Core Selection in Auctions and Exchanges*

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Abstract

This paper characterizes divisible good environments in which core outcomes are implementable using a Vickrey auction, offering a unified treatment of auctions where trade is necessarily one-sided and those where agents and the auctioneer can both buy and sell. We show that heterogeneity in either pre-auction marginal utilities or substitution patterns (as captured by utility Hessians) can independently challenge core selection. We introduce a joint condition on preferences and equilibrium allocations which ensures that the Vickrey design yields core outcomes. Our results point to the alignment between the bidders' incentives to substitute and the trades necessary to realize the available surplus as the key to core selection. In particular, substitutability *per se* is not essential for core selection, which may fail even with a single good.

JEL CODES: D44, D47. KEYWORDS: Core-selecting auction; Package auction; Package exchange; Divisible-good auction; Complements; Substitutes; Vickrey auction.

1 Introduction

The last decade has seen significant progress concerning allocation problems, in which traders seek to buy or sell multiple heterogeneous goods. Special attention has been given to a design's ability to select a core outcome – a strong efficiency criterion and a key challenge for practical market design. With efficiency as an objective, the literature has considered core selection with respect to the Vickrey mechanism, directly or as a benchmark; in fact, Goeree and Lien (2014) show that any core selecting auction is a Vickrey auction.[1] Fundamental for theory and practice, core selection (and the Vickrey auction, in general) is well understood in

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[1] Specifically, that any Bayesian incentive compatible core selecting mechanism that obeys participation constraints is a Vickrey auction. This result is proven for indivisible goods, but can easily be shown to hold for our divisible setting as well.

one-sided markets (e.g., auctions, supply chains), where all bidders are interested in buying, for *indivisible* goods. The literature has established a close link between core selection and a goods substitutes property. In many resource allocation problems, however, bidders are interested in selling or buying various goods. Likewise, the auctioneer may wish to acquire certain goods and sell others, or specifically desire to reallocate goods among the bidders.[2] Moreover, as is the case with spectrum, electricity, and emission permits, bidders typically have complex preferences over the goods being auctioned, and the same goods may be considered substitutes by some bidders and complements by others;[3] in particular, accommodating complementarities is often essential. In his Fisher-Schultz lecture, Milgrom (2007) emphasized that the theory of package exchanges has few predictive results. For markets with multiple heterogeneous divisible goods, this paper characterizes the possibilities and limits for a core-selecting Vickrey design in such *two-sided* settings – with the bidders and the auctioneer able to both buy and sell – which allow accommodation of auctions in which bidders enter with non-zero endowments of the goods auctioned, or reallocating goods among the bidders is part of the auctioneer's objective and bidders value (bundles of) goods differently.[4]

Main results. Two-sidedness has important implications for core selection. To begin, since gains from trade among agents may exist even in the absence of the auctioneer's quantity vector, a two-sided setting creates the possibility of additional deviations by coalitions that do not include the auctioneer. We show, however, that the Vickrey auction is immune to such deviations; that is, participation is not only individually, but coalitionally rational. For core-selection in two-sided divisible settings, we provide examples demonstrating that the Vickrey payoffs may not lie in the core, even when goods are substitutes for all bidders. In fact, core selection may fail even when a single good is being auctioned – that bidders hold non-zero endowments of the goods auctioned may critically impact efficiency. On the other hand, auctions in which bidder preferences over goods may exhibit rich complementarities, for all bidders, may yield a core outcome.

We establish a core selection condition as a *joint* restriction on allocations and demands. It involves the existence of implicit *packages* (bundles of goods) which are substitutes for all bidders.[5] Mathematically, the condition requires that each bidder's allocation must be

[2] Indeed, in applications like spectrum auctions, the auctioneer is not necessarily constrained to allocate each bidder a nonnegative quantity of each good; bidders may wish to switch to a different frequency. Reallocating the spectrum controlled by television stations into uses of higher-value is part of the the objective of the Federal Communications Commission in the upcoming U.S. spectrum auction scheduled for 2015; `http://www.fcc.gov/incentiveauctions`.

[3] For instance, spectrum licenses differ in their geographical coverage and technical characteristics such as interference with adjacent frequency bands; electricity contracts differ in duration and location characteristics; government bonds have different maturities; and emission permits are issued for different time periods and pollutants.

[4] Thus, we do not require that bidders are *single-minded* (i.e., that they wish to acquire one good, or bundle of goods, and care only about winning or losing and the payment).

[5] Other authors have considered substitutes under a particular reordering of the space of goods (e.g., Sun

contained in the cone generated by the packages. Intuitively, the joint condition captures that underlying the absence of incentives to deviate by coalitions of the bidders and the auctioneer is an order of packages, *common* among bidders, that aligns the bidders' incentives to substitute with the trades necessary for the surplus in the auction, created by the participants' pre-auction endowments and the quantity auctioned, to be realized. Our core selection condition reduces to the classic substitutes condition when the packages are just the primitive goods and the auction is one-sided. In general, the condition ensures that a two-sided auction for goods *functions like a one-sided auction for packages.*

We provide additional characterizations of core selection in terms of the primitives. Whether the Vickrey auction leads to the core outcomes depends on how different bidders' substitution patterns (as captured by utility convexities (Hessians)) and marginal utilities at the initial allocation are. With sufficient heterogeneity in substitution patterns in bidder preferences, core selection fails *independently of* heterogeneity in marginal utility at the initial (pre-auction) allocations; and with sufficient heterogeneity in the initial allocations, core selection fails *independently of* heterogeneity in the substitution patterns. A preference condition establishes when we can find packages that are substitutes. We can then characterize a set of auctioned quantity vectors for which core selection holds. This design aspect – that the core-selection property depends on the quantity vector – is unique to (perfectly and imperfectly) divisible goods. Hence, insofar as the quantity vector is a choice variable for the auctioneer, it can be selected to align the trades necessary to realize the surplus with the bidders' incentives to substitute, regardless of the heterogeneity in marginal utilities at the initial allocations. These results identify settings where the a designer's choice of the Vickrey auction will not cause the incentive problems associated with the absence of core selection.

Taken together, the core-selection results and the converses we introduce offer two general insights. First, it is not substitutability *per se* that enables core selection – the Vickrey auction yields a core outcome for a rich class of bidder valuations (utility Hessians) which may exhibit complementarities and substitutabilities and vary across bidders. Rather, a homogeneity property underlying the preference substitution patterns does. With it, each bidder's quantity demanded changes in a common set of directions in response to a price change. Second, unlike one-sided (divisible or indivisible) auctions, nor is substitutability (in goods or packages) sufficient in two-sided settings. While substitutes suffice for submodularity of agents' indirect utility functions, bidder submodularity requires a separate condition – a positive package allocation – which in turn restricts the direction of the quantity auctioned.

and Yang (2006); Ostrovsky (2008); Hatfield et al. (2013)) and, more generally, Baldwin and Klemperer (2013). Such transformations have been applied in the literature allowing for selling as well as buying. For instance, the gross substitutes and complements condition of Sun and Yang (2006) and the full substitutability condition (Ostrovsky (2008), Hatfield et al. (2013)) – versions of which are also used to guarantee existence in the the auctions model of Shioura and Yang (2013) and the supply chain model of Ostrovsky (2008), respectively – both effectively specify that the environment is a transformation of a one-sided setting with substitutes. (See also Example 4.)

Our analysis of substitutability is cast in the language of cones, which we use to provide a unified treatment of one- and two-sided divisible-good settings, and analytic and geometric arguments for all main results. Generated by packages, cones can induce partial orders in which the substitutes property holds, allowing us to formulate the notion of substitutability relevant for understanding core selection in divisible-good markets (i.e., package substitutability) and one that is a condition on the primitives; identify its homogeneity as central to core selection; demonstrate that the common substitutability in bidder preferences is insufficient for core selection in two-sided markets; and may be convenient for thinking about design.

Although the Vickrey auction possesses key advantages (truthful reporting as a dominant strategy, efficient allocation), its failure to select core allocations when goods are not substitutes for all bidders and revenue performance strictly below the smallest auctioneer's revenue at any core allocation, even when all bids are high, are seen as significant limitations in practice (e.g., Ausubel (2006); Milgrom (2007)). In the U.S. spectrum auctions, for example, both efficiency and revenue are mandates. We show that in two-sided divisible good settings, not only is goods substitutability not essential for the Vickrey auction to guarantee a core outcome, but the mechanism may outperform the uniform-price design in revenue terms, particularly in the evnironments where it is likely to be core-selecting.

The literature on package allocation problems for indivisible goods recognizes two central complexity challenges. First, finding core outcomes is NP-hard in the general case. In addition, with K goods, a bidder may be asked to report values for $2^K - 1$ (non-empty) packages – unrealistic, even if K is in the low double-digits.[6] For package allocation problems with divisible goods, the difficulty of computing core outcomes is largely absent, but the problem of asking bidders to consider a high-dimensional bid space is worsened: bidders may be asked to submit demand schedules for all goods as a function of all prices, $\mathbb{R}^K \to \mathbb{R}^K$, or, equivalently, their valuations over the entire space of feasible allocations. However, the demand structure (common package substitutability) we identify as facilitating core selection can enable a reduction in the dimensionality of the bid space by removing the requirement that agents make fully contingent bids. Instead, the designer can equivalently ask bidders to submit bids ($\mathbb{R} \to \mathbb{R}$) in K independent Vickrey auctions for packages. This is ultimately useful to the extent that these packages have a meaning in practical applications. We argue that this is the case in many high-stakes applications, including spectrum and electricity auctions.

Related literature. The literature on core selection has largely focused on settings with indivisible goods. This paper contributes to the analysis of core selection and the role of substitutes in auctions with multi-unit demands. An important predecessor is Milgrom and Strulovici (2009), who examine one-sided divisible good auctions.

In response to the low revenue problem of the Vickrey auction, the literature has explored

[6] De Vries and Vohra (2003) survey complexity issues associated with package designs.

core-selecting auctions (Day and Milgrom (2008)), which allow for compromising on strategy-proofness to attain revenue objectives by modifying transfers.[7] Milgrom and Segal (2014) require strategy-proofness but not the efficiency of the outcome given the bids.

In environments with indivisible or imperfectly divisible goods, the literature has demonstrated that substitutes properties are useful.[8] In a recent contribution, Baldwin and Klemperer (2013) provide a geometrically beautiful characterization of equilibrium existence in a very general framework which admits several canonical models as special cases. Like us, the authors also allow for negative allocations in indivisible-good settings, allow for complements, and recognize that when the substitutes condition holds for a transformed problem under a change of basis, many results continue to hold. However, they focus on existence and do not study core selection, which, as we show, requires conditions more stringent than substitutes when reallocation is feasible or when the property holds under a change of basis. Additionally, while the notion of demand types in Baldwin and Klemperer (2013) allows both substitutes and complements, the condition that the authors place on demand types in perfectly competitive environments with imperfectly divisible goods to get existence (not an order-theoretic result) is not order-theoretic.[9] Our analysis of core selection (an order-theoretic result) points to the importance of an order-theoretic property of preferences, which admits goods complementarity and substitutability.

Structure of the paper. Section 2 introduces the setting. Section 3 discusses the link between core selection and bidder submodularity in two-sided settings. Section 4 develops the main results: the motivating examples and characterization of core selection in two-sided settings. Section 5 presents the converses. Section 6 discusses extensions. Section 7 establishes complexity reduction and revenue results. All proofs are contained in the Appendix.

[7] Variants of core-selecting auctions include menu auctions (the pay-as-bid (Bernheim and Whinston (1986)), the ascending proxy auction (Ausubel and Milgrom (2002), Parkes and Ungar (2000)), the assignment auction (Milgrom (2009)), and the mechanisms resulting from the core computations in Day and Raghavan (2007), Day and Cramton (2012), Erdil and Klemperer (2010), and Ausubel and Baranov (2010). In light of the Vickrey auction's poor revenue performance outside of settings where goods are substitutes, the core-selecting auctions are viewed as a potential alternative to the Vickrey design in practice, if the problem of communication complexity can be addressed (e.g., Day and Milgrom (2008)). Articles on settings with divisible goods include Ausubel (2006), who proposed a dynamic auction design that achieves Pareto efficiency, and Ausubel and Cramton (2004) who examine design properties of simultaneous clock auctions.

[8] Integer programming problems, being over a nonconvex set, are not always amenable to the strong duality properties on which classical results from general equilibrium are dependent. Variations of the substitutes property can get around this problem; indeed, M-concavity, which is a strengthening of the substitutes property, guarantees that strong duality holds – see Murota (2003). Such properties guarantee existence of competitive equilibrium in indivisible Walrasian economies (Gul and Stacchetti (1999)), stable outcomes in matching models (Hatfield and Milgrom (2005), Ostrovsky (2008), Hatfield et al. (2013)), in addition to ensuring that the Vickrey auction is core-selecting (Ausubel and Milgrom (2002)).

[9] Rather, it is a condition on the integer basis for the full set (span) of changes in demand that can occur.

2 Setting

An auctioneer has a vector $Q \in \mathbb{R}^K$ of K perfectly divisible goods to sell to N heterogeneous agents who may participate in this auction. Throughout, we denote the set and its cardinality by the same symbol. Each agent i has a valuation for quantity obtained in the auction[10] q_i and payments (transfers) to the auctioneer x_i given by $u_i(q_i) - x_i$, where $u_i : \mathbb{R}^K \to \mathbb{R}$ is twice continuously differentiable and strictly concave and the images of the marginal utilities, $\nabla u_i(\mathbb{R}^K)$, are the same for all agents.[11] With respect to applications, our setting makes two innovations. We do not restrict the feasible allocations to be non-negative. This allows accommodation of environments in which bidders and the auctioneer can buy or sell; for instance, when bidders have non-zero endowments or short-selling is allowed. Additionally, this allows us to consider settings in which the auctioneer seeks to reallocate goods among the bidders, purchase goods (a procurement auction[12]) or has no quantity for sale (a proper exchange).

We refer to an auction in which each agent is only able to purchase goods from the auctioneer (who is looking to sell them) or only able to sell goods to the auctioneer (who is looking to buy them in a procurement auction) as *one-sided*. Formally, a one-sided auction is one where bidders' auction allocations q_i are constrained to lie in the positive or negative orthant.

(Two-sided) Vickrey auction. The most widely studied mechanism for efficient resource allocation in transferable utility environments such as ours is the (generalized) Vickrey auction. When $Q = 0$, the two-sided auction becomes what has been referred to as a *Vickrey exchange*.

A *two-sided (demand-implemented)*[13] *Vickrey auction* of K perfectly divisible goods asks bidders to submit (net) demand curves $b_i : \mathbb{R}^K \to \mathbb{R}^K$ that are C^1 and strictly downward-sloping (i.e., have negative definite Jacobians). We assume that if only one agent were present, the auction is cancelled and no goods are allocated;[14] thus, in the analysis to follow, there

[10] We can think of u_i as a reduced form describing the agents' initial endowments $\{q_i^0\}_i$ and preferences over total quantities $\hat{u}_i$; then $u_i(q_i) \equiv \hat{u}_i(q_i^0 + q_i) - \hat{u}_i(q_i^0)$ and, in particular, $u_i(0) = 0$. We work directly with $u_i(q_i)$, since the core-selection properties depend only on $u_i(q_i)$. Our information assumptions are minimal; we only assume that agents know their own preferences u_i.

[11] This ensures existence of a competitive equilibrium allocation in finite quantities and that aggregate demand is everywhere well-defined. This is far stronger than necessary for equilibrium existence, but allows us to focus on core selection using the language of demand and supply without running into issues related to invertibility of demands and supplies.

[12] While procurement auctions are one-sided, they have not to our knowledge been considered in the core selection literature. It turns out that in these auctions a more careful definition of core selection is necessary – see Section 3.

[13] It is useful to understand the Vickrey auction by thinking about its implementation by asking agents to submit demand functions $\mathbb{R}^K \to \mathbb{R}^K$, as we do here in our characterization of the payment rule. Given strict concavity, this is equivalent to implementation by requiring agents to submit utility functions $\mathbb{R}^K \to \mathbb{R}$, as is common in the literature.

[14] In one-sided settings, the convention in the literature is to assume that if only one bidder is present, he receives the entire quantity being auctioned for free. In a two-sided setting where $Q < 0$, this would imply

are at least two participating bidders. Otherwise, aggregating the demands of $j \neq i$, gives the (inverse) residual supply curve $p = R_i(q_i)$ of bidder i; bidders receive the market-clearing quantity, i.e., the unique solution to $R_i(q_i) = b_i(q_i)$, and are charged a payment of $\int_0^{q_i} R_i(\mathbf{r}) \cdot d\mathbf{r}$ (Figure 1). Hence, a bidder's payoff from receiving q_i is

$$\int_0^{q_i} (\nabla u_i(\mathbf{r}) - R_i(\mathbf{r})) \cdot d\mathbf{r}.$$

Geometrically, residual supply is the gradient of the value function of other agents in the coalition – the marginal cost of reallocating q_i to i; it is the price that would clear the market for $Q - q_i$ were i not present.[15] If bidder i could optimize ex post, his first-order condition would equalize his marginal utility and the marginal payment:

$$\nabla u_i(q_i) - R_i(q_i) = 0.$$

Hence, submitting the inverse demand function $b_i(q_i) = \nabla u_i(q_i)$ is a weakly dominant strategy for i, since it causes the ex post FOC to be satisfied for *any* realization of residual supply.

Together with market clearing, the bidders' demand functions $b_i(q_i)$ give us the Kuhn-Tucker conditions for the grand coalition's optimization problem, and thus the divisible-good Vickrey auction yields an efficient allocation. The reader may wonder: why run a two-sided auction instead of two one-sided Vickrey auctions, whose properties are well understood in the literature? We will show that while resulting in the same allocation, the standard design would compromise the auctioneer's revenue (Section 7.2).

Coalitional value function and the core. To characterize the *core* of the environment described above, let us label the auctioneer agent 0 and, for any *coalition* $W \subseteq N$, define the *surplus function* $v : 2^N \times \mathbb{R}^K \to \mathbb{R}$ as the maximum value of the goods allocation among the members of W,

$$v(W, Q) = \max_{\{q_i\}_{i \in W}} \sum_{i \in W} u_i(q_i) \text{ s.t. } \sum_{i \in W} q_i = Q.$$

For each i, the *efficient allocation* $q_i^* : 2^{N+1} \times \mathbb{R}^K \to \mathbb{R}^K$ is one that maximizes the total utility

that a single bidder whose preferences satisfy free disposal is better off in autarky than participating in the Vickrey auction. Therefore, we assume that with only one bidder present, the auctioneer abandons the Vickrey mechanism, since the knowledge of the agent's preferences that it is designed to reveal is then unnecessary to determine the efficient allocation. This is without loss for our results when $Q > 0$.

[15] The aggregate quantity supplied to i at p is $Q - \sum_{j \neq i} b_j^{-1}(p)$. Given the assumption that the $\nabla u_j(\mathbb{R}^K)$ are the same, this will be a surjective and upward-sloping function on $\nabla u_j(\mathbb{R}^K)$ under truthful reporting, and we can define $R_i : \mathbb{R}^K \to \mathbb{R}^K$ by $R_i(q_i) = \left(Q - \sum_{j \neq i} (\nabla u_j)^{-1}\right)^{-1}(q_i)$. To complete the definition of the game, we assume that if any (inverse) residual supply R_i is not well-defined, the auction is cancelled; this has no effect on incentive compatibility.

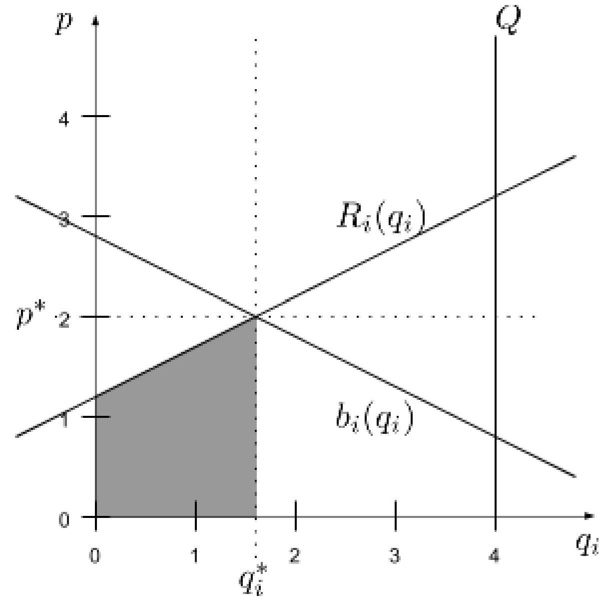

FIGURE 1: VICKREY AUCTION PAYMENT, ALLOCATION, RESIDUAL SUPPLY

Notes: The Vickrey auction can be implemented by soliciting downward-sloping (net) demand schedules and clearing the market as in a standard uniform price auction, but charging bidders the area under their residual supply curves, eliminating bidders' price impact and incentivizing them to bid sincerely. Bidder i's residual supply aggregates the other agents' quantity demanded, for every price and mirrors the result about the vertical line at Q. The equilibrium auction allocation q_i^* is then determined by the point where i's bid schedule intersects residual supply. Bidder i's payment corresponds to the area below his residual supply curve from 0 to q_i^*; this is the opportunity cost of not allocating q_i^* to the other agents.

of the bidders,

$$\{q_i^*(W,Q)\}_{i\in W} = \arg\max_{\{q_i\}_{i\in W}} \sum_{i\in W} u_i(q_i) \text{ s.t. } \sum_{i\in W} q_i = Q.$$

The surplus function allows us to characterize agents' payoffs in the Vickrey auction as their *marginal product to the grand coalition*, $v(N) - v(N \setminus i)$.[16,17] Now define the *coalitional value*

[16] This formulation is more commonly used in the literature, and indeed, as alluded to in an earlier footnote, the Vickrey auction can be equivalently implemented by soliciting agents' valuations, allocating the quantity vector efficiently, and charging a payment of $u_i(q_i^*(W,Q)) - v(N) + v(N \setminus i)$.

[17] It is useful to recall why this payment rule results in a payoff of $\pi_i = v(N,Q) - v(N \setminus i, Q)$: $R_i(q_i)$ is the marginal cost to the coalition $N \setminus i$ of assigning q_i to i, $-\nabla v(N \setminus i, Q - q_i)$. Hence,

$$v(N\setminus i,Q) - \int_0^{q_i^*(N,Q)} R_i(\mathbf{r})\cdot d\mathbf{r} = v(N\setminus i, Q - q_i^*(N,Q)) \Leftrightarrow \int_0^{q_i^*([0,N])} R_i(\mathbf{r})\cdot d\mathbf{r} = v(N\setminus i,Q) - v(N\setminus i, Q - q_i^*(N,Q)).$$

Since $v([0,N],Q) = v(N \setminus i, Q - q_i^*(N,Q)) + u_i(q_i^*(N,Q))$, we have $\pi_i = q_i(q_i^*(N,Q)) - \int_0^{q_i^*(N,Q)} R_i(\mathbf{r}) \cdot d\mathbf{r} = v(N,Q) - v(N \setminus i, Q)$.

function $V : 2^{N+1} \times \mathbb{R}^K \to \mathbb{R}$ by

$$V(W, Q) \equiv \begin{cases} v(W, Q), & 0 \in W \\ v(W, 0), & 0 \notin W \end{cases} .$$

The *core* is the set of payoff profiles such that no coalition can be better off by abandoning the mechanism and trading on its own. A payoff profile π is in the core for participants Z if $\sum_{i \in Z \cup 0} \pi_i = v(Z)$ and $\sum_{i \in W} \pi_i \geq V(W)$ for all $W \subset Z \cup 0$. If $\sum_{i \in W} \pi_i < V(W)$ we say that W *blocks* π. Note that we are not assuming that $v(W, Q) = 0$ if $0 \notin W$; that is, that goods cannot be redistributed among the bidders without the auctioneer. In the two-sided auction, unless reallocation of initial endowments among agents is unnecessary (e.g., initial allocation is efficient) or infeasible (e.g., the auction is one-sided) and the only surplus comes from allocating Q, agents may be able to realize gains from trade without benefit of the auctioneer's quantity vector.

3 Core Selection and Bidder Submodularity

Perhaps the main question that has returned attention to the Vickrey auction over the past decade concerns the conditions under which it is *core-selecting*, i.e., it yields a payoff profile that lies in the core, ensuring that no coalition can profitably persuade the auctioneer to cancel the auction and award them the good. This has been shown to reduce to the question of whether the coalitional value function $v(W, Q)$ is submodular in its first argument under the usual set order $\subseteq$ (Ausubel and Milgrom (2002)): $v(W, Q)$ is *bidder-submodular* for the quantity vector Q if, for all coalitions $W \subseteq N$ that include bidder j, $v(W, Q) - v(W \setminus j, Q)$ does not increase when more bidders are added to W. Unlike one-sided auctions, in two-sided settings, to ensure core selection, we need to consider coalitions that do not include the auctioneer: bidders may have a profitable joint deviation to cease their participation in the auction and trade among themselves. As it turns out, however, bidders always prefer to participate in the Vickrey auction than negotiate on their own.

Lemma 1 (**Coalitional Rationality of Participation**). *Coalitions which do not involve the auctioneer never block the Vickrey payoff profile.*

This new result shows that participation in the two-sided Vickrey auction is not only individually rational, but *coalitionally* rational as well. For intuition, observe that setting $Q = 0$ and considering the grand coalition yields the following corollary:

Corollary 1 (**Vickrey (1961)**). *The auctioneer's revenue in a Vickrey exchange is never positive.*

This classic no budget balance result by Vickrey (1961) implies that when $Q = 0$, and thus the mechanism is only reallocating, agents are collectively better off than without the auctioneer, as the auctioneer effectively subsidizes them. Lemma 1 generalizes this insight, showing that *every* coalition is so subsidized for *any* Q. This need not be read as a negative statement about Vickrey revenue: indeed, our analysis in section 7.3 will imply that when heterogeneity in the agents' marginal utilities at the initial endowment (i.e., with $Q = 0$) is sufficiently small relative to Q, the auctioneer will make a profit (indeed, he will make greater revenue than he would with the common design of the uniform-price auction).

The many reasons to care whether a design is core selecting are well understood from one-sided indivisible good settings (e.g., Ausubel and Milgrom (2002), Milgrom (2007)). Those most frequently mentioned are that (a) it prevents manipulation by the bidders: shill bidding is unprofitable (Yokoo and Matsubara (2004), Day and Milgrom (2008)); (b) the auctioneer's revenue is monotone in the number of bidders; and (c) no coalition of losing bidders can profitably collude. Note that the last of these does not readily carry over to the divisible-goods case, where the idea of a "losing bidder" is unnatural – typically, all bidders are allocated some quantity – and collusion among bidders can be profitable.[18] For the others, however, we can draw similar conclusions about two-sided divisible good auctions.

Lemma 2 (**Implications of Core Selection**). *The Vickrey auction is core selecting* regardless of which agents participate *if, and only if,* $v(\cdot, Q)$ *is submodular.*

Hence, if the Vickrey auction is core selecting for any participating agents,

(i) The auctioneer's revenue is increasing in the number of participating bidders;

(ii) If the potential shill bidder is not the only participant in the auction, shill bidding is unprofitable regardless of which agents participate.[19]

Since, in keeping with classical auction theory, we do not in general assume that the auctioneer has preferences over the goods he sells, the definition of the core requires the auctioneer to receive positive revenue so that $\{0\}$ does not block the payoff profile. This is unimportant in one-sided settings with $Q > 0$, where the auctioneer always receives non-negative revenue. To study core-selection in two-sided settings and in one sided settings for procurement auctions ($Q < 0$), we will also consider weaker conditions under which the payoff profile is *unblocked by all* $W \neq 0$. We say that the Vickrey auction is *core-selecting with respect to bidders* if the payoff profile is unblocked by every coalition except for $\{0\}$, the auctioneer alone.

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[18] Indeed, it is straightforward that upon learning the outcome of the auction or having some estimate where the auction would end, any coalition can profitably collude by submitting steeper demand curves that still clear the market at the same allocation so as to reduce its total payments. This contrasts with indivisible-good auctions, where profitable deviations by losing bidders are absent regardless of the bidder's knowledge of the market.

[19] More precisely, 'unprofitable shill bidding' means that if v is submodular *for any* utility profile $\{u_i\}_{i=1}^N$ in some type space U, then mimicking *any* set of agents with *any* types in U is unprofitable.

Corollary 2. *The Vickrey auction is core-selecting with respect to bidders regardless of which agents participate if, and only if, $v(\cdot, Q)$ is submodular on the sublattice $2^{N\backslash i} \times \{i\}$ for each i.*

Hence, if the Vickrey auction is core selecting with respect to bidders for any participating agents, when at least two bidders participate,

(i) The auctioneer's revenue is increasing in the number of participating bidders;

(ii) Shill bidding is unprofitable regardless of which agents participate.

Saying that $v(\cdot, Q)$ is submodular on each $2^{N\backslash i} \times \{i\}$ is equivalent to saying that $v(W, Q) + v(Z, Q) \geq v(W \cup Z, Q) + v(W \cap Z, Q)$ for all W, Z such that $W \cap Z \neq \emptyset$.[20]

4 Core Selection with Divisible Goods

This section contains the main results of the paper. We begin with examples that illustrate the key new aspects of core selection with divisible goods.

4.1 Motivating examples

The literature has established a close connection between core selection (bidder submodularity)[21] and the substitutes property of agents' demands.

Definition 1. Goods are *(weak) substitutes* if the quantity each agent demands of any good is nondecreasing in the price of every other good.

DO WE HAVE REASONS TO THINK THAT CORE SELECTION MIGHT *not* HOLD WITH DIVISIBLE GOODS? Existing results show that when goods are substitutes and allocations are restricted to be nonnegative, bidder submodularity holds and, thus, the Vickrey auction is core-selecting (e.g., Milgrom and Strulovici (2009, Theorem 31)). We first show by example that, in the context of a broader set of applications, this restriction on feasible allocations is crucial for core selection: when the bidders and the auctioneer can buy and sell, the Vickrey outcome may not lie in the core *even in settings with only one good.* In the examples that follow, we will be working with quadratic valuations

$$u_i(q_i, \theta_i, S_i) = \theta_i' q_i - \frac{1}{2} q_i' S_i q_i, \tag{1}$$

[20] This corollary then follows from the observation that for potential blocking coalitions other than the auctioneer alone, none of the terms in the proof of the first part take the empty set as an argument. The same is true in the second part when at least two bidders participate.

[21] Bidder submodularity is necessary when the objective is core selection for any agents as opposed to a given set of participants.

where S_i is positive definite, for which the coalitional value function and equilibrium quantities are given in closed form.[22]

Lemma 3 (**Coalitional Value Function: Quadratic Utility**). *With quadratic valuations, the coalitional value function is*

$$v(W,Q) = \frac{1}{2}\sum_{i\in W}\theta_i' S_i^{-1}\theta_i - \frac{1}{2}\left(\sum_{i\in W} S_i^{-1}\theta_i - Q\right)' H(W)\left(\sum_{i\in W} S_i^{-1}\theta_i - Q\right),$$

where $H(W) \equiv \left(\sum_{j\in W} S_j^{-1}\right)^{-1}$ *is a harmonic mean and the competitive equilibrium quantity of bidder* i *is* $q_i^*(W,Q) = S_i^{-1}\theta_i - S_i^{-1}H(W)\left(\sum_{j\in W} S_j^{-1}\theta_j - Q\right)$.

In particular, note that when $\theta_i = \theta$ for each $i \in N$, $v(W)$ simplifies to $\theta' Q - \frac{1}{2}Q' H(W) Q$.[23]

Example 1 (**No Core Selection with a Single Divisible Good**). An auctioneer has $Q = 1$ units of a divisible good for sale in a Vickrey auction. Consider bidders 1 and 2 with valuations $u_i(q_i) = 4q_i - \frac{1}{2}q_i^2$ and bidder 3 with valuation $u_3(q_3) = 2q_3 - \frac{1}{2}q_3^2$. Then, using Lemma 3, allocations are

$$\begin{aligned}
q_1^*(\{1,2,3\},Q) = q_2^*(\{1,2,3\},Q) &= 4 - \frac{1}{3}(4+4+2-1) = 1 & q_3^*(\{1,2,3\},Q) &= -1\\
q_1^*(\{1,3\},Q) = q_2^*(\{2,3\},Q) &= \frac{3}{2} & q_3^*(\{1,3\},Q) &= -\frac{1}{2}\\
q_1^*(\{1,2\},Q) = q_2^*(\{1,2\},Q) &= \frac{1}{2} & q_3^*(\{2,3\},Q) &= -\frac{1}{2},
\end{aligned}$$

and surpluses $v(W,Q)$ are

$$\begin{aligned}
v(\{1,2,3\},Q) &= \frac{1}{2}(16+16+4) - \frac{1}{6}(4+4+2-1)^2 = \frac{9}{2} & v(\{1,3\},Q) = v(\{2,3\},Q) &= \frac{15}{4}\\
v(\{1,2\},Q) &= \frac{15}{4} & v(\{1\},Q) &= \frac{7}{2}.
\end{aligned}$$

[22] The quadratic model is the commonly studied environment in the divisible good auctions and microstructure literature; the innovation here is allowing for heterogeneity in the utility convexitites $\{S_i\}_{i=1}^N$. By considering agents' preferences over auction allocations u_i instead of over total allocations $\hat{u}_i(q_i) \equiv \hat{\theta}_i{}' q_i - \frac{1}{2}q_i' S_i q_i$, (1) reduces heterogeneity in endowments $\{q_i^0\}_{i=1}^N$ and the constant term of marginal utility into heterogeneity in the latter alone (cf. Ft. 10): by letting $\theta_i = \hat{\theta}_i - S_i q_i^0$, we have

$$u_i(q_i,\theta_i,S_i) = \hat{u}_i(q_i+q_i^0,\hat{\theta}_i,S_i) - \hat{u}_i(0,\hat{\theta}_i,S_i) = \hat{\theta}_i'(q_i+q_i^0) - \frac{1}{2}(q_i+q_i^0)' S_i (q_i+q_i^0) - (\hat{\theta}_i' q_i^0 - \frac{1}{2}q_i^0{}' S_i q_i^0).$$

Hence, in the quadratic model, agent i is characterized by the tuple of characteristics (θ_i, S_i) (or equivalently, $(\hat{\theta}_i, q_i^0, S_i)$), where S_i is positive definite.

[23] Also note that the difference-in-difference $v(W\cup Z) + v(W\cap Z) - v(W) - v(Z)$ is just $\frac{1}{2}Q'(H(W) + H(Z) - H(W\cap Z) - H(W\cup Z))Q$, and that the central matrix in this expression is the difference-in-difference of the inverse slopes of coalitions' aggregate demands.

Thus, the sum of Vickrey payoffs for the coalition of the auctioneer and bidder 1 is $\pi_0 + \pi_1 = -v(\{0,1,2,3\}) + v(\{0,1,2\}) + v(\{0,1,3\}) = 3 < v(\{0,1\})$ and so the coalition $\{0,1\}$ blocks the Vickrey payoff imputation.[24]

We can glean from the literature the fact that if each of the agents' auction allocations were positive – that is, if the auction behaved like a one-sided one – that the payoff imputation would be in the core. Instead, bidder 3 ends up selling some of her initial endowment in the Vickrey auction. Since the Vickrey payment rule ensures that the auctioneer pays the bidder more for this amount than he ultimately receives from the other bidders from selling it, this makes accepting an offer to cancel the auction and avoid reallocation attractive; the auctioneer could award all of the quantity to bidder 1 to receive a payment larger than the Vickrey revenue.[25] However, if the *marginal* utilities at zero auction allocation were identical (i.e., initial allocations or marginal utility intercepts were identical), reallocation would be unnecessary and the outcome would be in the core (cf. Theorem 3).

Conversely, goods that are *not* substitutes are generally viewed as a challenge to core selection. We show that the absence of substitutes, even for all bidders, does not preclude Vickrey payoffs from being a core point.

Example 2 (Core Selection with Complement and Substitute Goods Valuations). Suppose a seller wishes to auction vector $[1\ 1]'$ of two goods using a Vickrey auction. Consider bidder 1 with valuation

$$[1,1]q_i - \frac{1}{2}q_i' \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix} q_i,$$

and bidders 2 and 3 with valuations

$$[1,1]q_i - \frac{1}{2}q_i' \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix} q_i.$$

Goods are complements for bidder 1, but substitutes for bidders 2 and 3. Let us calculate

[24] To the best of our knowledge, our point about reallocation being a challenge core selection is new. The divisible goods results of Milgrom and Strulovici (2009) focus on one-sided auctions and do not consider quantity vectors outside the positive orthant, as do the indivisible goods results of Ausubel and Milgrom (2002).

More specifically, in the argument for divisible goods in Milgrom and Strulovici (2009, Theorem 31), one step involves showing that descreasing differences in $v(\cdot, Q)$ are implied by $v(W,Q) - v(W,Q')$, $Q' \geq Q$, being nondecreasing in W. This implication follows from the grand coalition receiving fewer goods in aggregate when an additional agent participates, which is true only when that additional agent's allocation is positive.

[25] Note that the bidder who successfully forms a coalition with the auctioneer that blocks the Vickrey payoff need not be the one whose valuation is the highest. In Example 1, if the marginal utility intercepts of bidders 1 and 2 were $4-\varepsilon$ and $4+\varepsilon$, respectively, then $\{0,1\}$ would still form a blocking coalition.

allocations $q_i^*(W,Q)$,

$$q_2^*(\{1,2,3\},Q) = q_3^*(\{1,2,3\},Q) = \begin{bmatrix} 0.2857 \\ 0.2857 \end{bmatrix} \qquad q_1^*(\{1,2,3\},Q) = \begin{bmatrix} 0.4286 \\ 0.4286 \end{bmatrix}$$
$$q_2^*(\{1,2\},Q) = q_3^*(\{1,3\},Q) = \begin{bmatrix} 0.4 \\ 0.4 \end{bmatrix} \qquad q_1^*(\{1,2\},Q) = \begin{bmatrix} 0.6 \\ 0.6 \end{bmatrix}$$
$$q_2^*(\{2,3\},Q) = q_3^*(\{2,3\},Q) = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} \qquad q_1^*(\{1,3\},Q) = \begin{bmatrix} 0.6 \\ 0.6 \end{bmatrix},$$

and surpluses $v(W,Q)$,

$$v(\{1\},Q) = \frac{3}{2}, \quad v(\{2\},Q) = v(\{3\},Q) = \frac{5}{4},$$
$$v(\{2,3\},Q) = \frac{13}{8}, \quad v(\{1,2\},Q) = v(\{1,3\},Q) = \frac{17}{10}, \quad v(\{1,2,3\},Q) = \frac{25}{14}.$$

Then bidder 1 receives a payoff of $\frac{9}{56}$, bidders 2 and 3 receive a payoff of $\frac{3}{35}$, and thus the auctioneer receives $\frac{25}{14} - \frac{9}{56} - \frac{6}{35} = \frac{407}{280}$; this is positive, so the auctioneer alone does not block the Vickrey payoff profile. Trivially, no coalition of all but one agent blocks the Vickrey payoffs. We also have $\frac{3}{2} - \frac{9}{56} - \frac{407}{280} = -\frac{4}{35} < 0$ and $\frac{5}{4} - \frac{3}{35} - \frac{407}{280} = -\frac{81}{280} < 0$, so no coalition of one bidder and the auctioneer blocks the Vickrey payoffs. Since we know from Lemma 1 that no coalition which does not involve the auctioneer blocks the Vickrey payoff profile, it follows that the Vickrey payoffs are in the core. Clearly, bidders 2 and 3 could have the same complement valuation as bidder 1, and the Vickrey auction would still be core-selecting.[26]

Why does this example work? The necessity of the substitutes condition for the Vickrey payoffs to lie in the core has been demonstrated in a variety of environments. We will show that it is not the presence of complementarities *per se*, but instead a type of heterogeneity in *substitution patterns* – the second-order derivatives of agents' utility functions – that can challenge core selection. Anticipating our results, there is a sense in which substitution patterns are symmetric across bidders in Example 2: the two utility Hessian matrices commute, since both have the same eigenvectors $[1\ 1]'$, $[-1\ 1]'$.[27] The commutativity property allows transforming the problem from one with heterogeneous substitutability and complementarity between goods to one with substitutability between packages and, as in the example, achieves core selection. Importantly, the ability to transform the problem into one with strict substitutability does *not* generally, by itself, render the Vickrey auction core-selecting. Rather, what matters is that the substitution patterns are not too different across agents, as the next

[26] Baldwin and Klemperer (2013) provide an example with complements to illustrate equilibrium existence. Here, we analyze core selection and explore the role of (heterogeneity in) bidder preferences.

[27] Two matrices A and B commute if $AB = BA$. Diagonalizable matrices A and B commute if, and only if, they have the same eigenspace.

example demonstrates.

Example 3 (No Core Selection with Substitutes and Efficient Initial Allocation). An auctioneer wishes to sell 1 unit of good 1 and 0 units of good 2 using a two-sided Vickrey auction; that is, he wishes to auction the vector $Q = [1, 0]$.[28] Three bidders with the following valuations participate

$$[10\ 10]q_1 - \frac{1}{2}q_1' \begin{bmatrix} 0.25 & 0.19 \\ 0.19 & 1.49 \end{bmatrix} q_1, \quad [10\ 10]q_2 - \frac{1}{2}q_2' \begin{bmatrix} 5.8 & 0 \\ 0 & 5.8 \end{bmatrix} q_2, \quad [10\ 10]q_3 - \frac{1}{2}q_3' \begin{bmatrix} 3.07 & 3.47 \\ 3.47 & 4.47 \end{bmatrix} q_3.$$

Note that endowments and utility functions are such that no gains to trade exist prior to the auction (i.e., marginal utilities at zero trade are the same across bidders). By Lemma 3, since the initial allocation is efficient, if

$$Q'H(\{0,1,2\})Q + Q'H(\{0,1,3\})Q < Q'H(\{0,1\})Q + Q'H(\{0,1,2,3\})Q,$$

then

$$\pi_0 + \pi_1 = -v([0,3]) + v([0,2]) + v(\{0,1,3\}) < v(\{0,1\})$$

and the coalition $\{0, 1\}$ blocks the Vickrey allocation. This is exactly what happens, since

$$Q'\left(H(\{0,1\}) + H(\{0,1,2,3\}) - H(\{0,1,2\}) - H(\{0,1,3\})\right)Q = Q' \begin{bmatrix} -0.0015 & 0.0146 \\ 0.0146 & 0.2398 \end{bmatrix} Q = -0.0015.$$

In this example, a positive quantity is auctioned and goods are substitutes for all bidders. Thus, if the new quantity did not require reallocation of the goods among the bidders and only selling, we know from the literature that core selection would hold. Because the bidders' substitution patterns are heterogeneous, the new supply of good 1 creates gains to trade for the bidders from reallocating good 2 and, despite the absence of gains to trade before the auction, the reallocation is sufficient for core selection to break down.

These examples demonstrate that, in a two-sided Vickrey auction, goods substitutability is neither necessary (Example 2) nor sufficient (Examples 1 and 3) for a core-selecting design. Moreover, the heterogeneity in substitution patterns, as captured by utility convexities (Example 3), and *pre-auction gains from trade*, as determined jointly by marginal utilities at agents' initial allocations (Example 1), can separately challenge core selection.[29] In our

[28] Let us remark for a future reference that the logic of this example can be mimicked for the case when the auctioneer wishes to exchange 0.5 units of good 1 for 0.9 units of good 2 using a two-sided Vickrey auction; that is, he wishes to auction the vector $Q = [0.5, -0.9]$.

[29] That is, with sufficient heterogeneity in substitution patterns, core selection fails *regardless* of homogeneity in marginal utility at the initial endowment (Example 3); and with sufficient heterogeneity in the marginal utilities, core selection fails *regardless* of homogeneity in the substitution patterns (Example 1). Clearly, core selection can obtain with small enough heterogeneity in substitution patterns across bidders – e.g., when the

subsequent analysis, we work to characterize these challenges, and in so doing establish the reasons that the Vickrey payoffs are in or outside the core in the above examples.

We characterize the sense in which whether the Vickrey auction fails to be core-selecting depends on heterogeneity in substitution patterns, both across bidders and across allocations in the domain of $u_i(\cdot)$. In Section 4.2, we identify the notion of substitutability that is relevant for core selection in two-sided divisible good auctions; in Section 4.3, we add a condition on allocations.

4.2 Generalizing the substitutes property

The demand-theoretic substitutes property is a monotonicity property in the partial order $\geq$, relating changes in price to changes in quantity. When agents' preferences satisfy the substitutes condition, there are two related consequences:

- When the price of good ℓ increases, agents buy more of good j.
- When price vector for goods changes in a certain direction, agents' demand vectors for goods each move in a certain similar direction.

The literature has focused on the former, as is natural in one-sided auctions; indeed, as will be apparent later, in this case the weaker second condition is of limited use. The latter is, however, important in two-sided auctions, where we show that one can get additional mileage toward understanding when core selection holds from its weaker requirements.[30] Explaining the content of the latter at the primitive level is the focus of this section. The crux of our argument is that it is primarily the second of these that gives the substitutes condition its power, and hence that the choice to order the space of goods by $\geq$ does not constrain us when proving core selection. We show that one can use the substitutes monotonicity property in *any* lattice order on $\mathbb{R}^K$ to achieve core selection. In particular, we can endow $\mathbb{R}^K$ with a partial order generated by *packages* – bundles of goods – analogously to how $\geq$ is generated by goods. We show that when we can find such packages which are substitutes, they specify a change of basis which, when applied in either a one- or two-sided auction, allows us to aggregate bidders' indirect utility functions into a submodular objective function for the social planner's dual minimization problem, just as the usual substitutes condition does. We also give conditions on the primitives that establish precisely when we can find packages that are substitutes.

To develop these ideas, it will be useful to work with cones generated by packages; these cones will in turn generate partial orders in which the substitutes property holds. Recall that

common eigenvectors in Example 2 are perturbed slightly; and similarly for heterogeneity in marginal utilities at agents' initial allocations.

[30] We generalize the substitutes (monotonicity) property by dropping the first of these – the quantifier on goods.

a *cone* C is a subset of $\mathbb{R}^K$ such that for every vector $c \in C$, we also have $ac \in C$ for all $a > 0$. The *dual cone* of C, denoted C^*, is the set of all vectors in $x \in \mathbb{R}^K$ such that $x \cdot c \geq 0$ for all $c \in C$. Geometrically, C^* is the set of all vectors that form an acute angle with every vector in C. A cone is *pointed* if $-C \cap C = \{0\}$. A pointed cone C defines a partial order $\succeq_C$ on $\mathbb{R}^K$: $x \succeq_C x'$ if $x - x' \in C$. A set of K linearly independent vectors $\mathcal{C} = \{c_k\}_{k=1}^K$ is a *basis* for the cone C if C is the set of positive linear combinations of vectors in $\mathcal{C}$, i.e. $C = \{\sum_{k=1}^K c_k \lambda_k \mid \lambda_k \geq 0 \ \forall k\}$.

For any basis $\mathcal{C}$ of K linearly independent packages, the cone C they generate orders the space of allocations by the quantity of these packages they contain; likewise, the dual cone C^* orders the space of prices by how expensive these packages are. Given a basis of packages $\mathcal{C}$, let T_C^{-1} be their associated (pullback) change of basis matrix, that is, the matrix whose columns are the elements of $\mathcal{C}$. This gives alternative ways to define a cone and its dual

$$C = \{x | T_C x \geq 0\} = \{T_C^{-1} x | x \geq 0\}$$
$$C^* = \{T_C' x | x \geq 0\} = \{x | T_C^{-1\prime} x \geq 0\}$$

and so $C = C^{**}$. T_C, the (polyhedral) generator for C, maps quantity vectors to their representation as a vector of packages, and $T_C^{-1\prime}$ maps price vectors to their representation as a vector of package prices. We can also see from the above that the rows of T_C form a basis $\mathcal{C}^*$ for C^*; since $T_C T_C^{-1} = I_K$, this $\mathcal{C}^*$ consists of vectors that are orthogonal to all but one of the vectors of $\mathcal{C}$ – the package that it is the price of. Thus, when we talk about the price of a package increasing, we mean that the (goods) price vector moves in the direction in C^* orthogonal to each of the other packages in $\mathcal{C}$.[31] In the remainder of the paper, when we talk about packages $\mathcal{C}$ we mean a *basis* of packages $\mathcal{C}$, and when we talk about a cone C we mean a cone generated by such packages.

We argue that the key to the aggregation properties underlying the core selection results (i.e., bidder-submodularity) is that quantity demanded across bidders changes in a *common* set of directions in response to a price change. This leads us to the idea of package substitutability. Given packages $\mathcal{C}$, consider bidder i's utility maximization problem over packages

$$\max_{q_i^C} u_i(T_C^{-1} q_i^C) - p_C \cdot q_i^C,$$

[31] It is important for our analysis that the packages form a basis; otherwise the packages would define a partial order, but not a lattice order and there would be no corresponding lattice order on prices. It is immediate that when cone C is generated by a basis, $(\mathbb{R}^K, \succeq_C)$ is a lattice. Clearly, $x \vee_C y = T_C^{-1}(T_C x \vee T_C y)$ and $x \wedge_C y = T_C^{-1}(T_C x \wedge T_C y)$. This is important because we are interested in the submodularity of the indirect utility function for packages; for substitutes to guarantee this, we need the space of prices (which we need the indirect utility function to be submodular in) $(\mathbb{R}^K, \succeq_{C^*})$ to be a lattice. A finitely generated cone C is a lattice cone if, and only if, it is generated by a cone basis consisting of K linearly independent vectors (e.g., Aliprantis and Tourky (2007)).

where the price vector p_C specifies the price of each package and the quantity vector q_i^C is a vector of packages. It is immediate that $T_C^{-1} q_i^C(p_C) = q_i(T_C' p_C)$ and, equivalently, $q_i^C(T_C^{-1\prime} p) = T_C q_i(p)$.

Definition 2. *Packages are substitutes* if, for any k, the quantity $\{q_i^C(p_C)\}_k$ of package c_k demanded by i weakly increases when the price of any other package $j \neq k$ increases.

Intuitively, to characterize what package substitutability means in terms of primitives, we can relate demand behavior to the properties of the second order derivatives of the utility function through bidders' first-order conditions. We will use the following definition.

Definition 3. Matrix A is *C-positive* if $AC \subseteq C$. A is *C-quasipositive* if $A + \alpha I$ is C-positive for some $\alpha \in \mathbb{R}$. If A is $\mathbb{R}_+^K$-positive ($\mathbb{R}_+^K$-quasipositive), we say it is *positive (quasipositive).*

When A is the Hessian matrix of a function, quasipositivity means that the function has increasing differences. In particular, quasipositivity characterizing increasing differences for a *concave* function (one with a negative definite Hessian), Lemma 4 relates demand changes to marginal utility changes in the partial order defined by goods (i.e., $\mathbb{R}_+^K$).

Lemma 4 (**Goods Substitutability as a Condition on Primitives**)**.** *Goods are substitutes for i if, and only if, $(D^2 u_i(q_i))^{-1}$ is quasipositive for all q_i.*

Unlike the cone generated by goods (the positive orthant), the package cone and its dual need not be the same. Hence, we need to generalize the quasipositivity property in order to give an analogous statement about the primitives for packages.

Definition 4. A matrix A is *C-dual* if $AC \subseteq C^*$ and *C-quasidual* if $(A + \alpha T_C' T_C)C \subseteq C^*$ for some $\alpha \in \mathbb{R}$.

Quasipositivity and C-quasiduality of Hessians restrict the mapping from changes in prices to changes in quantity demanded, for goods and packages, respectively. Geometrically, C-quasipositivity maps a vector $x \in C$ to another vector in C minus a constant multiple of the first vector αx; similarly, C-quasiduality maps a vector in C to a vector in C^* minus a constant multiple of the dual representation of the first vector $\alpha T_C' T_C x$. Clearly, the C-(quasi)positivity and C-(quasi)duality properties coincide if C is generated by (goods or package) vectors that are orthogonal.[32] Theorem 1 relates demand changes to utility changes in the partial order C defined by packages.

Theorem 1 (**Package Substitutability as a Condition on Primitives**)**.** *Packages $\mathcal{C}$ are substitutes for i if, and only if, $(D^2 u_i(q_i))^{-1}$ is C^*-quasidual for all q_i.*

[32] $C = C^*$ corresponds to a rotation of the positive orthant, in which case C-duality is equivalent to C-positivity and C-quasiduality is equivalent to C-quasipositivity.

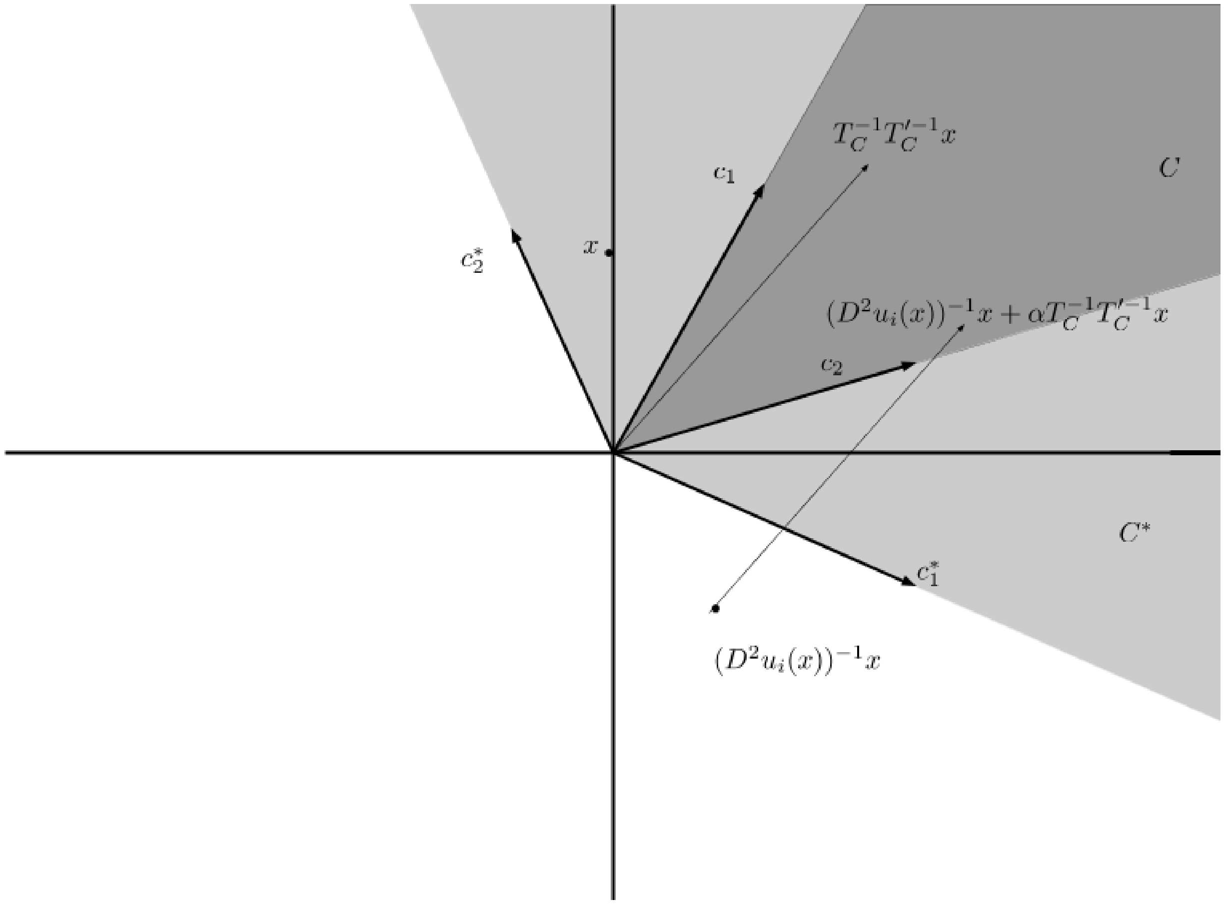

Figure 2: Cones and Substitutes

Notes: The figure depicts packages c_1 and c_2, the cone C they generate, its dual C^*, and the edges of C^*, which are the prices of packages c_1 and c_2, respectively. The substitutes condition says that the inverse Hessian matrix maps a point x in C^* to some point y in C, minus a scalar multiple of its dual representation $T_C^{-1}T_C'^{-1}x$.

Thus, like goods substitutability, package substitutability also corresponds to decreasing differences in indirect utility, but in the partial order defined by package prices rather than by prices of goods.[33] Theorem 1 completely characterizes the set of utility functions for which packages are substitutes; they correspond to (and, hence, are as restrictive as) the *submodularity of the indirect utility function in the package price partial order.*

What exactly are these packages in real-world terms? Consider selling two goods, wireless spectrum in Los Angeles and Chicago. Bidder demand for a unit of package of $[1\ 1]'$ corresponds to buying a unit of spectrum in each market, while a unit of package of $[1\ -1]'$ corresponds to buying a unit of spectrum in Los Angeles and selling a unit in Chicago. Bidders might differ

[33] The cone itself is important, because whether the packages are substitutes depends on the image of the cone that is dual to what they generate under the map defined by their inverse Hessian; specifically, on whether the image is a subset of a translation of the one generated by the packages.

in which market they value spectrum in relatively more; we need not assume that everybody's utility over packages is increasing in the same direction. Similarly, in electricity auctions, the goods could correspond to electricity in different locations and package $[1 - 1]'$ to a contract to transmit electricity from LA to Chicago. Alternatively, the package can represent contracts to move a TV station between different frequencies. We later discuss that the choice to characterize allocations as bundles of *goods* rather than *packages* is not only unimportant for core selection, but need not constrain implementation either. Generalizing substitutes from goods to packages by changing how we think of 'more' is thus natural in some applications when we know how agents are likely to substitute between these packages.

Let us make two final observations about the notion of substitutes. First, the demand-theoretic notion of *package* substitutability is not just a property of an agent's valuation; rather it is about the interaction of an agent's utility and the choice set (see Section 6.1 for its characterization in terms of valuations).[34] In addition, for the core selection analysis, the substitutes condition only needs to hold in a region of the space of goods where the auction allocation will be guaranteed by assumption to lie: precisely in the cube $C \cap (Q - C)$, since allocations lie in C and that the auction clears with allocations summing to Q (feasibility). We will say *packages $\mathcal{C}$ are substitutes on X* for bidder i if for a demanded quantity vector in a set X, a price change sufficiently small for the demand to be in X causes it to increase; more precisely, that is, if $Dq_i^C(p_C)$ is quasipositive whenever $q_i^C(p_C) \in X$, or equivalently in terms of the utility Hessian, using Theorem 1, if $(D^2 u_i(q_i))^{-1}$ is C^*-quasidual for all $q_i \in X$.

4.3 Bidder submodularity when packages are substitutes

As Examples 1 and 3 show, goods substitutability is neither sufficient nor necessary for core selection – allocations matter separately. Clearly, the same argument applies to package substitutability, and hence the condition in Theorem 1 does not guarantee core selection. As further hinted by the examples, (the direction of) the quantity vector Q matters. Theorem 2 establishes a joint condition on package substitutability and allocations as a condition for

[34] At the boundary of the choice set in a one-sided auction, what 'substitutes' mean at the primitive level (i.e., $u_i(\cdot)$) does not change, but what 'packages are substitutes' means in terms of primitives changes. Namely, unlike in the interior, when demand is on the choice set's boundary, there are directions in which demand is constrained not to move. Since the directions in which demand can change do not necessarily correspond to the span of a subset of the packages, an agent may only be able to increase quantity demanded of one package by changing the quantity he demands of the others. Hence, the conditions on the primitives that would cause him to do so in response to some change in price will in general differ, and the map between the properties of demand and the primitives is discontinuous at the boundary of the choice set. This is important only when we are interested in the substitutes property for packages, not goods (and, hence, is not a consideration in Milgrom and Strulovici (2009)). Clearly, in the indivisible goods literature, the distinction does not arise. Let us note that this observation is also relevant for the analysis in Baldwin and Klemperer (2013): constraining the allocation may change the (unimodal) demand types of an agent with the same valuation.

core selection.[35]

Theorem 2 (**Core Selection: Package Substitutability and Allocations**). *For a given Q, the Vickrey auction is core-selecting with respect to bidders for all profiles of participants if there exist packages $\mathcal{C}$ which are substitutes on $C \cap (Q - C)$ for all bidders, nonnegative quantities of which are allocated to each bidder, for each coalition $W \subset N$; that is, $q_i^*(W,Q) \in C$, for all i, where C is the cone generated by $\mathcal{C}$. If, in addition, for each W, the derivative of $v(W,Q)$ in the direction of each package in $\mathcal{C}$ is positive, then the Vickrey auction is core selecting for all profiles of participants.*

Relative to the literature, Theorem 2 extends sufficient conditions for core-selecting design to auctions in which bidders and the auctioneer can all buy and sell. The condition that the allocation must be contained in the cone generated by the packages – in particular, $q_i^*(W,Q) \succeq_C 0$, for all i – reduces to the classic substitutes condition when the packages are just the primitive goods and the auction is one-sided, since in this case allocations are automatically in the positive orthant. When the auction is two-sided or the packages which are substitutes are not simply goods, the quantity of goods need not always be non-negative in the efficient allocation, but the theorem requires that the quantity of packages must be. That is, bidders are all "net package buyers." Note that the direction in which Q lies from the initial allocations is essential for this, as it determines whether allocating nonnegative quantities of the packages to each bidder is even possible.[36]

Why is this additional condition necessary? The allocation and derivative conditions ensure that, in two different respects, *the two-sided auction for goods functions like a one-sided auction*

[35] The proof of Theorem 2 differs from the arguments used in the core selection result for one-sided divisible good auctions by Milgrom and Strulovici (2009, Theorem 31), based on the submodularity of the indirect utility function, employing which requires more assumptions. In a two-sided setting, submodularity of the social planner's dual objective function in (package) prices and bidders does not follow directly from submodularity of the bidders' indirect utility functions. Instead, to exploit the connection between submodularity of the indirect utility and value functions, we must make the additional assumption that bidders' allocations of the goods or packages which generate the lattice cone on the space of goods will be positive.

Additionally, the authors' lattice-theoretic arguments about the social planner's dual problem require that goods are substitutes – and hence, that bidders' indirect utility functions are submodular – everywhere. In a two-sided setting, making the identical argument more generally with respect to a package order would require considering prices outside of the positive orthant, because the package price order meets or joins of prices inside the positive orthant may lie there. Thus, the indirect utility function would need to be guaranteed to exist everywhere on $\mathbb{R}^K$, which will not be the case in many settings, such as those where free disposal holds. We show, however, that it is only necessary to require substitutability where the allocations are assumed to lie – namely, on $C \cap (Q - C)$. This guarantees that bidders' indirect utility functions are locally submodular on the subset of prices for which they demand quantities in $C \cap (Q - C)$; we show that the indirect utility function can then be extended to all of $\mathbb{R}^K$ in a submodular way. This extension is just the bidder's indirect utility when quantity demanded is constrained to lie in $C \cap (Q - C)$, which is where the equilibrium allocations lie by assumption.

[36] When an auctioneer can choose Q subject to a technical constraint or a cost of production, the auctioneer can make core selection easier by choosing a $Q \in C$; we formalize this in Section 5.2. Additionally, choosing Q can make the package substitutability condition easier to satisfy, since it need only apply on the consumption vectors up to Q – precisely on $C \cap (Q - C)$.

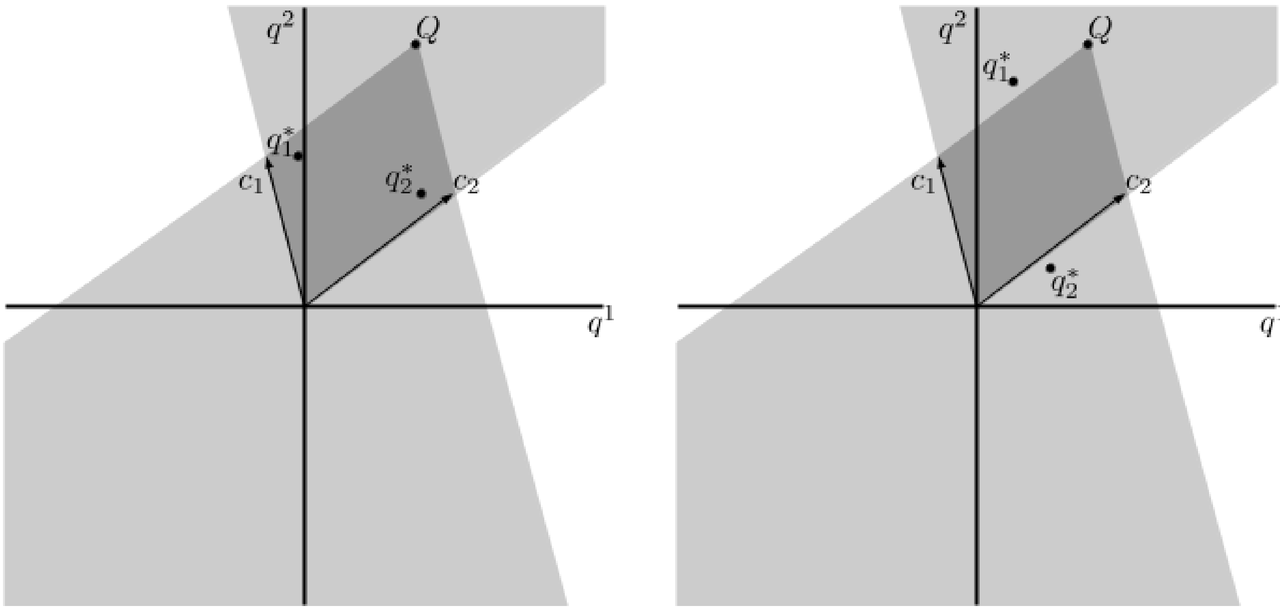

FIGURE 3: ALLOCATION CONDITION FOR THEOREM 2

Notes: Figure 3 shows the allocation condition of Theorem 2 with two bidders and two goods. In the left panel, both bidders receive allocations in C, and the condition is satisfied. In the right panel, this is no longer true.

for packages. First, the allocation condition ensures that we can restrict attention to the cone C – the positive orthant in the *package* space.[37] Second, the derivative condition ensures that packages function like *goods* for each coalition in that more packages is better for each, at least around Q. As we show, this ensures that the auctioneer will receive non-negative payments for packages from all bidders, just as in an one-sided auction.

Intuitively, underlying the absence of incentives to deviate by coalitions of the bidders and the auctioneer in one- and two-sided settings is an order of packages, common among bidders, that (i) aligns the bidders' incentives to substitute with the ways in which *gains from trade in the auction*, created by the available quantity Q and the differences between agents' marginal utilities at their initial endowments, can be realized; and (ii) defines packages that all coalitions desire, so that the auctioneer receives positive revenue.[38]

[37] Mathematically, the condition ensures that the coalitional value function is the minimum of a function which is submodular in (package) prices and coalitions. That is, we are reordering prices to show that the (social planner's) dual objective function is non-increasing in prices, which would not be the case if the condition were not satisfied. We need to show that each bidder's indirect utility function is decreasing in *package* prices. This is assured if agents consume positive quantities of each package – the indirect utility function must weakly increase as the price of any package weakly decreases.

[38] From Lemma 2 and Corollary 2, we know that the difference between bidder-submodularity everywhere and on the sublattices $2^{N-1} \times \{i\}$ is that positive revenue holds in the former so that the auctioneer does not block the payoff profile. The derivative condition on $v(\cdot)$ in Theorem 2 (new, compared to single-object or, more generally, one-sided auctions), when combined with the concavity of $v(\cdot)$, helps ensure that revenue is non-negative by guaranteeing that $v(W, Q) \geq 0$ for each W. This is necessary (but not, on its own, sufficient)

Note that the condition places no restrictions on how the auction allocation is to be broken up other than it being in C. Nor does the goods reordering logic used in Theorem 2 require reallocation of initial endowments among the bidders to be feasible: the package substitutability and allocation conditions give core selection also if the auctioneer cannot reallocate goods among the bidders (i.e., is restricted to run a one-sided auction). (See Section 6.1.)

Let us revisit the examples from Section 4.1 to see which conditions for core selection in Theorem 2 are violated; Figure 3 illustrates. In Example 1, marginal utilities differ at zero auction allocation (i.e., pre-auction gains from trade are non-zero) and core selection fails with one good, the cone is the half-line $\mathbb{R}_+$. Hence, for the condition from Theorem 2 to hold, all bidders would have to be net buyers or net sellers. In Example 2, core selection holds with complements and substitutes. As the core-selection condition in Theorem 2 entails, through a change of basis, one can find a common set of packages that are substitutes such that the direction defined by the auction quantities ought to be aligned with that of (package) substitutability. In Example 3, core selection fails with goods that are substitutes for all bidders and no gains from trade before the auction. This is because the substitution patterns among the agents are sufficiently different to create a need to reallocate goods with $Q \geq 0$.

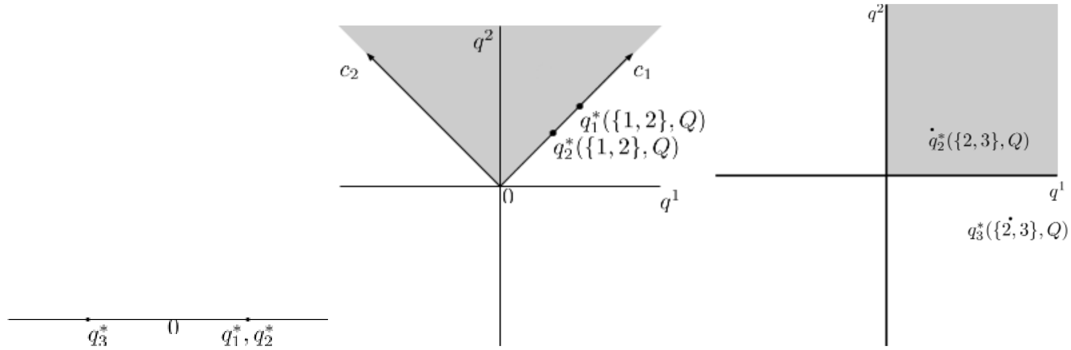

Figure 4: Core-selection condition in examples

Notes: These three panels illustrate Theorem 2 in Examples 1-3. First panel, Example 1: No cone in $\mathbb{R}$ can contain both auction allocations $q_3^*(\{1,2,3\},Q)$ and $q_1^* = q_2^*(\{1,2,3\},Q)$. Second panel, Example 2: Each of the quantity vectors lies in the cone formed by the common eigenvectors $c_1 = [1\ 1]'$ and $c_2 = [-1\ 1]'$, namely, along the line c_1. Third panel, Example 3: Auction allocation $q_3^*(\{2,3\},Q)$ is not in the positive orthant, $\mathbb{R}^2_+$,

Core selection beyond indivisible-good, one-sided auctions. Our results highlight two general observations about core selection in *divisible* good, *two-sided* auctions.

First, relative to (divisible or not) one-sided auctions, it is non-trivial in auctions where bidders are able to both buy and sell that the gains from trade in the auction cause the

for revenue to be positive for all profiles of participants: if some coalition W with $v(W,Q) < 0$ participated, revenue $v(W,Q) - \sum_{i \in W}(v(W,Q) - v(W \setminus i, Q))$ would be negative, since the sum terms are the agents' Vickrey payoffs, which are always positive.

equilibrium allocations to lie in a cone which satisfies the package substitutability property for each agent. Clearly, in one-sided auctions with substitutes, the alignment between substitution and gains from trade is assured; the core selection property is implied by the aggregation of substitutes alone.

Additionally, with divisible goods (i.e., multi-unit demands), substitutability is inessential for core selection in the Vickrey auction; instead, the relevant substitutability property is that for packages. Examples of *package* substitutability and the basis change idea have recently appeared in the literature: the *gross substitutes and complements* (*GSC*) property (Sun and Yang (2006),), *full substitutes* (Ostrovsky (2008), Hatfield et al. (2013)) and the notion of demand types in Baldwin and Klemperer (2013), which all admit complementarieties in goods. The authors apply these notions to establish equilibrium existence, whereas this paper points to the usefulness of a basis change in identifying the key conditions underlying core selection in the Vickrey auction. As we have shown, however, substitutability under a change of basis alone is insufficient for core selection in two-sided divisible good settings, as the equilibrium allocations must also lie in the cone (i.e., in the image of the positive orthant under that change of basis).[39] Example 4 illustrates how Theorem 2 contributes in the context of GSC which, to the best of our knowledge, has not been studied for core selection.

Example 4 (**Core Selection and Gross Substitutes and Complements**)**.** GSC implies that there are two sets of goods K_1 and K_2, and that all goods in each set K_l are substitutes for other goods in K_l but are complements for goods in K_{-l} (e.g., left and right shoes, production inputs, spectrum licences). It is plain to see (and has also been noted by Baldwin and Klemperer (2013)) that GSC is equivalent to substitutes under the basis change $T_{GSC} = I_{K_1} \oplus -I_{K_2}$ (i.e., substitutes with "negative goods quantities"). Unfortunately, it is only possible for GSC to work in Theorem 2 to prove that the Vickrey outcome is core-selecting when $Q_k < 0$ for each $k \in K_2$, since otherwise someone must receive a nonpositive allocation of $q_i^{GSC} = T_{GSC}^{-1} q_i$. In fact, even when the auctioneer seeks to acquire goods in K_2, GSC can be insufficient to ensure core selection. Suppose the auctioneer wishes to sell $Q = [2.6082\ 0.9203\ \ -0.0410]'$ to

[39] The basis change also suggests useful design properties (Section 7.1) and allows a unified treatment of the core selection conditions for one- and two-sided settigns. The full power of the basis change is in two-sided settings. In one-sided auctions, the ability to repackage goods when considering their substitutability is less likely to be useful for core selection in situations where the usual substitutes would not be, because the basis change would have to generate a cone with a significant overlap with the positive orthant, where the allocations lie. The overlap entails that, for one-sided auctions to have good core selection properties, goods must at least be *nearly* substitites. Since a basis change is needed that maps to substitutes, complementarities will generally be challenging.

three bidders with the quadratic valuations

$$u_1\ (q_1) = \begin{bmatrix} 42.4 & 21.6 & 1.6000 \end{bmatrix} q_1 - \frac{1}{2} q'_1 \begin{bmatrix} 9 & 2.52 & -3.1 \\ 2.52 & 2.11 & -1.355 \\ -3.1 & -1.355 & 3.6 \end{bmatrix} q_1$$

$$u_2(q_2) = \begin{bmatrix} 59.2 & 21 & 3.2 \end{bmatrix} q_2 - \frac{1}{2} q'_2 \begin{bmatrix} 14 & 2.08 & -2.22 \\ 2.08 & 3.54 & -2.37 \\ -2.22 & -2.37 & 3.14 \end{bmatrix} q_2$$

$$u_3(q_3) = \begin{bmatrix} 42.4 & 23.8 & 2.2 \end{bmatrix} q_3 - \frac{1}{2} q'_3 \begin{bmatrix} 13.2 & 6.74 & -5.1 \\ 6.74 & 6.19 & -3.03 \\ -5.1 & -3.03 & 4.14 \end{bmatrix} q_3.$$

That is, goods 1 and 2 are substitutes for each other and complements for good 3. The bidders' inverse Hessian matrices are given by

$$-S_1^{-1} = \begin{bmatrix} -0.1928 & 0.1631 & -0.1047 \\ 0.1631 & -0.7629 & -0.1467 \\ -0.1047 & -0.1467 & -0.4231 \end{bmatrix}, -S_2^{-1} = \begin{bmatrix} -0.0811 & 0.0187 & -0.0432 \\ 0.0187 & -0.5754 & -0.4210 \\ -0.0432 & -0.4210 & -0.6668 \end{bmatrix},$$

$$-S_3^{-1} = \begin{bmatrix} -0.2155 & 0.1631 & -0.1461 \\ 0.1631 & -0.3752 & -0.0737 \\ -0.1461 & -0.0737 & -0.4754 \end{bmatrix}.$$

Since these are also the Jacobian matrices of demand, we can see that each agent's preferences exhibits gross substitutes and complements; that is, the packages $\mathcal{C}_{GSC} = \{[1\ 0\ 0], [0\ 1\ 0], [0\ 0\ -1]\}$ are substitutes for these agents. However, even though $Q \in C_{GSC}$, we cannot use Theorem 2, since agent 2 does not receive an allocation in C_{GSC}, irrespective of which other agents participate in the auction:

$$q_2^*(\{1,2,3\},Q) = \begin{bmatrix} 1.8152 \\ -0.0591 \\ 0.5096 \end{bmatrix},\ q_2^*(\{1,2\},Q) = \begin{bmatrix} 1.9314 \\ 0.1140 \\ 0.4183 \end{bmatrix},\ q_2^*(\{2,3\},Q) = \begin{bmatrix} 2.0637 \\ 0.3990 \\ 0.2749 \end{bmatrix}.$$

However, consider the packages $\mathcal{C} = \{[0.8998\ -0.1367\ 0.5011]',\ [0.0683\ 1.0023\ -0.3417]',\ [0.1822\ -0.6606\ \ -0.9112]'\}$. We have

$$-T_C S_1^{-1} T'_C = \begin{bmatrix} -0.2516 & 0.1161 & 0.0258 \\ 0.1161 & -0.3613 & 0.1419 \\ 0.0258 & 0.1419 & -0.4129 \end{bmatrix}, -T_C S_2^{-1} T'_C = \begin{bmatrix} -0.125 & 0 & 0.125 \\ 0 & -0.1818 & 0.0909 \\ 0.125 & 0.0909 & -0.6705 \end{bmatrix},$$

$$-T_C S_3^{-1} T'_C = \begin{bmatrix} -0.2929 & 0.1414 & 0.0505 \\ 0.1414 & -0.1717 & 0.0101 \\ 0.0505 & 0.0101 & -0.3535 \end{bmatrix}$$

so $\mathcal{C}$ are *substitutes* for each agent. Then, we have $q_i^*(W,Q) \in C$ for each $W \subseteq \{0,1,2,3\}$ (we report the quantities in the Appendix). It then follows from Theorem 2 that $v(\cdot,Q)$ is submodular and, hence, the auction is core-selecting. Figure 5 illustrates.

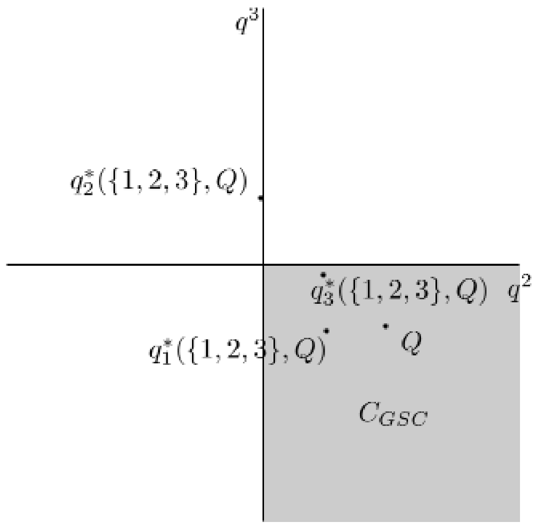

FIGURE 5: CORE SELECTION WITH GROSS SUBSTITUTES AND COMPLEMENTS (EXAMPLE 4)

Notes: Focusing on the first two goods for simplicity of illustration, auction allocation $q_2^*(\{1,2,3\},Q)$ of agent 2 is not in the GSC cone, C_{GSC}.

5 When Can One Expect Core Selection?

Theorem 2 pointed to the alignment between the cone generated by substitutable packages and the ways in which gains from trade in the auction are realized as the key to core selection. Intuitively, then, any conditions on primitives that guarantee its usefulness will restrict heterogeneity in bidder preferences. Theorem 3 presents one such condition: gains from trade before the auction are absent, and the utility Hessians commute for all bidders and quantities.

Theorem 3 (Core Selection: Sufficient Conditions on the Primitives I). *Suppose the bidders' initial endowments are efficiently allocated. Packages $\mathcal{C}$ which are substitutes for all bidders and such that $q_i^*(W,Q) \in C$ for all $W \subseteq N$ and any Q exists if the Hessians $D^2u_i(q_i)$ commute for all q_i and all i. In particular, the vectors in $\mathcal{C}$ are an eigenbasis for the Hessians $D^2u_i(q_i)$.*

This is intuitive: commutativity, equivalent to simultaneous diagonalizability, captures a homogeneity property of substitution patterns across bidders. It captures that there is some set of *implicit* packages in which agents' utility is *separable*:

Lemma 5 (**Eigenvector Condition for Substitution Symmetry**). *Utility $u_i(\cdot)$ is separable in packages $\mathcal{C}$ if, and only if, their scalar multiples form an orthonormal eigenbasis for utility Hessians $D^2u_i(q_i)$ for each q_i.*[40]

This explains why, contrary to what one might expect based on the literature, the Vickrey auction is core-selecting in the environment of Example 2: the agents' Hessians $-S_i$ commute, and their marginal utilities at zero are identical (their initial endowments are efficiently allocated among them).

More generally, we can canonically classify these eigenvector packages in a way that makes plain what they represent: we can think of vectors which have both positive and negative entries as *switching packages*, in addition to at most one[41] vector in the interior of either the positive or the negative orthant, which can be interpreted as a *buying package* or a *selling package*, respectively. In addition, there can be goods or bads (as, e.g., in the GSC cone C_{GSC} – see Example 4). For instance, in spectrum auctions, the buying and switching vectors can correspond to a license for nationwide coverage and one to relocate coverage between the East and West coast. In Example 2, c_1 is a buying vector and c_2 is a switching vector. If commutativity does not hold, it matters which way the packages are oriented.

It is important to note that these packages need not be explicitly incorporated into the design for our results to hold; they remain valid whether the auctioneer chooses to solicit bids for goods or for any arbitrary bundles, and we think of them in a strictly "as if" sense. However, they will in some cases be useful for implementation (see Section 7).

5.1 The role of preference heterogeneity

Theorems 2 and 3 motivate the converses to our core selection results. Indeed, in Examples 1 and 3, we showed that, in two-sided auctions, when pre-auction marginal utilities or substitution patterns differ significantly across agents, core selection may break down. We show that either type of heterogeneity in the environment can separately undermine the possibility of a core selecting design. Proposition 1 demonstrates that in fact, for *any* (possibly homogeneous) profile of bidders' utility functions over goods, there exist heterogeneous initial endowments such that the Vickrey outcome is not in the core for some profile of participants.

Proposition 1 (**No Core Selection: Heterogeneity in Initial Endowments I**). *Given Q, for any utility profile $\{u_i\}_{i\in N}$, $N \geq 3$, constant vectors $\{t_i\}_{i\in N}$ exist such that for the utility*

[40] E.g., quadratic models where the Hessians are constant, or environments with Hessians which have the same eigenvectors for all q_i but different eigenvalues.

[41] If one eigenvector is in the interior of the positive orthant, all other eigenvectors must be switching vectors, because they are orthogonal to it and any two vectors in the positive orthant form an acute angle unless they are both on its boundary. If none of the eigenvectors is in the positive or negative orthants, each is a switching vector.

profile $\{u_i(q_i + t_i)\}_{i \in N}$*, the Vickrey outcome is not in the core with respect to bidders for some profile of participants.*

Likewise, given some profile of utility functions in which the initial endowment is not efficiently allocated, there exists some quantity vector Q for which the Vickrey auction is not core selecting:

Proposition 2 (**No Core Selection: Heterogeneity in Initial Endowments II**). *For any utility profile* $\{u_i\}_{i \in N}$ *such that* $\nabla u_\ell(0) \neq \nabla u_j(0)$ *for some* $j, \ell \in N$*, there is a* Q *such that the Vickrey outcome is not in the core with respect to bidders for some profile of participants.*

As Theorem 3 shows, conditional on the efficiency of the pre-auction allocation, commutativity (homogeneity) is sufficient for core selection, for any quantity vector Q. Non-commuting utility Hessians or nonzero pre-auction gains to trade itself do *not* necessarily break core selection, as Theorem 2 shows – the outcome can be in the core for some quantity vectors Q; sufficient homogeneity in utility Hessians rather than separability is what matters. Isolating its effect by assuming that the initial allocation is efficient, when is homogeneity in Hessians insufficient for core selection? The next result characterizes this in a counterpart of Proposition 1. While we don't have a general converse, we provide one for the quadratic case: there, whenever there are no pre-auction gains to trade and the bidders' substitution patterns (Hessians) are sufficiently heterogeneous, there is a uniform-across-bidders rotation of the substitution patterns for which core selection breaks.

Proposition 3 (**No Core Selection: Heterogeneity in Substitution Patterns**). *Assume that agents have quadratic valuations* $u_i(q_i) = \theta_i' q_i - \frac{1}{2} q_i' S_i q_i$ *and* $\theta_i = \theta$ *for all* i*. For any* Q*, there exists a matrix* A *such that for the utility profile* $\{\theta' q_i - q_i' A' S_i A q_i\}_{i \in N}$*, the Vickrey outcome is not in the core for some profile of participants if* $S_\ell^{-1} H(Z \setminus \ell + Z \setminus j) S_j^{-1}$ *has a negative eigenvalue for some coalition* $Z \subset N$ *and agents* $\ell, j \in Z$*.*

To explain the role of the negative eigenvalue, let us first observe that in the quadratic environment with efficiently allocated initial endowments, nonnegativity of the difference-in-difference $v(Z, Q) + v(Z \setminus \{\ell \cup j\}, Q) - v(Z \setminus \ell, Q) - v(Z \setminus j, Q)$ is equivalent to its being non-increasing in the magnitude of Q, since it is a constant times the square of that magnitude. This difference-in-difference is just the incentive of coalition $0 \cup Z \setminus \{\ell \cup j\}$ to deviate and cancel the auction. From the discussion about common θ_i following Lemma 3, whenever the difference-in-difference of the inverses of the slopes of the coalitions' aggregate demand curves[42] $G(Z \setminus \ell, Z \setminus j) \equiv (H((Z \setminus \ell) \cup (Z \setminus j)) + H((Z \setminus \ell) \cap (Z \setminus j)) - H(Z \setminus \ell) - H(Z \setminus j))$, a *deviation matrix*, has a negative eigenvalue, there is some direction (indeed, some *set* of directions) in

[42] Or equivalently, of the slopes of the marginal coalitional value curves – see Section 5.2.

which this incentive to deviate is *increasing*. If Q lies in one of those directions from the origin, then bidder submodularity breaks.[43]

This gives us intuition about what went wrong in Example 3. Note that we can write the difference-in-difference in the surplus function as a constant times $Q'(S_1Q + H(\{1,2,3\})Q - H(\{1,2\})Q - H(\{1,3\})Q)$; in other words, since the harmonic means are the slopes of the marginal coalitional values (or equivalently, the inverses of the slopes of aggregate demands), as the dot product of Q and the difference-in-difference of changes in the market-clearing price between auction quantity of 0 and $Q = [1\ 0]'$.[44] These price changes are shown in Figure 6; due to the negative eigenvalue of $G(\{1,2\},\{1,3\})$, there is a set of quantity vectors, $Q = [1\ 0]'$ among them, which form an obtuse angle with the difference-in-difference of the price changes they induce.

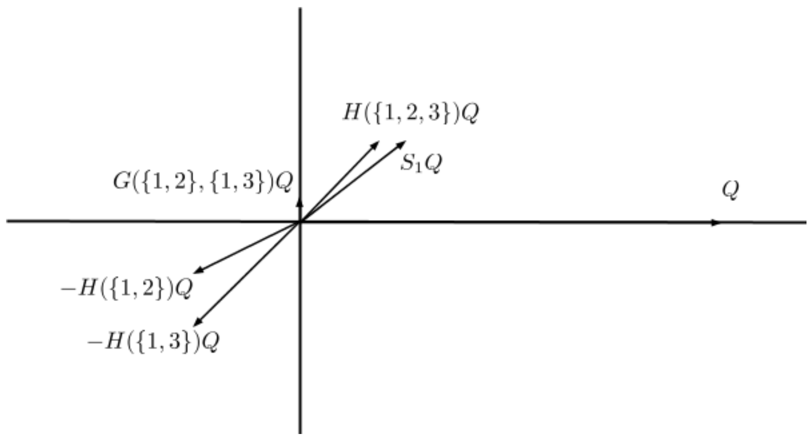

Figure 6: No core selection with heterogeneous substitution patterns (Example 3)

Lemma 10 in the Appendix shows that such a negative eigenvalue will appear whenever the product $S_\ell^{-1}H(Z\setminus j + Z\setminus \ell)S_j^{-1}$ has one. To interpret, diagonalize each S_i orthogonally as $L_i'D_iL_i$, where L_i's rows form an orthonormal eigenbasis for S_i and D_i is the diagonal matrix of its associated eigenvalues. L_i can be seen as a rotation matrix, rotating the orthant defined by the eigenvectors to the positive orthant – that is, changing coordinates to those in which $u_i(\cdot)$ is separable. When these rotation matrices – and thus the packages in which the bidders' utilities are separable – are the same, $S_\ell^{-1}H(Z\setminus \ell + Z\setminus j)S_j^{-1}$ is positive definite.[45] But when these packages are heterogeneous, $S_\ell^{-1}H(Z\setminus j + Z\setminus \ell)S_j^{-1}$ will in general be asymmetric and, if they are heterogeneous *enough*, possibly have negative eigenvalues. Proposition 3 says

[43] Mathematically, commutativity implies a lattice order on the Hessians. Without commutativity, the order of the positive definite Hessians is incomplete, and in the deviation matrix, the ranking of the inverse aggregate demand slope matrices defined by the harmonic means then depends on the vector Q they are multiplied with; if the matrices commute, the ranking is independent of Q (Theorem 3).

[44] We can only do this when the initial endowment is efficiently allocated; otherwise, surplus at zero would depend on the agents present.

[45] We can write $S_\ell^{-1}H(Z\setminus \ell + Z\setminus j)S_j^{-1} = L'D_\ell^{-1}\left(\sum_{Z\setminus\ell+Z\setminus j} D_i^{-1}\right)^{-1} D_j^{-1}L$.

that when the heterogeneity in these packages is sufficient for one of these triples to have a negative eigenvalue, there is a uniform rotation A of the L_i – that is, a rotation, common to all bidders, of the packages in which their valuations are separable – under which some coalition's incentive to deviate is increasing in the direction of the quantity vector.[46] Thus, the failure of positive definiteness in the difference-in-difference of inverse aggregate demand slopes results from insufficient symmetry – the negative eigenvalue condition corresponds to the heterogeneity in substitution patterns of the coalition members that is sufficient for no core selection.[47] Instead, this asymmetry creates an incentive to deviate for a group of bidders.

Example 5 (No Core Selection and Additive Valuations). One of the two best known no-core selection results for indivisible goods shows that when one agent's preferences do not satisfy the substitutes condition, there are some *additive* valuations for the other participating agents such that the auction is not core-selecting (Ausubel and Milgrom (2002)). What are additive valuations in our divisible good, strictly concave setting? One might mean that the valuation of any quantity vector is the sum of the values of the goods which comprise it, or add the stronger requirement that the utility function's second derivatives are the same for each good, so that the agents' marginal utility schedules for them differ by a constant. That is, with divisible goods, additivity could mean that valuations are separable in goods or that, in addition, bringing an agent's allocation of any good k from q to q' changes her marginal utility for good k by the same amount as bringing her allocation of any other good l from q to q' changes her marginal utility for l.

However, when we add the requirement that the marginal utilities at zero auction allocation are equal across agents, then with the second definition, Theorem 3 says that we will *always* have core selection when $N-1$ agents have additive preferences, again *regardless of the first agent's preferences.* Why? Because with this definition, the utility Hessian of each agent with additive preferences is always a scalar multiple of the identity matrix, and hence commutes with *any* $K \times K$ matrix; and since all but one agent has additive preferences, it follows that all agents' Hessians commute. This contrast between our result and that of Ausubel and Milgrom (2002) is due to the lack of a restriction of allocations to binary vectors – an agent can be allocated more than one or less than zero units of a good. Again, with divisible goods, substitutability matters as a *common monotonicity* condition; then, the result analogous to that in the literature (i.e., that the lack of substitute valuation for an agent is a problem) is

[46] If there were no negative eigenvalue, since the deviation matrix $G(W,Z) \equiv H(W \cup Z) + H(W \cap Z) - H(W) - H(Z)$ is symmetric, it is positive semidefinite; otherwise, a Q exists such that core selection fails. Suppose $\hat{e}$ is the unit eigenvector associated with the negative eigenvalue $\hat{\lambda}$ of $G(W,Z)$. Why is the difference-in-difference then negative when $Q = c\hat{e}$, $c \in \mathbb{R}$? Because it is symmetric, $G(W,Z)$ has orthonormal eigenvectors. Diagonalizing it as LD_iL', where $D = diag\{\lambda_k\}$ and L is the matrix whose columns are the eigenvectors of $G(W,Z)$, $c\hat{e}G(W,Z)c\hat{e} = c\hat{e}LDL'c\hat{e} = c^2\hat{\lambda} < 0$.

[47] If eigenvectors were the same (if separability in the same packages holds), than the outcome would be in the core – see Theorem 3.

Proposition 3.

Finally, let us note that the challenges for core selection that the analysis of this section brings attention to are not due to non-concavities or non-monotonicities in individual demands. Instead, well-behaved individual demands can result in the operation of aggregating agents' valuations (i.e., the coalitional value function) being nonconcave (non-submodular) on the space of bidders when their substitution patterns or initial marginal utilities are sufficiently heterogeneous.

5.2 The role of quantity auctioned and market size

The above converses to our results characterizing environments where the Vickrey auction yields core outcomes demonstrate that heterogeneity in either pre-auction marginal utilities (and hence the presence of gains to trade before the auction) or in substitution patterns (as determined by utility convexities) can challenge core selection – for a fixed number of participants N and given the quantity vector Q. This section shows first that the profitability of a multilateral deviation need not vanish as the number of participating bidders N becomes large; and second that, to the extent that Q is a choice variable (e.g., if the auctioneer aims for revenue maximization subject to a cost function or a technical constraint), the impact of heterogeneity in the initial allocations (or pre-auction marginal utilities) on core selection can, in fact, be mitigated by choosing the quantity vector Q large enough in an appropriate direction. Thus, core selection can be consistent with potentially large heterogeneity in the initial allocations. Indeed, a new aspect of design in divisible good settings is that the Vickrey auction's core-selecting property depends on the number of bidders as well as the quantity vector, separately from agents' primitive preferences.

Are the conditions for core selection easier to satisfy in large markets (i.e., $N \to \infty$)? One might expect that, holding Q constant, each agent's marginal product to the grand coalition will vanish in the limit, and so the auctioneer will capture all surplus, rendering the Vickrey payoffs ϵ-close to the core. This would indeed be the case if there was no need to reallocate goods among the bidders (e.g., in one-sided auctions, or with no bidders wanting to sell). It turns out that, in a two-sided auction, the auctioneer will not capture all surplus, in general.[48]

Example 6 (Core Selection in Large Auctions). Suppose bidders have quadratic utilities with $S_i = S$ for all i. This simplifies calculations considerably, implying $H(W) = \frac{1}{|W|}S$, and

[48] The converses for core selection in the literature typically hold N fixed and allow a rich class of potential valuations. This resonates with our converses in the previous section: While it is not true that a small perturbation in valuations will take us outside the core, if the number of *potential* bidders grows large, but the number of participating agents does not, bidder submodularity is harder to satisfy.

using Lemma 3, we can then compute the sum of the bidders' Vickrey payoffs:

$$\sum_{i\in N}\pi_i = \frac{N}{2(N-1)}\sum_{i\in N}\theta_i' S^{-1}\theta_i - \frac{1}{2(N-1)}(\sum_{j\in N}\theta_j)' S^{-1}(\sum_{j\in N}\theta_j) + \frac{1}{2(N-1)}Q'SQ.$$

(See Appendix for the derivation.) Unless $\{\theta_i\}_{i\in N}$ are identical (as in one-sided auctions with no pre-auction endowments), this need not become small as N becomes large. In fact, with i.i.d. private values $\{\theta_i\}_{i\in N}$, the Vickrey payoffs will be positive in expectation: letting $\{\lambda_k\}_{k\in K}$ be the (orthogonal) eigenvectors of S associated with unit eigenvalues,

$$E\left(\sum_{i\in N}\pi_i\right) = \frac{N}{2}\sum_{k\in K}\mathrm{Var}(\lambda_k'\theta_i) + \frac{1}{2(N-1)}Q'SQ.$$

Suppose that instead of encouraging bidder participation, the auctioneer wishes to increase the auctioned quantity. Core selection can continue to hold; however, not any increase will work. One insight from Theorem 2 is that the elements of Q must change in such a way that the equilibrium allocation vectors q_i^* are increasing in the order defined by the packages.

Proposition 4 (**Bidder-Submodularity and Quantity Auctioned**). *The equilibrium package allocation vectors $T_C q_i^*(W,Q)$ are each increasing in the direction x if, and only if, $x \in \bigcap_{i\in W}\left(\sum_{j\in W} D^2u_j(q_j^*(W,Q))^{-1}\right) D^2u_i(q_i^*(W,Q))C$.*

Of special note is the fact that this intersection is obviously nonempty if the substitution patterns intersection $\bigcap_{i\in W} D^2u_i(q_i^*(W,Q))C$ is. In particular, cones C generated by packages $\mathcal{C}$ which are substitutes are likely to result in this intersection being nonempty, since this implies that each $D^2u_i(\cdot)$ maps C to $-C^*$; additionally, when $\mathcal{C}$ are goods, the intersection is just that of the cones generated by the columns of the Hessians $D^2u_i(q_i^*(W,Q))$.

It follows that the designer can choose Q appropriately without risk of violating the $q_i^*(W,Q) \succeq_C 0$ condition for core selection in Theorem 2 (that is, without causing the auction to cease functioning like a one-sided auction for packages) if, and only if, the cone intersection

$$\bigcap_{i\in W}\left(\sum_{j\in W} D^2u_j(q_j^*(W,Q))^{-1}\right) D^2u_i(q_i^C(p_C))C$$

is nonempty. If the auctioneer wishes this to be true for all coalitions W, then clearly, the intersection over all $W \subseteq N$ of these intersections must be nonempty as well.

We can be specific about where Q needs to be: Applying the theorem for $Q = 0$ and equal marginal utilities at zero (so that the auction allocations are zero), and changing Q in a direction in which Proposition 4 shows will cause them to increase in the order defined by the packages $\succeq_C$, yields the following corollary.

Corollary 3 (Core Selection: Sufficient Conditions on the Primitives II). *Suppose that* $\nabla u_i(0) = \nabla u_j(0)$ *for all* $i, j \in N$. *If*

$$Q \in F(C) \equiv \bigcap_{W \subseteq N} \bigcap_{\{x_i\}_{i \in W} \in \mathbb{R}^{|W|K}} \bigcap_{\ell \in W} \left(\sum_{\ell \in W} D^2 u_j(x_j)^{-1} \right) D^2 u_i(x_i) C,$$

then $q_i^*(W, Q) \in C$ *for all* $W \subseteq N$.

$F(C)$ is the set of directions in which Q can move such that all bidders receive a larger amount of each package that generates C. This result also gives another set of sufficient conditions on primitives for core selection: knowing that $\mathcal{C}$ are substitutes for each bidder and marginal utilities are equal at zero, if $Q \in F(C)$, Corollary 3 and Theorem 2 imply that the auction is core selecting with respect to bidders. That is, the allocations are in the cone, there is no need to compute them, provided that Q in the the cone $F(C)$; unlike Theorem 3, separability is not required.

When it is a choice variable, the quantity auctioned gives a powerful instrument to assure core selection (that is, to make the auction function like a one-sided auction for packages).[49] The next result give conditions under which Q can be increased to bring allocations into the cone C. This holds regardless of the presence of pre-auction gains from trade.

Corollary 4 (Core Selection: Sufficient Conditions on the Primitives III). *For any quantity vector* Q *and any* $\Delta Q \in F(C + \epsilon T_C^{-1}\mathbf{1})$ *for some* $\epsilon > 0$, *there exists a scalar* $a > 0$ *such that* $q_i^*(W, Q + a\Delta Q) \in C$ *for all* W.

In particular, regardless of heterogeneity in marginal utilities at agents' initial allocations, whenever packages $\mathcal{C}$ are substitutes and $F(C + \epsilon T_C^{-1}\mathbf{1})$ is nonempty for some $\epsilon > 0$, Theorem 2 and Corollary 4 combine to show that there exists ΔQ such that for $Q + \Delta Q$, the Vickrey auction is core-selecting with respect to bidders. Note that this is particularly likely to hold in the quadratic environment, where the agents' Hessian matrices are constant. In fact, Corollary 4 assures that, for general utilities, Q can be increased in *any* direction in $F(C + \epsilon T_C^{-1}\mathbf{1})$ to ensure that the bidders will consume positive amounts of the packages, as long as it is increased sufficiently. This highlights another difference with indivisible good settings: With divisible goods, so long as bidders' substitution patterns are sufficiently aligned, a market designer can weaken the strategic effects of Proposition 1 by appropriately increasing the quantity vector.

[49] For instance, choosing the quantity vector when a range of spectrum and location combinations is feasible given engineering and interference restrictions or when the amount of nonmarket spectrum to make available is not fixed (the case in the upcoming FCC auction) can help ensure that a Vickrey auction is core selecting.

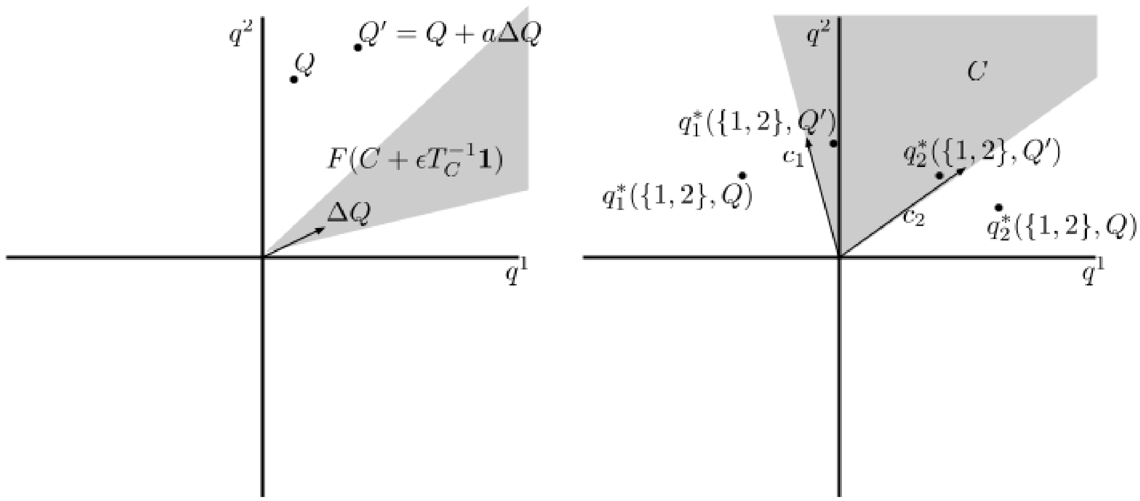

Figure 7: Core selection and quantity auctioned

Notes: The left panel shows the cone $F(C + \epsilon T_C^{-1}\mathbf{1})$ and the quantity vector before and after the change $\Delta Q \in F(C + \epsilon T_C^{-1}\mathbf{1})$. The right panel shows the resulting change in equilibrium allocations, bringing them inside the cone C.

6 Extensions

6.1 Infeasible reallocation

Often, the set of feasible allocations is restricted: for instance, when reallocation of quantity among bidders is infeasible and so bidders can only buy (or in procurement auctions, can only sell) or when goods are imperfectly divisible (so the allocations are restricted to integer vectors). When the former is the case and reallocation is infeasible, we want to know if the reallocation-constrained surplus function

$$\bar{v}(W, Q) = \max_{\{q_i\}_{i \in W}} \sum_{i \in W} u_i(q_i) \text{ s.t. } \sum_{i \in W} q_i = Q \text{ and } q_i \geq 0, \text{ for all } i,$$

is bidder-submodular when packages are substitutes. We show that this is the case – Theorem 2 still applies with an appropriately modified condition on the primitives necessary for packages to be substitutes when allocations are restricted to the positive orthant. We say that packages $\mathcal{C}$ are *reallocation-constrained substitutes* if the solution $\bar{q}_i^C(p_C)$ to $\bar{\Pi}_i^C = \max_{q_i^C}\{u_i(T_C^{-1}q_i^C) - p_C \cdot q_i^C \text{ s.t. } T_C^{-1}q_i^C \geq 0\}$ has the substitutes property.

Lemma 6 (Package Substitutability: Infeasible Reallocation). *If packages $\mathcal{C}$ are substitutes on $\mathbb{R}_+^K$ and for each $L \subset K$, $(0_L \oplus (D^2u_i(x)_{-L})^{-1})$ is C^*-quasidual for all x such that $x_L = 0$, then $\mathcal{C}$ are reallocation-constrained substitutes.*

When the packages are just goods, constraining reallocation does not cause the conditions on the primitives for substitutes to be different:

Corollary 5 (Goods Substitutability: Infeasible Reallocation). *When goods are substitutes, they are substitutes when reallocation is infeasible.*

Like Theorem 1, this yields a natural definition for packages $\mathcal{C}$ being *reallocation-constrained substitutes on* $X \subseteq \mathbb{R}^K_+$ if they are substitutes on X and, for each $L \subset K$, $(0_L \oplus (D^2 u_i(x)_{-L})^{-1})$ is C^*-quasidual for all $x \in X$ such that $x_L = 0$. The Schur complement condition on the Hessian is intuitive. When the reallocation constraint binds for goods L, the block of the Hessian of the indirect utility function corresponding to the other goods is naturally the same as its Hessian in an alternative problem where quantity of L is constrained to zero. It is plain to see that this is just $(D^2 u_i(x)_{-L})^{-1}$.

Corollary 6 (Core Selection: Infeasible Reallocation). *Given Q, when reallocation is constrained, the Vickrey auction is core selecting with respect to bidders for all profiles of participants if there exist packages $\mathcal{C}$, nonnegative quantities of which are allocated to each bidder in the efficient allocation for each $W \subset N$, which are reallocation-constrained substitutes on $\mathbb{R}^K_+ \cap C \cap (Q - C)$ for each bidder.*

6.2 Heterogeneous preference domains

Our analysis so far assumes that all bidders bid for all packages. One of the enduring problems in practical auction design is that not all bidders care about all goods; i.e., we have $u_i : \mathbb{R}^{K_i} \to \mathbb{R}$, $K_i \subset K$ for some i. We reformulate our assumption that the images of the marginal utilities are equal, which we used to ensure that residual supply is well defined, by assuming that they are *projections of the same* cube, that is, that there exist K intervals $I_k \subseteq \mathbb{R}$ such that $\nabla u_i(\mathbb{R}^{K_i}) = \prod_{k \in K_i} I_k$ for each i. Submitting the demand curve $b_i(q_i) = \nabla u_i(q_i)$ from $\mathbb{R}^{K_i}$ to itself is still a dominant strategy for i.[50] Together with market clearing, the bidders' demand curves give the Kuhn-Tucker conditions for the grand coalition's optimization problem

$$\max_{\{q_i\}_{i=1}^N \in \prod_{i=1}^N \mathbb{R}^{K_i}} \left\{ \sum_{i=1}^N u_i(q_i) \text{ s.t. } \sum_{i=1}^N (q_i \oplus \mathbf{0}_{K-K_i}) = Q \right\}$$

and, hence, the divisible-good Vickrey auction continues to yield an efficient allocation. Theorem 2 continues to hold when packages $\mathcal{C}$ are substitutes for all bidders and bidders get to choose the domain over which they get to bid. Importantly, core-selection holds even when the auctioneer does not know what goods each agent is interested in and, instead, agents define the bundles they are interested in through the demand curves they submit. In particular, bidders need not bid for packages they do not care about. Nontrivially, bidders need not be able to think about how much they value these packages independently of one another;[51] that

[50] See the proof of Lemma 7 in the Appendix.

[51] In particular, this would require (and none of our results assumes this) that each bidder's preference domain K_i is equal to the span of some subset of packages.

is, we can still consider a bidder's demand for each of them even if they include goods the bidder cannot buy or sell. The proof follows exactly the same logic as the homogeneous-domain version, with an appropriately modified substitutes condition in terms of the primitives, since agents' preference domains are no longer necessarily equal to the span of some set of packages.

Lemma 7 (**Package Substitutability: Heterogeneous Domains**). *Suppose preference domains are heterogeneous. Packages $\mathcal{C}$ are substitutes for i if, and only if, $((D^2u_i(x))^{-1} \oplus 0_{K\setminus K_i})$ is C^*-quasidual for all $x \in \mathbb{R}^{K_i}$.*

When each $\mathbb{R}^{K_i}$ is the span of some set of the packages, this simplifies to the familiar condition of Theorem 1: Suppose $\mathbb{R}^{K_i} = \operatorname{span} \mathcal{C}_i$ for some $\mathcal{C}_i \subseteq \mathcal{C}$. Then $\mathcal{C}$ are substitutes for i if, and only if, $(D^2u_i(x))^{-1}$ is C_i^*-quasidual for all $x \in \mathbb{R}^{K_i}$.

We say packages $\mathcal{C}$ are *substitutes on i's projection of X* if $Dq_i^C(p_C)$ is quasipositive whenever $T_C^{-1}q_i^C(p_C) \in X$, or equivalently if $(D^2u_i(x))^{-1} \oplus 0_{K\setminus K_i}$ is C^*-quasidual for all $x \in (X \cap (\mathbb{R}^{K_i} \oplus 0_{K\setminus K_i})) \setminus \mathbb{R}^{K\setminus K_i}$. The extension of the core selection Theorem 2 follows.

Corollary 7 (**Core Selection: Heterogeneous Domains**). *Suppose preference domains are heterogeneous. For a given Q, the Vickrey auction is core selecting with respect to bidders for all profiles of participants if there exist packages $\mathcal{C}$ which are substitutes on all bidders' projections of $C \cap (Q - C)$, nonnegative quantities of which are allocated to each bidder in the efficient allocation for each $W \subset N$; i.e., $q_i^*(W, Q) \in C$.*

6.3 Imperfect divisibility

For imperfectly divisible settings (i.e., when allocations are constrained to some subset of $\mathbb{Z}^K$), we contribute a converse similar to Proposition 1 to illustrate that the issues due to the two-sidedness that we examine, particularly those illustrated in Example 1, are not unique to divisible goods. We conjecture that results analogous to ours are available in indivisible good settings, with perhaps additional restrictions on the cones used to formulate the idea of package substitutability due to the integer choice set.

Lemma 8 (**No Core Selection with Indivisible Goods**). *Suppose allocations are constrained to be integer vectors. If $N \geq 4$, there exists a profile of agents with the same preferences but different initial allocations such that the Vickrey outcome is not in the core.*

7 Implementation

Clearly, our analysis applies no matter how the auctioneer chooses to characterize the set of bundles on offer, whether it be in terms of goods (a convention we have so far chosen to adopt) or in terms of an alternative set of packages. Our cone characterization of package substitutability (Theorem 1) and its application in giving sufficient conditions for core selection (Theorem 2) captures the fact that regardless of the packages employed in defining the auction, one may be able to find implicit packages through which the auctioneer can verify that core selection is to be expected. These implicit packages can be of more direct use in implementation.

7.1 Separability across packages

The requirement that bidders consider all combinations of goods when submitting their demand schedules to the auctioneer seems onerous in practice, as we mentioned in the introduction. Our analysis suggests that in the Vickrey auction, this imposition can be weakened considerably without changing the outcome of the auction when substitution patterns between goods are "the same" (i.e., commutative) across different bidders – they have the same eigenspace, and hence can be simultaneously diagonalized. Then we can write $D^2u_i(x) = T_C'M_i(x)T_C$ for each i where $M_i(x)$ is diagonal and T_C is orthogonal.[52] This means that we can define packages – the rows of T_C – in which each agent's payoff is additively separable (Lemma 5). One can then run separate auctions for each of the K packages, knowing that bidders have no incentive to change their bid in one auction upon learning the outcome of another. In what follows, when agents' utilities are additively separable in packages $\mathcal{C}$, assume (without loss) that the $\mathcal{C}$ are orthonormal and write $u_i(q_i) = \sum_{i=1}^{N} u_i^k(\{q_i^C\}_k)$. In the Appendix, we show that when $\mathcal{C}$ are auctioned separately, truthful bidding is still a dominant strategy.

Theorem 4 (**Equivalent Implementation**)**.** *Suppose that all agents' utilities are additively separable in the packages $\mathcal{C}$. When the packages $\mathcal{C}$ are sold in separate Vickrey auctions, payments and allocation are the same in the K-good Vickrey auction, and market-clearing prices are equivalent: $p = T_C'p_C$.*

Thus, knowing that utility Hessians commute is useful – apart from permitting core selection (under the conditions in Theorem 3), by Theorem 4 and Lemma 5, commutative Hessians allow complexity reduction.[53] In some markets, indeed, one can find packages such that all bidders' utilities are separable; e.g., packages such that how much spectrum one has does not affect the desirability of switching between licenses. Note that when reallocation is infeasible, Theorem 4 breaks down (unless, of course, the packages are just the K goods). The reason is that while utilities may be separable on their domain, feasibility constraints are not.

[52] Orthogonality follows from symmetry of the Hessians.

[53] One may wish that, additionally, the design does not require the bidders to submit more than is needed for computing the payment and the allocation – this complexity reduction can be accomplished through a dynamic implementation that induces valuations along the path one can integrate over to recover the Vickrey payments (e.g., as in Ausubel and Cramton (2004)).

When running *separate* auctions for packages, knowledge of the eigenbasis by the designer is critical for ensuring efficiency. Indeed, with packages other than those employed by Lemma 5, efficiency and the dominant strategy property can be compromised under auction separation.

7.2 Why run a two-sided auction instead of two one-sided auctions?

Throughout, we have assumed that a two-sided version of the Vickrey auction (that is, one that allows agents to both buy and sell) is being used – a one-sided Vickrey auction would only be constrained efficient, preventing reallocation of goods among bidders. When some agents are known to be buyers and some are known to be sellers, why not instead accommodate the designer's objective to both buy and sell by having him run *two* standard one-sided Vickrey auctions? This may seem especially tempting in light of our conclusion that the two-sided auction presents additional challenges for core selection. Let us call such a design choice a *split auction*.

Definition 5. A *split Vickrey auction* is two separate Vickrey auctions $\mathcal{M}_+, \mathcal{M}_-$ such that

(i) (Disjoint Participation) For each $i \in N$, either $i \in N_+$ and i participates in $\mathcal{M}_+$ or $i \in N_-$ and i participates in $\mathcal{M}_-$;

(ii) (Clearing Rule) The outcomes of the sub-mechanisms are related by $p_- = p_+$ and $Q_+ = Q - Q_-$.

If an agent is known to be a "seller," then he belongs in N_-; if he is known to be a "buyer," then he belongs in N_+. The split auction idea is exemplified by the proposed design of the upcoming FCC Incentive Auction for wireless spectrum licenses.[54] Such mechanisms have obvious disadvantages in relation to true two-sided auctions: the auctioneer needs to know ex ante who will be a buyer and who will be a seller; each agent must end up on the same side of the market for all goods; and two separate auctions must either be run simultaneously or one must be run repeatedly for different quantities of total trade. We add to these a result that a split Vickrey auction always generates less revenue than the two-sided version, despite resulting in the same allocation.

Proposition 5 (Revenue in Split vs. Two-Sided Vickrey Auction). *In a split Vickrey auction, revenue is weakly lower than in a two-sided Vickrey auction.*

7.3 Revenue

Since in divisible good settings, market clearing prices are well defined, employing a variant of the uniform-price auction would seem to be an attractive alternative. The Vickrey auction

[54] The auction is currently scheduled for 2015. The proposed design is dynamically implemented with forward and reverse components, and a clearing rule that equalizes price across the two one-sided mechanisms.

is often viewed as having unfavorable revenue properties, especially when compared to the commonly used uniform-price auction. However, this is not always the case. One can show that while the Vickrey auction creates a higher surplus compared to uniform pricing (since the latter has an inherent inefficiency with divisible goods), it gives the auctioneer a smaller share of it (Figure 8). Which effect dominates then determines the two auctions' revenue ranking.

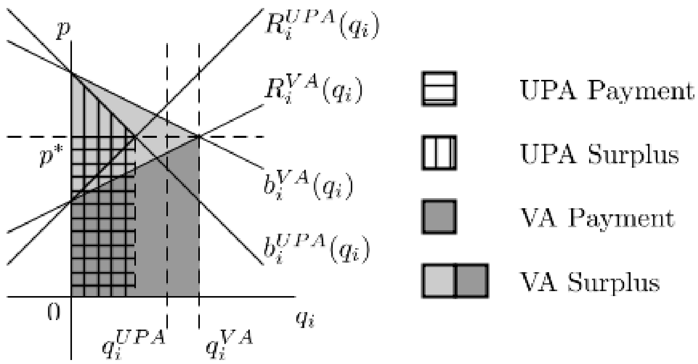

FIGURE 8: REVENUE AND SURPLUS IN VICKREY AND UNIFORM-PRICE AUCTIONS

Notes: The figure shows a bidder i's payment and surplus generated (i.e., $u_i(q_i)$) in the Vickrey and uniform-price auctions for a single good. In the uniform-price auction, agents shade their bids b_i^{UPA} inward from their marginal utility schedules b_i^{VA}; the market clears at the same price p^* as in the Vickrey auction, but at a quantity of smaller magnitude q_i^{UPA}.

When bidders have identical quadratic utilities – in particular, common valuations (i.e., $\theta_i = \theta$, for all i) that are unknown to the seller – and one good, Ausubel et al. (2014) show that the Vickrey auction outperforms the uniform-price auction. To allow for heterogeneity in $\{\theta_i\}_{i \in N}$, we compare the revenues in the two formats in the independent private values environment with quadratic utilities.[55] Proposition 6 shows that the Vickrey auction dominates uniform pricing in revenue terms when heterogeneity is small, thus in the environments where core selection is more likely to be expected.

First, consider expected revenue in the Vickrey auction. When agents' substitution patterns are common, i.e., $S_i = S$, expected Vickrey revenue is[56]

$$E(\pi_0^{VA}) = E(v(N)) - E\left(\sum_{i \in N} \pi_i\right) = -\frac{1}{2}\sum_{k \in K} \mathrm{Var}(\lambda_k' \theta_i) - \frac{2N-1}{2N(N-1)} Q'SQ + E(\theta_i)'Q.$$

where $\{\lambda_k\}_{k \in K}$ are the (orthogonal) eigenvectors of S^{-1} associated with unit eigenvalues.

[55] In a working paper version of Ausubel et al. (2014), the authors compare Vickrey and UPA revenue in the independent private values case with one good. Our technical contribution is the extension to the multiple good case (a simple one, due to the package separability created by common substitution patterns). More importantly, we use the result to provide an intuitive link between the set of environments where UPA is outperformed by the Vickrey auction and those in which core selection is likely to hold.

[56] For the derivation, see the proof of Proposition 6 in the Appendix.

Immediately we can see that Vickrey revenue is decreasing in expected heterogeneity (i.e., the variance term). When the auctioneer buys from some agents and sells to others in the course of the auction, he is losing money on every unit he needs to reallocate (i.e., when the auctioneer is buying, he pays more than the market clearing price, whereas he receives less than that price when he is selling; see Figure 9). The need to reallocate arises precisely when valuations are heterogenous; hence, he loses money when he needs to realize gains to trade among the agents. With a large enough quantity auctioned, however, the auctioneer will make non-negative profit – see Section 5.2.

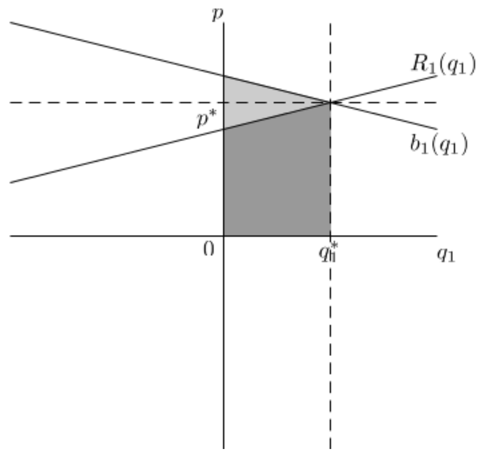

FIGURE 9: WHEN HE NEEDS TO REALLOCATE, THE AUCTIONEER LOSES MONEY

Notes: To see why reallocating means the auctioneer loses money (i.e., pays more than and receives less than price p), consider what happens when two bidders participate in a Vickrey exchange. In Figure 1, bidder 1's residual supply curve is just bidder 2's bid mirrored around the y-axis. The auction clears at q_1^*, p^*; bidder 1 makes a payment equal to the area of the smaller, darker-shaded region, while the auctioneer must pay bidder 2 that area plus that of the larger, lighter-shaded region.

Next, let us compute equilibrium in the uniform-price auction. Assume that agents' Hessians $-S_i$ are common knowledge (an assumption which was unnecessary in the analysis of the Vickrey auction). When submitting demand schedules, bidders equalize their marginal utility with price *plus price impact* $\Delta_i q_i$. Hence, bidders' first order conditions that must be satisfied for every price by their demand schedules b_i are

$$\theta_i - S_i b_i(p) = p + \Delta_i b_i(p),$$

where price impact Δ_i is the slope of the inverse residual supply curve that they face, which is given by

$$R_i(q_i) = \left(\sum_{j\neq i}(\Delta_j + S_j)^{-1}\right)^{-1} \left(\sum_{\ell\neq i}(\Delta_\ell + S_\ell)^{-1}\theta_\ell + q_i - Q\right).$$

Hence, $\Delta_i = \left(\sum_{j\neq i}(\Delta_j + S_j)^{-1}\right)^{-1}$. Assuming again that $S_i = S$ for all i, we have $\Delta_i = \Delta =$

$\frac{1}{N-2}S$. Market-clearing prices can then be computed as $\frac{1}{N}\sum_{i=1}^{N}\theta_i - \frac{N-1}{N(N-2)}SQ$. Thus, the expected revenue in the uniform-price auction is

$$E(\pi_0^U) = E(\theta_i)'Q - \frac{N-1}{N(N-2)}Q'SQ. \tag{2}$$

Proposition 6 (**Revenue in Vickrey vs. Uniform-Price Auctions**). *With quadratic utilities $u_i(\cdot)$, when $S_i = S$ for all $i \in N$ and θ_i are i.i.d., the expected revenue is higher in the Vickrey auction than the uniform-price auction if, and only if,*

$$\sum_{k\in K}\mathrm{Var}(\lambda_k'\theta_i) < \frac{1}{(N-1)(N-2)}Q'SQ.$$

The performance of the Vickrey auction vis-á-vis the uniform-price auction in terms of revenue depends on (i) the variance of the valuations of the "implicit packages" $\{\lambda_k\}_{k\in K}$ (the quadratic utility term is constant), with a higher variance making the uniform-price auction more attractive; (ii) the number of participating bidders N, with a larger number making uniform pricing more attractive; and (iii) the sum of the quantities of the packages $\{\lambda_k/||\lambda_k^2||\}_{k\in K}$ supplied by the auctioneer, since these are the eigenvectors of S corresponding to unit eigenvalues, with a larger supply of them rendering the Vickrey auction more attractive.

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Appendix[57]

Proof of Lemma 1 (Coalitional Rationality of Participation): Suppose that Z is the set of agents who participate. We want to show that

$$|W|v(Z,Q) - \sum_{i\in W} v(Z\setminus i, Q) \geq v(W,0),$$

for all W. Denote by $|W|Z$ the set of agents formed by replicating Z, $|W|$ times. Note further that $|W|v(Z,Q) = v(|W|Z, |W|Q)$: the same price clears each of the replica economies, so joining them together cannot change the efficient allocation and, thus, leaves total surplus unchanged. Now note that $v(|W|Z, |W|Q) \geq v(W,0) + \sum_{i\in W} v(Z\setminus i, Q)$, since in the social planner's problem for the grand coalition $|W|Z$, we could always allocate Q to the members

[57] The cones in this paper are convex and pointed. Quasipositivity of A is equivalent to nonnegativity of the off-diagonal entries of A, i.e. $-A$ is an Z-matrix (and so also an M-matrix when A is negative definite).

of each $Z \setminus i$ and reallocate goods efficiently between the members of the residual coalition W. The assertion follows. $\square$

Proof of Lemma 2 (**Implications of Core Selection**): (*Submodularity of* $v(\cdot)\Leftrightarrow$ *core selection*) First, note that since 2^N is a product of chains, submodularity is equivalent to decreasing differences. Let Z be the set of participating agents. Since $\sum_{i\in Z}\pi_i = v(Z)$ by allocative efficiency, the payoff imputation is unblocked by the union of 0 and $W \subset Z$ if, and only if,

$$\begin{aligned}
& v(Z,Q) - \sum_{i\notin W}\pi_i \geq v(W,Q)\ \forall W \subset Z \\
\Leftrightarrow & v(Z,Q) - v(W,Q) \geq \sum_{i\notin W} v(Z,Q) - v(Z\setminus i, Q) \\
\Leftrightarrow & \sum_{i\notin W} v(Z\setminus\{j\notin W|j<i\},Q) - v(Z\setminus\{j\notin W|j\leq i\},Q) - v(Z,Q) + v(Z\setminus i,Q) \geq 0 \\
\Leftrightarrow & \sum_{i\notin W} v(Z\setminus\{j\notin W|j<i\},Q) + v(Z\setminus i,Q) - v((Z\setminus i)\cap(Z\setminus\{j\notin W|j<i\}),Q)\ - v((Z\setminus i)\cup(Z\setminus\{j\notin W|j<i\}),Q) \geq 0.
\end{aligned}$$

All of the sum terms are positive whenever $v(\cdot)$ is submodular in bidders. When it is not, some difference-in-difference must be positive, and so the Vickrey payoff profile for some set of participants must be blocked by some coalition formed by excluding two participating bidders.

($\Rightarrow$ (1)) We want to show that when an agent acquires some quantity by submitting more than one bid, he can acquire the same quantity for a smaller payment by submitting one bid. If agent $i \in Z$ makes two bids b_α and b_β, which are the truthful bids of some admissible types u_α and u_β, and acquires q_α and q_β, then the market must clear at the same price as it would if agent i acquired $q_\alpha + q_\beta$ through a single bid, since the other agents must still be consuming a total of $Q - q_\alpha - q_\beta$ and R_i is injective. Then (from the calculations in footnote 17) shill α pays

$$x_\alpha = v((Z\cup\beta)/i) - v(Z\cup\beta/i, Q-q_\alpha) = v((Z\cup\beta)/i) - v((Z\cup\beta\cup\alpha)/i) + u_\alpha(q_\alpha)$$

and shill β pays

$$x_\beta = v((Z\cup\alpha)/i) - v((Z\cup\alpha)/i, Q-q_\beta) = v((Z\cup\alpha)/i) - v((Z\cup\beta\cup\alpha)/i) + u_\beta(q_\beta).$$

Now suppose agent i instead made the bid $b_{\alpha\beta} = \nabla v(\alpha\cup\beta,\cdot)$. This bid would yield i a quantity of $q_\alpha + q_\beta$ and cause him to pay

$$\begin{aligned}
x_{\alpha\beta} &= v(Z/i) - v((Z\cup\alpha\cup\beta)/i, Q-q_\alpha-q_\beta) \\
&= v(Z/i) - v((Z\cup\beta\cup\alpha)/i) + v(\alpha\cup\beta, q_\alpha+q_\beta) \\
&= v(Z/i) - v((Z\cup\beta\cup\alpha)/i) + u_\alpha(q_\alpha) + u_\beta(q_\beta).
\end{aligned}$$

Then, by bidder-submodularity, the following inequality holds

$$x_{\alpha\beta} - x_\alpha - x_\beta = v(Z/i) + v((Z \cup \beta \cup \alpha)/i) - v((Z \cup \beta)/i) - v((Z \cup \alpha)/i) \leq 0$$

and so it costs less to acquire $q_\alpha + q_\beta$ with one bid than with two when coalition Z/i is present. It follows by induction that any quantity that can be acquired with some collection of shill bids can be acquired more cheaply with a single bid.

($\Rightarrow$ (2)) When the agents in Z participate, seller revenue is

$$\sum_{i \in Z} v(Z/i) - (|Z| - 1)v(Z) = v(Z/j) + \sum_{i \in Z/j} (v(Z/i) - v(Z))$$

choosing some $j \in Z$. Then the difference between revenue when $Z \cup \ell$ participate and when Z participate is $v(Z) - v(Z \cup \ell) + v((Z \cup \ell)/j) - v(Z/j) + \sum_{i \in Z/j}(v((Z \cup \ell)/i) - v(Z \cup \ell) - v(Z/i) + v(Z))$. Each term of the sum is weakly positive by bidder-submodularity, as is the part outside the sum. $\square$

For Corollary 2, note that the more stringent requirement of bidder-submodularity does not yield a stronger conclusion about shill bidding, since the empty set would only arise in the proof if the shill bidder was the only participant in the auction, a case in which incentives to shill are different and primarily concern fooling the auctioneer into running the Vickrey auction instead of canceling it.

Proof of Lemma 3 (**Coalitional Value Function**): The Lagrangian is

$$\mathcal{L}(q_W, \mu) = \sum_{i \in W} \left(\theta_i' q_i - \frac{1}{2} q_i' S_i q_i \right) - \mu' (\sum_{i \in W} q_i - Q).$$

The first-order condition (which is sufficient, by concavity of the objective) is

$$\theta_i - S_i q_i = \mu, \ \forall i \in W.$$

Solving for μ in terms of primitives by finding q_i and then summing across the i:

$$\mu = H(W) \left(\sum_{i \in W} S_i^{-1} \theta_i - Q \right).$$

Substituting back into the FOC:

$$\theta_i - S_i q_i = H(W)\left(\sum_{j\in W} S_j^{-1}\theta_j - Q\right),\ \forall i \in W$$
$$q_i = S_i^{-1}\theta_i - S_i^{-1}H(W)\left(\sum_{j\in W} S_j^{-1}\theta_j - Q\right),\ \forall i \in W.$$

Now substituting our optimal choice of quantity back into the coalitional payoff:

$$\begin{aligned}
v(W,Q) =& \sum_{i\in W}\theta_i' S_i^{-1}\theta_i - \left(\sum_{i\in W}\theta_i' S_i^{-1}\right) H(W)\left(\sum_{i\in W} S_i^{-1}\theta_i - Q\right)\\
&- \frac{1}{2}\sum_{i\in W}\theta_i' S_i^{-1}\theta_i + \left(\sum_{i\in W}\theta_i' S_i^{-1}\right) H(W)\left(\sum_{i\in W} S_i^{-1}\theta_i - Q\right)\\
&- \frac{1}{2}\left(\sum_{i\in W} S_i^{-1}\theta_i - Q\right)' H(W)\left(\sum_{i\in W} S_i^{-1}\theta_i - Q\right)\\
=& \frac{1}{2}\sum_{i\in W}\theta_i' S_i^{-1}\theta_i - \frac{1}{2}\left(\sum_{i\in W} S_i^{-1}\theta_i - Q\right)' H(W)\left(\sum_{i\in W} S_i^{-1}\theta_i - Q\right).
\end{aligned}$$

□

Proof of Lemma 4 (**Goods Substitutability as a Condition on Primitives**): Agent i's first-order condition is $\nabla u_i(q_i(p)) - p = 0$. By the implicit function theorem, then, we have $D_p q_i(p) = (D^2 u_i(q_i(p)))^{-1}$. Hence, goods are substitutes for i exactly when the off-diagonal entries of $(D^2 u_i(q_i(p)))^{-1}$ are nonnegative, that is, when $(D^2 u_i(q_i(p)))^{-1}$ is quasipositive. □

Proof of Theorem 1 (**Package Substitutability as a Condition on Primitives**): ((1) ⇔ (2)) In the package pricing UMP, the agent's first-order condition is $T_C^{-1\prime}\nabla u_i(T_C^{-1}q_i^C) - p_C = 0$. By the implicit function theorem,

$$Dq_i^C(p_C) = \left(T_C^{-1\prime} D^2 u_i(T_C^{-1}q_i^C)T_C^{-1}\right)^{-1} = T_C(D^2 u_i(T_C^{-1}q_i^C))^{-1}T_C'.$$

This matrix is quasipositive everywhere if, and only if, for some α we have

$$\begin{aligned}
&T_C(D^2 u_i(x))^{-1}T_C' y + \alpha y \geq 0, \quad \forall y \geq 0, x \in \mathbb{R}^K\\
\Leftrightarrow\ &T_C(D^2 u_i(x))^{-1}z + \alpha T_C^{-1\prime} z \geq 0, \quad \forall z \in C^*, x \in \mathbb{R}^K\\
\Leftrightarrow\ &T_C(D^2 u_i(x))^{-1}z + \alpha T_C T_C^{-1} T_C^{-1\prime} z \geq 0, \quad \forall z \in C^*, x \in \mathbb{R}^K\\
\Leftrightarrow\ &(D^2 u_i(x))^{-1}z + \alpha T_C^{-1} T_C^{-1\prime} z \in C, \quad \forall z \in C^*, x \in \mathbb{R}^K
\end{aligned}$$

as desired. □

Proof of Theorem 2 (Core Selection: Package Substitutability and Allocations): We want to deal with the problem where allocations are constrained to the region where we know packages $\mathcal{C}$ are substitutes, because we know this problem will have the same solution as that without such a constraint, since by assumption $q_i^*(W,Q) \in C \cap (Q-C)$ for all $W \subseteq N$. In order to do so, we need the following lemma, which essentially says that when packages are locally substitutes in the original, unconstrained problem, they are still (globally) substitutes in the new constrained one.

Lemma 9 (Constrained Submodularity)**.** *If packages are substitutes on $C \cap (Q-C)$, then*

$$\tilde{\Pi}_i^C(p_C) = \max_{q_i^C}\{u_i(T_C^{-1}q_i^C) - p_C' q_i^C \ \textit{s.t.}\ \{q_i^C\}_k \in [0, \{T_C Q\}_k]\}$$

is submodular.

Proof of Lemma 9 (Constrained Submodularity): Let $\tilde{q}_i^C(p_C) \equiv \arg\max_{q_i^C}\{u_i(T_C^{-1}q_i^C) - p_C' q_i^C \text{ s.t. } \{q_i^C\}_k \in [0, \{T_C Q\}_k]\}$. Since $u_i(\cdot)$ is strictly concave, $\tilde{q}_i^C(p_C)$ is single-valued. So by Corollary 4 in Milgrom and Segal (2002), $\nabla\tilde{\Pi}_i^C(p_C) = -\tilde{q}_i^C(p_C)$ everywhere. By the maximum theorem, $\tilde{q}_i^C(p_C)$ is continuous, so $\{\tilde{q}_i^C\}_k^{-1}((0,\{T_C Q\}_k))$ is open. For each pair of disjoint $K_1, K_2 \subseteq K$, let

$$S_{K_1,K_2} \equiv \bigcap_{k\in K_1} \operatorname{int}\{\tilde{q}_i^C\}_k^{-1}(0) \cap \bigcap_{k\in K_2} \operatorname{int}\{\tilde{q}_i^C\}_k^{-1}(\{T_C Q\}_k) \cap \bigcap_{k\notin K_1+K_2} \{\tilde{q}_i^C\}_k^{-1}((0,\{T_C Q\}_k)).$$

On each of these open sets in price space, the agent's first-order condition is

$$\left(T_C^{-1\prime}\nabla u_i\left(T_C^{-1}\left(\{\tilde{q}_i^C(p_C)\}_{K\setminus(K_1+K_2)} \oplus 0_{K_1} \oplus \{T_C Q\}_{K_2}\right)\right) - \{p_C\}\right)_{K\setminus(K_1+K_2)} = 0_{K\setminus(K_1+K_2)}.$$

By the implicit function theorem, then,

$$\left(D^2\tilde{q}_i^C(p_C)\right)_{K\setminus(K_1+K_2)} = \left(\left(T_C^{-1\prime}D^2u_i(T_C^{-1}\tilde{q}_i^C(p_C))T_C^{-1}\right)_{K\setminus(K_1+K_2)}\right)^{-1} \ \forall p_C \in S_{K_1,K_2}.$$

Because $\{\tilde{q}_i^C(p_C)\}_{K_1+K_2}$ is constant on S_{K_1,K_2}, we have

$$D^2\tilde{q}_i^C(p_C) = \left(\left(\left(T_C^{-1\prime}D^2u_i(T_C^{-1}\tilde{q}_i^C(p_C))T_C^{-1}\right)_{K\setminus(K_1+K_2)}\right)^{-1} \oplus 0_{K_1+K_2}\right) \ \forall p_C \in S_{K_1,K_2}.$$

Since $\left(T_C^{-1\prime}D^2u_i(T_C^{-1}\tilde{q}_i^C(p_C))^{-1}T_C^{-1}\right)^{-1}$ is quasipositive, and the class of quasipositive matrices is closed under Schur complementation, $D^2\tilde{q}_i^C(p_C)$ has nonnegative off-diagonal entries on each S_{K_1,K_2}. The fundamental theorem of calculus for line integrals then implies that $\{\tilde{q}_i^C(p_C)\}_k$ is nondecreasing in $\{p_C\}_j$, $j \neq k$, on each of these sets' path components.[58] Since $\tilde{q}_i^C(p_C)$ is

[58] These are open since Euclidean space is locally path connected.

continuous, this must hold on those path components' closures as well; since $\bigcup_{K_1,K_2} S_{K_1,K_2}$ is by construction dense in $\mathbb{R}^K$, it holds everywhere. Another application of the fundamental theorem for line integrals tells us that $\tilde{\pi}_i^C$ has decreasing differences, and so is submodular. $\square$

Note that

$$
\begin{aligned}
v(W,Q) &= \max_{\{q_i\}_{i\in W}} \left\{ \sum_{i\in W} u_i(q_i) \text{ s.t. } \sum_{i\in W} q_i = Q \right\} \\
&= \max_{\{q_i \in C\cap(Q-C)\}_{i\in W}} \left\{ \sum_{i\in W} u_i(q_i) \text{ s.t. } \sum_{i\in W} q_i = Q \right\} \qquad (3) \\
&= \max_{\{q_i \in C\cap(Q-C)\}_{i\in W}} \left\{ \sum_{i\in W} u_i(q_i) \text{ s.t. } \sum_{i\in W} T_C q_i = T_C Q \right\} \qquad (4) \\
&= \min_{p_C} \max_{\{q_i \in C\cap(Q-C)\}_{i\in W}} \left\{ \sum_{i\in W} u_i(q_i) - p_C \cdot \left(\sum_{i\in W} T_C q_i - T_C Q \right) \right\} \qquad (5) \\
&= \min_{p_C} \left\{ \sum_{i\in W} \max_{q_i \in C\cap(Q-C)} \{ u_i(q_i) - p_C \cdot T_C q_i \} + p_C \cdot T_C Q \right\} \\
&= \min_{p_C} \left\{ \sum_{i\in W} \max_{\{q_i^C\}_k \in [0,T_C Q]} \{ u_i(T_C^{-1} q_i^C) - p_C \cdot q_i^C \} + p_C \cdot T_C Q \right\} \\
&= \min_{p_C} \left\{ \sum_{i\in W} \tilde{\Pi}_i^C(p_C) + p_C \cdot T_C Q \right\}
\end{aligned}
$$

We write the social planner's problem as (3) using our assumption about the efficient allocation. The primal (4) and dual (5) optimization problems are equivalent since the primal problem is maximization of a concave function over a convex region under a feasible linear constraint, and hence Slater's condition ensures that strong duality holds. From here we follow the proofs of Theorems 2.6.2 and 2.7.6 in Topkis (1998):

The objective $\sum_{i\in W} \tilde{\Pi}_i^C(p_C) + p_C \cdot x$ has decreasing differences in (W, p_C): $\sum_{i\in W} (\tilde{\Pi}_i^C(p_C''') - \tilde{\Pi}_i^C(p_C''))$, $p_C''' > p_C''$, is decreasing in W since $(\tilde{\Pi}_j^C(p_C''') - \tilde{\Pi}_j^C(p_C'')) < 0$: the allocation chosen by j at p_C''' is cheaper at p_C'' (because $q_i^C(p_C''') \geq 0$) so profit must be at least as high at the smaller price vector. Hence for all $p_C, p_C' \in \mathbb{R}^K$ we have

$$
\begin{aligned}
&\textstyle\sum_{i\in W\cup Z}\tilde{\Pi}_i^C(p_C'\vee p_C)-\sum_{i\in W}\tilde{\Pi}_i^C(p_C) \\
&\textstyle= \sum_{i\in W\cup Z}\tilde{\Pi}_i^C(p_C'\vee p_C)-\sum_{i\in W\cup Z}\tilde{\Pi}_i^C(p_C) \\
&\textstyle\quad+\sum_{i\in W\cup Z}\tilde{\Pi}_i^C(p_C)-\sum_{i\in W}\tilde{\Pi}_i^C(p_C) \\
&\textstyle\leq \sum_{i\in Z}\tilde{\Pi}_i^C(p_C'\vee p_C)-\sum_{i\in Z}\tilde{\Pi}_i^C(p_C) && \text{(decreasing differences)} \\
&\textstyle\quad+\sum_{i\in Z}\tilde{\Pi}_i^C(p_C)-\sum_{i\in W\cap Z}\tilde{\Pi}_i^C(p_C) && (W\cup Z-W=Z-W\cap Z) \\
&\textstyle\leq \sum_{i\in Z}\tilde{\Pi}_i^C(p_C')-\sum_{i\in Z}\tilde{\Pi}_i^C(p_C\wedge p_C') && \text{(Lemma 9)} \\
&\textstyle\quad+\sum_{i\in Z}\tilde{\Pi}_i^C(p_C\wedge p_C')-\sum_{i\in W\cap Z}\tilde{\Pi}_i^C(p_C\wedge p_C') && \text{(decreasing differences)} \\
&\textstyle= \sum_{i\in Z}\tilde{\Pi}_i^C(p_C')-\sum_{i\in W\cap Z}\tilde{\Pi}_i^C(p_C'\wedge p_C).
\end{aligned}
$$

We also have $p_C'\vee p_C-p_C=p_C'-p_C'\wedge p_C$. So for all $p_C,p_C'\in\mathbb{R}^K$,

$$
\begin{aligned}
\sum_{i\in Z}\tilde{\Pi}_i^C(p_C')+p_C'\cdot T_CQ+&\sum_{i\in W}\tilde{\Pi}_i^C(p_C)+p_C\cdot T_CQ \\
&\geq \sum_{i\in W\cup Z}\tilde{\Pi}_i^C(p_C'\vee p_C)+p_C'\vee p_C\cdot T_CQ+\sum_{i\in W\cap Z}\tilde{\Pi}_i^C(p_C'\wedge p_C)+p_C'\wedge p_C\cdot T_CQ \\
&\geq v(W\cup Z,Q)+v(W\cap Z,Q).
\end{aligned} \tag{6}
$$

for any W,Z such that $W\cap Z\neq\emptyset$. Taking minimums over prices on the left-hand side yields

$$v(W\cup Z,Q)+v(W\cap Z,Q)\leq v(Z,Q)+v(W,Q)$$

as desired, and $v(\cdot)$ is submodular on the sublattice $2^{N-1}\times\{i\}$ for each $i\in N$. Now we consider $W\cap Z=\emptyset$. When $p_C,p_C'\geq 0$, $p_C\wedge p_C'\geq 0$ and so from (6) we have

$$
\begin{aligned}
\sum_{i\in Z}\tilde{\Pi}_i^C(p_C')+p_C'\cdot T_CQ+\sum_{i\in W}\tilde{\Pi}_i^C(p_C)+p_C\cdot T_CQ &\geq v(W\cup Z,Q)+p_C'\wedge p_C\cdot T_CQ \\
&\geq v(W\cup Z,Q)+0=v(W\cup Z,Q)+v(W\cap Z,Q)
\end{aligned} \tag{7}
$$

Note that, from the envelope theorem, $\nabla v(W,Q)=T_C'\arg\min_{p_C}\left\{\sum_{i\in W}\Pi_i^C(p_C)+p_C\cdot T_CQ\right\}$, and we have

$$T_C'^{-1}\nabla v(W,Q)=\arg\min_{p_C}\left\{\sum_{i\in W}\Pi_i^C(p_C)+p_C\cdot T_CQ\right\}\Rightarrow T_C'^{-1}\nabla v(W,Q)\in\arg\min_{p_C}\left\{\sum_{i\in W}\tilde{\Pi}_i^C(p_C)+p_C\cdot T_CQ\right\}.$$

The derivative of $v(W,Q)$ in the direction of c_k is positive means that $c_k'\nabla v(W,Q)\geq 0$. When this is true for each $c_k\in C$ and for all W, then $T_C'^{-1}\nabla v(W,Q)\geq 0$ since the c_k form the columns of T_C^{-1}. It follows that we can minimize the left-hand side of (7) over the positive orthant to get $v(W\cup Z,Q)+v(W\cap Z,Q)\leq v(Z,Q)+v(W,Q)$ and $v(\cdot)$ is bidder-submodular.

The assertions follow by Lemma 2 and Corollary 2. □

Calculations from Example 4 (Core Selection and Gross Substitutes and Complements) Using the formula for $q_i^*(W,Q)$ from Lemma 3, we have

$$T_C q_1^*(\{1,2,3\},Q) = \begin{bmatrix}0.4410\\0.8217\\0.4190\end{bmatrix} \quad T_C q_2^*(\{1,2,3\},Q) = \begin{bmatrix}1.9171\\0.4240\\0.3362\end{bmatrix} \quad T_C q_3^*(\{1,2,3\},Q) = \begin{bmatrix}0.2419\\0.5543\\0.0448\end{bmatrix}$$
$$T_C q_1^*(\{1,2\},Q) = \begin{bmatrix}0.5850\\1.1454\\0.3963\end{bmatrix} \quad T_C q_2^*(\{1,2\},Q) = \begin{bmatrix}2.0150\\0.6546\\0.4037\end{bmatrix} \quad T_C q_3^*(\{1,3\},Q) = \begin{bmatrix}1.2443\\0.4284\\0.3567\end{bmatrix}$$
$$T_C q_1^*(\{1,3\},Q) = \begin{bmatrix}1.3557\\1.3716\\0.4433\end{bmatrix} \quad T_C q_2^*(\{2,3\},Q) = \begin{bmatrix}2.1187\\1.0073\\0.4859\end{bmatrix} \quad T_C q_3^*(\{2,3\},Q) = \begin{bmatrix}0.4813\\0.7927\\0.3141\end{bmatrix},$$

and, hence, $q_i^*(W,Q) \in C$ for each $W \subseteq \{0,1,2,3\}$.

Proof of Lemma 5 (Eigenvector Condition for Substitution Symmetry): Let T_C be a polyhedral generator for the cone generated by $\mathcal{C}$. It follows from the chain rule that $D^2_{q_i^C} u_i(T_C^{-1} q_i^C) = T_C^{-1\prime} D^2_{q_i} u_i(q_i) T_C^{-1}$. Since $\mathbb{R}$ is a chain, $u_i(\cdot)$ is separable in q_i^C if, and only if, it is modular in q_i^C; that is, if, and only if, $D_x^2 u_i(T_C^{-1}x)$ is diagonal. This is true if, and only if, we can write $D_x^2 u_i(x) = T_C' M_i(x) T_C$ for diagonal $M_i(x)$, which we can do if, and only if, scalar multiples of $\mathcal{C}$ are an orthonormal eigenbasis for $D^2 u_i(x)$. □

Proof of Theorem 3 (**Core Selection: Sufficient Conditions on the Primitives**): Matrices commute if, and only if, they have the same eigenspace; let $\mathcal{C}$ be their orthonormal eigenvectors, oriented so that Q is in the cone C they generate. (Such orthogonal eigenvectors exist because the matrices are symmetric.) By Lemma 5, $u_i(\cdot)$ is separable in the $\mathcal{C}$ for each i, implying they are substitutes for each bidder. Hence we can write $u_i(q_i) = \sum_{k=1}^K u_i^k(\{q_i^C\}_k)$ and optimize in each package separately when computing $v(\cdot)$ and q_i^*. For each k, if the initial endowment is efficiently allocated among N, we have $u_i^{k\prime}(0) = d_k$ for each $i \in N$ and some d_k. Given $W \subseteq N$, from the first-order conditions of the coalitional optimization we have $u_i^{k\prime}(\{q_i^{*C}(W,Q)\}_k) = \mu_k(W,Q)$ for each i and some $\mu_k(W,Q)$. Then for each $k \in K$, either $\mu_k(W) \leq d_k$ and thus strict concavity implies $\{q_i^{*C}(W,Q)\}_k \geq 0$ for each i, or $\mu_k(W,Q) > d_k$ and thus $\sum_{i \in W} \{q_i^{*C}(W,Q)\}_k < 0$ for each i. The latter is impossible since $\{q_i^{*C}(W,Q)\}_k = T_C Q \geq 0$. □

Proof of Proposition 1 (**No Core Selection: Heterogeneity in Initial Endowments I**): Let agents 1, 2, and 3 participate; choose

$$t_1 = 0,\ t_2 = (\nabla u_2)^{-1}(\nabla u_1(Q)),\ t_3 = \begin{cases} 0, & \nabla u_1(Q) \neq \nabla u_3(0) \\ \mathbf{1}, & \nabla u_1(Q) = \nabla u_3(0) \end{cases}.$$

So for coalitions $\{0,1\}, \{0,1,2\}$, the price $\nabla u_1(Q)$ clears the market, hence allocating the entire quantity vector to bidder 1 is efficient and v is the same for each of them. Now $v(\{0,1,2,3\}) \geq v(\{0,1,3\})$ because we could always allocate nothing to bidder 2 in the grand coalition. The inequality is strict: for $q_2 = 0$ to be efficient in the grand coalition, the market would have to clear at $\nabla u_1(Q)$ and the entire quantity vector would have to again be allocated to bidder 1; but at such a price, bidder 3 would demand some amount different than zero. Then, $\pi_0 + \pi_1 = v(\{0,1,2\}) + v(\{0,1,3\}) - v(\{0,1,2,3\}) < v(\{0,1\})$, and the coalition $\{0,1\}$ blocks the Vickrey payoff profile. □

Proof of Proposition 2 (**No Core Selection: Heterogeneity in Initial Endowments II**): Let agents 1, j, and ℓ participate; choose $Q = (\nabla u_1)^{-1}(\nabla u_j(0))$. The rest of the proof follows exactly the same as that of Proposition 1, with agents 2 and 3 replaced by agents j and ℓ respectively. □

Proof of Proposition 3 (**No Core Selection: Heterogeneity in Substitution Patterns**): We first prove the following result.

Lemma 10 (**No Core Selection: Quadratic Utilities**). *When agents have quadratic valuations $u_i(q_i) = \theta_i' q_i - \frac{1}{2} q_i' S_i q_i$, if $\theta_i = \theta$ for each i and $S_\ell^{-1} H(Z \setminus \ell + Z \setminus j) S_j^{-1}$ has a negative eigenvalue for some coalition $Z \subset N$ and agents $\ell, j \in Z$, then $v(\cdot, Q)$ is not bidder-submodular for some Q. In particular, this implies that $(H(Z) + H(Z \setminus \{\ell \cup j\}) - H(Z \setminus j) - H(Z \setminus \ell))$ has a negative eigenvalue, and that $v(\cdot, Q)$ is not bidder-submodular when Q is a scalar multiple of the eigenvector associated with the negative eigenvalue.*

Proof of Lemma 10 (**No Core Selection: Quadratic Utilities**): In the quadratic environment, when agents' marginal utilities at zero are equal – and thus $\theta_i = \theta$ for each i – we know from Lemma 3 that decreasing differences, and thus bidder submodularity, holds if and only if

$$Q'(H(Z) + H(Z \setminus \{\ell \cup j\}) - H(Z \setminus \ell) - H(Z \setminus j))Q > 0 \tag{8}$$

Because $(H(Z) + H(Z \setminus \{\ell \cup j\}) - H(Z \setminus \ell) - H(Z \setminus j))$ is symmetric, this is true *for all* Q if, and only if, it is positive semidefinite; that is, if $H(Z) + H(Z \setminus \{\ell \cup j\}) \succeq H(Z \setminus \ell) + H(Z \setminus j)$ in the positive semidefinite order. Otherwise, $H(Z) + H(Z \setminus \{\ell \cup j\}) - H(Z \setminus \ell) - H(Z \setminus j)$ has a negative eigenvalue and thus (8) is negative whenever Q is a scalar multiple of the associated eigenvector.

Matrix algebra gives that

$$\sum_{i\in Z\setminus\ell} S_i^{-1} + \sum_{i\in Z\setminus j} S_i^{-1} = \sum_{i\in Z\setminus\ell} S_i^{-1} + \sum_{i\in Z\setminus\{\ell\cup j\}} S_i^{-1} + S_\ell^{-1}$$

$$\begin{aligned}I_K =& \frac{1}{2}\left(\sum_{i\in Z\setminus\ell} S_i^{-1} + \sum_{i\in Z\setminus j} S_i^{-1}\right)^{-1}\left(\sum_{i\in Z\setminus\ell} S_i^{-1} + \frac{1}{2}\sum_{i\in Z\setminus\{\ell\cup j\}} S_i^{-1}\right) + \left(\sum_{i\in Z\setminus\ell} S_i^{-1} + \sum_{i\in Z\setminus j} S_i^{-1}\right)^{-1} S_\ell^{-1} \\ &+ \frac{1}{2}\left(\sum_{i\in Z\setminus\ell} S_i^{-1} + \frac{1}{2}\sum_{i\in Z\setminus\{\ell\cup j\}} S_i^{-1}\right)\left(\sum_{i\in Z\setminus\ell} S_i^{-1} + \sum_{i\in Z\setminus j} S_i^{-1}\right)^{-1} + S_\ell^{-1}\left(\sum_{i\in Z\setminus\ell} S_i^{-1} + \sum_{i\in Z\setminus j} S_i^{-1}\right)^{-1}\end{aligned}$$

$$\begin{aligned}\sum_{i\in Z\setminus\ell} S_i^{-1} - \sum_{i\in Z\setminus\{\ell\cup j\}} S_i^{-1} =& \left(\sum_{i\in Z\setminus\ell} S_i^{-1}\right)\left(\sum_{i\in Z\setminus\ell} S_i^{-1} + \sum_{i\in Z\setminus j} S_i^{-1}\right)^{-1}\left(\sum_{i\in Z\setminus\ell} S_i^{-1}\right) \\ &- \left(\sum_{i\in Z\setminus\{\ell\cup j\}} S_i^{-1}\right)\left(\sum_{i\in Z\setminus\{\ell\cup j\}} S_i^{-1} + \sum_{i\in Z} S_i^{-1}\right)^{-1}\left(\sum_{i\in Z\setminus\{\ell\cup j\}} S_i^{-1}\right) \\ &+ \frac{1}{2}S_j^{-1}\left(\sum_{i\in Z\setminus\ell} S_i^{-1} + \sum_{i\in Z\setminus j} S_i^{-1}\right)^{-1} S_\ell^{-1} + \frac{1}{2}S_\ell^{-1}\left(\sum_{i\in Z\setminus\ell} S_i^{-1} + \sum_{i\in Z\setminus j} S_i^{-1}\right)^{-1} S_j^{-1},\end{aligned}$$

where the last equality is by difference of squares and since $\sum_{i\in Z\setminus\{\ell\cup j\}} S_i^{-1} + \sum_{i\in Z} S_i^{-1} = \sum_{i\in Z\setminus j} S_i^{-1} + \sum_{i\in Z\setminus\ell} S_i^{-1}$. Then, from the harmonic mean identity

$$(A^{-1} + B^{-1})^{-1} = A - A(A+B)^{-1}A,$$

we have

$$\begin{aligned}(H(Z\setminus\ell) + H(Z\setminus j))^{-1} =& (H(Z) + H(Z\setminus\{\ell\cup j\}))^{-1} + \frac{1}{2}S_\ell^{-1}H(Z\setminus\ell + Z\setminus j)S_j^{-1} \\ &+ \frac{1}{2}S_j^{-1}H(Z\setminus\ell + Z\setminus j)S_\ell^{-1}.\end{aligned}$$

Since $A \succeq B$ if, and only if, $B^{-1} \succeq A^{-1}$, we can now deduce that $H(Z) + H(Z\setminus\{\ell\cup j\}) \succeq H(Z\setminus\ell) + H(Z\setminus j)$ if, and only if,

$$S_\ell^{-1}H(Z\setminus\ell + Z\setminus j)S_j^{-1} + S_j^{-1}H(Z\setminus\ell + Z\setminus j)S_\ell^{-1}$$

is positive semidefinite. If $S_\ell^{-1}H(Z\setminus\ell + Z\setminus j)S_j^{-1}$ has a negative eigenvalue, though, then this cannot be the case: clearly, $x'(S_\ell^{-1}H(Z\setminus\ell + Z\setminus j)S_j^{-1} + S_j^{-1}H(Z\setminus\ell + Z\setminus j)S_\ell^{-1})x < 0$ when x is the associated eigenvector. □

Back to the assertion of Proposition 3, let A be the rotation matrix that maps Q to the eigenvector associated with the negative eigenvalue of $(H(Z) + H(Z \setminus \{\ell \cup j\}) - H(Z \setminus \ell) - H(Z \setminus j))$ that, by the previous lemma, must exist. Since $\left(\sum_{i \in W}(A'S_iA)^{-1}\right)^{-1} = A'H(W)A$, we then have

$$v(Z, Q) + v(Z \setminus \{\ell \cup j\}, Q) > v(Z \setminus \ell, Q) + v(Z \setminus j, Q)$$

and there is some profile of participants such that the Vickrey outcome is not in the core by Lemma 2. Hence, if the difference-in-difference of the harmonic mean has a negative eigenvalue, then there is some cone which gives no core selection when it contains Q. □

Calculations from Example 6 (Sum of Vickrey Payoffs): We have

$$\begin{aligned}
\sum_{i \in N} \pi_i =& \frac{1}{2} \sum_{i \in N} \theta_i' S^{-1} \theta_i - \frac{1}{2N} \sum_{i \in N} (\sum_{j \in N} \theta_j - SQ)' S^{-1} (\sum_{j \in N} \theta_j - SQ) \\
& + \frac{1}{2(N-1)} \sum_{i \in N} (\sum_{j \in N} \theta_j - \theta_i - SQ)' S^{-1} (\sum_{j \in N} \theta_j - \theta_i - SQ) \\
=& \frac{1}{2} \sum_{i \in N} \theta_i' S^{-1} \theta_i - \frac{1}{2} (\sum_{j \in N} \theta_j - SQ)' S^{-1} (\sum_{j \in N} \theta_j - SQ) \\
& + \frac{N}{2(N-1)} (\sum_{j \in N} \theta_j - SQ)' S^{-1} (\sum_{j \in N} \theta_j - SQ) \\
& - \frac{1}{N-1} \sum_{i \in N} \theta_i' S^{-1} (\sum_{j \in N} \theta_j - SQ) + \frac{1}{2(N-1)} \sum_{i \in N} \theta_i' S^{-1} \theta_i \\
=& \frac{N}{2(N-1)} \sum_{i \in N} \theta_i' S^{-1} \theta_i + \frac{1}{2(N-1)} (\sum_{j \in N} \theta_j - SQ)' S^{-1} (\sum_{j \in N} \theta_j - SQ) \\
& - \frac{1}{N-1} \sum_{i \in N} \theta_i' S^{-1} (\sum_{j \in N} \theta_j - SQ) \\
=& \frac{N}{2(N-1)} \sum_{i \in N} \theta_i' S^{-1} \theta_i - \frac{1}{2(N-1)} (\sum_{j \in N} \theta_j)' S^{-1} (\sum_{j \in N} \theta_j) + \frac{1}{2(N-1)} Q' S Q.
\end{aligned}$$

Letting $\{\lambda_k\}_{k\in K}$ be the (orthogonal) eigenvectors of S associated with unit eigenvalues,

$$\begin{aligned}\sum_{i\in N}\pi_i =& \frac{1}{2(N-1)}\sum_{i\in N}\theta_i' S^{-1}\left(N\theta_i-\sum_{j\in N}\theta_j\right)+\frac{1}{2(N-1)}Q'SQ\\ =& \frac{1}{2(N-1)}\sum_{i\in N}\theta_i' S^{-1}\left((N-1)\theta_i-\sum_{j\neq i}\theta_j\right)+\frac{1}{2(N-1)}Q'SQ\\ =& \frac{1}{2}\sum_{i\in N}\theta_i' S^{-1}\left(\theta_i-\frac{1}{N-1}\sum_{j\neq i}\theta_j\right)+\frac{1}{2(N-1)}Q'SQ\\ \Rightarrow E\left(\sum_{i\in N}\pi_i\right) =& \frac{N}{2}\sum_{k\in K}\mathrm{Var}(\lambda_k'\theta_i)+\frac{1}{2(N-1)}Q'SQ.\end{aligned}$$

Proof of Proposition 4 (**Bidder-Submodularity and Quantity Auctioned**) Let $p_C^*(W,Q)$ be the market-clearing package price vector when W participate in a Vickrey auction for Q. The envelope conditions for agents' primal (profit-maximization) problems yield

$$\nabla\Pi_i^C(p_C)=q_i^C \Rightarrow \sum_{i\in W}\nabla\Pi_i^C(p_C^*(W,Q))=T_CQ$$

The inverse function theorem yields

$$\begin{aligned}D_Q p_C^*(W,Q) &= \left(\sum_{i\in W}D^2\Pi_i^C(p_C^*(W,Q))\right)^{-1}T_C\\ &= \left(\sum_{i\in W}T_C D^2u_i(q_i^*(W,Q))^{-1}T_C'\right)^{-1}\end{aligned}$$

From the proof of Theorem 1, $Dq_i^C(p_C)=T_CD^2u_i(q_i^C(p_C))^{-1}T_C'$, so applying the chain rule yields

$$\begin{aligned}D_QT_Cq_i^*(W,Q) &= T_CD^2u_i(q_i^*(W,Q))^{-1}T_C'\left(\sum_{j\in W}T_CD^2u_j(q_j^*(W,Q))^{-1}T_C'\right)^{-1}T_C\\ &= T_CD^2u_i(q_i^*(W,Q))^{-1}\left(\sum_{j\in W}D^2u_j(q_j^*(W,Q))^{-1}\right)^{-1}\end{aligned}$$

It follows that $T_Cq_i^*(W,Q)$ are each increasing in exactly the directions in the polyhedral cone $\bigcap_{i\in W}\left(\sum_{j\in W}D^2u_j(q_j^*(W,Q))^{-1}\right)D^2u_i(q_i^*(W,Q))T_C^{-1}\mathbb{R}_+^K$. □

Proof of Corollary 3 (**Core Selection: Sufficient Conditions on the Primitives II**): The assertion follows from Propostion 4. □

Proof of Corollary 4 (**Core Selection: Sufficient Conditions on the Primitives III**): From Proposition 4, we have

$$T_C q_i^*(W, Q + a\Delta Q) = T_C q_i^*(W, Q)$$
$$+a \int_0^1 T_C D^2 u_i(r q_i^*(W, Q + a\Delta Q) + (1-r) q_i^*(W, Q))^{-1} \left(\sum_{\ell \in W} D^2 u_j(r q_j^*(W, Q + a\Delta Q) + (1-r) q_j^*(W, Q))^{-1} \right)^{-1} \Delta Q dr$$

Then let[59]

$$a = \frac{\max_{k \in K, i \in W, W \subseteq N} -\{T_C q_i^*(W, Q)\}_k}{\inf_{k \in K, \{x_j\}_{j \in W} \in \mathbb{R}^{|W|K}, i \in W, W \subseteq N} \{T_C (D^2 u_i(x_i))^{-1} \left(\sum_{\ell \in W} D^2 u_j(x_j)^{-1} \right)^{-1} \Delta Q\}_k}$$

It follows that $q_i^*(W, Q + a\Delta Q) \in C$ for all W and all $i \in W$. □

Proof of Lemma 6 (**Package Substitutability: Infeasible Reallocation**): The proof is similar to that of Lemma 9, but now each constraint need not involve only one dimension of q_i^C. Again, since $u_i(\cdot)$ is strictly concave, $\bar{q}_i^C(p_C) = T_C \bar{q}_i(T_C' p_C)$ is single-valued; by the maximum theorem, $\bar{q}_i(p)$ is continuous, so $\{\bar{q}_i\}_k^{-1}((0, \infty))$ is open for each k. Let $U_{K_3} = \bigcap_{k \in K_3} \text{int}\{\bar{q}_i\}_k^{-1}(0) \cap \bigcap_{k \notin K_3} \{\bar{q}_i\}_k^{-1}((0, \infty))$. Then on each U_{K_3} we can use the implicit function theorem as in Lemma 9 to get

$$D\bar{q}_i(p) = \left(\left(D^2 u_i(\bar{q}_i(p))_{K \setminus K_3} \right)^{-1} \oplus 0_{K_3} \right) \forall p \in U_{K_3}$$

and, by the chain rule,

$$D\bar{q}_i^C(p_C) = T_C \left(\left(D^2 u_i(T_C^{-1} \bar{q}_i^C(p_C))_{K \setminus K_3} \right)^{-1} \oplus 0_{K_3} \right) T_C' \forall p \in U_{K_3}, q_i(p) < \infty,$$

for each $K_3 \subseteq K$. A similar argument to that which ends Lemma 9 shows that quasipositivity of this matrix implies the packages $\mathcal{C}$ are reallocation-constrained substitutes. □

Proof of Corollary 5 (**Goods Substitutability: Infeasible Reallocation**): $(D^2 u_i(\bar{q}_i(p))_{-L})^{-1}$ is quasipositive for each L because it is a Schur complement of $(D^2 u_i(\bar{q}_i(p)))^{-1}$ and the class of M-matrices (and hence that of quasipositive matrices) are closed under Schur complementation. □

Proof of Lemma 7 (**Package Substitutability: Heterogeneous Domains**): First, we show that the familiar equilibrium in dominant strategies still exists when agents' preference domains are heterogeneous. In the two-sided demand-implemented Vickrey auction of K perfectly divisible goods, bidders with heterogeneous domains submit demand curves

[59] The reason we require $\Delta Q \in F(\epsilon T_C^{-1} \mathbf{1} + C)$ and not merely $\Delta Q \in F(\text{int}(C))$ is so that the denominator of this expression is nonzero. Specifically, it is at least ϵ.

$B_i : \mathbb{R}^{K_i} \to \mathbb{R}^{K_i}$ that are C^1 and have negative definite Jacobians. Adding demand curves horizontally gives the bidders' *total* residual supply curves, $R_i^{-1}(p) = Q - \sum_{j \neq i} B_j^{-1}(p_{K_j}) \oplus \mathbf{0}_{K-K_j}$. The inverses exist and are C^1 with positive definite Jacobian by the inverse function theorem as long as at least two bidders bid on each good.Bidders receive the market-clearing quantity – the unique solution to $R_i(\mathbf{0}_{K-K_i} \oplus q_i)_{K_i} = B_i(q_i)$, and are charged a payment of $\int_0^{q_i} R_i(\mathbf{0}_{K-K_i} \oplus \mathbf{r}) \cdot d\mathbf{r}$. Hence, similarly to the case of homogeneous preference domains, a bidder's payoff from receiving q_i is

$$\int_0^{q_i} (\nabla u_i(\mathbf{r}) - R_i(\mathbf{0}_{K-K_i} \oplus \mathbf{r})) \cdot d\mathbf{r}$$

and, thus, i's ex post first-order condition is still $\nabla u_i(q_i) - R_i(\mathbf{0}_{K-K_i} \oplus q_i)_{K_i} = 0$.

Now we have $q_i^C(p_C) = T_C(q_i(T_C' p_C) \oplus 0_{K \setminus K_i})$. From the inverse function theorem,

$$Dq_i(p) = \begin{bmatrix} (D^2 u_i(q_i(p)))^{-1} \\ 0_{K \setminus K_i} \end{bmatrix}.$$

Then, from the chain rule, $Dq_i^C(p_C) = T_C((D^2 u_i(q_i(p)))^{-1} \oplus 0_{K \setminus K_i}) T_C'$. It follows from the proof of Theorem 1 that this matrix is quasipositive everywhere if, and only if, $((D^2 u_i(x))^{-1} \oplus 0_{K \setminus K_i})$ is C^*-quasidual for all $x \in \mathbb{R}^{K_i}$. □

Proof of Lemma 8 (No Core Selection with Indivisible Goods): Choose $u_1 \in U$ and some point $z \leq Q - 4 \cdot \mathbf{1}$. Now choose

$$u_2(x) = u_3(x) = u_1(x + 2z - Q) - u_1(2z - Q) \text{ and } u_4(x) = u_1(x + z) - u_1(z).$$

For coalitions $\{1, 2\}, \{1, 3\}, \{1, 2, 4\}, \{1, 3, 4\}$ any price in the subdifferential of $u_1(z)$ clears the market, hence the allocations $\{z, Q - z\}, \{z, Q - z\}, \{z, Q - z, 0\}, \{z, Q - z, 0\}$ are efficient. It is clear, then, that $v(\cdot)$ is the same for each of them.

For $\{1, 2, 3\}$, since $z + 2(Q - z) > Q + 3 \cdot \mathbf{1}$, some agent must receive a quantity of some good at least 2 units smaller than he did in any of the two-person coalitions he was a member of. It follows from strict concavity that the new market-clearing price p' is such that $z \notin q_1(p') = q_2(p') + (2z - Q) = q_3(p') + (2z - Q) = q_4(p') + z$. Since $0 \notin q_4(p')$, it follows that in the grand coalition, allocating 0 to agent 4 is strictly suboptimal, and hence that $v(\{1, 2, 3, 4\}) > v(\{1, 2, 3\})$. □

Lemma 11 (Dominant Strategy Property under Separability). *Suppose that all agents' utilities are additively separable in the packages $\mathcal{C}$. When the $\mathcal{C}$ are sold in separate Vickrey auctions, submitting the demand curves $b_i^k(p_k) = (u_i^{k\prime})^{-1}(p_k)$ for each package $c_k \in \mathcal{C}$ is a dominant strategy and therefore optimal ex post.*

Proof of Lemma 11 (**Dominant Strategy Property under Separability**): Each agent i's utility is separable in the $\mathcal{C}$, and by design, payments are separable in the $\mathcal{C}$. Hence, the optimality conditions in each auction are unaffected by the outcome of the others. By the logic of Section 2.3, then, submitting $b_i^k(p_k) = (u_i^{k\prime})^{-1}(p_k)$ is a dominant strategy. $\square$

Proof of Theorem 4 (**Equivalent Implementation**): *(Equivalent prices and identical allocation)* Market clearing in each of the separate Vickrey auctions is given by

$$c_k'Q = \sum_{i=1}^{N} \{q_i^C\}_k \ \forall k \in K \text{ and } b_i^k(p_k) = \{q_i^C\}_k \ \forall \ i \in N, \ k \in K.$$

Since untilities $u_i(\cdot)$ are separable, for equilibrium bids this is equivalent to

$$T_C Q = \sum_{i=1}^{N} q_i^C \text{ and } p_C = D_{q_i^C} u_i(T_C' q_i^C) \ \forall \ i \in N,$$

by the previous lemma. Applying the chain rule yields $p_C = T_C \nabla u_i(T_C' q_i^C) \ \forall \ i \in N.$. Since $p_C = T_C p$ and $q_i^C = T_C q_i$, this gives us $Q = \sum_{i=1}^{N} q_i$ and $p = \nabla u_i(q_i) \ \forall \ i \in N$, which are just the market clearing conditions for the K-good Vickrey auction, given equilibrium bids.
(Equal payments) In the K-good Vickrey auction, payments are given by $\int_0^{q_i} R_i(\mathbf{r}) \cdot d\mathbf{r}$. This integral is path-independent since DR_i is symmetric, and so can alternatively be written

$$\sum_{k=1}^{K} \int_0^{c_k' q_i} c_k' R_i(rc_k + \sum_{\ell < k} (c_\ell' q_i) c_\ell) dr,$$

since $q_i = T_C' T_C q_i = \sum_{k \in K} (c_k' q_i) c_k$. We want to show that $c_k' R_i(q_i + \epsilon c_\ell) = c_k' R_i(q_i)$ for all $k, q_i, \epsilon, \ell \neq k$ so that we can get rid of the sum inside R_i. The easiest way to do this is to show that $D_t T_C R_i(T_C' t)$ is diagonal. The inverse function theorem and the chain rule gives us

$$\begin{aligned} D_t T_C R_i(T_C' t) &= T_C D R_i(T_C' t) T_C' \\ &= -T_C \left(\sum_{j \neq i} D u_j(t_j)^{-1} \right)^{-1} T_C' \text{ for some } t_j \\ &= -\left(\sum_{j \neq i} M_j(t_j)^{-1} \right)^{-1} \quad \text{for some } t_j, \end{aligned}$$

for some diagonal $M_j(t_j)$, by Lemma 5. Thus, we can write payments as

$$\sum_{k=1}^{K} \int_0^{c_k' q_i} c_k' R_i(rc_k) dr$$

which is equivalent to the total payment $\sum_{k=1}^{K} \int_0^{\{q_i^C\}_k} R_i^k(r)dr$ when packages are sold separately.[60] □

Proof of Proposition 5 (**Revenue in Split vs. Two-Sided Vickrey Auction**): In the split auction, if $i \in N_+$, i receives payoff $v(N_+, Q_+) - v(N_+ - i, Q_+)$; if $i \in N_-$, i receives payoff $v(N_-, Q_-) - v(N_- - i, Q_-)$. By price equalization, marginal utilities equalize across the two submechanisms and the allocation is efficient; hence $v(N \cup \{0\}, Q) = v(N_+, Q_+) + v(N_-, Q_-)$. We also have, by the definition of $v(\cdot)$ and disjoint participation, $v(W_+, Q_+) + v(W_-, Q_-) \leq v(W_+ \cup W_-, Q)$. So, if $i \in N_+$, i receives payoff $\pi_i = v(N_+, Q_+) - v(N_+ - i, Q_+) = v(N_+, Q_+) + v(N_-, Q_-) - (v(N_+ - i, Q_+) + v(N_-, Q_-)) \;\geq v(N \cup 0, Q) - v((N \setminus i) \cup 0, Q)$; and if $i \in N_-$, i receives payoff $\pi_i = v(N_-, Q_-) - v(N_- - i, Q_-) = v(N_+, Q_+) + v(N_-, Q_-) - (v(N_- - i, Q_-) + v(N_+, Q_-)) \geq v(N \cup 0, Q) - v((N \setminus i) \cup 0, Q)$. Since in either the split or two-sided auction, the auctioneer receives revenue $\pi_0 = v(N \cup 0, Q) - \sum_{i=1}^{N} \pi_i$, weakly higher payoffs for bidders implies weakly lower revenue for the auctioneer. □

Proof of Proposition 6 (Revenue in Vickrey vs. Uniform-Price Auctions): The expected revenue in the uniform price auction is computed in the main text. For the expected Vickrey revenue, $E(\pi_0^{VA}) = E(v(N)) - E\left(\sum_{i \in N} \pi_i\right)$, from Example 6 we have

$$E\left(\sum_{i \in N} \pi_i\right) = \frac{N}{2} \sum_{k \in K} \text{Var}(\lambda_k' \theta_i) + \frac{1}{2(N-1)} Q' S Q,$$

Computing $E(v(N))$:

$$\begin{aligned}
v(N) &= \frac{1}{2} \sum_{i \in N} \theta_i' S^{-1} \theta_i - \frac{1}{2N} \left(\sum_{i \in N} \theta_i - SQ\right)' S^{-1} \left(\sum_{i \in N} \theta_i - SQ\right) \\
&= \frac{1}{2} \sum_{i \in N} \theta_i' S^{-1} \theta_i - \frac{1}{2N} \left(\sum_{i \in N} \theta_i\right)' S^{-1} \left(\sum_{i \in N} \theta_i\right) - \frac{1}{2N} Q' S Q + \frac{1}{N} \left(\sum_{i \in N} \theta_i\right)' Q \\
&= \frac{1}{2N} \sum_{i \in N} \theta_i' S^{-1} \left(N\theta_i - \sum_{j \in N} \theta_j\right) - \frac{1}{2N} Q' S Q + \frac{1}{N} \left(\sum_{i \in N} \theta_i\right)' Q \\
&= \frac{N-1}{2N} \sum_{i \in N} \theta_i' S^{-1} \left(\theta_i - \frac{1}{N-1} \sum_{j \neq i} \theta_j\right) - \frac{1}{2N} Q' S Q + \frac{1}{N} \left(\sum_{i \in N} \theta_i\right)' Q \\
\Rightarrow E(v(N)) &= \frac{N-1}{2} \sum_{k \in K} \text{Var}(\lambda_k' \theta_i) - \frac{1}{2N} Q' S Q + E(\theta_i)' Q
\end{aligned}$$

[60] That $c_k' R_i(rc_k) = R_i^k(r)$ is implied by the equivalence result of part I of the proof.

and so expected revenue in the Vickrey auction is

$$E(\pi_0^{VA}) = -\frac{1}{2}\sum_{k\in K} \mathrm{Var}(\lambda_k'\theta_i) - \frac{2N-1}{2N(N-1)}Q'SQ + E(\theta_i)'Q. \tag{9}$$

The ranking then follows from (2) and (9). □