$ for any $\mathbf{p}_{-i}$ in the support of the equilibrium. Using the same argument as above this implies that player i must play a pure strategy. By induction, all players must play pure strategies. ■

4.5.2 Symmetric Equilibria

The usual approach to derive a symmetric equilibrium is to change the strategic variable from p_i to λ_i and note that if all other sellers post the same price, then it must be that the arrival probability is $\frac{n-\lambda_i}{m-1}$ for any $j \neq i$. That is, the indifference condition (11) implies that p_i can be expressed in terms of λ_i and p, the price charged by all other sellers as

$$p_i = 1 - (1-p) \frac{\lambda_i}{1 - e^{-\lambda_i}} \frac{\left(1 - e^{-\frac{n-\lambda_i}{m-1}}\right)(m-1)}{n - \lambda_i}. \tag{17}$$

Eliminating p_i from the objective function in (9) we obtain the reduced form optimization problem in λ_i given by

$$\max_{\lambda_i}\left[1-e^{-\lambda_i}-(1-p)\frac{\lambda_i\left(1-e^{-\frac{n-\lambda_i}{m-1}}\right)}{\frac{n-\lambda_i}{m-1}}\right]. \quad (18)$$

One can solve (18) and then impose the equilibrium condition $\lambda_i=\frac{n}{m}$. The result is the exact same candidate equilibrium price as what I derive below. However, the symmetric equilibrium is as easily derived by directly maximizing (9) over p_i. By strict concavity (and the fact that corners can be ruled out) any equilibrium must satisfy the first order condition

$$0=1-e^{-\lambda_i(\mathbf{p})}+p_i e^{-\lambda_i(\mathbf{p})}\frac{\partial\lambda_i(\mathbf{p})}{\partial p_i}. \quad (19)$$

If $p_j=p^*$ for all j it follows that $\lambda_i(\mathbf{p})=\frac{n}{m}$ and

$$\frac{\partial\lambda_i(\mathbf{p})}{\partial p_i}=\frac{g\left(\frac{n}{m}\right)}{(1-p_i)\,g'\left(\frac{n}{m}\right)}\frac{\sum_{j\neq i}\frac{g\left(\frac{n}{m}\right)}{g'\left(\frac{n}{m}\right)}}{\sum_{j=1}^{m}\frac{g\left(\frac{n}{m}\right)}{g'\left(\frac{n}{m}\right)}}=-\frac{\frac{1-e^{-\frac{n}{m}}}{\frac{n}{m}}}{(1-p_i)\frac{1-e^{-\frac{n}{m}}\left(1+\frac{n}{m}\right)}{\left(\frac{n}{m}\right)^2}}\frac{m-1}{m}, \quad (20)$$

and substituting into (19) and rearranging gives

$$p^*=\frac{1-e^{-\frac{n}{m}}\left(1+\frac{n}{m}\right)}{1-e^{-\frac{n}{m}}\left[1+\frac{n}{m^2}\right]}\rightarrow\frac{1-e^{-\rho}(1+\rho)}{\left[1-e^{-\rho}\right]}=1-\frac{\rho}{e^{\rho}-1} \quad (21)$$

as $n,m\rightarrow\infty$ and $\frac{n}{m}\rightarrow\rho$. The reader should note that Proposition 2 guarantees that $\mathbf{p}^*=(p^*,...,p^*)$ is a symmetric equilibrium. Additionally, using the fact that corners can be ruled out, (19) is necessary, so there can be no other equilibria. Hence, since p^* is the unique price such that (19) holds when $p_j=p^*$ for all j we can conclude that there exists a unique symmetric equilibrium in the model where each seller posts price p^*.

4.5.3 Prices are Strategic Complements

A price $p_i=0$ can be ruled out as the profit is 0 and that a slightly higher price must attract buyers with positive probability. A price p_i such that $\lambda_i(p_i,\mathbf{p}_{-i})=0$ can be ruled out as there must be some sellers attracting customers, implying that mimicking such a seller will earn a positive profit (unless all sellers charge 0 in which case a slightly higher price will earn a profit)..

As the seller profit function is strictly concave in the relevant range, the unique best response to any $\mathbf{p}_{-1}\in[0,1]^{m-1}$, denoted $\beta_i(\mathbf{p}_{-i})$, must satisfy the first order condition.

$$0=\frac{\partial\pi_i(\mathbf{p}_{-i},\beta_i(\mathbf{p}_{-i}))}{\partial p_i}=1-e^{-\lambda_i(\mathbf{p}_{-i},\beta_i(\mathbf{p}_{-i}))}+\beta_i(\mathbf{p}_{-i})\,e^{-\lambda_i(\mathbf{p}_{-i},\beta_i(\mathbf{p}_{-i}))}\frac{\partial\lambda_i(\mathbf{p}_{-i},\beta_i(\mathbf{p}_{-i}))}{\partial p_i} \quad (22)$$

Differentiating the first order condition for optimality with respect to $p_k\neq p_i$ we have that

$$\frac{\partial^2\pi_i(\mathbf{p}_{-i},\beta_i(\mathbf{p}_{-i}))}{\partial p_i^2}\frac{\partial\beta_i(\mathbf{p}_{-i})}{\partial p_k}+\frac{\partial^2\pi_i(\mathbf{p}_{-i},\beta_i(\mathbf{p}_{-i}))}{\partial p_i\partial p_k}=0, \quad (23)$$

where $\frac{\partial^2 \pi_i(\mathbf{p}_{-i},\beta_i(\mathbf{p}_{-i}))}{\partial p_i^2} < 0$ because the profit function is strictly concave in p_i. Hence, in order to sign the effect on p_i from changing $p_k \neq p_i$ we need to be able to sign

$$\begin{aligned}\frac{\partial^2 \pi_i\left(\mathbf{p}_{-i}, \beta_i\left(\mathbf{p}_{-i}\right)\right)}{\partial p_i \partial p_k} &= e^{-\lambda_i(p)}\left(1-\beta_i\left(\mathbf{p}_{-i}\right) \frac{\partial \lambda_i\left(\mathbf{p}_{-i}, \beta_i\left(\mathbf{p}_{-i}\right)\right)}{\partial p_i}\right) \frac{\partial \lambda_i\left(\mathbf{p}_{-i}, \beta_i\left(\mathbf{p}_{-i}\right)\right)}{\partial p_k} \qquad (24)\\ &+\beta_i\left(\mathbf{p}_{-i}\right) e^{-\lambda_i\left(\mathbf{p}_{-i}, \beta_i\left(\mathbf{p}_{-i}\right)\right)} \frac{\partial \lambda_i^2\left(\mathbf{p}_{-i}, \beta_i\left(\mathbf{p}_{-i}\right)\right)}{\partial p_i \partial p_k}.\end{aligned}$$

The first term in (24) simplifies as we only need to evaluate at a best response $p_i = \beta_i(p)$, so by using the first order condition (22) we have that

$$\begin{aligned}& e^{-\lambda_i\left(\mathbf{p}_{-i}, \beta_i\left(\mathbf{p}_{-i}\right)\right)}\left(1-\beta_i\left(\mathbf{p}_{-i}\right) \frac{\partial \lambda_i\left(\mathbf{p}_{-i}, \beta_i\left(\mathbf{p}_{-i}\right)\right)}{\partial p_i}\right) \frac{\partial \lambda_i\left(\mathbf{p}_{-i}, \beta_i\left(\mathbf{p}_{-i}\right)\right)}{\partial p_k} \qquad (25)\\ =\ & e^{-\lambda_i\left(\mathbf{p}_{-i}, \beta_i\left(\mathbf{p}_{-i}\right)\right)}\left(1-\frac{1-e^{-\lambda_i\left(\mathbf{p}_{-i}, \beta_i\left(\mathbf{p}_{-i}\right)\right)}}{e^{-\lambda_i\left(\mathbf{p}_{-i}, \beta_i\left(\mathbf{p}_{-i}\right)\right)}}\right) \frac{\partial \lambda_i\left(\mathbf{p}_{-i}, \beta_i\left(\mathbf{p}_{-i}\right)\right)}{\partial p_k}=\frac{\partial \lambda_i\left(\mathbf{p}_{-i}, \beta_i\left(\mathbf{p}_{-i}\right)\right)}{\partial p_k}>0 \qquad (26)\end{aligned}$$

Hence, the first term in (24) can be signed with no effort at all. Unfortunately, the same is not true for the second term in (24).Differentiating (15) with respect to p_k one obtains

$$\begin{aligned}\frac{\partial^2 \lambda_i(\mathbf{p})}{\partial p_i \partial p_k} &= \frac{1}{1-p_i}\left[\frac{\left[g_i'\left(\lambda_i(\mathbf{p})\right)\right]^2-g(\lambda_i(\mathbf{p})) g''\left(\lambda_i(\mathbf{p})\right)}{\left[g_i'\left(\lambda_i(\mathbf{p})\right)\right]^2}\right] \frac{\sum_{j \neq i} \frac{g(\lambda_j(\mathbf{p}))}{g'(\lambda_j(\mathbf{p}))}}{\sum_{j=1}^m \frac{g(\lambda_i(\mathbf{p}))}{g'(\lambda_j(\mathbf{p}))}} \frac{\partial \lambda_i(\boldsymbol{p})}{\partial p_k} \qquad (27)\\ &+\frac{g\left(\lambda_i(\mathbf{p})\right)}{\left(1-p_i\right) g'\left(\lambda_i(\mathbf{p})\right)} \frac{d}{d p_k}\left[\frac{\sum_{j \neq i} \frac{g(\lambda_j(\mathbf{p}))}{g'(\lambda_j(\mathbf{p}))}}{\sum_{j=1}^m \frac{g(\lambda_i(\mathbf{p}))}{g'(\lambda_j(\mathbf{p}))}}\right],\end{aligned}$$

where one can show that the first term is negative, whereas the second term is a rather complicated object that is hard to sign for general choices of p_{-i}. However, in the appendix we establish that the terms in (27) are small enough so that they are dominated by the direct effect on the arrival rates from a change in p_k, implying that $\frac{\partial^2 \pi_i(\mathbf{p}_{-i},\beta_i(\mathbf{p}_{-i}))}{\partial p_i \partial p_k}$ is strictly positive. The critical step in this argument is to use the buyer indifference conditions to show that the second term in (27) is not too large in absolute value. That is, I show that:

Lemma 2 *For any* $\mathbf{p}$ *such that* $\lambda_k(\mathbf{p}) > 0$ *we have that* $\frac{d}{dp_k}\left[\frac{\frac{g(\lambda_i(\mathbf{p}))}{g'(\lambda_i(\mathbf{p}))}}{\sum_{j=1}^m \frac{g(\lambda_i(\mathbf{p}))}{g'\left(\lambda_j(\mathbf{p})\right)}}\right] > \frac{1}{\sum_{j=1}^m \frac{g\left(\lambda_j(\mathbf{p})\right)}{g'\left(\lambda_j(\mathbf{p})\right)}} \frac{\partial \lambda_i(\mathbf{p})}{\partial p_k}$

The key steps in the argument uses Lemma 1 and the relation between the first derivatives and cross-derivatives in (16). Combining Lemma 2, which holds regardless of whether prices are consistent with equilibrium or not, with the optimality conditions we can prove that prices are strategic complements.

Proposition 4 *Pick any* $\mathbf{p}$ *such that* $\lambda_k(\mathbf{p}) > 0$ *and* $p_i = \beta_i(\mathbf{p}_{-i})$ *is an optimal response to* $\mathbf{p}_{-i}$*. Then* $\frac{\partial^2 \pi_i(\mathbf{p}_{-i},\beta_i(\mathbf{p}_{-i}))}{\partial p_i \partial p_k} > 0$*, implying that*

$$\frac{\partial \beta_i\left(\mathbf{p}_{-i}\right)}{\partial p_k}=-\frac{\frac{\partial^2 \pi_i\left(\mathbf{p}_{-i}, \beta_i\left(\mathbf{p}_{-i}\right)\right)}{\partial p_i \partial p_k}}{\frac{\partial^2 \pi_i\left(\mathbf{p}_{-i}, \beta_i\left(\mathbf{p}_{-i}\right)\right)}{\partial p_i^2}}>0.$$

The proof can be found in the appendix.

Using symmetry and the fact that prices are strategic complements we can rule out any asymmetric equilibria. The reasoning is simple. If seller i posts a lower price than seller j then seller i faces a distribution of competing prices that is identical to the one for seller j except that p_j is replaced with p_i in the best response problem for firm j. As $p_i < p_j$ and the best reply is increasing in the price of any competitor it follows that firm j should optimally post a lower price, which is a contradiction. It has already been shown by a direct calculation that there can be only one symmetric equilibrium, so this proves uniqueness of of equilibria:

Proposition 5 *There exists no asymmetric equilibrium in the model. Hence, the unique equilibrium is the one in which every seller posts the price in (21).*

Proof. Suppose that there exists an equilibrium p^* in which $p_i^* > p_j^*$. Let $p_{-i,,j}$ denote all prices but p_i and p_j and write the equilibrium conditions for i and j as

$$\begin{aligned} p_i^* &= \beta_i\left(p_{-i,j}^*, p_j^*\right) \\ p_j^* &= \beta_j\left(p_{-i,j}^*, p_i^*\right). \end{aligned} \tag{28}$$

But, the best response functions β_i and β_j are identical so we may drop the index and write $\beta = \beta_i = \beta_j$ and note that from the fundamental theorem of calculus

$$p_j^* = \beta\left(p_{-i,j}^*, p_i^*\right) = p_j^* = \beta\left(p_{-i,j}^*, p_j^*\right) + \int_{p_j^*}^{p_i^*} \frac{\partial\beta\left(p_{-i,j}^*, p_j\right)}{\partial p_j} = p_i^* + \int_{p_j^*}^{p_i^*} \frac{\partial\beta\left(p_{-i,j}^*, p_j\right)}{\partial p_j} > p_i^*, \tag{29}$$

a contradiction. ■

5 Entry

5.1 Failure of Existence of Pure Strategy Entry Equilibria

Naively, it would seem that adding an entry stage to the model should be easy. After all, we have that the unique equilibrium profit of a seller when m sellers compete is

$$\Pi(m,n) = \left(\frac{1 - e^{-\frac{n}{m}}\left(1 + \frac{n}{m}\right)}{1 - e^{-\frac{n}{m}}\left[1 + \frac{n}{m^2}\right]}\right)\left(1 - e^{-\frac{n}{m}}\right) \to 1 - e^{-\rho}(1+\rho) \tag{30}$$

as $m, n \to \infty$ and $\frac{n}{m} \to \rho$. Clearly, $1 - e^{-\rho}(1+\rho)$ is increasing in ρ. It takes some work, but one can check that $\Pi(m,n)$ is strictly decreasing in m for finite m, n as one would expect. Hence, it seems that an equilibrium entry profile would be characterized by some m such that

$$\Pi(m+1,n) \le K \le \Pi(m,n), \tag{31}$$

where K is the the seller valuation of the good or the cost of entry. The problem with this is that if m sellers enter and an additional seller considers a deviation, the relevant optimization problem is to solve

$$\max_{p_i} p_i \left(1 - e^{-\lambda_i(p_i, \mathbf{p}_m(m))}\right), \tag{32}$$

where $\mathbf{p}_m(m)$ is the price profile with m sellers posting the price in (21) for the equilibrium with m sellers, that is

$$\mathbf{p}_m(m) = \underbrace{\left(\frac{1 - e^{-\frac{n}{m}}\left(1 + \frac{n}{m}\right)}{1 - e^{-\frac{n}{m}}\left[1 + \frac{n}{m^2}\right]}, \ldots\ldots, \frac{1 - e^{-\frac{n}{m}}\left(1 + \frac{n}{m}\right)}{1 - e^{-\frac{n}{m}}\left[1 + \frac{n}{m^2}\right]} \right)}_{m \text{ identical coordinates}} \tag{33}$$

In contrast, we have that

$$\Pi(m+1, n) = \max_{p_i} p_i \left(1 - e^{-\lambda_i(p_i, \mathbf{p}_m(m+1))}\right), \tag{34}$$

where $\mathbf{p}_m(m+1)$ is the price profile with m sellers posting the price in (21) for the equilibrium with m sellers, that is

$$\mathbf{p}_m(m+1) = \underbrace{\left(\frac{1 - e^{-\frac{n}{m+1}}\left(1 + \frac{n}{m+1}\right)}{1 - e^{-\frac{n}{m+1}}\left[1 + \frac{n}{(m+1)^2}\right]}, \ldots\ldots, \frac{1 - e^{-\frac{n}{m+1}}\left(1 + \frac{n}{m+1}\right)}{1 - e^{-\frac{n}{m+1}}\left[1 + \frac{n}{(m+1)^2}\right]} \right)}_{m \text{ identical coordinates}}. \tag{35}$$

It is easy to check (and it is also a consequence of Proposition 4) that the equilibrium price with $m+1$ sellers is strictly lower than the equilibrium price with m sellers. Hence, $\max_{p_i} p_i \left(1 - e^{-\lambda_i(p_i, \mathbf{p}_m(m))}\right) > \Pi(m+1, n)$. If in addition

$$\Pi(m+1, n) < K < \max_{p_i} p_i \left(1 - e^{-\lambda_i(p_i, \mathbf{p}_m(m))}\right), \tag{36}$$

then it is impossible to have a pure strategy equilibrium in the entry stage.

A similar issue arises also in models without entry, but where different sellers (or firms) take different actions targeting different types. For example, in Galenianos and Kircher (2009) firms specialize as high wage or low wage firms. They use a market utility framework with a continuum of firms, implying that one of the equilibrium conditions is an indifference condition between posting a high and a low wage.

In a finite version the same problem arises as with entry. In equilibrium, firms know that there are, say, m high wage firm, so that high wage firms would be strictly worse off than low wage firms if one additional firm switched to a high wage. However, to check whether the wage postings are consistent with equilibrium it must be that a firm that would *unexpectedly* switch to a high wage would be worse off, and for the same reasons as above, the profit is higher for a firm that switches unexpectedly than if the other firms can react. Hence, there will not be pure strategy equilibria with finite numbers in these type models either.

5.2 Equilibrium Analysis with Randomly Drawn Sellers

I now consider the full model as specified in Section 2, in which buyers and sellers are distributed in accordance with independent Poisson distributions. Each seller is now also going to attach a strictly positive valuation K to the object, which will be set in a way so as to guarantee non-trivial entry decisions. Consider first a symmetric entry strategy where each potential entrant enters with probability α. Conditional on potential k entrants the probability distribution is a binomial with parameters a and k implying that the probability that v sellers enter is a Poisson with expected number of entrants αm. This observation follows from the same derivation as in Section A.1 in the appendix.

5.2.1 Symmetric Seller Strategies are Without Loss of Generality

It was already shown in the context of a fixed set of sellers that it is without loss of generality to restrict attention to equilibria where all buyers follow the same mixed continuation strategy. This argument trivially extends to this setting as buyers are assumed to be able to distinguish sellers based on prices posted. Consequently, we will assume a symmetric continuation strategy for the buyers and the continuation game from this point on is on the same form as with a known set of sellers.

Next, we will ask whether entry can be type dependent and to what extent it matters. Assume that sellers are distinguished by color, where, for simplicity, they can be either green or blue. Let green sellers be drawn from a Poisson with expectation m_g and blue sellers be drawn from a Poisson with expectation $m_b = m - m_g$. Furthermore, suppose that green and blue sellers randomize with probabilities α_g and α_b, which implies that the probability that v sellers enter is

$$\begin{aligned}\sum_{k=0}^{v} \frac{e^{-(\alpha_g m_g)} \left(\alpha_g m_g\right)^k}{k!} \frac{e^{-(\alpha_b m_b)} \left(\alpha_b m_b\right)^{v-k}}{(v-k)!} &= \frac{e^{-(\alpha_g m_g + \alpha_b m_b)}}{v!} \sum_{k=0}^{v} \frac{v!}{k!\,(v-k)!} \left(\alpha_g m_g\right)^k \left(\alpha_b m_b\right)^{v-k} \\ &= \frac{e^{-(\alpha_g m_g + \alpha_b m_b)}}{v!} \left(\alpha_g m_g + \alpha_b m_b\right)^v, \qquad (37)\end{aligned}$$

by use of the binomial theorem. This shows that the probability of v players entering is a Poisson with expected queue length $\alpha_g m_g + \alpha_b m_b$. Hence, if all active sellers post the same price (I will show that this must be the case in the proof of Proposition 6 below), then it is irrelevant whether different groups choose the same randomization as only the expected number of entrants matter. That is, all randomizations (α_g, α_b) satisfying $\alpha_g m_g + \alpha_b (m - m_g) = \alpha m$ are equivalent in terms of the incentives to enter.

In general, recall that $t \in T = \{1, ..., T\}$ denotes the (payoff-irrelevant) type of a seller where m_t is the expected number of type t sellers. As T is arbitrary it is possible to make the probability of two identical clones arbitrary small. The result is that these types can be used to decide who enters and who does not, but that payoffs cannot be affected by use of them.

Proposition 6 *In any equilibrium there exists a unique equilibrium price p^* such that $p^*_t = p^*$ in any equilibrium. Moreover, if $(a^*_1, ..., a^*_T)$ is an equilibrium entry profile there exists α^* such that*

$\alpha_t^* = \alpha^*$ *for each* t *such that all buyers and sellers earn the same payoffs as in the equilibrium with asymmmetric entry.*

Proof. Let $\mathbf{p}^* = (p_1^*, ..., p_T^*)$ and $\boldsymbol{\alpha}^* = (\alpha_1^*, ..., \alpha_T^*)$ be an equilibrium. We need to introduce notation for the Poisson arrivals of buyers conditional on entry vector $k = (k_1, ..., k_T)$ and to so we write $\lambda(p, \mathbf{p}^*, k) = \lambda(p, \mathbf{p}^*, k_1, .., k_T)$, which denotes the expected queue length at a seller (of any type) who posts p given that for each $t \in T$ there are k_t *other* sellers who posts p_t^*. The queue length is uniquely pinned down by the indifference conditions in (11) given any $p, \mathbf{p}^*, k$. The expected profit for the seller posting p is thus

$$\pi(p, \mathbf{p}^*, \boldsymbol{\alpha}^*) = \left[\sum_{k_1=0}^{\infty} ..... \sum_{k_T=0}^{\infty}\right] \frac{e^{-(\alpha_1^* m_1)} (\alpha_1^* m_1)^{k_1}}{k_1!} \times ... \times \frac{e^{-(\alpha_T^* m_T)} (\alpha_T^* m_T)^{k_T}}{k_T!} p \left(1 - e^{-\lambda(p, \mathbf{p}^*, k)}\right) - K. \tag{38}$$

From Proposition 2 we know that each $p\left(1 - e^{-\lambda(p, \mathbf{p}^*, k)}\right)$ is strictly concave in p on some range $[0, \overline{p}(\mathbf{p}^*)]$, but has a non-concave range where no buyers visit the firm. However, all active sellers face the same optimization problem with the same set of maximizers. Let $p_{\min}^*$ be the smallest maximizer. Obviously, a seller posting $p_{\min}^*$ must be visited with strictly positive probability regardless of which sellers enter, so $p_{\min}^*$ is in the range where the all profit functions in the support of the expected profit function is strictly concave, implying that (38) is strictly concave in a range around $p_{\min}^*$. It follows that $p_{\min}^*$ must be the unique global maximum. It follows that there is a unique price $p(p^*, \alpha^*)$ that is a best response, which implies that in any equilibrium there must be a unique price p^* such that $p_t^* = p^*$ for each t. Since all types post the same price only the total number of entrants matter. This is distributed Poisson with parameter $\sum_{t \in T} \alpha_t^* m_t$ implying that there is a symmetric randomization probability

$$\alpha^* = \frac{\sum_{t \in T} \alpha_t^* m_t}{\sum_{t \in T} m_t} \tag{39}$$

that is consistent with equilibrium. ■

5.2.2 Solving for a Symmetric Equilibrium

Since any equilibrium is payoff equivalent to a symmetric equilibrium I now simplify the profit function by expressing it in terms of the own price p_i, the common equilibrium price p, and the common entry probability α as

$$\pi(p_i, p, \alpha) = e^{-(\alpha m)} p_i + \sum_{k=1}^{\infty} \frac{e^{-(\alpha m)} (\alpha m)^k}{k!} p_i \left(1 - e^{-\lambda_k(p_i, p)}\right) - K, \tag{40}$$

where $\lambda_k(p_i, p)$ denotes the buyer queue at a seller posting p_i given that k sellers post p. Notice that $e^{-(\alpha m)}$ is the probability that no other seller enters in case the seller attracts a buyer for sure. An equilibrium price must be symmetric and satisfy the first order condition

$$e^{-(\alpha m)} + \sum_{k=1}^{\infty} \frac{e^{-(\alpha m)} (\alpha m)^k}{k!} \left(1 - e^{-\lambda_k(p^*, p^*)} + p^* e^{-\lambda_k(p^*, p^*)} \frac{\partial \lambda_k(p^*, p^*)}{\partial p_i}\right) = 0 \tag{41}$$

Using symmetry, we have that $\lambda_k(p^*,p^*) = \frac{n}{k+1}$ for each $k \geq 1$ and that

$$\frac{\partial \lambda_k(p^*,p^*)}{\partial p_i} = \frac{g\left(\frac{n}{k+1}\right)}{(1-p^*)\, g'\left(\frac{n}{k+1}\right)} \frac{k}{k+1} = -\frac{\frac{n}{k+1}\left(1-e^{-\frac{n}{k+1}}\right)}{(1-p^*)\left(1-e^{-\frac{n}{k+1}}\left(1+\frac{n}{k+1}\right)\right)} \frac{k}{k+1}, \tag{42}$$

we can solve for the candidate equilibrium price in closed form,

$$p^*(\alpha) = \frac{e^{-\alpha m} + \sum_{k=1}^{\infty} \frac{e^{-(\alpha m)}(\alpha m)^k}{k!}\left(1-e^{-\frac{n}{k+1}}\right)}{e^{-\alpha m} + \sum_{k=1}^{\infty} \frac{e^{-(\alpha m)}(\alpha m)^k}{k!} \frac{\left(1-e^{-\frac{n}{k+1}}\right)}{\left(1-e^{-\frac{n}{k+1}}\left(1+\frac{n}{k+1}\right)\right)}\left[1-e^{-\frac{n}{k+1}}\left(1+\frac{n}{(k+1)^2}\right)\right]}. \tag{43}$$

Given any $\alpha \in [0,1]$ we have that $p^*(\alpha)$ is uniquely defined, so we can immediately conclude that there is a unique candidate equilibrium price $p^*(\alpha)$ for every entry probability α. However, in order to show that there is a unique equilibrium price given any cost of selling K we need to establish that $p^*(\alpha)$ is monotone in α.

Let

$$G(k) = \begin{cases} 1 & \text{for } k = 0 \\ 1-e^{-\frac{n}{k+1}} & \text{for } k \geq 0 \end{cases}, \tag{44}$$

and let $H : R_+ \to R_+$ be defined as

$$H(k) = \begin{cases} 1 & \text{for } 0 \leq k < 1 \\ \frac{\left(1-e^{-\frac{n}{k+1}}\right)}{\left(1-e^{-\frac{n}{k+1}}\left(1+\frac{n}{k+1}\right)\right)}\left[1-e^{-\frac{n}{k+1}}\left(1+\frac{n}{(k+1)^2}\right)\right] & \text{for } k \geq 1 \end{cases} \tag{45}$$

To demonstrate that $p^*(\alpha)$ is monotone in α we show that.

Lemma 3 *$G(\cdot)$ is strictly decreasing and $H(\cdot)$ is increasing on R_+ and strictly increasing at any $k \geq 1$.*

The proof is relegated to the appendix. Using the monotonicity of G and H we are in a position where we can demonstrate that the price is strictly monotone in the entry probability, as one would expect.

Lemma 4 *$p^*(\cdot)$ defined in (43) is strictly decreasing in α.*

Proof. Suppose that $a' < a''$. Then the Poisson with probability mass density $\frac{e^{-(\alpha' m)}(\alpha' m)^k}{k!}$ is first order stochastically dominated by the Poisson with probability mass density $\frac{e^{-(\alpha'' m)}(\alpha'' m)^k}{k!}$. Since $G(k)$ is strictly decreasing and $H(k)$ is increasing (also when restricted to integers) it follows that

$$\begin{aligned} E_{\alpha'}(G(k)) &= e^{-\alpha' m} + \sum_{k=1}^{\infty} \frac{e^{-(\alpha' m)}(\alpha m)^k}{k!}\left(1-e^{-\frac{n}{k+1}}\right) \\ &> e^{-\alpha'' m} + \sum_{k=1}^{\infty} \frac{e^{-(\alpha'' m)}(\alpha m)^k}{k!}\left(1-e^{-\frac{n}{k+1}}\right) = E_{\alpha''}(G(k)) \end{aligned} \tag{46}$$

and

$$\begin{aligned}
E_{\alpha'}\left(H\left(k\right)\right) &= e^{-\alpha' m}+\sum_{k=1}^{\infty}\frac{e^{-(\alpha' m)}\left(\alpha m\right)^{k}}{k!}\frac{\left(1-e^{-\frac{n}{k+1}}\right)}{\left(1-e^{-\frac{n}{k+1}}\left(1+\frac{n}{k+1}\right)\right)}\left[1-e^{-\frac{n}{k+1}}\left(1+\frac{n}{\left(k+1\right)^{2}}\right)\right] \\
&\leq e^{-\alpha'' m}+\sum_{k=1}^{\infty}\frac{e^{-(\alpha'' m)}\left(\alpha m\right)^{k}}{k!}\frac{\left(1-e^{-\frac{n}{k+1}}\right)}{\left(1-e^{-\frac{n}{k+1}}\left(1+\frac{n}{k+1}\right)\right)}\left[1-e^{-\frac{n}{k+1}}\left(1+\frac{n}{\left(k+1\right)^{2}}\right)\right] \\
&= E'_{\alpha'}\left(H\left(k\right)\right), \qquad (47)
\end{aligned}$$

which implies that $p^{*}\left(a'\right)>p^{*}\left(a''\right)$. ■

Corollary 1 *For any $K>0$ there is a unique entry probability $\alpha\left(K\right)$ that is consistent with an equilibrium in symmetric entry strategies. Moreover if*

$$\frac{\left[e^{-m}+\sum_{k=1}^{\infty}\frac{e^{-(m)}(m)^{k}}{k!}\left(1-e^{-\frac{n}{k+1}}\right)\right]^{2}}{e^{-m}+\sum_{k=1}^{\infty}\frac{e^{-(m)}(m)^{k}}{k!}\frac{\left(1-e^{-\frac{n}{k+1}}\right)}{\left(1-e^{-\frac{n}{k+1}}\left(1+\frac{n}{k+1}\right)\right)}\left[1-e^{-\frac{n}{k+1}}\left(1+\frac{n}{(k+1)^{2}}\right)\right]}<K<1, \qquad (48)$$

then $\alpha\left(K\right)\in\left(0,1\right)$ and every seller that enters is indifferent between selling and no selling.

The proof is immediate and left to the reader. If sellers have different characteristics to condition on one can construct equilibria with different types entering with different probabilities, but the average probability of entry is uniquely defined, so the equilibrium is essentially unique.

5.2.3 Convergence Towards the Market Utility Benchmark

With some further notational abuse I write $\alpha\left(K,n,m\right)$ for the unique (average) randomization probability when the cost of selling is K and there are n buyers and m potential sellers. While $p^{*}\left(\alpha\left(K,n,m\right)\right)$ in (43) is a deterministic number that can be thought of as an expectation (and not a random variable) it turns out that it is useful to note that $G\left(k\right)$ and $H\left(k\right)$ converges in probability in order to derive the (deterministic) limit of $p^{*}\left(\alpha\left(K,n,m\right)\right)$ as $n,m\to\infty$ and $\frac{n}{m}\to\rho$.

That is, we use the fact that the number of active sellers, $\frac{k}{m}$, converges in probability to $\alpha\left(K,n,m\right)$ in order to derive the limiting price and expected equilibrium profit. More formally, Chebyshev's inequality implies that

$$\Pr\left[\left|k-\alpha\left(K,n,m\right)m\right|>\delta m\right]\leq\frac{\sigma^{2}}{\delta^{2}m^{2}}=\frac{\alpha\left(K,n,m\right)}{\delta^{2}m} \qquad (49)$$

holds for any m, because $\sigma^{2}=\alpha\left(K,n,m\right)m$ is the variance of the number of active sellers k as $\alpha\left(K,n,m\right)m$ is both the first and the second moment of the Poisson generated when the expected number of potential sellers is m and every seller enters with probability $\alpha\left(K,n,m\right)$.It follows that if $\frac{n}{m}=\rho$ we have that

$$\Pr\left[\left|\frac{k}{n}-\frac{\alpha\left(K,n,m\right)}{\rho}\right|>\frac{\delta}{\rho}\right]\leq\frac{\alpha\left(K,n,m\right)}{\delta^{2}m}\to0 \qquad (50)$$

as $m \to \infty$. More generally, if $n, m \to \infty$ and $\frac{n}{m} \to \rho$ and $\alpha(K, n, m) \to \alpha^*$ (taking a subsequence if necessary) we have that $\frac{k}{n}$ converges in probability to $\frac{\alpha^*}{\rho}$. Hence,

$$p^*(\alpha(K, n, m)) \to \frac{\left(1 - e^{-\frac{\rho}{\alpha^*}}\right)}{\left(1 - e^{-\frac{\rho}{\alpha^*}}\right)\left[\frac{1 - e^{-\frac{\rho}{\alpha^*}}}{\left(1 - e^{-\frac{\rho}{\alpha^*}}\left(1 + \frac{\rho}{\alpha^*}\right)\right)}\right]} = \frac{1 - e^{-\frac{\rho}{\alpha^*}}\left(1 + \frac{\rho}{\alpha^*}\right)}{1 - e^{-\frac{\rho}{\alpha^*}}} = 1 - \frac{\frac{\rho}{\alpha^*}}{e^{\frac{\rho}{\alpha^*}} - 1}, \tag{51}$$

as $n, m \to \infty$ and $\frac{n}{m} \to \rho$ and $\alpha(K, n, m) \to \alpha^*$. Also note that as $\frac{k}{n}$ converges in probability to $\frac{\alpha^*}{\rho}$ we have that $e^{-\frac{k}{n}}$ converges in probability to $e^{-\frac{\rho}{\alpha^*}}$ implying that the equilibrium expected profit

$$\pi\left(p^*(\alpha(K, n, m)), p^*(\alpha(K, n, m)), \alpha(K, n, m)\right) \to 1 - e^{-\frac{\rho}{\alpha^*}}\left(1 + \frac{\rho}{\alpha^*}\right) = K \tag{52}$$

along any convergent subsequence. As $1 - e^{-x}(1 + x)$ is strictly increasing there is a unique solution α^* to $1 - e^{-\frac{\rho}{\alpha^*}}\left(1 + \frac{\rho}{\alpha^*}\right) = K$ implying that is a $\alpha(K, n, m)$ convergent sequence. Hence, the right hand side of (52) is the limiting equilibrium profit. Also notice that the right hand side of (52) is identical with the profit (4) in the market utility model and that the limiting price in (51) corresponds to the equilibrium price (3) for the equilibrium active buyer to seller ratio. We conclude that the market utility benchmark can indeed be seen as the limit a large finite model in which both buyers and sellers are distributed in accordance with a Poisson.

6 Concluding Remarks

The results in this paper suggests that drawing players randomly from a Poisson distribution can be a useful compromise between a market utility model and a standard finite directed search model. In the simplest possible setup I demonstrated that there is no longer any need to make an equilibrium selection and that an otherwise untractable entry problem can be handled without too much difficulty. The added tractability should be helpful also in richer models where buyers or sellers have payoff relevant types, and or in models where contracts are more elaborate than posting a single price, but I don't explore this in the current paper.

References

[1] Bland, J. and S. Loertscher (2012), "Monotonicity, Non-Participation, and Directed Search Equilibria," *mimeo*, University of Melbourne.

[2] Burdett, K. Shi S., and R. Wright, (2001). "Pricing and Matching with Frictions," *Journal of Political Economy*, 109(5), October 2001, 1060-1085.

[3] Galenianos, M. and P. Kircher, (2009) "Directed Search with Multiple Job Applications," *Journal of Economic Theory*, 144(2), 2009, 445-471.

[4] Galenianos, M. and P. Kircher, (2012) "On the Game-theoretic Foundations of Competitive Search Equilibrium," *International Economic Review*, 53(1), 1-21.

[5] Galenianos, M. P. Kircher and G. Virag (2011), “Market power and efficiency in a search model,” mimeo, London School of Economics.

[6] Geromichalos, A. (2012) “Directed search and optimal production,” *Journal of Economic Theory,* 47(6), Pages 2303–2331

[7] Julien, B., J. Kennes and I. King, (2000), “Bidding for labor,” *Review of Economic Dynamics*, 3(4), 619-649.

[8] Kircher, P., (2009), “Efficiency of simultaneous search,” *Journal of Political Economy*, 117(5), 861- 913.

[9] Kim, J. and G. Camera (2014). “Uniqueness of equilibrium in directed search models,” *Journal of Economic Theory* 151, 248-267.

[10] Lester, B., (2011) "Information and Prices with Capacity Constraints," *American Economic Review*, 101(4), 1591-1600.

[11] Lester, B, Visschersz L., and R. Wolthoff, “Meeting Technologies and Optimal Trading Mechanisms in Competitive Search Markets,” *mimeo*, September 2014.

[12] McAfee P. (1993), “Mechanism design by competing sellers,” *Econometrica* 61(6), 1281- 1312.

[13] Moen, E. R. (1997). “Competitive Search Equilibrium.” *Journal of Political Economy*, 105(2), 385–411.

[14] Myerson, Roger B. (2000). “Large Poisson Games,” *Journal of Economic Theory* 94 (1): 7–45. doi:10.1006/jeth.1998.2453

[15] Peters, M. (1984). “Bertrand Equilibrium with Capacity Constraints and Restricted Mobility.”, *Econometrica*, 52(5): 1117–27.

[16] Peters, M. (2000), “Limits of Exact Equilibria for Capacity Constrained Sellers with Costly Search,” *Journal of Economic Theory*, 95(2), 139-168.

[17] Shimer, R., (2005), “The assignment of workers to jobs in an economy with coordination frictions,” *Journal of Political Economy*, 113(5), 996-1025.

A Appendix

A.1 A Reminder about a Poisson Distribution Property

Claim *If every buyer visits seller i with probability θ_i and if buyers are distributed in accordance with a Poisson with parameter n, then the number of buyers visiting seller i is a Poisson distribution with parameter $n\theta_i$.*

Proof. Conditional on there being b buyers the number of buyers at seller i follows a binomial with parameters θ_i and b, so that

$$\Pr\left[v \text{ buyers at } i | b \text{ buyers}\right] = \frac{b!}{v!\,(b-v)!}\theta_i^v\left[1-\theta_i\right]^{b-v} \tag{A1}$$

The unconditional probability of v buyers arriving at seller i is thus

$$\begin{aligned} \Pr\left[v \text{ buyers at } i\right] &= \sum_{b=v}^{\infty}\frac{e^{-n}n^b}{b!}\frac{b!}{v!\,(b-v)!}\theta_i^v\left[1-\theta_i\right]^{b-v} = \frac{e^{-n}}{v!}\left[n\theta_i\right]^v\sum_{b=v}^{\infty}\frac{1}{(b-v)!}\left[n\left(1-\theta_i\right)\right]^{b-v} \\ &= \frac{e^{-n}}{v!}\left(n\theta_i\right)^v\sum_{r=0}^{\infty}\frac{1}{r!}\left[n\left(1-\theta_i\right)\right]^r = \frac{e^{-n\theta_i}}{v!}\left(n\theta_i\right)^v, \end{aligned} \tag{A2}$$

a Poisson distribution with parameter $n\theta_i$. ■

A.2 Proof of Lemma 1

Proof. The first inequality holds as

$$\begin{aligned} & \left[g'(x)\right]^2 - g(x)\,g''(x) \\ = & \left[\frac{1-e^{-x}(1+x)}{x^2}\right]^2 - \left[\frac{1-e^{-x}}{x}\right]\left[2\frac{1-e^{-x}\left[1+x+\frac{x^2}{2}\right]}{x^3}\right] \\ = & \frac{1}{x^4}\left[\left(1-e^{-x}(1+x)\right)^2 - 2\left(1-e^{-x}\right)\left(1-e^{-x}\left(1+x+\frac{x^2}{2}\right)\right)\right] \\ = & \frac{1}{x^4}\left[\left(1-e^{-x}(1+x)\right)^2 - \left(1-e^{-x}\right)\left(1-e^{-x}\left(1+x+\frac{x^2}{2}\right)\right)\right] - \frac{1}{x^4}\left[\left(1-e^{-x}\right)\left(1-e^{-x}\left(1+x+\frac{x^2}{2}\right)\right)\right] \\ = & \frac{1}{x^4}\left[\left(1-e^{-x}(1+x)\right)\left(1-e^{-x}\left(1+x+\frac{x^2}{2}\right)+e^{-x}\frac{x^2}{2}\right) - \left(1-e^{-x}\right)\left(1-e^{-x}\left(1+x+\frac{x^2}{2}\right)\right)\right] \\ & -\frac{1}{x^4}\left[\left(1-e^{-x}\right)\left(1-e^{-x}\left(1+x+\frac{x^2}{2}\right)\right)\right] \\ = & \frac{1}{x^4}\left[\left(1-e^{-x}\left(1+x+\frac{x^2}{2}\right)\right)\left(1-e^{-x}(1+x)-1+e^{-x}\right)\right] + \frac{1}{x^4}\left[\left(1-e^{-x}(1+x)\right)e^{-x}\frac{x^2}{2}\right] \\ & -\frac{1}{x^4}\left[\left(1-e^{-x}\right)\left(1-e^{-x}\left(1+x+\frac{x^2}{2}\right)\right)\right] \\ = & -\frac{1}{x^4}\left[\left(1-e^{-x}\left(1+x+\frac{x^2}{2}\right)\right)e^{-x}x\right] + \frac{1}{x^4}\left[\left(1-e^{-x}(1+x)\right)e^{-x}\frac{x^2}{2}\right] \\ & -\frac{1}{x^4}\left[\left(1-e^{-x}\right)\left(1-e^{-x}\left(1+x+\frac{x^2}{2}\right)\right)\right] \\ = & -\frac{1}{x^4}\left[\left(1-e^{-x}\left(1+x+\frac{x^2}{2}\right)\right)e^{-x}x\right] + \frac{1}{x^4}\left[\left(1-e^{-x}\left(1+x+\frac{x^2}{2}\right)\right)e^{-x}\frac{x^2}{2} - \left(e^{-x}\frac{x^2}{2}\right)^2\right] \\ & -\frac{1}{x^4}\left[\left(1-e^{-x}\right)\left(1-e^{-x}\left(1+x+\frac{x^2}{2}\right)\right)\right] \end{aligned}$$

$$= \frac{1}{x^4}\left[\left(1-e^{-x}\left(1+x+\frac{x^2}{2}\right)\right)\left[e^{-x}\frac{x^2}{2}-e^{-x}x-\left(1-e^{-x}\right)\right]\right]-\frac{1}{x^4}\left(e^{-x}\frac{x^2}{2}\right)^2$$

$$= \frac{1}{x^4}\left[\left(1-e^{-x}\left(1+x+\frac{x^2}{2}\right)\right)\left[\underbrace{e^{-x}\left[1+x+\frac{x^2}{2}\right]-1}_{<0}-2e^{-x}x\right]\right]-\frac{1}{x^4}\left(e^{-x}\frac{x^2}{2}\right)^2<0,$$

and the second follows from

$$\begin{aligned}
2\left[g'(x)\right]^2-g(x)g''(x) &= 2\left[\frac{1-e^{-x}(1+x)}{x^2}\right]^2-\left[\frac{1-e^{-x}}{x}\right]\left[2\frac{1-e^{-x}\left[1+x+\frac{x^2}{2}\right]}{x^3}\right]\\
&= \frac{2}{x^4}\left[\left(1-e^{-x}(1+x)\right)^2-\left(1-e^{-x}\right)\left(1-e^{-x}\left(1+x+\frac{x^2}{2}\right)\right)\right]\\
&= \frac{2}{x^4}\left[\left(1-e^{-x}(1+x)\right)^2-\left(1-e^{-x}(1+x)+e^{-x}x\right)\left(1-e^{-x}(1+x)-e^{-x}\frac{x^2}{2}\right)\right]\\
&= \frac{2}{x^4}\left[e^{-x}\frac{x^2}{2}\left(1-e^{-x}(1+x)\right)-e^{-x}x\left(1-e^{-x}(1+x)-e^{-x}\frac{x^2}{2}\right)\right]\\
&= \frac{2e^{-x}}{x^3}\left[\frac{x}{2}\left(1-e^{-x}(1+x)\right)-\left(1-e^{-x}(1+x)-e^{-x}\frac{x^2}{2}\right)\right]\\
&= \frac{2e^{-x}}{x^3}\left[\frac{x}{2}\left(1-e^{-x}\right)-e^{-x}\frac{x^2}{2}-\left(1-e^{-x}(1+x)-e^{-x}\frac{x^2}{2}\right)\right]\\
&= \frac{2e^{-x}}{x^3}\left[\frac{x}{2}\left(1-e^{-x}\right)-\left(1-e^{-x}(1+x)\right)\right]\\
&= \frac{2e^{-x}}{x^3}\left[\frac{x}{2}-1+e^{-x}\left(1+\frac{x}{2}\right)\right]>0
\end{aligned}$$

for every $x>0$. ■

A.3 Proof of Proposition 2

Proof. Differentiating (15) we obtain

$$\begin{aligned} \frac{\partial^2 \lambda_i(\boldsymbol{p})}{\partial p_i^2} &= \frac{g(\lambda_i(\boldsymbol{p}))}{(1-p_i)^2 g'(\lambda_i(\boldsymbol{p}))} \frac{\sum_{j\neq i} \frac{g(\lambda_j(\boldsymbol{p}))}{g'(\lambda_j(\boldsymbol{p}))}}{\sum_{j=1}^m \frac{g(\lambda_j(\boldsymbol{p}))}{g'(\lambda_j(\boldsymbol{p}))}} \\ &+ \frac{1}{(1-p_i)} \frac{[g'(\lambda_i(\boldsymbol{p}))]^2 - g(\lambda_i(\boldsymbol{p}))g''(\lambda_i(\boldsymbol{p}))}{[g'(\lambda_i(\boldsymbol{p}))]^2} \frac{\sum_{j\neq i} \frac{g(\lambda_j(\boldsymbol{p}))}{g'(\lambda_j(\boldsymbol{p}))}}{\sum_{j=1}^m \frac{g(\lambda_j(\boldsymbol{p}))}{g'(\lambda_j(\boldsymbol{p}))}} \frac{\partial \lambda_i(\mathbf{p})}{\partial p_i} \\ &+ \frac{g(\lambda_i(\boldsymbol{p}))}{(1-p_i)^2 g'(\lambda_i(\boldsymbol{p}))} \frac{d}{dp_i} \left[\frac{\sum_{j\neq i} \frac{g(\lambda_j(\boldsymbol{p}))}{g'(\lambda_j(\boldsymbol{p}))}}{\sum_{j=1}^m \frac{g(\lambda_j(\boldsymbol{p}))}{g'(\lambda_j(\boldsymbol{p}))}} \right]. \end{aligned} \tag{A3}$$

By substituting $\frac{\partial \lambda_i(\mathbf{p})}{\partial p_i} = \frac{g(\lambda_i(\boldsymbol{p}))}{(1-p_i)g'(\lambda_i(\boldsymbol{p}))} \frac{\sum_{j\neq i} \frac{g(\lambda_j(\boldsymbol{p}))}{g'(\lambda_j(\boldsymbol{p}))}}{\sum_{j=1}^m \frac{g(\lambda_j(\boldsymbol{p}))}{g'(\lambda_j(\boldsymbol{p}))}}$ we write this as

$$\begin{aligned} \frac{\partial^2 \lambda_i(\boldsymbol{p})}{\partial p_i^2} &= \frac{g(\lambda_i(\boldsymbol{p}))}{(1-p_i)^2 g'(\lambda_i(\boldsymbol{p}))} \frac{\sum_{j\neq i} \frac{g(\lambda_j(\boldsymbol{p}))}{g'(\lambda_j(\boldsymbol{p}))}}{\sum_{j=1}^m \frac{g(\lambda_j(\boldsymbol{p}))}{g'(\lambda_j(\boldsymbol{p}))}} \\ &+ \frac{1}{(1-p_i)} \underbrace{\frac{[g'(\lambda_i(\boldsymbol{p}))]^2 - g(\lambda_i(\boldsymbol{p}))g''(\lambda_i(\boldsymbol{p}))}{[g'(\lambda_i(\boldsymbol{p}))]^2}}_{<0} \left[\frac{\sum_{j\neq i} \frac{g(\lambda_j(\boldsymbol{p}))}{g'(\lambda_j(\boldsymbol{p}))}}{\sum_{j=1}^m \frac{g(\lambda_j(\boldsymbol{p}))}{g'(\lambda_j(\boldsymbol{p}))}} \right]^2 \underbrace{\frac{g(\lambda_i(\boldsymbol{p}))}{(1-p_i)g'(\lambda_i(\boldsymbol{p}))}}_{<0} \\ &+ \frac{g(\lambda_i(\boldsymbol{p}))}{(1-p_i) g'(\lambda_i(\boldsymbol{p}))} \frac{d}{dp_i} \left[\frac{\sum_{j\neq i} \frac{g(\lambda_j(\boldsymbol{p}))}{g'(\lambda_j(\boldsymbol{p}))}}{\sum_{j=1}^m \frac{g(\lambda_j(\boldsymbol{p}))}{g'(\lambda_j(\boldsymbol{p}))}} \right], \end{aligned} \tag{A4}$$

where we used the first part of Lemma 1 to sign the second term. Noting that $0 < \frac{\sum_{j\neq i} \frac{g(\lambda_j(\boldsymbol{p}))}{g'(\lambda_j(\boldsymbol{p}))}}{\sum_{j=1}^m \frac{g(\lambda_j(\boldsymbol{p}))}{g'(\lambda_j(\boldsymbol{p}))}} < 1$, which implies that $0 < \left[\frac{\sum_{j\neq i} \frac{g(\lambda_j(\boldsymbol{p}))}{g'(\lambda_j(\boldsymbol{p}))}}{\sum_{j=1}^m \frac{g(\lambda_j(\boldsymbol{p}))}{g'(\lambda_j(\boldsymbol{p}))}} \right]^2 < \frac{\sum_{j\neq i} \frac{g(\lambda_j(\boldsymbol{p}))}{g'(\lambda_j(\boldsymbol{p}))}}{\sum_{j=1}^m \frac{g(\lambda_j(\boldsymbol{p}))}{g'(\lambda_j(\boldsymbol{p}))}} < 1$ and that the second term in (A4) is positive we get an upper bound on the second derivative given by

$$\begin{aligned} \frac{\partial^2 \lambda_i(\boldsymbol{p})}{\partial p_i^2} &< \frac{g(\lambda_i(\boldsymbol{p}))}{(1-p_i)^2 g'(\lambda_i(\boldsymbol{p}))} \underbrace{\left[\frac{2[g'(\lambda_i(\mathbf{p}))]^2 - g(\lambda_i(\mathbf{p}))g''(\lambda_i(\mathbf{p}))}{[g'(\lambda_i(\mathbf{p}))]^2} \right]}_{>0} \frac{\sum_{j\neq i} \frac{g(\lambda_j(\boldsymbol{p}))}{g'(\lambda_j(\boldsymbol{p}))}}{\sum_{j=1}^m \frac{g(\lambda_j(\boldsymbol{p}))}{g'(\lambda_j(\boldsymbol{p}))}} \\ &+ \frac{g(\lambda_i(\boldsymbol{p}))}{(1-p_i) g'(\lambda_i(\boldsymbol{p}))} \frac{d}{dp_i} \left[\frac{\sum_{j\neq i} \frac{g(\lambda_j(\boldsymbol{p}))}{g'(\lambda_j(\boldsymbol{p}))}}{\sum_{j=1}^m \frac{g(\lambda_j(\boldsymbol{p}))}{g'(\lambda_j(\boldsymbol{p}))}} \right], \end{aligned} \tag{A5}$$

where we have usedthe second part of Lemma 1 to sign the bracketed expression. Clearly, the first term in (A5) is negative. Moreover,

$$
\begin{aligned}
\frac{d}{dp_i}\left[\frac{\sum_{j\neq i}\frac{g(\lambda_j(\boldsymbol{p}))}{g'(\lambda_j(\boldsymbol{p}))}}{\sum_{j=1}^{m}\frac{g(\lambda_j(\boldsymbol{p}))}{g'(\lambda_j(\boldsymbol{p}))}}\right] &= \frac{d}{dp_i}\left[1-\frac{\frac{g(\lambda_i(\mathbf{p}))}{g'(\lambda_i(\mathbf{p}))}}{\sum_{j=1}^{m}\frac{g(\lambda_j(\boldsymbol{p}))}{g'(\lambda_j(\boldsymbol{p}))}}\right] \qquad \text{(A6)}\\
&= -\frac{\frac{[g'(\lambda_i(\boldsymbol{p}))]^2-g(\lambda_i(\boldsymbol{p}))g''(\lambda_i(\boldsymbol{p}))}{[g'(\lambda_i(\boldsymbol{p}))]^2}\frac{\partial\lambda_i(\mathbf{p})}{\partial p_i}\left[\sum_{j=1}^{m}\frac{g(\lambda_j(\boldsymbol{p}))}{g'(\lambda_j(\boldsymbol{p}))}\right]}{\left[\sum_{j=1}^{m}\frac{g(\lambda_j(\boldsymbol{p}))}{g'(\lambda_j(\boldsymbol{p}))}\right]^2}\\
&\quad+\frac{\frac{g(\lambda_i(\mathbf{p}))}{g'(\lambda_i(\mathbf{p}))}\sum_{j=1}^{m}\frac{[g'(\lambda_j(\mathbf{p}))]^2-g(\lambda_j(\mathbf{p}))g''(\lambda_j(\mathbf{p}))}{[g'(\lambda_j(\mathbf{p}))]^2}\frac{\partial\lambda_j(\boldsymbol{p})}{\partial p_i}}{\left[\sum_{j=1}^{m}\frac{g(\lambda_j(\boldsymbol{p}))}{g'(\lambda_j(\boldsymbol{p}))}\right]^2}\\
&= -\frac{\overbrace{\frac{[g'(\lambda_i(\boldsymbol{p}))]^2-g(\lambda_i(\boldsymbol{p}))g''(\lambda_i(\boldsymbol{p}))}{[g'(\lambda_i(\boldsymbol{p}))]^2}}^{<0}\overbrace{\frac{\partial\lambda_i(\mathbf{p})}{\partial p_i}}^{<0}\left[\overbrace{\sum_{j\neq i}\frac{g(\lambda_j(\mathbf{p}))}{g'(\lambda_j(\boldsymbol{p}))}}^{<0}\right]}{\left[\sum_{j=1}^{m}\frac{g(\lambda_j(\boldsymbol{p}))}{g'(\lambda_j(\boldsymbol{p}))}\right]^2}\\
&\quad+\frac{\overbrace{\frac{g(\lambda_i(\mathbf{p}))}{g'(\lambda_i(\mathbf{p}))}}^{<0}\sum_{j=1}^{m}\overbrace{\frac{[g'(\lambda_j(\mathbf{p}))]^2-g(\lambda_j(\mathbf{p}))g''(\lambda_j(\mathbf{p}))}{[g'(\lambda_j(\mathbf{p}))]^2}}^{<0}\overbrace{\frac{\partial\lambda_j(\boldsymbol{p})}{\partial p_i}}^{>0}}{\left[\sum_{j=1}^{m}\frac{g(\lambda_j(\boldsymbol{p}))}{g'(\lambda_j(\boldsymbol{p}))}\right]^2} > 0.
\end{aligned}
$$

Combining (A6) and (A5) establishes the result. ■

A.4 Proof of Lemma 2

Proof. Taking the derivative, we note that

$$\begin{aligned}
\frac{d}{dp_k}\left[\frac{\frac{g(\lambda_i(\mathbf{p}))}{g'(\lambda_i(\mathbf{p}))}}{\sum_{j=1}^m \frac{g(\lambda_i(\mathbf{p}))}{g'(\lambda_j(\mathbf{p}))}}\right] &= \frac{\overbrace{\frac{[g'(\lambda_i(\mathbf{p}))]^2-g(\lambda_i(\mathbf{p}))g''(\lambda_j(\mathbf{p}))}{[g'(\lambda_i(\mathbf{p}))]^2}}^{<0}\overbrace{\frac{\partial\lambda_i(\mathbf{p})}{\partial p_k}}^{>0}\overbrace{\left(\sum_{j=1}^m \frac{g(\lambda_j(\mathbf{p}))}{g'(\lambda_j(\mathbf{p}))}\right)}^{<0}}{\left(\sum_{j=1}^m \frac{g(\lambda_j(\mathbf{p}))}{g'(\lambda_j(\mathbf{p}))}\right)^2} \\
&\quad - \frac{\frac{g(\lambda_i(\mathbf{p}))}{g'(\lambda_i(\mathbf{p}))}\left(\sum_{j=1}^m \frac{[g'(\lambda_j(\mathbf{p}))]^2-g(\lambda_j(\mathbf{p}))g''(\lambda_j(\mathbf{p}))}{[g'(\lambda_j(\mathbf{p}))]^2}\frac{\partial\lambda_j(\mathbf{p})}{\partial p_k}\right)}{\left(\sum_{j=1}^m \frac{g(\lambda_j(\mathbf{p}))}{g'(\lambda_j(\mathbf{p}))}\right)^2} \\
&> - \frac{\frac{g(\lambda_i(\mathbf{p}))}{g'(\lambda_i(\mathbf{p}))}\left(\sum_{j=1}^m \frac{[g'(\lambda_j(\mathbf{p}))]^2-g(\lambda_j(\mathbf{p}))g''(\lambda_j(\mathbf{p}))}{[g'(\lambda_j(\mathbf{p}))]^2}\frac{\partial\lambda_j(\mathbf{p})}{\partial p_k}\right)}{\left(\sum_{j=1}^m \frac{g(\lambda_j(\mathbf{p}))}{g'(\lambda_j(\mathbf{p}))}\right)^2} \\
&= \frac{\left|\frac{g(\lambda_i(\mathbf{p}))}{g'(\lambda_i(\mathbf{p}))}\right|\left(\sum_{j=1}^m \overbrace{\frac{[g'(\lambda_j(\mathbf{p}))]^2-g(\lambda_j(\mathbf{p}))g''(\lambda_j(\mathbf{p}))}{[g'(\lambda_j(\mathbf{p}))]^2}}^{<0}\overbrace{\frac{\partial\lambda_j(\mathbf{p})}{\partial p_k}}^{\substack{<0 \text{ for } j=k \\ >0 \text{ for } j\neq k}}\right)}{\left(\sum_{j=1}^m \frac{g(\lambda_j(\mathbf{p}))}{g'(\lambda_j(\mathbf{p}))}\right)^2} \\
&> \frac{\left|\frac{g(\lambda_i(\mathbf{p}))}{g'(\lambda_i(\mathbf{p}))}\right|\left(\sum_{j\neq k} \frac{[g'(\lambda_j(\mathbf{p}))]^2-g(\lambda_j(\mathbf{p}))g''(\lambda_j(\mathbf{p}))}{[g'(\lambda_j(\mathbf{p}))]^2}\frac{\partial\lambda_j(\mathbf{p})}{\partial p_k}\right)}{\left(\sum_{j=1}^m \frac{g(\lambda_j(\mathbf{p}))}{g'(\lambda_j(\mathbf{p}))}\right)^2}
\end{aligned}$$

Moreover, as $\frac{[g'(x)]^2-g(x)g''(x)}{[g'(x)]^2} > -1$ for any $x > 0$ we can further bound the derivative as follows

$$\begin{aligned}
\frac{d}{dp_k}\left[\frac{\frac{g(\lambda_i(\mathbf{p}))}{g'(\lambda_i(\mathbf{p}))}}{\sum_{j=1}^m \frac{g(\lambda_i(\mathbf{p}))}{g'(\lambda_j(\mathbf{p}))}}\right] &> -\frac{\left|\frac{g(\lambda_i(\mathbf{p}))}{g'(\lambda_i(\mathbf{p}))}\right|\sum_{j\neq k}\frac{\partial\lambda_j(\mathbf{p})}{\partial p_k}}{\left(\sum_{j=1}^m \frac{g(\lambda_j(\mathbf{p}))}{g'(\lambda_j(\mathbf{p}))}\right)^2} \\
\Bigg/ \begin{array}{c}\sum_{j=1}^m \frac{\partial\lambda_j(\mathbf{p})}{\partial p_k} = 0 \\ \text{for buyer randomization to add up}\end{array} \Bigg/ \quad &= \frac{\left|\frac{g(\lambda_i(\mathbf{p}))}{g'(\lambda_i(\mathbf{p}))}\right|\frac{\partial\lambda_k(\mathbf{p})}{\partial p_k}}{\left(\sum_{j=1}^m \frac{g(\lambda_j(\mathbf{p}))}{g'(\lambda_j(\mathbf{p}))}\right)^2} \\
/\text{using (16)}/ \quad &= -\frac{\left|\frac{g(\lambda_i(\mathbf{p}))}{g'(\lambda_i(\mathbf{p}))}\right|}{\left(\sum_{j=1}^m \frac{g(\lambda_j(\mathbf{p}))}{g'(\lambda_j(\mathbf{p}))}\right)^2}\frac{\sum_{j\neq k}\frac{g(\lambda_j(\mathbf{p}))}{g'(\lambda_j(\mathbf{p}))}}{\frac{g(\lambda_i(\mathbf{p}))}{g'(\lambda_i(\mathbf{p}))}}\frac{\partial\lambda_i(\mathbf{p})}{\partial p_k} \\
&= \frac{\sum_{j\neq k}\frac{g(\lambda_j(\mathbf{p}))}{g'(\lambda_j(\mathbf{p}))}}{\left(\sum_{j=1}^m \frac{g(\lambda_j(\mathbf{p}))}{g'(\lambda_j(\mathbf{p}))}\right)^2}\frac{\partial\lambda_i(\mathbf{p})}{\partial p_k} \\
&> \frac{1}{\sum_{j=1}^m \frac{g(\lambda_j(\mathbf{p}))}{g'(\lambda_j(\mathbf{p}))}}\frac{\partial\lambda_i(\mathbf{p})}{\partial p_k}
\end{aligned}$$

■

A.5 Proof of Proposition 4

Proof. For notational convenience I will use $\mathbf{p}$ as an argument rather that $\left(\beta_{-i}\left(\mathbf{p}_{-i}\right), \mathbf{p}_{-i}\right)$ in all calculations that follows, but the reader should keep in mind that some steps do rely on player i best responding to $\mathbf{p}_{-i}$. Since $\frac{[g'(x)]^2 - g(x)g''(x)}{[g'(x)]^2} > -1$ for any x (Lemma 1) we can bound the first term (27) as follows

$$\begin{aligned}
& \frac{1}{1-p_i}\left[\frac{\left[g_i'\left(\lambda_i\left(\mathbf{p}\right)\right)\right]^2 - g(\lambda_i\left(\mathbf{p}\right))g''\left(\lambda_i\left(\mathbf{p}\right)\right)}{\left[g_i'\left(\lambda_i\left(\mathbf{p}\right)\right)\right]^2}\right]\frac{\sum_{j\neq i}\frac{g(\lambda_j(\mathbf{p}))}{g'(\lambda_j(\mathbf{p}))}}{\sum_{j=1}^m \frac{g(\lambda_i(\mathbf{p}))}{g'(\lambda_j(\mathbf{p}))}}\frac{\partial\lambda_i\left(\boldsymbol{p}\right)}{\partial p_k} \qquad \text{(A7)}\\
& > -\frac{1}{1-p_i}\frac{\sum_{j\neq i}\frac{g(\lambda_j(\mathbf{p}))}{g'(\lambda_j(\mathbf{p}))}}{\sum_{j=1}^m \frac{g(\lambda_i(\mathbf{p}))}{g'(\lambda_j(\mathbf{p}))}}\frac{\partial\lambda_i\left(\boldsymbol{p}\right)}{\partial p_k}\\
& = -\frac{g\left(\lambda_i\left(\mathbf{p}\right)\right)}{\left(1-p_i\right)g'\left(\lambda_i\left(\mathbf{p}\right)\right)}\frac{g'\left(\lambda_i\left(\mathbf{p}\right)\right)}{g\left(\lambda_i\left(\mathbf{p}\right)\right)}\frac{\sum_{j\neq i}\frac{g(\lambda_j(\mathbf{p}))}{g'(\lambda_j(\mathbf{p}))}}{\sum_{j=1}^m \frac{g(\lambda_i(\mathbf{p}))}{g'(\lambda_j(\mathbf{p}))}}\frac{\partial\lambda_i\left(\boldsymbol{p}\right)}{\partial p_k}\\
/\text{using (15)}/ & = -\frac{\partial\lambda_i\left(\boldsymbol{p}\right)}{\partial p_i}\frac{g'\left(\lambda_i\left(\mathbf{p}\right)\right)}{g\left(\lambda_i\left(\mathbf{p}\right)\right)}\frac{\partial\lambda_i\left(\boldsymbol{p}\right)}{\partial p_k}\\
/\text{using (22)}/ & = \frac{1-e^{-\lambda_i(\boldsymbol{p})}}{\beta_i\left(\mathbf{p}_{-i}\right)e^{-\lambda_i(\boldsymbol{p})}}\frac{g'\left(\lambda_i\left(\mathbf{p}\right)\right)}{g\left(\lambda_i\left(\mathbf{p}\right)\right)}\frac{\partial\lambda_i\left(\boldsymbol{p}\right)}{\partial p_k} > -\frac{1-e^{-\lambda_i(\boldsymbol{p})}}{2\beta_i\left(\mathbf{p}_{-i}\right)e^{-\lambda_i(\boldsymbol{p})}}\frac{\partial\lambda_i\left(\boldsymbol{p}\right)}{\partial p_k},
\end{aligned}$$

where the last inequality follows as $\frac{g'(x)}{g(x)} = \frac{\frac{e^{-x}(1+x)-1}{x^2}}{\frac{1-e^{-x}}{x}} \in \left(-\frac{1}{2}, 0\right)$ for every $x > 0$. For the second part of (27) we use Lemma 2 to conclude that

$$\frac{d}{dp_k}\left[\frac{\sum_{j\neq i}\frac{g(\lambda_j(\mathbf{p}))}{g'(\lambda_j(\mathbf{p}))}}{\sum_{j=1}^m \frac{g(\lambda_i(\mathbf{p}))}{g'(\lambda_j(\mathbf{p}))}}\right] = \frac{d}{dp_k}\left[1 - \frac{\frac{g(\lambda_i(\mathbf{p}))}{g'(\lambda_i(\mathbf{p}))}}{\sum_{j=1}^m \frac{g(\lambda_i(\mathbf{p}))}{g'(\lambda_j(\mathbf{p}))}}\right] < -\frac{1}{\sum_{j=1}^m \frac{g(\lambda_j(\mathbf{p}))}{g'(\lambda_j(\mathbf{p}))}}\frac{\partial\lambda_i\left(\mathbf{p}\right)}{\partial p_k},$$

so that

$$\begin{aligned}
& \frac{g\left(\lambda_i\left(\mathbf{p}\right)\right)}{\left(1-p_i\right)g'\left(\lambda_i\left(\mathbf{p}\right)\right)}\frac{d}{dp_k}\left[\frac{\sum_{j\neq i}\frac{g(\lambda_j(\mathbf{p}))}{g'(\lambda_j(\mathbf{p}))}}{\sum_{j=1}^m \frac{g(\lambda_i(\mathbf{p}))}{g'(\lambda_j(\mathbf{p}))}}\right] > -\frac{g\left(\lambda_i\left(\mathbf{p}\right)\right)}{\left(1-p_i\right)g'\left(\lambda_i\left(\mathbf{p}\right)\right)}\frac{1}{\sum_{j=1}^m \frac{g(\lambda_j(\mathbf{p}))}{g'(\lambda_j(\mathbf{p}))}}\frac{\partial\lambda_i\left(\mathbf{p}\right)}{\partial p_k} \qquad \text{(A8)}\\
& = \left|\frac{g\left(\lambda_i\left(\mathbf{p}\right)\right)}{\left(1-p_i\right)g'\left(\lambda_i\left(\mathbf{p}\right)\right)}\right|\frac{1}{\sum_{j=1}^m \frac{g(\lambda_j(\mathbf{p}))}{g'(\lambda_j(\mathbf{p}))}}\frac{\partial\lambda_i\left(\mathbf{p}\right)}{\partial p_k}\\
& = \left|\frac{g\left(\lambda_i\left(\mathbf{p}\right)\right)}{\left(1-p_i\right)g'\left(\lambda_i\left(\mathbf{p}\right)\right)}\right|\frac{\sum_{j\neq i}\frac{g(\lambda_j(\mathbf{p}))}{g'(\lambda_j(\mathbf{p}))}}{\sum_{j=1}^m \frac{g(\lambda_j(\mathbf{p}))}{g'(\lambda_j(\mathbf{p}))}}\frac{1}{\sum_{j\neq i}\frac{g(\lambda_j(\mathbf{p}))}{g'(\lambda_j(\mathbf{p}))}}\frac{\partial\lambda_i\left(\mathbf{p}\right)}{\partial p_k}\\
/\text{using (15)}/ & = -\frac{\partial\lambda_i\left(\mathbf{p}\right)}{\partial p_i}\frac{1}{\sum_{j\neq i}\frac{g(\lambda_j(\mathbf{p}))}{g'(\lambda_j(\mathbf{p}))}}\frac{\partial\lambda_i\left(\mathbf{p}\right)}{\partial p_k}\\
/\text{using (22)}/ & = \frac{1-e^{-\lambda_i(\boldsymbol{p})}}{\beta_i\left(\mathbf{p}_{-i}\right)e^{-\lambda_i(\boldsymbol{p})}}\frac{1}{\sum_{j\neq i}\frac{g(\lambda_j(\mathbf{p}))}{g'(\lambda_j(\mathbf{p}))}}\frac{\partial\lambda_i\left(\mathbf{p}\right)}{\partial p_k} > -\frac{1-e^{-\lambda_i(\boldsymbol{p})}}{2\beta_i\left(\mathbf{p}_{-i}\right)e^{-\lambda_i(\boldsymbol{p})}}\frac{\partial\lambda_i\left(\mathbf{p}\right)}{\partial p_k}
\end{aligned}$$

as $\frac{1}{\sum_{j\neq i}\frac{g\left(\lambda_j(\mathbf{p})\right)}{g'\left(\lambda_j(\mathbf{p})\right)}} > -\frac{1}{2(m-1)} > -\frac{1}{2}$. Substituting (A7) and (A8) into (27) and then substituting (27) into (24) we find (again suppressing $\beta_i(\mathbf{p}_{-i})$ from the arguments)

$$\begin{aligned}\frac{\partial^2 \pi_i(\mathbf{p})}{\partial p_i \partial p_k} &= \frac{\partial \lambda_i(\mathbf{p})}{\partial p_k} + \beta_i(\mathbf{p}_{-i})\, e^{-\lambda_i(\mathbf{p})} - 2\left(\frac{1-e^{-\lambda_i(\boldsymbol{p})}}{2\beta_i(\mathbf{p})\, e^{-\lambda_i(\boldsymbol{p})}}\frac{\partial \lambda_i(\mathbf{p})}{\partial p_k}\right). \qquad \text{(A9)}\\ &= e^{-\lambda_i(\mathbf{p})}\frac{\partial \lambda_i(\mathbf{p})}{\partial p_k} > 0,\end{aligned}$$

which completes the proof. ■

A.6 Proof of Lemma 3

Proof. It is left to the reader to differentiate $G(k)$ with respect to k to show that $G(\cdot)$ is strictly decreasing. It is more work to demonstrate monotonicity of $H(\cdot)$. As a first step, we check that $H(k)$ is increasing at at $k = 1$ which is true as

$$\begin{aligned}\lim_{k\downarrow 1} H(k) &= \frac{\left(1-e^{-\frac{n}{2}}\right)}{\left(1-e^{-\frac{n}{2}}\left(1+\frac{n}{2}\right)\right)}\left[1-e^{-\frac{n}{2}}\left(1+\frac{n}{4}\right)\right]\\ &= \frac{\left(1-e^{-z}\right)}{\left(1-e^{-z}(1+z)\right)}\left[1-e^{-z}\left(1+\frac{z}{2}\right)\right] > 1,\end{aligned}$$

where $z = n/2 > 0$. For $0 \le k < 1$ is constant by construction, so it remains to check that $H(k)$ is increasing at $k > 1$. To do this, first notice that

$$\begin{aligned}\frac{d}{dx}\frac{1-e^{-x}}{1-e^{-x}(1+x)} &= \frac{e^{-x}}{1-e^{-x}(1+x)} - \frac{\left(1-e^{-x}\right)e^{-x}x}{\left[1-e^{-x}(1+x)\right]^2}\\ \frac{d}{dx}\left[1-e^{-x}\left(1+\frac{n}{(k+1)^2}\right)\right] &= e^{-x}\left(1+\frac{n}{(k+1)^2}\right)\end{aligned}$$

Hence,

$$
\begin{aligned}
& \frac{d}{dx}\left[\frac{1-e^{-x}}{1-e^{-x}(1+x)}\left[1-e^{-x}\left(1+\frac{n}{(k+1)^2}\right)\right]\right] \\
= & \left[\frac{e^{-x}}{1-e^{-x}(1+x)}-\frac{(1-e^{-x})e^{-x}x}{[1-e^{-x}(1+x)]^2}\right]\left[1-e^{-x}\left(1+\frac{n}{(k+1)^2}\right)\right] \\
& +\frac{1-e^{-x}}{1-e^{-x}(1+x)}e^{-x}\left(1+\frac{n}{(k+1)^2}\right) \\
= & \frac{e^{-x}}{1-e^{-x}(1+x)}\left[1-\frac{(1-e^{-x})x}{1-e^{-x}(1+x)}\right]\left[1-e^{-x}\left(1+\frac{n}{(k+1)^2}\right)\right] \\
& +\frac{e^{-x}}{1-e^{-x}(1+x)}\left(1-e^{-x}\right)\left(1+\frac{n}{(k+1)^2}\right) \\
= & \frac{e^{-x}}{1-e^{-x}(1+x)}\underbrace{\left[\frac{1-e^{-x}-x}{1-e^{-x}(1+x)}\right]}_{<0}\left[1-e^{-x}\left(1+\frac{n}{(k+1)^2}\right)\right] \\
& +\underbrace{\frac{e^{-x}}{1-e^{-x}(1+x)}\left(1-e^{-x}\right)}_{>0}\left(1+\frac{n}{(k+1)^2}\right).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
& \frac{d}{dx}\left[\frac{1-e^{-x}}{1-e^{-x}(1+x)}\left[1-e^{-x}\left(1+\frac{n}{(k+1)^2}\right)\right]\right] \\
< & \frac{e^{-x}}{1-e^{-x}(1+x)}\left(\frac{1-e^{-x}-x}{1-e^{-x}(1+x)}\right)\left[1-e^{-x}\right]+\frac{e^{-x}}{1-e^{-x}(1+x)}\left(1-e^{-x}\right) \\
= & \frac{e^{-x}(1-e^{-x})}{1-e^{-x}(1+x)}\left(\frac{1-e^{-x}-x}{1-e^{-x}(1+x)}+1\right) \\
= & \frac{e^{-x}(1-e^{-x})}{[1-e^{-x}(1+x)]^2}\left(2\left(1-e^{-x}\right)-x\left(1+e^{-x}\right)\right).
\end{aligned}
$$

Let $A(x)=2(1-e^{-x})-x(1+e^{-x})$ and note that $A(0)=2\left(1-e^{-0}\right)-0\left(1+e^{0}\right)=0$ and that

$$A'(x)=2e^{-x}-\left(1+e^{-x}\right)+xe^{-x}=e^{-x}(1+x)-1<0.$$

We conclude that $A(x)=A(0)+\int_0^x A'(z)\,dz<0$, implying that

$$\frac{d}{dx}\left[\frac{1-e^{-x}}{1-e^{-x}(1+x)}\left[1-e^{-x}\left(1+\frac{n}{(k+1)^2}\right)\right]\right]<0.$$

In conclusion we have that for $k>1$

$$
\begin{aligned}
H'(k) = & \underbrace{\left.\frac{d}{dx}\right|_{x=\frac{n}{k+1}}\left[\frac{1-e^{-x}}{1-e^{-x}(1+x)}\left[1-e^{-x}\left(1+\frac{n}{(k+1)^2}\right)\right]\right]}_{<0}\left(-\frac{n}{(k+1)^2}\right) \\
& +\frac{1-e^{-x}}{1-e^{-x}(1+x)}\left[e^{-x}\left(1+\frac{2n}{(k+1)^3}\right)\right] > 0,
\end{aligned}
$$

which completes the proof. ■