The Income Effect under Uncertainty: A Slutsky-like Decomposition with Risk Aversion

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Abstract

We study the effect of changing income on optimal decisions in the multidimensional expected utility framework with strongly separable preferences. Using the Kihlstrom and Mirman (1974) (KM) utility representation, we show that the effect of changing income can be decomposed into a modified income effect linked to the classical income effect and an effect representing attitudes to risk, modified by income.

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1 Introduction

The use of comparative statics is at the foundation of the study of behavior in economics. Under certainty, the income and substitution effects are building blocks for the understanding of the effects of parameter changes on optimal behavior. Moreover, the Slutsky equation which establishes a sharp decomposition for changes in the demand for a good, caused by changing parameters, is an important tool for understanding the comparative analysis. However, the notions of income and substitution effects have had little impact on the comparative analysis for optimal behavior under uncertainty with the exception of Kihlstrom and Mirman (1974) (KM) and Mirman and Santugini (2013) (MS). Indeed, there is no Slutsky-like decomposition separating attitude toward risk (i.e., risk aversion) and tastes (i.e., ordinal preferences with the marginal rate of substitution).

Comparative statics under uncertainty began with Arrow (1965) and Pratt (1964) in the portfolio problem with a one-dimensional utility function.[1] In their papers (as well as all applications with a one-dimensional utility function in the literature), there is no income effect. Hence, the comparative analysis in the portfolio problem depends only on the substitution effect.[2] This is not the case with a multidimensional utility function (i.e., several goods) as shown by KM.[3] It is in KM that the notions of income and substitution effects are first used to study the effect of risk aversion in the multidimensional case. Then, MS use the income and substitution effects to study the behavior of optimal decisions due to changes in risk aversion in a general multidimensional setting under uncertainty.

Since virtually all aspects of behavior deal with uncertainty (e.g., consumers facing uncertainty about prices and income, firms facing uncertainty

[1] Specifically, Arrow (1965) considers the effect of changing income in the portfolio problem. Pratt (1964), on the other hand, deals with changes in risk aversion on optimal decisions.

[2] Although dealing with two different issues, the effect of risk aversion (Pratt) and the effect of increasing income (Arrow) have identical results, that is, the Arrow and Pratt theorems are equivalent in the portfolio case.

[3] KM uses the notion of a concave transformation to define increases in risk aversion in the multidimensional case, establishing the fact that the effect of risk aversion and the effect of tastes (i.e., marginal rate of substitution) cannot be in general separated.

about the amount of money they can raise and the input prices), it is important to understand the comparative analysis under uncertainty. Moreover, although the comparative analysis in the portfolio problem under uncertainty in the one-dimensional case is a natural first step, the comparative statics properties in the general multidimensional expected utility framework must be studied. In the multidimensional case, the effect of uncertainty on optimal behavior combines both tastes (i.e., ordinal preferences) and attitudes toward risk (i.e., risk aversion). In order to study the effect of risk aversion on optimal behavior, the effects of tastes and risk aversion must be identified. This is done by KM which generalizes the notion of risk aversion to the multidimensional case by introducing utility representations that are concave transformations. It is precisely this definition of risk aversion that MS used to characterize the effect of risk aversion on optimal behavior.

It is the purpose of this paper to provide a Slutsky-like decomposition which takes account of the effect of tastes and of attitudes toward risk. Specifically, we study the effect of changing income on optimal decisions in the multidimensional expected utility framework.[4] To that end, we use the KM utility representation to highlight the role of risk aversion on the comparative analysis. Indeed, the introduction of uncertainty is always implicitly accompanied by assumptions regarding attitudes toward risk, otherwise, risk-neutrality vitiates the effect of uncertainty. In the general utility representation, it is not very difficult to do comparative statics and obtain income and substitution expressions that are expectations of the classical income and substitution effects. However, these expectations do not reveal the role played by risk aversion in determining optimal behavior. In other words, with the general utility representation, risk aversion is implicit.

The general utility representation is thus inappropriate to present comparative statics results under uncertainty since it hides the role of risk aversion. In order to decompose the effect of changing income and identify the role of attitudes toward risk and the role of ordinal preferences, we use the KM utility representation. We show that the basic principles of classical de-

[4]Our analysis could be extended to changes in prices under different sources of uncertainty.

mand theory can be used to understand the comparative analysis of income changes under uncertainty. In particular, preferences interact with risk aversion to give a modified separately identified role for risk aversion, and risk preferences interact with the ordinal income effect to give a modified income effect. This is the basis of the decomposition that we show is possible.

More specifically, we study the effect of changing income in the consumption-saving problem when the rate of return is random, making the consumption of the good subject to saving risky. We first decompose the effect of a change in income on both the sure good and the risky good. We show that the effect of changing income depends on the now explicit change in risk aversion. Specifically, the effect of changing income can be decomposed into a modified income effect and a hybrid risk-aversion effect. The modified income effect captures the effect of changing income through uncertainty and the hybrid risk-aversion effect captures the effect of changing risk aversion due to changes in income. The hybrid risk-aversion effects are related to the pure risk aversion effect contained in MS.

Using the decomposition, we study the effect of a change in income. The sign of the modified income effect depends on the normality (tastes) of the goods whereas the sign of the hybrid risk-aversion effect depends on both tastes and attitudes toward risk. We consider both constant and decreasing absolute risk aversion. When risk preferences exhibit constant absolute risk aversion, it is shown that an increase in income always increases the amount of the sure good. However, the effect of changing income is ambiguous under decreasing absolute risk aversion. In particular, suppose that both goods are normal. On the one hand, the normality of the risky good induces more consumption of the risky good when income increases. On the other hand, as income increases, the individual becomes less risk averse, which, under certain conditions regarding the income and substitution effects, induces less consumption of the risky good. Hence, the overall effect of increasing income is ambiguous. One implication of our results is that there is no equivalence between Pratt's portfolio theorem (Pratt, 1964) and Arrow's portfolio theorem (Arrow, 1965) in the multidimensional case.

The paper is organized as follows. In Section 2, we introduce the model and discuss the KM approach. Section 3 presents the comparative analysis.

2 KM Framework

The effect of uncertainty on optimal behavior in the multidimensional case combines both tastes (i.e., ordinal preferences) and attitudes toward risk (i.e., risk aversion). In order to study the effect of risk aversion on optimal behavior, the effects of tastes and risk aversion must be identified. This issue does not arise for the class of one-dimensional strictly increasing utility functions since tastes are represented by the natural ordering on the real line, i.e., $x_A > x_B$ means that $x_A \succ x_B$. However, the relationship between the utility representation, uncertainty, risk aversion, and tastes is much more delicate in the multidimensional case since there is no natural order. In other words, different utility functions incorporate different tastes as well as different attitudes toward risk so that the link between risk aversion and risk averse behavior cannot be clearly identified.[5]

In this section, we use the approach established by KM for the study of the effect of risk aversion on optimal behavior under uncertainty in the multidimensional case. When going from certainty to uncertainty, risk aversion is always implicit. Hence, uncertainty and risk aversion are naturally entangled with each other. In other words, the effect of uncertainty on behavior cannot be studied without taking account of the role played by risk aversion.

To show the intricacies of optimal behavior under uncertainty, it is useful to begin with a general utility representation. In that case, risk aversion, implicit in optimal behavior, cannot in general be recognized. To remedy that problem, we consider utility functions that are concave transformations of each other as studied in KM. Studying optimal behavior under uncertainty using the KM framework makes the role of risk aversion explicit. Moreover, this representation clarifies the relationship between uncertainty and risk aversion as well as their distinct roles for optimal behavior. We apply the KM framework to the consumption-saving problem under uncertainty and

[5] For instance, KM provides an example in which the preference between a sure outcome and a gamble depends solely on tastes and not on risk aversion. See Appendix A.

derive optimal behavior, highlighting the role of risk aversion for the effect of changing income on optimal behavior in the consumption-saving problem under uncertainty.

General Utility Representation. Consider an individual making decisions under uncertainty. Let the consumption profile $(x, \tilde{y}) \in \mathbb{R}^2_+$ have utility representation $V(x, \tilde{y})$. In the stochastic environment, x is the *sure* good and $\tilde{y}$ is the *risky* good due to the presence of randomness in the budget constraint. Specifically, the maximization problem under uncertainty is

$$\max_x \mathbb{E}_{\tilde{\varepsilon}} V(x, Z(x, \tilde{\varepsilon}, I)), \tag{1}$$

where $\mathbb{E}_{\tilde{\varepsilon}}$ is the expectation operator over a random shock $\tilde{\varepsilon}$. The risky good depends on the sure good x, the random shock $\tilde{\varepsilon}$ and the income I through a budget constraint, i.e., $\tilde{y} = Z(x, \tilde{\varepsilon}, I)$. Assuming that the second-order condition is satisfied, optimal consumption is defined by the first-order condition corresponding to (1), i.e.,

$$\mathbb{E}_{\tilde{\varepsilon}} \frac{\partial V(x, Z(x, \tilde{\varepsilon}, I))}{\partial x} + \mathbb{E}_{\tilde{\varepsilon}} \frac{\partial V(x, Z(x, \tilde{\varepsilon}, I))}{\partial \tilde{y}} \frac{\partial Z(x, \tilde{\varepsilon}, I)}{\partial x} = 0 \tag{2}$$

evaluated at $x = x^*$.

Although implicit, expression (2) is uninformative regarding the effect of risk aversion on optimal behavior under uncertainty. In particular, the general utility representation confounds risk aversion and tastes. Indeed, consider the effect of changing income on the sure good. That is, using (2),

$$\begin{aligned} \frac{\partial x^*}{\partial I} \overset{S}{=} & \mathbb{E}_{\tilde{\varepsilon}} \frac{\partial^2 V(x, Z(x, \tilde{\varepsilon}, I))}{\partial x \partial \tilde{y}} \frac{\partial Z(x, \tilde{\varepsilon}, I)}{\partial I} + \mathbb{E}_{\tilde{\varepsilon}} \frac{\partial^2 V(x, Z(x, \tilde{\varepsilon}, I))}{\partial \tilde{y}^2} \frac{\partial Z(x, \tilde{\varepsilon}, I)}{\partial x} \frac{\partial Z(x, \tilde{\varepsilon}, I)}{\partial I} \\ & + \mathbb{E}_{\tilde{\varepsilon}} \frac{\partial V(x, Z(x, \tilde{\varepsilon}, I))}{\partial \tilde{y}} \frac{\partial^2 Z(x, \tilde{\varepsilon}, I)}{\partial x \partial I} \end{aligned} \tag{3}$$

evaluated at $x = x^*$ where $\overset{S}{=}$ means *of the same sign as.* Expression (3) does not make risk aversion explicit and provides no information on how risk aversion influences the comparative analysis. In fact, from expression (3), it appears that uncertainty has minimal effect on optimal behavior, i.e., removing the expectation operator in (3) yields the effect of changing income on the sure good in a deterministic environment. It looks as if there is no risk aversion effect although the introduction of uncertainty cannot occur without regard to risk aversion. Hence, the general utility representation hides the intricacies of optimal behavior under uncertainty.

KM utility representation. In order to clarify the relationship between uncertainty and risk aversion as well as their distinct roles on optimal behavior, the KM utility representation is adopted. Formally, let $V(x, \tilde{y}) \equiv \varphi\left(U(x, \tilde{y})\right)$ be the utility associated with the consumption profile $(x, y) \in \mathbb{R}_{+}^{2}$. Here, φ is a strictly increasing and concave function, $\varphi' > 0, \varphi'' \leq 0$ and $U(x, \tilde{y})$ is a quasiconcave function. Under the KM utility representation, $U(x, \tilde{y})$ refers to tastes as well as attitudes toward risk whereas φ reflects *changes* in risk aversion. Specifically, a more concave φ (and, thus, a more concave V) means that the individual is more risk-averse. Concave transformations of the utility function alter the expected marginal rate of substitution in a way that is consistent with ordinal preferences, but do not alter the deterministic marginal rate of substitution. In particular, with the KM approach, attitudes towards risk (i.e., the concavity of φ) is independent of any gamble.

Using the KM utility representation, the maximization problem under uncertainty defined by (1) is rewritten as

$$\max_{x} \mathbb{E}_{\tilde{\varepsilon}} \varphi\left(U(x, Z(x, \tilde{\varepsilon}, I))\right). \tag{4}$$

Using (4), optimal consumption is defined by the first-order condition

$$\mathbb{E}_{\tilde{\varepsilon}} \varphi'\left(U(x, Z(x, \tilde{\varepsilon}, I))\right) \cdot MU(x, Z(x, \tilde{\varepsilon}, I)) = 0 \tag{5}$$

evaluated at $x = x^{*}$. Here,

$$MU(x, Z(x, \tilde{\varepsilon}, I)) \equiv \frac{\partial U(x, Z(x, \tilde{\varepsilon}, I))}{\partial x} + \frac{\partial U(x, Z(x, \tilde{\varepsilon}, I))}{\partial \tilde{y}} \frac{\partial Z(x, \tilde{\varepsilon}, I)}{\partial x} \tag{6}$$

is the marginal utility of consumption for the sure good. Unlike (2), expression (5) makes risk aversion explicit through the term $\varphi'\left(U(x, Z(x, \tilde{\varepsilon}, I))\right)$. It is also clear that risk aversion and tastes are entwined. In particular, changing income has an effect on both attitudes toward risk and the marginal utility of consumption. That is, for the sure good, using (5),

$$\frac{\partial x^*}{\partial I} \stackrel{S}{=} \mathbb{E}_{\tilde{\varepsilon}}\varphi''\left(U(x, Z(x, \tilde{\varepsilon}, I))\right) \cdot \frac{\partial U(x, Z(x, \tilde{\varepsilon}, I))}{\partial \tilde{y}} \cdot \frac{\partial Z(x, \tilde{\varepsilon}, I)}{\partial I} \cdot MU(x, Z(x, \tilde{\varepsilon}, I))$$
$$+ \mathbb{E}_{\tilde{\varepsilon}}\varphi'\left(U(x, Z(x, \tilde{\varepsilon}, I))\right) \cdot \frac{\partial MU(x, Z(x, \tilde{\varepsilon}, I))}{\partial \tilde{y}} \cdot \frac{\partial Z(x, \tilde{\varepsilon}, I)}{\partial I} \tag{7}$$

evaluated at $x = x^*$. Unlike expression (3), expression (7) highlights the importance of the role of risk aversion for the comparative analysis through the concavity of φ. However, while (7) is more informative than (3), it remains to analyze the influence of attitudes toward risk (e.g., constant or decreasing absolute risk aversion) and tastes (e.g., normal or income-neutral goods) on the comparative analysis. To that end, we turn to a classical application.

Application to Consumption-Saving Problem. We now apply the KM approach to the consumption-saving problem under uncertainty. We present the model and derive optimal behavior. In the next section, we perform a comparative analysis of the effect of changing income on optimal behavior.

Consider an individual making consumption and saving decisions under uncertainty. As noted, in the stochastic environment, x is the sure good and $\tilde{y}$ is the risky good due to the presence of randomness in the budget constraint. Specifically, the individual is endowed with income $I > 0$ from which $x \in (0, I)$ is consumed and the remaining $s \equiv I - x$ is saved in a risky asset with the random gross return $\tilde{R}$. Given the random budget constraint, it follows that $\tilde{y} = \tilde{R}(I - x)$.

To facilitate the discussion, we adopt a binary distribution for the rate of return of the risky asset and consider strongly separable preferences.

Assumption 2.1. $\tilde{R} \sim (\pi \circ \overline{R}, (1-\pi) \circ \underline{R})$ *such that* $\pi \in (0, 1)$ *and* $0 < \underline{R} < \overline{R}$.

Assumption 2.2. $U(x, \tilde{y}) = u_1(x) + u_2(\tilde{y})$ *such that* $u_1', u_2' > 0, u_1'', u_2'' \leq 0$.

Additive preferences allows us to study several types of tastes. If $u_1'', u_2'' < 0$, then both goods are normal. If $u_1'' = 0, u_2'' < 0$ or $u_1'' < 0, u_2'' = 0$, then preferences are quasilinear. Finally, $u_1'' = u_2'' = 0$ refers to a situation in which both goods are income-neutral as in the Arrow-Pratt portfolio problem.[6]

Given Assumptions 2.1 and 2.2 and $\tilde{y} = \tilde{R}(I - x)$, (4) is rewritten as[7]

$$\max_{x \in (0,I)} \pi\varphi\left(u_1(x) + u_2(\overline{R}(I-x))\right) + (1-\pi)\varphi\left(u_1(x) + u_2(\underline{R}(I-x))\right). \quad (8)$$

Optimal consumption is defined by the first-order condition corresponding to (8), i.e.,

$$\pi\varphi'\left(u_1(x^*) + u_2(\overline{R}(I-x^*))\right) \cdot \left[u_1'(x^*) - u_2'(\overline{R}(I-x^*))\overline{R}\right] \\ + (1-\pi)\varphi'\left(u_1(x^*) + u_2(\underline{R}(I-x^*))\right) \cdot \left[u_1'(x^*) - u_2'(\underline{R}(I-x^*))\underline{R}\right] = 0 \quad (9)$$

evaluated at $x = x^*$ so that the amount allocated to the risky good is $s^* \equiv I - x^*$. Here, for $R \in \{\overline{R}, \underline{R}\}$, $u_1'(x^*) - u_2'(R(I-x^*))R$ is the deterministic marginal utility of consumption for the sure good. From (9), the KM utility representation makes risk aversion explicit in optimal behavior under uncertainty. Specifically, since $\varphi'' < 0$, it follows that

$$0 < \varphi'\left(u_1(x^*) + u_2(\overline{R}(I-x^*))\right) < \varphi'\left(u_1(x^*) + u_2(\underline{R}(I-x^*))\right). \quad (10)$$

Hence, risk aversion adds more weight to the deterministic marginal utility corresponding to the lowest rate of return.

In the next section, we use (9) to analyze the effect of changing income on optimal behavior. Under certainty, the effect of income is a pure income effect. However, under uncertainty, the income and substitutions effects play an additional part in the comparative statics because they are also involved in signing the effect of risk aversion. This relationship between the income and substitution effects and the risk aversion effect was pointed out in MS. Specifically, MS characterizes the effect of changing risk aversion on the basis of the income and substitution effects under different sources of uncertainty.

[6] In the Arrow-Pratt portfolio problem, $V(x, \tilde{y}) = \varphi(x + \tilde{y})$.

[7] Note that, in this formulation, $V(x, \tilde{y}) = \varphi(u_1(x) + u_2(\tilde{y}))$ cannot be additive.

In other words, MS studies the *pure* risk aversion effect, i.e., the change in both the sure good and the risky good due to a change in risk aversion. In this paper, for the case of a random price of the risky good (i.e., the rate of return on the risky good), we show that the effect of changing risk aversion is part of the effect of changing income. Note that, for the pure risk aversion effect, income remains constant so that the change in the risky good offsets the change in the sure good. In the problem of changing income studied in this paper, the pure risk aversion effect must be modified to take account of the change in income.

Before proceeding, note that the income and substitution effects (related to a deterministic change in the rate of return) order the deterministic marginal utility of consumption for the sure good. To simplify the discussion, we hereafter refer to income and substitution effects without mentioning that these effects are related to a deterministic change in the rate of return. Formally,[8]

Remark 2.3. $u_1'(x^*) - u_2'(\overline{R}(I-x^*))\overline{R} > 0 > u_1'(x^*) - u_2'(\underline{R}(I-x^*))\underline{R}$ *if and only if the income effect for the sure good is stronger than the substitution effect.*[9]

3 Comparative Analysis

We study, in this section, the effect of changing income on the sure good x^*, as well as the risky good through the amount saved, i.e., $s^* \equiv I -$

[8]Let $MU(x, R; I) \equiv u_1'(x) - u_2'(R(I-x))R$ so that $\partial MU(x, R; I)/\partial R|_{x=x^*} = -u_2''(R(I-x^*))R(I-x^*) - u_2'(R(I-x^*))$ where $-u_2''(R(I-x^*))R^2 > 0$ and $-u_2'(R(I-x^*)) < 0$ are proportional to and of the same sign as the income effect and the substitution effect, respectively, related to a deterministic change in R. Note that with no uncertainty, x^* is optimal for some value of $R \in (\underline{R}, \overline{R})$, An increase in R means a relative price increase for the sure good and a higher income. The income effect means a higher amount of the sure good and the substitution effect means a lower amount. Hence, if there is an increase in the amount of the sure good, the income effect dominates and the first part of the inequality must hold and similarly for the other side.

[9]Note that we implicitly ignore the case in which the income and substitution effects cancel each other since, in this case, uncertainty has no effect on optimal behavior and risk aversion is thus irrelevant.

x^*.[10] From (9), a change in income affects optimal behavior through the influence on risk aversion as well as the marginal utility of consumption. We proceed in several steps. We first decompose the effect of a change in income for the sure good and the risky good. We show that the effect of changing income depends on the effect of changing risk aversion. We then study the effect of changing income under different assumptions regarding attitudes toward risk and tastes. Finally, we show that in general there is no equivalence between Pratt's portfolio theorem (Pratt, 1964) and Arrow's portfolio theorem (Arrow, 1965).

Decomposition of the Effect of Changing Income. We begin by decomposing the effect of changing income on the sure good. Propositions 3.1 states that the effect of changing income is determined by the sum of two terms, i.e., $\mathcal{MIE}_{x^*} + \mathcal{HE}_{x^*}$. The term $\mathcal{MIE}_{x^*}$ corresponds to the *modified income effect* on the sure good, i.e., the income effect modified by uncertainty. It has the same characteristics as the deterministic income effect, however it is modified to take account of the fact that the rate of return is random. The term $\mathcal{HE}_{x^*}$ is the *hybrid risk-aversion effect* that contains the effect of a change in risk aversion modified by the change in income. In other words, $\mathcal{MIE}_{x^*}$ captures the effect of changing income on the sure good through uncertainty whereas $\mathcal{HE}_{x^*}$ captures the effect of changing risk aversion due to changes in income. As noted, the symbol $\overset{S}{=}$ means *of the same sign as.*

Proposition 3.1. *From* (9),

$$\frac{\partial x^*}{\partial I} \overset{S}{=} \mathcal{MIE}_{x^*} + \mathcal{HE}_{x^*} \tag{11}$$

where

$$\begin{aligned} \mathcal{MIE}_{x^*} \equiv \; & -\pi\varphi'\left(u_1(x^*) + u_2(\overline{R}(I - x^*))\right) u_2''(\overline{R}(I - x^*))\overline{R}^2 \\ & -(1-\pi)\varphi'\left(u_1(x^*) + u_2(\underline{R}(I - x^*))\right) u_2''(\underline{R}(I - x^*))\underline{R}^2 \end{aligned} \tag{12}$$

[10] Specifically, a change in income changes the amount allocated to the risky good (i.e., savings), which induces a change in the distribution of the risky good. The effect of changing income on the risky good simply refers to the effect of changing income on savings.

and

$$\begin{aligned}\mathcal{HE}_{x^*} \equiv\ & -\pi\varphi'\left(u_1(x^*)+u_2(\overline{R}(I-x^*))\right)\left[u_1'(x^*)-u_2'(\overline{R}(I-x^*))\overline{R}\right] \\ & \cdot\left(\frac{-\varphi''\left(u_1(x^*)+u_2(\overline{R}(I-x^*))\right)}{\varphi'\left(u_1(x^*)+u_2(\overline{R}(I-x^*))\right)}u_2'(\overline{R}(I-x^*))\overline{R}\right. \\ & \left. -\frac{-\varphi''\left(u_1(x^*)+u_2(\underline{R}(I-x^*))\right)}{\varphi'\left(u_1(x^*)+u_2(\underline{R}(I-x^*))\right)}u_2'(\underline{R}(I-x^*))\underline{R}\right). \qquad (13)\end{aligned}$$

Proof. See Appendix B. □

Proposition 3.2 complements Proposition 3.1 by decomposing the effect of increasing income on s^*. As in the case of the sure good, the effect of changing income on the risky good is determined by both a modified income effect and a hybrid risk-aversion effect. However, these effects are different, i.e., $\mathcal{MIE}_{x^*} \neq \mathcal{MIE}_{s^*}$ and $\mathcal{HE}_{x^*} \neq \mathcal{HE}_{s^*}$.

Proposition 3.2. *From* (9),

$$\frac{\partial s^*}{\partial I} \overset{S}{=} \mathcal{MIE}_{s^*} + \mathcal{HE}_{s^*} \qquad (14)$$

where

$$\begin{aligned}\mathcal{MIE}_{s^*} \equiv\ & -\pi\varphi'\left(u_1(x^*)+u_2(\overline{R}(I-x^*))\right)u_1''(x^*) \\ & -(1-\pi)\varphi'\left(u_1(x^*)+u_2(\underline{R}(I-x^*))\right)u_1''(x^*) \qquad (15)\end{aligned}$$

and

$$\begin{aligned}\mathcal{HE}_{s^*} \equiv\ & \pi\varphi'\left(u_1(x^*)+u_2(\overline{R}(I-x^*))\right)u_1'(x^*)\left[u_1'(x^*)-u_2'(\overline{R}(I-x^*))\overline{R}\right] \\ & \cdot\left(\frac{-\varphi''\left(u_1(x^*)+u_2(\overline{R}(I-x^*))\right)}{\varphi'\left(u_1(x^*)+u_2(\overline{R}(I-x^*))\right)}-\frac{-\varphi''\left(u_1(x^*)+u_2(\underline{R}(I-x^*))\right)}{\varphi'\left(u_1(x^*)+u_2(\underline{R}(I-x^*))\right)}\right). \qquad (16)\end{aligned}$$

Proof. See Appendix B. □

Two comments are in order for the case of strongly separable preferences.

First, from (12) and (15), $\mathcal{MIE}_{x^*} \geq 0$ and $\mathcal{MIE}_{s^*} \geq 0$. Whether the modified income effects are zero or strictly positive depends on the normality of the goods. Specifically, $\mathcal{MIE}_{x^*} > 0$ when the sure good is normal and $\mathcal{MIE}_{s^*} > 0$ when the risky good is normal. Hence, the signs of the modified income effects are independent of changes in risk aversion, and depend solely on ordinal preferences, i.e., tastes. Second, from (13) and (16), the signs of $\mathcal{HE}_{x^*}$ and $\mathcal{HE}_{s^*}$ depend on both attitudes toward risk and tastes. In other words, the hybrid risk-aversion effects combine the effect of risk aversion with the effect of changes in income on risk aversion. Although the functional forms of the hybrid risk-aversion effects differ between the two goods, both $\mathcal{HE}_{x^*}$ and $\mathcal{HE}_{s^*}$ are due to risk aversion, which is made explicit in our formulation.

Note that these hybrid risk-aversion effects are related to the pure risk aversion effect contained in MS. Indeed, in MS, the pure risk aversion effect is shown to depend on the income and substitution effects, i.e., the sign of $u_1'(x^*) - u_2'(\overline{R}(I - x^*))\overline{R}$. Formally, let $a > 0$ be a coefficient of risk aversion such that an increase in a implies an increase in risk aversion. That is, for any z, $\partial\left(-\varphi''(z)/\varphi'(z)\right)/\partial a > 0$. Hence, from MS,[11]

$$\frac{\partial x^*}{\partial a} \overset{S}{=} -\left[u_1'(x^*) - u_2'(\overline{R}(I - x^*))\overline{R}\right]. \tag{17}$$

However, with changes in income, the hybrid risk-aversion effect contains both changes in risk aversion and changes in income. In other words, the effect of changing income depends on the pure risk aversion effect through the hybrid risk-aversion effects. However, the hybrid risk-aversion effects are not completely analogous to the pure risk aversion effects because they are modified to take account of the changes in income, i.e., the changes in the level of utility, and thus alters the impact of the *pure* risk aversion effect.

[11] Since there is no increase in income with the pure risk aversion effect, it follows that $\frac{\partial x^*}{\partial a} = -\frac{\partial s^*}{\partial a}$.

To see this, rewrite the hybrid risk-aversion effects. For the sure good,

$$\mathcal{HE}_{x^*} \overset{S}{=} \frac{\partial x^*}{\partial a} \cdot \left(\frac{-\varphi''\left(u_1(x^*) + u_2(\overline{R}(I - x^*))\right)}{\varphi'\left(u_1(x^*) + u_2(\overline{R}(I - x^*))\right)} u_2'(\overline{R}(I - x^*))\overline{R} - \frac{-\varphi''\left(u_1(x^*) + u_2(\underline{R}(I - x^*))\right)}{\varphi'\left(u_1(x^*) + u_2(\underline{R}(I - x^*))\right)} u_2'(\underline{R}(I - x^*))\underline{R} \right) \tag{18}$$

where $\frac{\partial x^*}{\partial a}$ is defined by (17). For a change in income, the terms $u_2'(\overline{R}(I - x^*))\overline{R}$ and $u_2'(\underline{R}(I - x^*))\underline{R}$ representing changes in the marginal rates of substitution are weights for the change in risk aversion due to income. These terms combined determines the strength of the pure risk aversion effect for the hybrid risk-aversion effect. In other words, as income changes, not only does risk aversion change but tastes are also distorted so that the pure risk aversion effect is altered. The terms in parenthesis in (18) take account of the influence of both changes in risk aversion and changes in tastes on the pure risk aversion effect. Hence, the sign of the hybrid risk-aversion effect depends on the interaction between risk aversion, income and substitution effects. From (18), the income and substitution effects play a dual role in determining the sign of the effect of changing income. First, they order the marginal utilities explicit in (18) which are used as weights for the effect of income on risk aversion. Here, the weights depend on the different levels of utility consistent with different levels of income. Second, they determine the sign of the pure risk aversion effect in (17) embedded in (18).

For the risky good, the income and substitution effects influence the sign of $\mathcal{HE}_{s^*}$ (and thus $\partial s^*/\partial I$) only through the pure risk aversion effect. As in (18), the effect of changing risk aversion influences the effect of changing income in a multiplicative way. In this case, the hybrid risk-aversion effect is the product of the effect of risk aversion and the effect of income on risk aversion without additional weights. That is, (16) is equivalent to

$$\mathcal{HE}_{s^*} \overset{S}{=} -\frac{\partial x^*}{\partial a} \cdot \left(\frac{-\varphi''\left(u_1(x^*) + u_2(\overline{R}(I - x^*))\right)}{\varphi'\left(u_1(x^*) + u_2(\overline{R}(I - x^*))\right)} - \frac{-\varphi''\left(u_1(x^*) + u_2(\underline{R}(I - x^*))\right)}{\varphi'\left(u_1(x^*) + u_2(\underline{R}(I - x^*))\right)} \right) \tag{19}$$

where $\frac{\partial x^*}{\partial a}$ is defined by (17). Note finally that the hybrid risk-aversion effects in (18) and (19) can also be rewritten in terms of the effect of risk aversion on the risky good since $\frac{\partial x^*}{\partial a} = -\frac{\partial s^*}{\partial a}$.[12]

Direction of a Change in Income. Using Propositions 3.1 and 3.2 for the case of strongly separable preferences, we now proceed to determine the direction of the effect of a change in income on optimal behavior under uncertainty. We first consider the case of constant absolute risk aversion and then analyze the case of decreasing absolute risk aversion. For each case, we discuss four situations. We begin with the case in which both goods are income-neutral as in the Arrow-Pratt portfolio problem. We then continue with quasilinear preferences and finish with both goods being normal.

Suppose that risk preferences exhibit constant absolute risk aversion. First, Proposition 3.3 shows that the hybrid risk-aversion effect is different for the sure good and the risky good.[13] Proposition 3.3 states that under constant absolute risk aversion, the hybrid risk-aversion effect is present for the sure good but absent for the risky good. Specifically, the hybrid risk-aversion effect for the sure good is strictly positive due to the presence of risk aversion (i.e., $\varphi'' < 0$). In other words, the effect of changing income for the sure good in this case dominates the effect of risk aversion, and thus the amount of the sure good increases. However, the hybrid risk-aversion effect for the risky good is zero because, under constant absolute risk-aversion, an increase in income has no effect on the individual's risk aversion.

Proposition 3.3. *Suppose that risk preferences exhibit constant absolute risk-aversion. i.e.,*

$$-\frac{\varphi''\left(u_1(x^*) + u_2(\overline{R}(I - x^*))\right)}{\varphi'\left(u_1(x^*) + u_2(\overline{R}(I - x^*))\right)} = -\frac{\varphi''\left(u_1(x^*) + u_2(\underline{R}(I - x^*))\right)}{\varphi'\left(u_1(x^*) + u_2(\underline{R}(I - x^*))\right)}. \qquad (20)$$

Then,

[12]Note also that the term representing changes in tastes as income changes seems to disappear. However, it can be seen from (18) to be the term $u_2'(x^*)$, which, due to the representation of preferences by additive utility, factors out.

[13]Note that for the case of pure risk aversion from MS, the effect of a change in risk aversion is the same (except for the signs) for both the sure good and the risky good.

1. *From* (13),

$$\mathcal{HE}_{x^*} \equiv -\pi\varphi''\left(u_1(x^*) + u_2(\overline{R}(I - x^*))\right)\left[u_1'(x^*) - u_2'(\overline{R}(I - x^*))\overline{R}\right]^2 \\ -(1-\pi)\varphi''\left(u_1(x^*) + u_2(\underline{R}(I - x^*))\right)\left[u_1'(x^*) - u_2'(\underline{R}(I - x^*))\underline{R}\right]^2 > 0. \tag{21}$$

2. *From* (16), $\mathcal{HE}_{s^*} = 0$.

Proof. See Appendix B. □

Under constant absolute risk aversion, we now consider differences regarding the normality of the goods. Proposition 3.4 states that with constant absolute risk aversion, when the risky good is income-neutral (i.e., $u_1'' = 0$), a change in income is entirely allocated to the sure good, regardless of whether the sure good is normal (i.e., $u_2'' < 0$) or income-neutral (i.e., $u_2'' = 0$). In particular, under constant absolute risk aversion, a change in income does not affect risk aversion. Hence, since the risky good is income-neutral, the individual has no incentive to increase the amount of the risky good.

Proposition 3.4. *Suppose that risk preferences exhibit constant absolute risk aversion and that the risky good is income-neutral. Then,*

$$\frac{\partial x^*}{\partial I} = 1, \tag{22}$$

$$\frac{\partial s^*}{\partial I} = 0. \tag{23}$$

Proof. Suppose that the risky good is income neutral, i.e., $u_1'' = 0$. Then, from (14), (15), (16), and (20), $\frac{\partial s^*}{\partial I} = 0$. Since $s^* \equiv I - x^*$, it follows that $\frac{\partial x^*}{\partial I} = 1$. □

Consider next quasilinear preferences with the sure good income-neutral and the risky good normal. In that case, an increase in income induces an increase in both goods. Since the risky good is normal, the amount of the risky good increases because the modified income effect is strictly positive. As in Proposition 3.4, the sure good increases through the hybrid risk-aversion effect, although not as much since part of the new income is used for the risky good. While consumption of both goods increase with an increase in income, the reason for the increases are different. Indeed, the sure good increases because of risk aversion and the change in income through the hybrid risk-aversion effect whereas the risky good increases only because of the change in income.

Proposition 3.5. *Suppose that risk preferences exhibit constant absolute risk aversion. If the sure good is income-neutral and the risky good is normal, then*

1. $\frac{\partial x^*}{\partial I} \in (0,1)$ *due only to the hybrid risk-aversion effect, i.e.,* $\mathcal{MIE}_{x^*} = 0, \mathcal{HE}_{x^*} > 0.$

2. $\frac{\partial s^*}{\partial I} \in (0,1)$ *due only to the modified income effect, i.e.,* $\mathcal{MIE}_{s^*} > 0, \mathcal{HE}_{s^*} = 0.$

Finally, Proposition 3.6 states that when both goods are normal, an increase in income increases both the sure good and the risky good. Going from an income-neutral sure good to a normal sure good amplifies the positive effect of increasing income on the sure good.

Proposition 3.6. *Suppose that risk preferences exhibit constant absolute risk aversion. If both goods are normal, then*

1. $\frac{\partial x^*}{\partial I} \in (0,1)$ *due to the modified income and hybrid risk-aversion effects, i.e.,* $\mathcal{MIE}_{x^*} > 0, \mathcal{HE}_{x^*} > 0.$

2. $\frac{\partial s^*}{\partial I} \in (0,1)$ *due only to the modified income effect, i.e.,* $\mathcal{MIE}_{s^*} > 0, \mathcal{HE}_{s^*} = 0.$

Next, suppose that risk preferences exhibit decreasing absolute risk aversion. Proposition 3.7 states that under decreasing absolute risk aversion, the hybrid risk-aversion effect for the risky good is positive or negative depending on the income and substitution effects. Moreover, the hybrid risk-aversion effect for the sure good is strictly positive if the income effect is stronger than the substitution effect. However, if the substitution effect is stronger than the income effect, the hybrid risk-aversion effect for the sure good cannot be signed.

Proposition 3.7. *Suppose that risk preferences exhibit decreasing risk aversion, i.e.,*

$$-\frac{\varphi''\left(u_1(x^*)+u_2(\overline{R}(I-x^*))\right)}{\varphi'\left(u_1(x^*)+u_2(\overline{R}(I-x^*))\right)} < -\frac{\varphi''\left(u_1(x^*)+u_2(\underline{R}(I-x^*))\right)}{\varphi'\left(u_1(x^*)+u_2(\underline{R}(I-x^*))\right)}. \tag{24}$$

Then,

1. *From* (13), $\mathcal{HE}_{x^*} > 0$ ***if****, for the sure good, the income effect is stronger than the substitution effect.*

2. *From* (16), $\mathcal{HE}_{s^*} > 0$ ***if and only if****, for the sure good, the substitution effect is stronger than the income effect.*

Proof. See Appendix B. □

We now consider decreasing absolute risk aversion with different assumptions regarding the normality of the goods. Proposition 3.8 states that when both goods are income-neutral, then an increase in income induces the individual to increase the risky good. To understand this, note that from MS, due to the substitution effect when the price of the risky good is random, a reduction in risk aversion induces the individual to decrease the sure good and increase the risky good. Under decreasing absolute risk aversion, an increase in income also induces a reduction in risk aversion, which then implies an increase in the risky good through the hybrid risk-aversion effect. The effect for the sure good is however ambiguous. On the one hand, the reduction in risk aversion (induced by an increase in income) leads to a decrease in the sure good. On the other hand, the increased income must be allocated between the two goods, which yields more consumption of the sure good. The overall effect on the sure good depends on the relative strength of these two forces.

Proposition 3.8. *Suppose that risk preferences exhibit decreasing absolute risk aversion and that both goods are income-neutral. Then,* $\mathcal{MIE}_{x^*} = \mathcal{MIE}_{s^*} = 0$*, and the substitution effect for the sure good is stronger than the income effect, which is zero. Hence,*

1. $\frac{\partial x^*}{\partial I} \stackrel{S}{=} \mathcal{HE}_{x^*}$ *cannot be signed.*

2. $\frac{\partial s^*}{\partial I} \stackrel{S}{=} \mathcal{HE}_{s^*} > 0$ *due only to the hybrid risk-aversion effect.*

Suppose that preferences are quasilinear with the sure good income-neutral (i.e., $u_2'' = 0$) and the risky good normal (i.e., $u_1'' < 0$). As in Proposition 3.8, the ambiguous effect of increasing income on the sure good is due to the hybrid risk-aversion effect. For the risky good, the hybrid risk-aversion effect is positive due to the substitution effect since there is no income effect (i.e., $u_2'' = 0$). Compared to an income-neutral risky good, the positive effect of increasing income on the risky good is accentuated now that the risky good is normal. Indeed, for the risky good, both the modified income effect and the hybrid risk-aversion effect push in the direction of more consumption of the risky good.

Proposition 3.9. *Suppose that risk preferences exhibit decreasing absolute risk and that the sure good is income-neutral and the risky good is normal. Then,* $\mathcal{MIE}_{x^*} = 0$*, and the substitution effect for the sure good is stronger than the income effect, which is zero. Hence,*

1. $\frac{\partial x^*}{\partial I} \stackrel{S}{=} \mathcal{HE}_{x^*}$ *cannot be signed.*

2. $\frac{\partial s^*}{\partial I} > 0$ *due to the modified income and hybrid risk-aversion effects, i.e.,* $\mathcal{MIE}_{s^*} > 0, \mathcal{HE}_{s^*} > 0$.

Suppose next that preferences are quasilinear but now the sure good is normal (i.e., $u_2'' < 0$) and the risky good is income-neutral (i.e., $u_1'' = 0$). Here, the normality of the sure good makes the sign of the hybrid risk-aversion effect for the risky good ambiguous through the effect of increasing risk aversion. When the substitution effect for the sure good is stronger than the income effect, the new income is either allocated between the two goods or only to the risky good along with substitution from the sure good to the risky good. However, when the income effect is stronger than the substitution effect, the hybrid risk-aversion effect for the risky good is negative, which reduces consumption for the risky good, and thus increases consumption of the sure good.

Proposition 3.10. *Suppose that risk preferences exhibit decreasing absolute risk aversion and that the sure good is normal and the risky good is income-neutral.*

1. *If the substitution effect for the sure good is stronger than the income effect, then* $\frac{\partial x^*}{\partial I}$ *cannot be signed and* $\frac{\partial s^*}{\partial I} \overset{S}{=} \mathcal{HE}_{s^*} > 0$*.*
2. *If the income effect for the sure good is stronger than the substitution effect, then* $\frac{\partial x^*}{\partial I} > 1$ *with* $\mathcal{MIE}_{x^*} > 0, \mathcal{HE}_{x^*} > 0$*, and* $\frac{\partial s^*}{\partial I} \overset{S}{=} \mathcal{HE}_{s^*} < 0$*.*

From Propositions 3.9 and 3.10, the normality of the good in quasilinear preferences has a profound effect on the comparative analysis. In the case of an income-neutral sure good, an increase in income increases consumption of the risky good. due to a pure income effect. In the case of an income-neutral risky good, the normality of the sure good makes it possible for the hybrid risk-aversion effect corresponding to the risky good to be negative. In this case, an increase in income induces an increase in the sure good (due to the presence of more income) as well as a reallocation from the risky good to the sure good due to the pure risk aversion effect contained in the hybrid risk-aversion effect for the risky good.

Suppose finally that both goods are normal, (i.e., $u_1'', u_2'' < 0$). In that case, the effect of increasing income on the risky good is ambiguous. On the one hand, $u_1'' < 0$ makes the modified income effect positive, which increases savings. On the other hand, the sign of the hybrid risk-aversion effect depends on the relative size of the income and substitution effects, and is negative if the income effect for the sure good is stronger, which reduces saving.

Proposition 3.11. *Suppose that risk preferences exhibit decreasing absolute risk aversion and that both goods are normal. Then,*

1. *If the substitution effect for the sure good is stronger than the income effect, then*
   - *(a)* $\frac{\partial x^*}{\partial I}$ *cannot be signed.*
   - *(b)* $\frac{\partial s^*}{\partial I} > 0$ *due to the modified income and hybrid risk-aversion effect, i.e.,* $\mathcal{MIE}_{s^*} > 0, \mathcal{HE}_{s^*} > 0$*.*

2. *If the income effect for the sure good is stronger than the substitution effect, then*

   (a) $\frac{\partial x^*}{\partial I} > 0$ *due to the modified income and hybrid risk-aversion effect, i.e.,* $\mathcal{MIE}_{x^*} > 0, \mathcal{HE}_{x^*} > 0$.

   (b) $\mathcal{HE}_{s^*} < 0$ *and* $\mathcal{MIE}_{s^*} > 0$ *so that* $\frac{\partial s^*}{\partial I} < 0$ *if and only if* $\mathcal{MIE}_{s^*} + \mathcal{HE}_{s^*} < 0$.

On Equivalence. Finally, we show that in general there is no equivalence between Pratt's portfolio theorem (Pratt, 1964) and Arrow's portfolio theorem (Arrow, 1965). Indeed, the equivalence depends on the assumption regarding tastes, i.e., whether goods are normal or income-neutral. It also depends on the fact that it is the price of the risky good that is random. Indeed, if income or the price of the sure good were random, the result would change.

We begin with the Arrow-Pratt portfolio problem in which both goods are income-neutral, i.e., $V(x, \tilde{y}) = \varphi(x + \tilde{y})$. For Arrow (1965), increasing income when risk preferences exhibit decreasing absolute risk aversion makes the individual less risk averse and thus the amount allocated to the risky good is increased while, for Pratt (1964), increasing risk aversion decreases the amount of the risky good, so that a decrease in risk aversion increases the risky good.

In the case of income-neutral goods, the modified income effect is absent and thus the effect of changing income is entirely linked to the effect of changing risk aversion through the hybrid risk-aversion effect, i.e., using (19) and $\frac{\partial x^*}{\partial a} = -\frac{\partial s^*}{\partial a}$, decreasing absolute risk aversion implies that

$$\frac{\partial s^*}{\partial I} \overset{S}{=} \mathcal{HE}_{s^*} \tag{25}$$

$$\frac{\partial s^*}{\partial a} \overset{S}{=} -\mathcal{HE}_{s^*} \tag{26}$$

Moreover, since the sure good is income-neutral, the substitution effect is dominant so that $\mathcal{HE}_{s^*} > 0$. Proposition 3.12 follows immediately.

Proposition 3.12. *(Arrow-Pratt) Suppose that both goods are income-neutral. Then, the following two statements are* ***equivalent.***

1. *An decrease in risk aversion increases the amount allocated to the risky good.*

2. *When risk preferences exhibit decreasing absolute risk aversion, an increase in income increases the amount allocated to the risky good.*

Suppose now that both goods are normal. The normality of the risky good implies that an increase in income induces more consumption of the risky good through the modified income effect (i.e., $\mathcal{MIE}_{s^*} > 0$). However, the normality of the sure good implies that an increase in income induces less consumption of the risky good through the hybrid risk-aversion effect (i.e., $\mathcal{HE}_{s^*} < 0$) if the income effect is stronger than the substitution effect. Note that the source of the nonequivalence is solely due to the normality of the risky good. Indeed, because the normality of the sure good affects the sign only through the hybrid risk-aversion effect which is directionally equivalent to the sign of the effect of risk aversion, the normality of the sure good reinforces the equivalence result in the case of two income-neutral goods while the normality of the risky good pulls in the opposite direction.

Proposition 3.13. *Suppose that both goods are normal. Then, the following two statements are* ***correct*** *but* ***not equivalent.***

1. *A decrease in risk aversion decreases the amount allocated to the risky good if and only if the income effect for the sure good is stronger than the substitution effect.*

2. *When risk preferences exhibit decreasing absolute risk aversion, an increase in income increases the amount allocated to the risky good when the income effect for the sure good is stronger than the substitution effect if and only if the modified income effect is stronger than the hybrid risk-aversion effect.*

A KM Example

To show that attitudes toward risk and tastes are not separated, we recall the example stated in KM. Let $V^1(x, y)$ and $V^2(x, y)$ be two distinct utility functions yielding indifference curves of the type IC_1 and IC_2, respectively, as depicted in Figure 1. Let (x_A, y_A) and (x_B, y_B) be two distinct consumption bundles such that $V^1(x_A, y_A) > V^1(x_B, y_B)$ and $V^2(x_A, y_A) < V^2(x_B, y_B)$. Consider choosing between the sure outcome yielding (x_A, y_A) and a gamble yielding (x_A, y_A) with probability $\pi \in (0, 1]$ and (x_B, y_B) with probability $1 - \pi$.

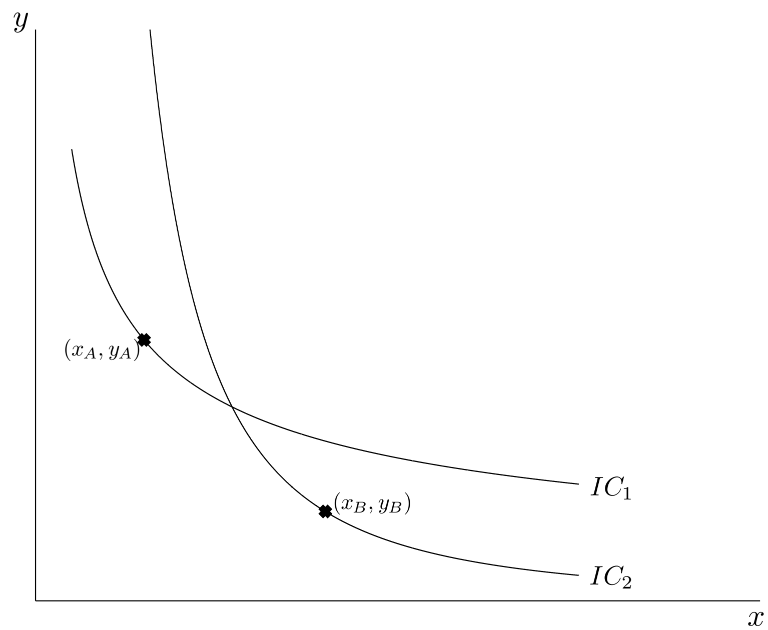

Figure 1: KM Example

Consistent with Figure 1, an individual with preferences $V^1(x, y)$ prefers the sure outcome, while an individual with preferences $V^2(x, y)$ prefers the gamble.[14] The individual with preferences $V^2(x, y)$ acts in a seemingly more

[14]In other words, $V^1(x_A, y_A) > \pi V^1(x_A, y_A) + (1 - \pi)V^1(x_B, y_B)$ and $V^2(x_A, y_A) < \pi V^2(x_A, y_A) + (1 - \pi)V^2(x_B, y_B)$.

risk-averse way than the individual with preferences $V^1(x, y)$, but is not more risk-averse. Rather, it is the composition of goods in the gamble that is preferred.

B Proofs

Proof of Propositions 3.1 and 3.2. From (9),

$$\frac{\partial x^*}{\partial I} = \frac{\Omega}{-\Delta} \tag{27}$$

and, since $s^* \equiv I - x^*$,

$$\frac{\partial s^*}{\partial I} = 1 - \frac{\Omega}{-\Delta}, \tag{28}$$

where

$$\begin{aligned}
\Omega \equiv &\left(\frac{\varphi''\left(u_1(x^*) + u_2(\overline{R}(I - x^*))\right)}{\varphi'\left(u_1(x^*) + u_2(\overline{R}(I - x^*))\right)} u_2'(\overline{R}(I - x^*))\overline{R} \right. \\
&\left. - \frac{\varphi''\left(u_1(x^*) + u_2(\underline{R}(I - x^*))\right)}{\varphi'\left(u_1(x^*) + u_2(\underline{R}(I - x^*))\right)} u_2'(\underline{R}(I - x^*))\underline{R} \right) \\
&\cdot \pi\varphi'\left(u_1(x^*) + u_2(\overline{R}(I - x^*))\right)\left[u_1'(x^*) - u_2'(\overline{R}(I - x^*))\overline{R}\right] \\
&- \pi\varphi'\left(u_1(x^*) + u_2(\overline{R}(I - x^*))\right) u_2''(\overline{R}(I - x^*))\overline{R}^2 \\
&- (1 - \pi)\varphi'\left(u_1(x^*) + u_2(\underline{R}(I - x^*))\right) u_2''(\underline{R}(I - x^*))\underline{R}^2
\end{aligned} \tag{29}$$

and

$$\Delta \equiv \left(\frac{\varphi''\left(u_1(x^*) + u_2(\overline{R}(I - x^*))\right)}{\varphi'\left(u_1(x^*) + u_2(\overline{R}(I - x^*))\right)} \left[u_1'(x^*) - u_2'(\overline{R}(I - x^*))\overline{R}\right] \right.$$
$$\left. - \frac{\varphi''\left(u_1(x^*) + u_2(\underline{R}(I - x^*))\right)}{\varphi'\left(u_1(x^*) + u_2(\underline{R}(I - x^*))\right)} \left[u_1'(x^*) - u_2'(\underline{R}(I - x^*))\underline{R}\right] \right)$$
$$\cdot \pi\varphi'\left(u_1(x^*) + u_2(\overline{R}(I - x^*))\right) \left[u_1'(x^*) - u_2'(\overline{R}(I - x^*))\overline{R}\right]$$
$$+ \pi\varphi'\left(u_1(x^*) + u_2(\overline{R}(I - x^*))\right) \left[u_1''(x^*) + u_2''(\overline{R}(I - x^*))\overline{R}^2\right]$$
$$+ (1 - \pi)\varphi'\left(u_1(x^*) + u_2(\underline{R}(I - x^*))\right) \left[u_1''(x^*) + u_2''(\underline{R}(I - x^*))\underline{R}^2\right]. \tag{30}$$

Note that the second-order condition implies that $\Delta < 0$. Rearranging (27) and (28) yields the decomposition stated in Propositions 3.1 and 3.2.

Proof of Proposition 3.3. We show that $\mathcal{HE}_{x^*}$ can be rewritten as (21) when condition (20) holds. To that end, let

$$\Gamma = -\pi\varphi''\left(u_1(x^*) + u_2(\overline{R}(I - x^*))\right) \left[u_1'(x^*) - u_2'(\overline{R}(I - x^*))\overline{R}\right]^2$$
$$- (1 - \pi)\varphi''\left(u_1(x^*) + u_2(\underline{R}(I - x^*))\right) \left[u_1'(x^*) - u_2'(\underline{R}(I - x^*))\underline{R}\right]^2 \tag{31}$$

which is positive since $\varphi'' < 0$. Expression (31) can be rewritten as

$$\Gamma = \frac{-\varphi''\left(u_1(x^*) + u_2(\overline{R}(I - x^*))\right)}{\varphi'\left(u_1(x^*) + u_2(\overline{R}(I - x^*))\right)} \pi\varphi'\left(u_1(x^*) + u_2(\overline{R}(I - x^*))\right) \left[u_1'(x^*) - u_2'(\overline{R}(I - x^*))\overline{R}\right]^2$$
$$+ \frac{-\varphi''\left(u_1(x^*) + u_2(\underline{R}(I - x^*))\right)}{\varphi'\left(u_1(x^*) + u_2(\underline{R}(I - x^*))\right)} (1 - \pi)\varphi'\left(u_1(x^*) + u_2(\underline{R}(I - x^*))\right)$$
$$\cdot \left[u_1'(x^*) - u_2'(\underline{R}(I - x^*))\underline{R}\right]^2. \tag{32}$$

Multiplying both sides of (9) by $[u_1'(x^*) - u_2'(\underline{R}(I - x^*))\underline{R}]$ yields

$$(1 - \pi)\varphi'\left(u_1(x^*) + u_2(\underline{R}(I - x^*))\right) \cdot \left[u_1'(x^*) - u_2'(\underline{R}(I - x^*))\underline{R}\right]^2$$
$$= -\pi\varphi'\left(u_1(x^*) + u_2(\overline{R}(I - x^*))\right) \cdot \left[u_1'(x^*) - u_2'(\overline{R}(I - x^*))\overline{R}\right] \left[u_1'(x^*) - u_2'(\underline{R}(I - x^*))\underline{R}\right]. \tag{33}$$

Plugging (33) into (32) and using (20) yields

$$\Gamma = \frac{-\varphi''\left(u_1(x^*) + u_2(\overline{R}(I-x^*))\right)}{\varphi'\left(u_1(x^*) + u_2(\overline{R}(I-x^*))\right)}\pi\varphi'\left(u_1(x^*) + u_2(\overline{R}(I-x^*))\right)\left[u_1'(x^*) - u_2'(\overline{R}(I-x^*))\overline{R}\right]^2$$

$$- \frac{-\varphi''\left(u_1(x^*) + u_2(\underline{R}(I-x^*))\right)}{\varphi'\left(u_1(x^*) + u_2(\underline{R}(I-x^*))\right)}\pi\varphi'\left(u_1(x^*) + u_2(\overline{R}(I-x^*))\right)$$

$$\cdot \left[u_1'(x^*) - u_2'(\overline{R}(I-x^*))\overline{R}\right]\left[u_1'(x^*) - u_2'(\underline{R}(I-x^*))\underline{R}\right], \tag{34}$$

$$= \frac{-\varphi''\left(u_1(x^*) + u_2(\overline{R}(I-x^*))\right)}{\varphi'\left(u_1(x^*) + u_2(\overline{R}(I-x^*))\right)}\left(u_2'(\underline{R}(I-x^*))\underline{R} - u_2'(\overline{R}(I-x^*))\overline{R}\right)$$

$$\pi\varphi'\left(u_1(x^*) + u_2(\overline{R}(I-x^*))\right) \cdot \left[u_1'(x^*) - u_2'(\overline{R}(I-x^*))\overline{R}\right], \tag{35}$$

which is identical to (13) when condition (20) holds. Hence, $\mathcal{HE}_{x^*} = \Gamma$ and $\mathcal{HE}_{x^*}$ is equal to (31) as stated in (21).

Proof of Proposition 3.7. Suppose that the income effect is stronger than the substitution effect. Then, from Remark 2.3,

$$u_1'(x^*) - u_2'(\overline{R}(I-x^*))\overline{R} > 0 > u_1'(x^*) - u_2'(\underline{R}(I-x^*))\underline{R} \tag{36}$$

so that

$$u_2'(\underline{R}(I-x^*))\underline{R} > u_2'(\overline{R}(I-x^*))\overline{R}. \tag{37}$$

Using (24), (36), and (37) implies that (13) is positive, i.e., $\mathcal{HE}_{x^*} > 0$ if the income effect is stronger than the substitution effect. If the substitution effect is stronger than the income effect, the sign of (13) is ambiguous. From (16), (24) implies that $\mathcal{HE}_{s^*} > 0$ if and only if the substitution effect is stronger than the income effect.

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