Unobservable Investments and Search Frictions

Yujing Xu*

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Abstract

This paper studies investment incentives in a dynamic random search environment with exogenous entry, where a seller entrant can make unobservable and selfish investments to reduce his production cost before searching for buyers. We showed that in the unique steady state equilibrium, sellers play mixed strategy over investments and buyers play mixed strategy over price offers. When buyers have all the bargaining power, although sellers make positive investments, the players' equilibrium payoffs and the social welfare are a) constant given any search friction and b) equal to the equilibrium values when investments were observable (hence no investment in equilibrium). As search frictions diminish, the investment strategy converges in distribution to a point mass at the socially optimal level. In contrast, the equilibrium outcome fails to converge to the first best. The delay in trade completely dissipates the benefit from investments. I then demonstrate that the equilibrium properties preserve in more general settings with two-sided investments or two-sided offers.

*School of Economics and Finance, University of Hong Kong. E-mail: yujingxu@hku.hk.
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1 Introduction

Sellers oftentimes need to decide how much to invest to reduce their production costs before searching for buyers. In the simple holdup model where investments are relation-specific and observable, sellers have little incentive to invest and hence the welfare is low. If the environment favors sellers more, in the sense that investments are no longer relation-specific and that buyers cannot perfectly observe the investments, will the investments become more efficient and improve the social welfare?

This question is particularly relevant when we focus on investment incentives in large markets. For example, a contractor can invest to reduce its cost before approaching to a procurer. Such investments are not relation-specific, since the benefit of trade increases in investments with any procurers in the market. Moreover, a procurer is unlikely to be perfectly informed about the cost of a contractor whom he/she randomly meet. Another example is the labor market. New graduates, even with the same academic degree, have different costs of labor, partly due to their different levels of costly self-training.

This paper will therefore focus on investment incentives in a large market with search frictions and investigate the total and the division of the resulting social welfare. The main result is that sellers do invest positive amount, but such positive investments do not necessarily lead to welfare improvement under some realistic settings.

To be more precise, I propose a discrete time infinite horizon random search model with exogenous entry and pre-entry investments. At the beginning of each period, there are one unit mass of ex ante identical sellers and buyers entering the market. A seller entrant is endowed with a technology to produce one unit of output at cost x_0 and he can invest to reduce the production cost before entry. Denote the production cost resulting from the socially efficient investments by x^*. Meanwhile, a buyer demands one unit of the output and gets utility $y_0 > x_0$ from consuming it. After the investment stage, all agents (entrants and incumbents) are randomly paired and the buyer in each pair makes a take-it-or-leave-it offer, without observing the investment. If the offer is accepted, then production takes place and both agents leave the market permanently. Otherwise, the pair is dissolved and both agents search in the next period. We will call agents in the model as buyers and sellers. But as shown earlier, agents could also be procurers and contractors, or firms and workers, etc.

We have three main results with this base line model. First of all, focusing on steady-state equilibrium, I showed that the equilibrium exists and is unique. In equilibrium, the investment distribution of ex-ante identical sellers is non-degenerate. So is the price offer distribution of ex-ante identical buyers. In other words, sellers choose to become ex-post heterogeneous, given that investments are unobservable and that price distribution is non-degenerate. Buyers offer diverse prices when they face the trade-off between a lower price and a higher probability of trade.

The second main result is that although investments are always positive, agents' ex-ante payoffs as well as the social welfare are constant given any search friction and equal to the equilibrium values when investments are observable (hence no investment). There will be ex-post values created by positive investments. However, the ex-ante values also depend on the speed of trade. In equilibrium, the ex-post values are completely dissipated by delay in trade for any search friction, which results from the possibility of mismatch between a seller's reserve price and a buyer's offer. This result demonstrates that more investments not necessarily lead to a higher social welfare.

Thirdly, as search frictions disappear, the investment strategy becomes efficient. But the no welfare gain result still holds. Some comparative statics exercises could help us understand this result. As search frictions decreases, buyers price more aggressively. Underinvested sellers are therefore less able to trade quickly and congest the market even if there are fewer of them among entrants. In the limit, the stationary cost distribution, i.e. the cost distribution of incumbents, converges in distribution to a point mass at the initial cost x_0 and the price offer distribution converges in distribution to a point mass at the reserve price of a type x^* seller. This delay in trade caused by severe mismatch between offers and reserve prices completely dissipates all the surplus from the efficient investments in the limit.

The baseline model is extended along two dimensions to incorporate the possibilities of two-sided investments and two-sided offers. In the first extension, buyer entrants could also invest before entry to raise their valuations of the good. In the context of the contractor-procurer relationship, a procurer could have an opportunity to costly boost the value of his/her project before searching for a contractor. The results of the baseline model regarding the sellers' investment strategy continue to hold. Moreover, we have a set of new results on the buyer's investment strategy. Perhaps most significantly, although buyers have all the bargaining power, surprisingly they still underinvest and use a mixed strategy over an interval of investments given any search friction.

I then extend the model to allow sellers to make take-it-or-leave-it offers occasionally. The result of welfare equivalence between observable and unobservable investments still holds. Among sellers who invest, the least efficient seller invests positive amount, since now he/she makes offer with positive probability and he/she is a residual claimant when he/she makes the offer. In the limit, this minimum investment level as well as the social welfare converge to the first best.

1.1 Related Literature

The paper is related to random search models with heterogeneous agents. Albrecht and Vroman (1992) demonstrates in a random search model that when seller entrants are exogenously heterogeneous, a single price can never be an equilibrium. My paper complements their paper in the sense that I showed that sellers' heterogeneity can emerge in equilibrium because of the diverse prices.

The searching stage of the current model is similar to settings of voluminous works on search and bargaining games with asymmetric information (for instance, Rubinstein and Wolinsky (1990), Satterthwaite and Shneyerov (2007), Shneyerov and Wong (2010a), Lauermann (2012) (2013), etc). One of the central topics in this literature is to understand how frictions influence efficiency of equilibrium. For instance, Lauermann (2013) shows that under reasonable assumptions, equilibrium outcomes converge to those of competitive equilibrium as frictions disappear, i.e., there is no delay in trade in the limit. In contrast, it is not the case in the currents setting: the delay in trade is the most severe in the limit and the equilibrium social welfare is constant over any search friction. This drastically different prediction results from the fact that the entrants type distribution arises endogenously in my model but is exogenously given in his paper. In other words, in my model as search frictions change, the type distribution also varies and that is why the trade does not converge to the efficient.

This paper also relates to the literature on the hold-up problem, dating back to the seminal works Grossman and Hart (1986) and Hart and Moore (1990). Our setting differs from the existing works along some dimensions. We assume that investments are purely selfish, whereas other works investigate investment incentives with cooperative investments (Che and Hausch (1999), Mailath Postlewaite and Samuelson (2012), Hermalin (2013), etc). Moreover, investments are assumed to be unobservable. In other environments, the non-investing party might have some information about investments or the outcome of investments (Rogerson (1992), Lau (2008), Hermalin and Katz (2009), etc). Finally, instead of free entry (Acemoglu (1996), Davis (2001), Acemoglu and Shimer (1999), etc), we assume that entry is exogenous.

Investment incentive with unobservable investments has also been studied in Gul (2001), with a Coasian setting where a buyer's valuation is determined by his/her unobservable investments prior to the game. Similar to our model, equilibrium investment strategy is a mixed strategy and it becomes efficient as the time between two rounds shrinks to zero. The difference is that in the setting of Gul (2001) the social welfare converges to the first best, resulting from the fact that, as also emphasized in Lau (2008), there is no "bargaining delay" in the limit. In our model, although the investment strategy is efficient, almost all incumbent sellers are the underinvested types. The per-period trading probability of any price except for the highest price is therefore close to zero.

The rest of the paper is organized as follows. The model is introduced in section 2 where we also solve the first best and the observable investments benchmark cases. Section 3 derives equilibrium conditions and shows the existence and the uniqueness of the steady state equilibrium. The equilibrium is characterized in section 4. Section 5 examines the two-sided investments extension and section 6 considers another extension with two-sided offers. Discussions on robustness and other extensions are in section 7. Finally, section 8 concludes the paper.

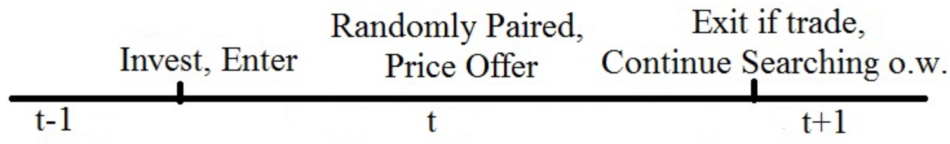

Figure 1: Timeline

2 The Model and Benchmark Specifications

2.1 The Model

We consider a discrete time infinite horizon random search model with pre-entry investments. The timeline of this game is illustrated in Figure 1.

Player: The players are seller and buyers. In each period, there are one unit mass of new sellers and one unit mass of new buyers entering the market. A buyer demands one unit of the output with valuation $y_0 > 0$; a seller entrant is endowed with a technology that produces one unit of output at cost $x_0 > 0$. We consider the "gap" case in this paper, that is, the (minimum) surplus from trade $y_0 - x_0$ is assumed to be strictly positive.

Strategy: Before entering the market, a seller can invest $c(x)$ to reduce the production cost to $x \geq 0$. We assume $c(x_0) = c'(x_0) = 0$ and $c'(0) < -1$, and for any $x < x_0$, $c(x)$ is of class C^1, strictly decreasing and strictly convex. All entrants on both sides then join the incumbents who did not exit in the last period. The market sizes on both sides are assumed to be the same. In each period, one buyer is randomly matched with one seller and vice versa. The buyer in each pair makes a take-it-or-leave-it offer p, which is a monetary transfer from the buyer to the seller, and the seller decides whether or not to accept it.

Therefore, a seller's strategy consists of two components: a investment strategy CDF $F_e(x)$ and a reserve price mapping $r_S(x)$, where $F_e(x)$ measures the probability that the investment is weakly larger than $c(x)$ and $r_S(x)$ is the lowest price that a seller with cost x is willing to accept. A buyer's strategy is a price offer CDF $H(p)$, where $H(p)$ equals the probability of offering a price weakly lower than p.

Preference: If the offer is accepted, one unit of output is produced and sold, which leaves the seller payoff $p - x$ and the buyer payoff $y_0 - p$. Both agents exit the market permanently. Otherwise, the pair is dissolved and both agents search in the next period. All agents have the same discount factor $\beta \in [0, 1)$, which is used to model search frictions in the model.

Information: A crucial assumption is that buyers have no information about investments. In addition, the matching is anonymous.

2.2 Benchmark Specifications

The First Best

We first characterize the efficient allocation, which consists of both efficient investment and efficient trade.

At the search stage, a social planner would find it optimal to always conduct trade between any two agents given any cost distribution, since the surplus from trade is always positive and postponing trading is costly due to discounting.

Given that trades take place immediately at the search stage, if a seller invests to reduce his cost to x, he increases the social welfare by $x_0 - x$ with investment cost $c(x)$. A social planner will thus choose $x^* > 0$ implicitly defined by $c'(x^*) = -1$ so that the marginal cost of investment equals the marginal benefit.

Observable Investments

Next consider the situation where investments are observable. Following the same logic in Diamond (1971), because buyers have all the bargaining power, a seller with any cost x gets zero search stage payoff. The reason is that the buyer in the current match and buyers in all future matches will offer exactly the seller's production cost plus the discounted continuation payoff, since buyers can observe the seller's production cost. Because the discount factor is strictly less than 1, this infinitely repeated discounting drives the search stage payoff down to zero. The sellers therefore have no incentive to invest.

Therefore, in the unique equilibrium, no seller invests and all buyers offer price x_0. Investments are inefficient while trades are efficient.

This analysis shows why unobservability is necessary to restore investment incentives. When buyers cannot observe (or perfectly infer in equilibrium) investments, it is possible that a seller will receive a price offer larger than his/her reservation price. Such possibility creates rents for sellers. We will illustrate this intuition in detail in the next section.

3 The Steady State Equilibrium

Let us now solve the steady-state equilibrium in the decentralized market. A steady state equilibrium consists of a seller's investment strategy CDF $F_e(x)$ when he is an entrant and reserve price function $r_S(x)$ when he is an incumbent, a buyer's price offer distribution CDF $H(p)$ and a stationary cost distribution $F(x)$.

3.1 The Seller's Problem

At the search stage, a seller with production cost x chooses the lowest price he/she is willing to accept, i.e. the reserve price $r_S(x)$, to maximize his/her search stage payoff $U(x)$. Given a price offer distribution $H(p)$, his trading probability is $1 - H(r_S(x)) + Pr(\tilde{p} = r_S(x))$, which is decreasing in $r_S(x)$. The maximization problem of a type x seller can be summarized as follows,

$$U(x) = \max_r \{(E(\tilde{p} \mid \tilde{p} \geq r) - x)(1 - H(r) + Pr(\tilde{p} = r)) + (H(r) - Pr(\tilde{p} = r))\beta U(x)\} \quad (1)$$

Solving the above problem, the reserve price $r_S(x)$ equals:

$$r_S(x) = x + \beta U(x) \quad (2)$$

A seller is willing to accept any price that is enough to cover his/her opportunity cost of trading: production cost x plus the discounted continuation payoff.

A more efficient seller should have a higher search stage payoff $U(x)$. He/she should be willing to accept lower price offers in equilibrium, since the opportunity cost of trade delay is higher. Moreover, the least efficient seller should receive zero search stage payoff, since no buyer would offer prices higher than his/her reserve price. Denote the highest production cost on the support by $\bar{x}$. The following lemma confirms this conjecture.

Lemma 1. *In any steady-state equilibrium, $U(x)$ is strictly decreasing and continuous in x, with $U(\bar{x}) = 0$. $r_S(x)$ is strictly increasing and continuous in x.*

Otherwise mentioned, all proofs of this paper are gathered in the appendix.

Lemma 1 implies that $\hat{x}(p)$, the inverse function of $r_S(x)$, is well-defined, continuous and strictly increasing. Function $\hat{x}(p)$ specifies the highest type of a seller who is willing to accept price p.

Lemma 1 also implies that the highest cost on the support equals the initial cost, i.e. $\bar{x} = x_0$. A seller with the highest cost gets zero search stage payoff and therefore has no incentive to invest ex ante.

Corollary 1. *In any steady-state equilibrium, the least efficient sellers invest zero,* $\bar{x} = x_0$.

The least efficient sellers' ex-ante payoff therefore equals 0. In equilibrium, ex-ante identical sellers must be indifferent over any x on the support of the investment strategy, and weakly prefer these x to any other x that is not on the support. The search stage payoff $U(x)$ depends on the price offer distribution $H(p)$. In equilibrium, $H(p)$ must be such that

$$U(x) - c(x) = 0, \text{ for any } x \text{ on the support of } F_e(x) \tag{3}$$

$$U(x) - c(x) \leq 0, \text{ for any } x \text{ not on the support of } F_e(x) \tag{4}$$

3.2 The Buyer's Problem

A buyer's strategy is a price offer CDF $H(p)$. We know from the last section that if p is offered, any seller with cost lower than $\hat{x}(p)$ will agree to trade. Recall that the probability of meeting a buyer with cost weakly lower than x is $F(x)$. The probability of trade therefore equals $F(\hat{x}(p))$.

In equilibrium, $F(x)$ must be such that it makes a buyer indifferent over any p on the support of $H(p)$. That is, any p on the support solves the following maximization problem,

$$\pi = \max_p \{(y_0 - p)F(\hat{x}(p)) + (1 - F(\hat{x}(p)))\beta\pi\} \tag{5}$$

3.3 The Seller's Investment Strategy

The last piece of the model is the seller entrants' cost distribution, which is also the seller's investment strategy[1]. In a steady state equilibrium, the measure of outflow of any type must equal the measure of inflow of the same type to preserve the stationary distribution over time. A seller with type x leaves the market if he gets an offer that is weakly higher than $r_S(x)$ (which happens with probability $1 - H(r_S(x)))$[2]. Meanwhile, the measure of entrants with type lower than x is $F_e(x)$. Denotes the most efficient sellers on the support by $\underline{x}$. The steady-state equilibrium requires that for any x on the support,

$$F_e(x) = \frac{F(x) - \int_{\underline{x}}^{x} H(r_S(\tilde{x}))dF(\tilde{x})}{1 - \int_{\underline{x}}^{x_0} H(r_S(\tilde{x}))dF(\tilde{x})} \tag{6}$$

3.4 Equilibrium Existence and Uniqueness

Let us first summarize the dynamic of a steady-state equilibrium. At any period, the stationary cost distribution $F(x)$ is such that it keeps buyers indifferent over price offers. Players trading strategy then determines the cost distribution of those who exit. At the beginning of the next period, the new generation of seller entrants, since they are indifferent over these investments given price distribution $H(p)$, will choose the investments so that they exactly replace who exit. This way, the stationary cost distribution is preserved over time.

Proposition 1. *In any steady state equilibrium, the price offer distribution $H(p)$, the seller's investment strategy $F_e(x)$ and the stationary cost distribution $F(x)$ have the following properties,*

1. *$F(x)$ and $F_e(x)$ have support $[x^*, x_0]$ with the unique point mass at x^*.*

2. *$H(p)$ has support $[r_S(x^*), r_S(x_0)]$ and is atomless;*

[1]Since there are one unit mass of entrants, we get the equivalence between the entrants' cost distribution and the investment strategy when we abuse the law of large number as usual.

[2]The probability of receiving an offer weakly higher than $r_S(x)$ should be $1 - H(r_S(x)) + Pr(\tilde{p} = r_S(x))$. However, as will be proved in the next section, there is no point mass on the support of $H(p)$. Therefore, to simplify the notation, we write down the probability as if there is no point mass.

Although all agents are identical ex-ante, proposition 1 shows that the stationary price offer distribution and the investment strategy are non-degenerate. The unobservability of investments is the key behind this result. Suppose all buyers offer the same price. Because investments are unobservable, sellers become residual claimants facing the single price and will either invest efficiently or invest zero. If the price is not enough to cover the investment cost plus the production cost ($c(x^*) + x^*$), then sellers will invest zero and reject the offer after entering the market. This leaves buyers zero profit. Therefore to earn a positive profit, buyers must offer a price higher than $c(x^*) + x^*$ so that sellers invest efficiently ex-ante and agree to accept the price. However, given that all other buyers offer the price, a buyer will find it profitable to deviate to a slightly lower price: because of discounting, the matched seller is willing to accept the lower price. Hence, a single price can never be an equilibrium. Then the investment distribution has to be non-degenerate as well to support the price distribution.

We might also suspect that there could be a gap on the support of the price distribution and the stationary cost distributions. For instance, it is possible that while two prices p_1 and p_2 are on the support, prices on the interval (p_1, p_2) are not offered, because sellers with types $x \in (\hat{x}(p_1), \hat{x}(p_2))$ are not in the market and hence there is no gain from offering their reserve prices. Meanwhile, no sellers choose to become type $x \in (\hat{x}(p_1), \hat{x}(p_2))$ because their reserve prices are not offered. Unfortunately, this intuition neglects the indifference condition: a seller must be indifferent between $\hat{x}(p_1)$ and $\hat{x}(p_2)$, and weakly prefer them to any x in the interval. Because no price between p_1 and p_2 is offered, two sellers having these two costs respectively trade with the same probability. Therefore, $U(x)$ increases linearly on the interval $(\hat{x}(p_1), \hat{x}(p_2))$. On the other hand, the investment cost function $c(x)$ is strictly convex. Hence,the indifference condition can never hold with a gap.

Proposition 1 claims that there is no point mass on the price distribution either. Any point mass will result in a jump in the probability of trade, which in turn leads to a kink in $U(x)$. However, the investment cost function $c(x)$ is of class C^1. This again contradicts the indifference condition.

Of course, the above reasoning is based on the assumption that the investment cost function $c(x)$ is of class C^1, continuous and strictly convex. If $c(x)$ is not so well-behaved, then some properties of the price distribution will change accordingly.

This proposition also shows that the lowest production cost on the market is the efficient cost x^*. Intuitively, this is because a seller with the lowest cost trades immediately with probability one: any price offer in the market is weakly higher than his/her reserve price. Since his/her investments are unobservable to the buyer, this seller becomes the residual claimant and invests efficiently.

Finally, the measure of type x^* sellers must be positive, since a buyer who offers $r_S(x^*)$ can only trade with the type x^* sellers and he/she must get strictly positive equilibrium payoff. Moreover, there is no other point mass in the investment strategy and the stationary cost distribution, because any other point mass would lead to a jump in the buyer's payoff as a function of p. This contradicts the buyer's indifference condition.

Notice that the above results and intuitions hold even when $\beta = 0$, in which case the buyer in each pair is a monopolist. The monopolist is indifferent over the interval of prices because the stationary cost distribution is adjusted so that the demand function has elasticity equals -1 at any price in the interval.

In the rest of this section, we will first solve $H(p)$ and $F(x)$, and then show the existence and the uniqueness of the steady state equilibrium.

$H(p)$ can be solved from the envelope condition of $U(x)$. It is legitimate to take the derivative of $U(x)$ because we have proved that the support is an interval, and that $U(x) - c(x) = 0$ for any x on the support. $c(x)$ is of class C^1 then implies that $U(x)$ is also of class C^1. The envelope condition is

$$U'(x) = -(1 - H(r_S(x))) + H(r_S(x))\beta U'(x)$$

Using the equilibrium restriction that $U'(x) = c'(x)$, $H(p)$ can be solved,

$$H(p) = \frac{1 + c'(\hat{x}(p))}{1 + \beta c'(\hat{x}(p))} \tag{7}$$

$F(x)$ is solved from the buyer's indifference condition. If a buyer offers the highest reserve price $r_S(x_0) = x_0 + \beta U(x_0) = x_0$, he can trade with probability 1. Therefore $\pi = y_0 - x_0$.

Any other price on the support must yield the same expected profit. In other words,

$$(y_0 - p)F(\hat{x}(p)) + [1 - F(\hat{x}(p))]\beta\pi = y_0 - x_0$$

Therefore, the stationary cost distribution $F(x)$ equals,

$$F(x) = \begin{cases} 0, \text{ if } x \in (-\infty, x^*), \\ \frac{y_0-\beta\pi-x_0}{y_0-\beta\pi-x-\beta c(x)}, \text{ if } x \in [x^*, x_0], \\ 1, \text{ if } x \in (x_0, +\infty). \end{cases} \tag{8}$$

We have shown that $F(x)$ and $H(p)$ exist and are unique. The investment strategy $F_e(x)$ then also exists and is uniquely determined by (6).

Proposition 2. *Steady state equilibrium exists and is unique.*

4 Equilibrium Characterization

4.1 Investment Strategy and Stationary Cost Distribution

There are two distributions about sellers' type. One is the investment strategy $F_e(x)$, which is the type distribution of entrants. The other is the stationary cost distribution $F(x)$, which is the type distribution of incumbents. We have already shown that the per-period trading probability increases in the investment level. Therefore, a more efficient seller trades and exits the market in a faster speed. In other words, the type distribution of sellers who exit is more efficient than the stationary cost distribution. This in turn implies that the investment strategy $F_e(x)$ is always more efficient than $F(x)$.

Proposition 3. *$F(x)$ has first order stochastic dominance over $F_e(x)$.*

4.2 Constant Payoffs and Social Welfare

As shown in the previous section, sellers always invest with positive probability given any search friction. We would expect the social welfare to be higher than what we could get in the benchmark case with observable investments, where sellers have no incentive to invest. Unfortunately, this is not the case as shown in the following theorem.

Theorem 1. *For any $\beta \in [0, 1)$, the seller's ex-ante payoff is 0, the buyer's ex-ante payoff equals the social welfare, which is $y_0 - x_0$.*

Proof. we know that $v = U(x) - c(x) = 0$ and $\pi = y_0 - x_0$. The social welfare therefore equals $v + \pi = y_0 - x_0$. $\square$

We can easily verify that the agents' payoffs and the social welfare are the same in both observable and unobservable cases. This is because the unobservability, while incentivizes investments, also causes trading inefficiency. The welfare gain generated from investments could be realized fully only if seller entrants and buyer entrants agree to trade immediately after entering the market. However, this is impossible with the presence of information and search frictions: because of the unobservability, both the cost distribution and the price distribution are non-degenerate. Profitable trades are therefore conducted only probabilistically. In other words, there is expected delay in trade for any buyer whose price offer is strictly lower than $r_S(x_0)$. The welfare loss due to the delay in trade exactly offsets the welfare gain from the more efficient investments.

We have shown that the social welfare are constant over search frictions. Since the social welfare depends on both the investment efficiency and trade efficiency, the constant welfare result could arise from the constant investment and constant trade efficiency, or it could arise from a more efficient investments and a less efficient trade, or vise versa. To determine which is the case, we would like to see the change of investment strategy and efficiency of trade as we vary search frictions.

4.3 Comparative Statics and the Limiting Case

Consider any seller with cost $x \in (x^*, x_0)$. As meetings become more frequent, the seller trades with higher probability per unit of time if the price distribution remains constant. Consequently, the marginal benefit of investment, which strictly increases in the probability of trade, will be strictly larger than the marginal cost of investment. Therefore, to keep a seller indifferent across investments when search frictions decrease, buyers must price more aggressively. That is, the per-period trading probability $1 - H(r_S(x))$ must strictly decreases in β. Indeed,

$$\frac{\partial(1 - H(r_S(x)))}{\partial\beta} = \frac{(1 + c'(x))c'(x)}{(1 + \beta c'(x))^2} < 0, \text{ for any } x \in (x^*, x_0)$$

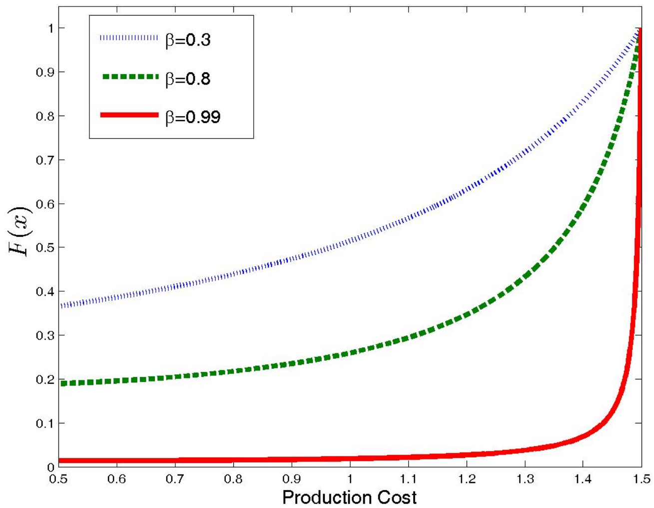

Figure 2: Stationary Seller Type Distribution

(*In this example:* $x_0 = 1.5$, $c(x) = \frac{1}{2}(x - x_0)^2$, $y_0 = 2.2$.)

As search frictions vanish, the probability of trade $1 - H(r_S(x))$ must converges to 0 for any $x \in (x^*, x_0)$. Or equivalently, buyers price extremely aggressively in the limit: the price offer distribution must converge in distribution to a point mass at $r_S(x^*)$. Otherwise, any seller trades almost for sure within any small amount of time if the trading probability is bounded away from zero. The marginal benefit of investment therefore becomes 1 and a seller cannot be indifferent across investment levels. This is a contradiction. We can also mathematically verify this intuition from equation (7) by taking the limit $\beta \to 1$.

Next we show that the stationary cost distribution $F(x)$ is less efficient as β increases. If $F(x)$ stays constant when meetings become more frequent, a buyer who is originally indifferent over price offers would strictly prefer to offer the lowest price $r_S(x^*)$. Therefore, the probability of trade of a buyer offering a price lower than $r_S(x_0)$ must decrease in β. That is, the stationary cost distribution $F(x)$ with a larger β has first order stochastic dominance over an $F(x)$ with a smaller β. Indeed,

$$\frac{\partial F(x)}{\partial \beta} = \frac{(y_0 - x_0)(x - x_0 + c(x))}{[y_0 - \beta\pi - x - \beta c(x)]^2} < 0$$

In the limit, we can verify that $F(x)$ converges in distribution to a point mass at x_0 from equation (8). That is, almost all incumbents are composed of sellers who invested zero.

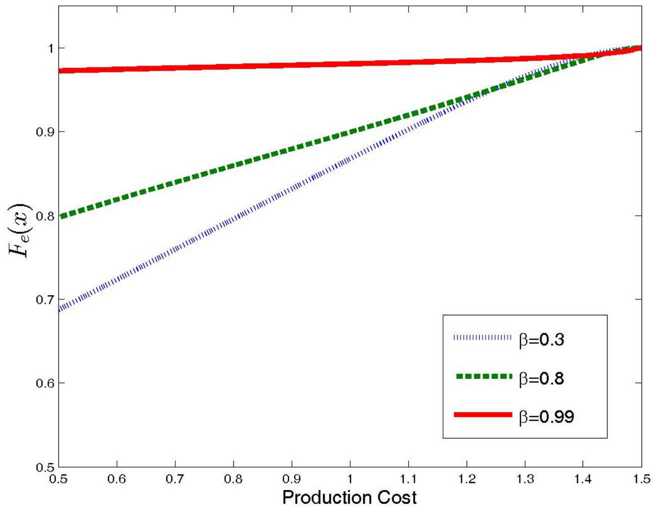

Figure 3: Seller's Investment Strategy

(*In this example:* $x_0 = 1.5$, $c(x) = \frac{1}{2}(x - x_0)^2$, $y_0 = 2.2$.)

This result is shown graphically in Figure 2. Intuitively, if $F(x)$ is bounded away from 0 in the limit for some $x < x_0$, then the per-period trading probability of a buyer offering $r_S(x)$ is strictly positive. Again, as the time between successive meetings shrinks to zero, if the per-period trading probability is strictly positive, then the buyer could trade immediately. Hence, a buyer who offers $r_S(x_0)$ would find it optimal to lower the price offer to $r_S(x)$ without changing trading probability, leading to a contradiction.

In steady state, the cost distribution of entrants is the same as that of exits to preserve the stationary cost distribution. Since the cost distribution of incumbents becomes less efficient as β increases, we might expect that the distribution of exits and hence the investment strategy $F_e(x)$ becomes less efficient.

It turns out that the opposite result holds: the measure of the point mass $F_e(x^*)$ strictly increases in β, i.e., an entrant invests efficiently with strictly higher probability. In the limit, $F_e(x^*)$ increases to 1 and hence the investment strategy becomes efficient. We already know that $F_e(x)$ is always more efficient than $F(x)$. Our limiting result further shows that each distribution converges in distribution to a point masses at one polar point. The investment strategies with different discount factors is plotted in Figure 3.

To understand this result, first notice that a seller who invests efficiently always exits the market immediately independent of β. In addition, we have three observations from the previous analysis: as β increases, 1) larger proportion of incumbents are underinvested type; 2) per-period trading probabilities of underinvested sellers strictly decrease; 3) the average cost of underinvested sellers strictly increases, which implies that a seller with the average cost exits with lower probability. The first effect raises the proportion of exits who underinvest, as captured by the previously mentioned casual intuition. The rest two effects explain why $1 - F_e(x^*)$ decrease in β: any underinvested seller exits less often and the composition of the underinvested sellers becomes more inefficient.

We summarizethe above comparative statics and the limiting results in the following propositions.

Theorem 2. *As β increases to 1, in the steady state equilibrium*

1. *$H(r_S(x))$ strictly increases for any $x \in (x^*, x_0)$ and converges in distribution to a point mass at $r_S(x^*)$;*

2. *$F(x)$ strictly decreases for any $x \in [x^*, x_0)$ and converges in distribution to a point mass at x_0;*

3. *$F_e(x^*)$ strictly increases and $F_e(x)$ converges in distribution to a point mass at x^*.*

From this proposition, we can better understand the mechanism behind the constant social welfare result. As β increases, new entrants invest more efficiently and this could potentially generate additional social welfare if the trading efficiency remains constant. Unfortunately at the same time trades become more inefficient. As β increases, the stationary cost distribution has more mass on high costs. Consequently, for a buyer who offers a given price, it takes more periods in expectation to find a seller who is willing to accept the price.

4.4 Convergence from the initial time to the steady state

So far we have focused on the steady-state equilibrium. In this section, let us investigate if there exist a path starting from date zero and converging to the steady state, such that the equilibrium properties in the steady-state are preserved along the path.

Because we want to construct an equilibrium path that preserves the equilibrium properties, we assume that agents in any cohort (correctly) expect to get the same search stage payoffs as in the steady-state equilibrium. At the beginning of date $t = 1$, one unit mass of the first cohort of sellers and buyers are born. Sellers are indifferent over investment levels given their expected search stage payoffs. Therefore, we can let them invest according to $F(x)$. $F(x)$ then also becomes the incumbents' cost distribution at date $t = 1$, since the market is empty originally. Given the incumbents' cost distribution and continuation payoff π, buyers are indifferent over reserve prices and therefore we can let me use mixed strategy $H(p)$. In the end of date $t = 1$, only part of agents (denoted by $e_1 < 1$) exit the market because of the mismatch between investments and price offers, and the cost distribution of exiting sellers is exactly $F_e(x)$.

At the beginning of date $t = 2$, the new cohort of sellers are again indifferent over investment levels. Let measure e_1 of them invest according to $F_e(x)$ and measure $1 - e_1$ of them invest according to $F(x)$. This way, the incumbents' type distribution at $t = 2$ will be $F(x)$. The price distribution of buyers are the same. We can show that the size of exits at $t = 2$, e_2, is still smaller than 1.

In general, we can show that at any time t along the path, the size of the exits is always smaller than the size of the entrants. Therefore, the investment distribution of sellers in any cohort can be constructed in the way specified above to preserve the incumbents' cost distribution $F(x)$. Eventually, the size of the market will grow to the steady-state equilibrium size and the size of exits will be equal to the size of entrants, so we converge to the steady-state equilibrium.

As we can see from the above construction, the agents' ex-ante payoffs, the social welfare, the incumbents' cost distribution and the price distribution are all constant along the path.

Proposition 4. *There exists an equilibrium path starting from the initial time and converging to the steady state, such that,*

1. *At any point along the path, the incumbents' cost distribution is $F(x)$ as specified in (8) and the price distribution is $H(p)$ as specified in (7).*

2. *Within any cohort, the seller's ex-ante payoff is* 0*, the buyer's ex-ante payoff is* $y_0 - x_0$ *and the social welfare is* $y_0 - x_0$*.*

3. *The investment strategy becomes more efficient as time increases.*

The rest of the paper extends the baseline model along two directions. Section 5 considers the situation where a buyer could also invest to raise the valuation. Section 6 examines a two-sided offering case where a seller makes a take-it-or-leave-it offer with positive probability.

5 Two-Sided Investments

In many cases both sides of the market can invest before searching for trading partners. For instance, a firm can invest in technology to raise the output per unit of labor.

Following this idea, we assume that before entering, a buyer can increase the value from y_0 to y with investment $e(y)$, where $e(y_0) = e'(y_0) = 0$ and $e(y)$ is strictly increasing and strictly convex for any $y > y_0$. The surplus from trade between a type x seller and a type y buyer is assumed to be $y - x$. Notice that this assumption implies no complementarity between investments, which simplifies the analysis a lot[3]. Finally, the observability of buyer's investments could be arbitrary, since there is no complementarity by assumption and buyers have all the bargaining power.

We can again consider the benchmark cases of a) the first best, b) observable investments. In the first best benchmark case, since there is no complementarity, socially optimal trading strategy requires all agents to trade upon their first meetings. Given this trading strategy, a social planner would invest efficiently on both sides, i.e., all sellers invest to reduce the production cost to x^* and all buyers invest to raise the value to y^*, where y^* is defined implicitly by $e'(y^*) = 1$.

[3] In the case with one-sided investment, we could impose this assumption without loss of generality, since all buyers are identical. In the two-sided investment case, however, this assumption excludes some interesting scenarios that we could observe in many industries. For instance, the surplus from trade could be supermodular, that is, a higher seller's investment level leads to a larger marginal benefit of the buyer's investment.

In the benchmark case with observable investments, all sellers invest zero, and all buyers offer price x_0 and invest efficiently. In equilibrium, a seller gets payoff 0 and a buyer gets payoff $y^* - x_0 - e(y^*)$, which is also the social welfare.

In the rest of this section, we will first derive optimality conditions of the steady state equilibrium and then characterize the equilibrium.

5.1 The Steady State Equilibrium

Because of no complementarity, sellers can not benefit from buyers' investments directly and they only care about the price distribution. Therefore, the seller's problem is exactly the same as the one in the baseline model and the previous equilibrium conditions for sellers must continue to hold.

We will therefore focus on the buyer's problem in the rest of the section. A buyer's strategy consists of his investment strategy $G_e(y)$ and the price offer $p(y)$. $p(y)$ maximizes the search stage payoff of a type y buyer, which is denoted as $\Pi(y)$,

$$\Pi(y) = \max_p \{(y - p)F(\hat{x}(p)) + [1 - F(\hat{x}(p))]\beta\Pi(y)\} \tag{9}$$

Moreover, the following indifference conditions must hold, which essentially require a buyer to be indifferent across any y on the support of the investment strategy $G_e(y)$ and weakly prefer those y to any other y that is not on the support,

$$\Pi(y) - e(y) = \pi \geq 0, \text{ for any } y \text{ on the support of } G_e(y) \tag{10}$$

$$\Pi(y) - e(y) \leq \pi, \text{ for any } y \text{ not on the support of } G_e(y) \tag{11}$$

Proposition 5. *In any steady state equilibrium with two-sided investments, $p(y)$ is single-valued and increases in y for any y on the support of $G_e(y)$.*

Proposition 5 shows that the price offer increases in buyer's type. Intuitively, the waiting cost of a buyer with a higher valuation is larger. Therefore, he is willing to offer a higher price to ensure a higher trading probability. Consequently, a buyer with the highest valuation $\bar{y}$ on the support of $G_e(y)$ offers the highest price, which is the reserve price of a type x_0

seller, i.e., $p(\bar{y}) = x_0$ and $F(\hat{x}(p(\bar{y}))) = 1$. Therefore, $\Pi(\bar{y}) = \bar{y} - x_0$ and $\pi = \Pi(\bar{y}) - e(\bar{y}) = \bar{y} - x_0 - e(\bar{y}) > 0$.

Proposition 5 also demonstrates that given his ex-ante investments, a buyer will never play mixed pricing strategy at the search stage. Suppose that there are two buyers investing the same $e(y)$ but offer different prices. In particular, buyer 1 offers price p_1 and buyer 2 offers price p_2, with $p_1 > p_2$. Given the non-degenerate production cost distribution, buyer 1 trades faster in expectation and therefore has larger marginal benefit of investment. On the other hand, since they choose the same investment level, the marginal cost of investments are the same for both. We have a contradiction.

More importantly, the pure pricing strategy implies that buyers, although they have all the bargaining power, will play mixed investment strategy and hence underinvest with strictly positive probability.

Proposition 6. *In any steady state equilibrium with two-sided investments, the seller's investment strategy $F_e(x)$, the stationary cost distribution $F(x)$, the buyer's investment strategy $G_e(y)$ and the stationary valuation distribution $G(y)$ have the following properties,*

1. *$F_e(x)$ and $F(x)$ have support $[x^*, x_0]$ with the unique point mass at x^*.*

2. *$G_e(y)$ and $G(y)$ have support $[\underline{y}, y^*]$ and is atomless, where $\underline{y}$ is uniquely determined by*

$$y^* - x_0 - e(y^*) = [\underline{y} - x^* - \beta c(x^*)]e'(\underline{y}) - e(\underline{y}) \tag{12}$$

The steady-state condition now also need to hold for buyers. That is, the investment strategy $G_e(y)$ must equal the valuation distribution of buyers who exit the market. A buyer exits when his offer is accepted, which happens with probability $F(\hat{x}(p(y)))$. Combined with $G(y)$, the distribution of exits is determined. Equating the entrant and exit distributions, we have the following equilibrium condition.

$$G_e(y) = \frac{\int_{\underline{y}}^{y} F(\hat{x}(p(\tilde{y})))dG(\tilde{y})}{\int_{\underline{y}}^{y^*} F(\hat{x}(p(\tilde{y})))dG(\tilde{y})} \tag{13}$$

We are now ready to solve the equilibrium. The convex supports and the indifference conditions imply that both $U(x)$ and $\Pi(y)$ are differentiable with x and y on the support.

We can therefore use the envelope conditions to solve for the stationary distributions and the price offer function $p(y)$. The derivation also proves the existence and uniqueness of the steady-state equilibrium. We summarize the results in the following proposition.

Proposition 7. *The steady state equilibrium with two-sided investment exists. The stationary cost distribution CDF $F(x)$ is defined by (15), the sellers' investment strategy CDF $F_e(x)$ is defined by (6), the reserve price $r_S(x)$ is defined by (2), the stationary valuation distribution CDF $G(y)$ is defined by (16), the buyers' investment strategy CDF is defined by (13) and the price offer $p(y)$ is defined by (14).*

Moreover, the steady state equilibrium is unique.

$$p(y) = y - \frac{e(y) + y^* - x_0 - e(y^*)}{e'(y)} \tag{14}$$

$$F(x) = \begin{cases} 0, \text{ if } x \in (-\infty, x^*), \\ \frac{(1-\beta)e'(\hat{y}(r_S(x)))}{1-\beta e'(\hat{y}(r_S(x)))}, \text{ if } x \in [x^*, x_0], \text{ where } \hat{y}(p) \text{ is the inverse of } p(y) \\ 1, \text{ if } x \in (x_0, +\infty). \end{cases} \tag{15}$$

$$G(y) = \begin{cases} 0, \text{ if } y \in (-\infty, \underline{y}) \\ \frac{1+c'(\hat{x}(p(y)))}{1+\beta c'(\hat{x}(p(y)))}, \text{ if } y \in [\underline{y}, y^*], \\ 1, \text{ if } y \in (y^*, +\infty). \end{cases} \tag{16}$$

In the baseline model, we demonstrated that the equilibrium payoffs and the social welfare are the same as if investments were observable. This result still holds in the two-sided investments extension.

Theorem 3. *In the steady state equilibrium with two-sided investment, the seller's ex-ante payoff v equals 0, the buyer's ex-ante payoff π and the social welfare equals $y^* - x_0 - e(y^*)$.*

5.2 Comparative Statics and the Limiting Case

The comparative statics and the limiting results regarding $H(p)$, $F(x)$ and $F_e(x)$ in theorem 2 can be extended here. Therefore, We will not repeat the results but instead devote this section to the comparative statics exercise with the buyers' investment strategy $G_e(y)$ and the stationary valuation distribution $G(y)$.

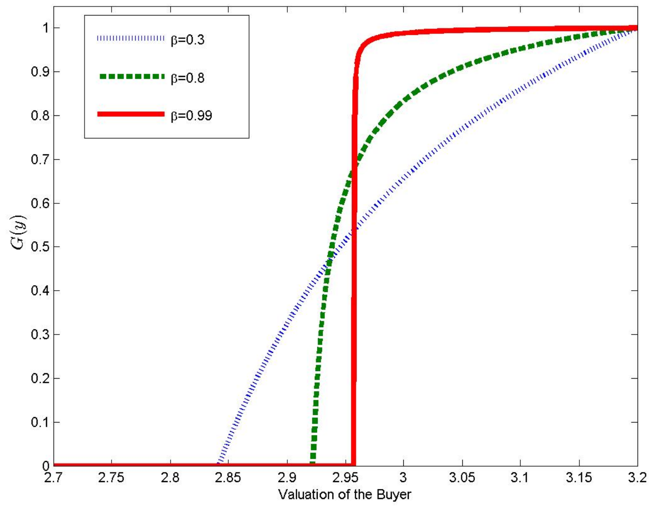

Figure 4: Stationary Buyer Type Distribution

(*In this example:* $x_0 = 1.5$, $c(x) = \frac{1}{2}(x - x_0)^2$, $y_0 = 2.2$, $e(y) = \frac{1}{2}(y - y_0)^2$.)

First of all, the lower bound $\underline{y}$ as defined in condition (12) strictly increases in β. To understand this result, we know that a type $\underline{y}$ buyer offers the price which equals a type x^* seller's reserve price $x^* + \beta c(x^*)$. Therefore, his price offer strictly increases in β. Meanwhile, the trading probability is strictly lower, as $F(x^*)$ strictly decreases in β. Therefore, $\underline{y}$ must strictly increases in β to keep the buyer's ex ante payoff constant ($\pi = y^* - x_0 - e(y^*)$). Furthermore, the limit of $\underline{y}$ as $\beta \to 1$ is strictly less than y^*. In other words, the investment strategy is non-degenerate even when search frictions vanish.

The stationary valuation distribution $G(y)$ also adjusts as β changes. We know from the baseline model that the mass of the price distribution shifts towards lower reserve prices as β increases. Combined with the fact that the price offer $p(y)$ strictly increases in y, there must be larger mass of buyers who invest small amount and offer low prices. $G(y)$ indeed converges in distribution to a point mass at $\underline{y}$ in the limit. Figure 4 graphically shows the above two results using a numerical example.

The buyers' investment strategy $G_e(y)$ also converges in distribution to a point mass at $\underline{y}$ as β goes to 1. To understand why, we know that almost all buyers offer the lowest price in the limit. Therefore, almost all trades take place at the lowest price offered by the type

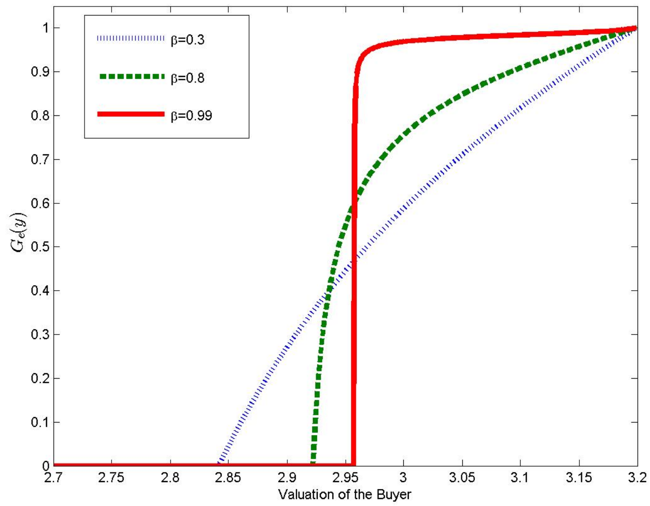

Figure 5: Buyer's Investment Strategy

(*In this example:* $x_0 = 1.5$, $c(x) = \frac{1}{2}(x - x_0)^2$, $y_0 = 2.2$, $e(y) = \frac{1}{2}(y - y_0)^2$.)

$\underline{y}$ buyers. Hence, entrants who replace these exits in the steady state also consists of almost all type $\underline{y}$ buyers.[4] The investment strategy with the same set of parameters is plotted in Figure 5.[5]

The above discussions are summarized in proposition (8).

Proposition 8. *In the steady-state equilibrium with two-sided investments,*

1. *As* β *increases to* 1, *(i) The lowest valuation* $\underline{y}$ *strictly increases; (ii) A buyer with valuation that equals the* $t * 100th$ *percentile of* $G(y)$ *offers the reserve price of a more efficient seller, for any* $t \in (0, 1)$.

2. *As* $\beta \to 1$, *(i)* $\underline{y}$ *is still bounded away from* y^*, *i.e.,* $\lim_{\beta \to 1} \underline{y} < y^*$; *(ii)* $G_e(y)$ *and* $G(y)$ *converge in distribution to a point mass at* $\underline{y}$.

[4]Notice that the above argument will not hold if there is a point mass at y^*, like what we had for the seller's limiting investment strategy. The reason is that, when there is positive measure of agents who exit the market with probability 1, the average exit type might be higher than $\underline{y}$ or even approaches y^*.

[5]To clarify, although the investment strategy has more mass on lower types, we cannot conclude that the investment strategy becomes less efficient, because the lowest valuation $\underline{y}$ strictly increases in β.

6 Two-Sided Offers

In some situations a seller also has the opportunity to make offers. Specifically, in each meeting, nature randomly selects the seller to make a take-it-or-leave-it offer with probability α that is bounded above zero and selects the buyer with the complementary probability. Therefore, a seller's strategy now also includes price offer $p_S(x)$ and a buyer's strategy also includes reserve price r_R.

We can verify that $r_R = y_0 - \beta\pi$, that is, a buyer is willing to pay the price if it leaves him at least his discounted continuation payoff. Therefore, a seller with any cost will propose $p_S(x) = r_R$ if $r_R - x$ is weakly larger than $\beta U(x)$.

6.1 Benchmark Case: Observable Investments

As a benchmark, let us first characterize the steady-state equilibrium with observable investments. Conditional on investing positive amount, a seller will choose $\bar{x}$ to maximize his/her ex-ante payoff.

$$\max_{x}\{\frac{\alpha(y_0 - x - \beta\pi)}{1-\beta(1-\alpha)} - c(x)\}$$

Therefore, $\bar{x}$ is uniquely defined by,

$$c'(\bar{x}) = \frac{-\alpha}{1-\beta(1-\alpha)} \tag{17}$$

Proposition 9. *In any stationary equilibrium with two-sided offers and observable investments, all seller entrants invest to become type $\bar{x}$. Either all incumbents are type $\bar{x}$ and trade immediately, or some incumbents are type x_0 and never trade.*

Therefore, depending on parameters, one of the following two equilibria will arise. In the first equilibrium, the entrants in any period invest to become type $\bar{x}$ and they are the only incumbents. In other words, the market size equals the entrant size. All agents trade immediately. Alternatively, a seller can invest zero and get zero payoff. Therefore, this equilibrium requires the search stage payoff, which depends on the bargaining power α and the shape of $c(x)$, to be larger than the investment cost.

When this condition fails to hold, we have the second equilibrium. In the second equilibrium, incumbents consist of not only the new entrants, but also some sellers who invest zero. All seller entrants trade immediately, while a buyer entrant only trades if he/she meets an invested incumbents. All non-invested incumbents never trade and stay in the market forever. Because of the presence of the non-invested sellers, the buyer's payoff is lower. Hence an invested seller gets a larger share of the investment benefit in expectation, and is indifferent between $\bar{x}$ and x_0. One way to understand the presence of these non-invested incumbents is to use the convergence analysis from the initial time to the steady-state: before reaching the steady-state, entrants mix between invest and not invest, and non-investing sellers never leave the market.

Proposition 10. *The steady-state equilibrium with two-sided offer and observable investments takes one of the following two forms:*

1. *If $\alpha(y_0 - \bar{x}) \geq c(\bar{x})$, both $F(x)$ and $F_e(x)$ is a point mass at $\bar{x}$. The seller's ex-ante payoff is $\alpha(y_0 - \bar{x}) - c(\bar{x})$ and the buyer's ex-ante payoff is $(1-\alpha)(y_0 - \bar{x})$*

2. *If $\alpha(y_0 - \bar{x}) < c(\bar{x})$, $F_e(x)$ is a point mass at $\bar{x}$ and $F(x)$ has two point mass at x_0 and $\bar{x}$, with $F(\bar{x}) = \frac{\alpha(y_0-\bar{x})}{\beta(1-\alpha)c(\bar{x})} - \frac{1-\beta(1-\alpha)}{\beta(1-\alpha)}$. The seller's ex-ante payoff is 0 and the buyer's ex-ante payoff is $\frac{\alpha(y_0-\bar{x})-(1+\alpha\beta-\beta)c(\bar{x})}{\alpha\beta}$.*

From (17) we can see that $\bar{x}$ decreases in β and converges to x^* in the limit. Therefore, the investment strategy becomes efficient in the steady-state equilibrium. Moreover, in the first equilibrium, trade is efficient. In the second equilibrium, trade also becomes efficient when $\beta \to 1$ since buyer's per-period trading probability $F_e(\bar{x})$ is bounded away from zero. Therefore, the social welfare converges to the first best in the limit.

Proposition 11. *In the two-sided offer case with observable investments, as β converges to 1, $F_e(x)$ converges in distribution to a point mass at x^*. In addition, the social welfare converges to the first best.*

Proposition 11 shows that even if investments are observable, when sellers have some bargaining power (could be arbitrarily small), investments become efficient in the limit. To understand this result, note that the buyer's reserve price is independent of his opponent's investments. Hence a seller is the residual claimant of his investments when he makes the offer. As the time between two meetings shrinks to zero, any positive α implies that the seller has the chance to make the offer almost immediately after entry. Therefore he becomes the full residual claimant and will invest efficiently, although he only gets α share of the total surplus from trade.

Moreover, the social welfare also convergence to the first best, since both the investments and the trades are efficient. Note that this limiting result does not imply any discontinuity around $\alpha = 0$. A smaller α requires a larger β for the social welfare to fall in a given neighborhood of the first-best level.

6.2 Steady-State Equilibrium

Let us turn to the steady-state equilibrium with unobservable investments. For any x on the support of $F_e(x)$, trade always takes place if the seller is selected to make the offer, since the surplus from trade $y_0 - x - \beta\pi - \beta U(x)$ must be non-negative to ensure a positive search stage payoff $U(x)$.

We can follow the same logic as in the baseline model to verify that the support of $F_e(x)$ is $[x^*, \bar{x}]$ and that buyers play mixed pricing strategy over $[r_S(x^*), r_S(\bar{x})]$.

In equilibrium, a seller's ex-ante payoff v and a buyer's ex-ante payoff π must be non-negative. Similar to the observable benchmark case, the equilibrium could take one of the two forms, depending on whether $\alpha(y_0 - \bar{x})$ is larger than $c(\bar{x})$. When $\alpha(y_0 - \bar{x}) \geq c(\bar{x})$, using the indifference conditions, to solve for v and π, we only need to focus on a seller with cost $\bar{x}$ and a buyer who offers $r_S(\bar{x})$. Combining their value functions,

$$
\begin{aligned}
U(\bar{x}) &= \frac{\alpha(y_0 - \bar{x} - \beta\pi)}{1 + \alpha\beta - \beta} = c(\bar{x}) + v \\
\frac{1 - \alpha\beta}{1 - \alpha}\pi &= y_0 - \bar{x} - \beta c(\bar{x}) - \beta v
\end{aligned}
$$

we can solve v and π:

$$v = \alpha(y_0 - \bar{x}) - c(\bar{x}) \tag{18}$$

$$\pi = (1-\alpha)(y_0 - \bar{x}) \tag{19}$$

By the assumption $\alpha(y_0 - \bar{x}) - c(\bar{x}) \geq 0$, both v and π are positive.

When $\alpha(y_0 - \bar{x}) < c(\bar{x})$, positive fraction of incumbent sellers are type x_0 and they never trade. We can verify that the fraction $F(x_0)$ as well as the equilibrium payoffs are the same as in the observable investments benchmark case.

To sum up, the equilibrium outcome equivalence between observable and unobservable investments still holds in this extension.

6.3 Comparative Statics and the Limiting Case

From (17) we know that the highest cost on the market $\bar{x}$ is determined by (17). $\bar{x}$ strictly decreases in α and β, i.e., it is closer to the efficient cost when a seller has a larger probability to make the offer within one unit of time. In the limit as $\beta \to 1$ or as $\alpha \to 1$, $\bar{x}$ converges to x^* and hence the investment strategy becomes efficient. Although in the baseline model the investment strategy also converges to the first best, the mechanisms behind the result are quite different.

With two-sided offers, the equivalence between observable and unobservable investments still holds. Therefore, as shown in the observable benchmark case, the social welfare converges to the first best when search frictions vanish.

We summarize the above discussion in the following proposition.

Proposition 12. *In the unique steady-state equilibrium with two-sided offers, the highest production cost $\bar{x}$ strictly decreases in β and α. Moreover, the investment strategy converges in distribution to a point mass at x^* as $\beta \to 1$ or $\alpha \to 1$.*

The equilibrium social welfare converges to the first best as $\beta \to 1$ or $\alpha \to 1$.

7 Robustness and Other Extensions

7.1 Robustness

As we can see, the intuition for mixed strategy does not rely too much on the specific set up of the model. Moreover, as long as agents are still using mixed strategy, most of the main results will continue to hold. In this section, we will check the robustness of results of the baseline model to some alternative assumptions.

General one-to-one Matching Technology. In the baseline model, I assume that each player is paired with one player from the other side for sure in each period. The main results are robust if instead we have a general one-to-one matching technology so that the probability of not being paired in one period is positive.

To be more precise, the equilibrium investment strategy $F_e(x)$ is still non-degenerate with convex support $[\underline{x}, x_0]$ and a buyer plays mixed pricing strategy over reserve prices of these types. The difference is that $\underline{x}$ is higher than the efficient cost if a seller cannot be paired with probability 1 each period. Moreover, because the indifference conditions still hold, the equilibrium payoffs and the social welfare equal the values generated with observable investments. Finally, as β converges to 1, $\underline{x}$ converges to x^*: as meetings become more frequent, it is as if the most efficient sellers could trade immediately and hence they will invest efficiently. The convergence results of $F(x)$ and $F_e(x)$ can also be extended, with different rates of convergence that depend on the matching technology.

Exogenous Death Shock. Suppose each player experiences an exogenous death shock with positive probability in each unit of time. For the most part of the analysis, this is equivalent to redefining a smaller discount factor which also converges to 1 in the limit. The only complication is that now the group of exit sellers also include those who experienced the death shock. So we need to rewrite the stationary distribution condition accordingly.

If we denote the discount rate by r_1 and the rate of the death shock by r_2, then the limit of $F_e(x^*)$ is continuous and decreasing in r_2 and is bounded above zero for any r_2.

Proposition 13. *As the time between two consecutive meetings shrinks to zero,*

$$F_e(x^*) \to [1 + \frac{x_0 - x^* - c(x^*)}{y_0 - x_0} \frac{r_2}{r_1 + r_2}]^{-1}$$

Therefore, the limit of $F_e(x^*)$ decreases in the rate of death shock r_2 and is larger than $[1 + \frac{x_0 - x^* - c(x^*)}{y_0 - x_0}]^{-1}$.

7.2 Other Extensions

Investments are Observable with Positive Probability. As the previous analysis shows, the social welfare that could be generated from positive investments is completely dissipated because of the inefficiency of trade. If in the search stage investments are observed with positive probability q per period, then profitable trades could be conducted with no delay when investments are observable. We can show that given any search friction, partial information yields strictly higher social welfare than no information (and full information).

To be more precise, seller entrants still use a mixed investment strategy. Its support is now $[\underline{x}, x_0]$, with $\underline{x} > x^*$ pinned down by

$$c'(\underline{x}) = \frac{-(1-q)}{1-q\beta}$$

Since x_0 is still on the support, the seller's ex-ante payoff equals zero. However, the buyer's payoff is strictly higher compared to that in the baseline model.

$$\pi = (y_0 - x_0)(1-q) + q \int_{\underline{x}}^{x_0} [y_0 - x - \beta c(x)] dF(x) > y_0 - x_0$$

Buyers Costly Verify Sellers Type.

As a related extension, we can also examine how equilibrium outcomes change when a buyer can costly verify the opponent's production cost once they meet.

Assume before making the offer, a buyer can pay verification cost $A > 0$ to perfectly observe the seller's production cost. To make the extension non-trivial, assume that the cost A is not too large so that a buyer is willing to verify in some situations. Now the buyer's strategy also includes the probability of verification. Let us use $a \in [0, 1]$ to denote the probability of verification.

First of all, we can see that the net surplus from trade $y_0 - x - \beta U(x) - \beta\pi$ must be positive for any x on the support. Therefore, after paying the verification cost, a buyer will offer his/her opponent's reserve price $x + \beta U(x)$. This is exactly why a buyer might be willing to pay the verification cost: instead of paying a high price x_0 (or a lower price but with smaller trading probability), he/she can trade with a lower price. In addition, we can see that a must be strictly less than 1. Otherwise, none of seller entrants will invest, which makes it optimal to not to verify.

Following the same argument, the investment strategy and the pricing strategy without verification must be mixed strategies. In each period, a seller receives his/her reserve price with probability a and receives a random price offer with probability $1 - a$. The type of the most efficient sellers on the support $\underline{x}$ is determined by

$$c'(\underline{x}) = \frac{-(1-a)}{1-a\beta}$$

For a positive a, $\underline{x}$ is higher than the socially efficient production cost and it is increasing in a.

If a buyer decides not to verify, he/she must be indifferent between reserve prices. Like before, the indifference condition pins down the stationary cost distribution $F(x)$ with support $[\underline{x}, x_0]$, where $\underline{x}$ is to be determined.

If a buyer decides to verify, then his/her expected payoff is

$$\pi = y_0 - (\underline{x} + \beta c(\underline{x}))F(\underline{x}) - \int_{\underline{x}}^{x_0} (x + \beta c(x))dF(x) - A$$

Therefore, if the buyer is indifferent between verify and not verify, it must be true that

$$x_0 = (\underline{x} + \beta c(\underline{x}))F(\underline{x}) + \int_{\underline{x}}^{x_0} (x + \beta c(x))dF(x) + A \tag{20}$$

The right hand side of (20) strictly increases in $\underline{x}$ and is strictly larger than the left-hand side when $\underline{x} = x_0$. Therefore, the condition either 1) uniquely determines $\underline{x}$ or 2) shows that the buyer strictly prefers not to verify if the equation has no solution. In the first case, the mixed strategy parameter a can be uniquely solved from $c'(\underline{x})$. In the second case, the equilibrium outcomes are the same as that in the baseline model.

After solving the model, we can see that most of equilibrium properties preserve after allowing for costly verification. For instance, the agents' ex-ante payoffs as well as the social welfare are the same as before. Moreover, we can show that for any cost A, buyers will choose not to verify with a large enough discount factor and hence we will go back to the baseline model. Intuitively, we know that the stationary cost distribution becomes less "uncertain" as search frictions vanish. Hence the value of information is smaller.

Proposition 14. *In the steady-state equilibrium with costly verification, the seller's ex-ante payoff is* 0*, the buyer's ex-ante payoff and the social welfare is* $y_0 - x_0$*.*

Moreover, given any $A > 0$*, there exist a* $\hat{\beta}$*, such that for any* $\beta > \hat{\beta}$*, buyers choose not to verify and the equilibrium outcomes are the same as that in the baseline model.*

8 Conclusion

This paper examines the investment incentive and its welfare consequences in an infinite horizon random search and bargaining game with unobservable and selfish investments.

We demonstrated that in the unique steady state equilibrium, both the investment strategy and the price offer distribution are non-degenerate with convex supports. Unobservability generates rent for high investment and therefore incentivizes investment even if sellers have no bargaining power.

However, positive investments above the minimum level fail to generate any social welfare for any search friction. Trading inefficiency caused by unobservability erodes the welfare gain that could be created.

Moreover, we showed that if buyers have all the bargaining power, as meetings become more frequent, quite interestingly, the investment distributions of incumbents and entrants shift in the opposite directions: incumbent investment distribution converges to a point mass at no investment whereas an entrant's investment becomes efficient.

Appendices

A Proofs for the Baseline Model

Appendix A.A Proof of Lemma 1

1. $U(x)$ **Strictly Decreasing.** Since type x seller can always choose the reserve price of type $x + \epsilon$, for some $\epsilon > 0$, $U(x)$ must be strictly decreasing in x.

2. $r_S(x)$ **Strictly Increasing.** Multiply both sides of the seller's value function by β and add x, we get the following equation after rearranging,

$$\begin{aligned} &r_S(x) - \beta[E(\tilde{p} \mid \tilde{p} \geq r_S(x))(1 - H(r_S(x)) + Pr(\tilde{p} = r_S(x))) \\ &+ r_S(x)(H(r_S(x)) - Pr(\tilde{p} = r_S(x)))] = (1 - \beta)x \end{aligned} \tag{21}$$

   The left hand side strictly increases in $r_S(x)$ while the right hand side strictly increases in x. Therefore, $r_S(x)$ must be strictly increasing in x.

3. $U(\bar{x}) = 0$. Since $r(\bar{x})$ is the highest price that a buyer is willing to offer, equation (1) for $x = \bar{x}$ becomes $U(\bar{x}) = \beta U(\bar{x})$. Therefore $U(\bar{x}) = 0$.

4. **Continuity.** No buyer will offer a price that is higher than $r_S(\bar{x})$. Therefore, for any type $x \geq > x$, $H(r_S(x)) = 1$ and $U(x) = 0$.

   We know from the previous step that $U(\bar{x}) = 0$.

   For $x < \bar{x}$, since $U(x)$ decreases in x, $U(x)$ can only have downward jump. Suppose $U(x)$ jumps down at point x. Because $r_S(x) = x + \beta U(x)$, $r_S(x)$ must also jump downwards. However, this contradicts $r_S(x)$ being strictly increasing.

   Therefore, for any x, $U(x)$ is continuous. $r_S(x)$ is also continuous as a result.

Appendix A.B Proof of Proposition 1

1. Supports of $F(x)$, $F_e(x)$ and $H(p)$.

The closeness is obtained from the assumption that sellers and buyers will trade in the case of indifference.

Before showing the convexity, it is worth noticing that a price offer p being on the support of $H(p)$ implies $\hat{x}(p)$ being on the support. Otherwise, the buyer offering price p is not optimizing because he can lower the price to $r_S(x')$ without affecting the probability of trade, where x' is the highest seller type that is smaller than x and on the support.

Now suppose that there exist p_1, p_2 on the support of $H(p)$, such that any $p \in (p_1, p_2)$ is not on the support. Since p_1 and p_2 are on the support, there exist worker type x_1 and x_2 on the support such that $r_S(x_i) = p_i$, $i = 1, 2$. For any $x \in (x_2, x_1)$, $U'(x)$ is a constant since

$$U'(x) = \frac{-1 + H(p_2)}{1 - \beta H(p_2)}$$

On the other hand, $c'(x)$ strictly increases in x. Together with the continuity of $U(x)$, it is impossible to have $U(x_1) - c(x_1) = U(x_2) - c(x_2) \geq U(x) - c(x)$, for any $x \in (x_2, x_1)$. Therefore, the support of $H(p)$ is convex. By the continuity of $r_S(x)$, the support of $F(x)$ and $F_e(x)$ is also convex.

The lowest price offer in equilibrium is never lower than the reserve price of the most efficient sellers, i.e., $H(r_S(\underline{x})) = 0$. Plugging it into the envelope condition, $U'(\underline{x}) = c'(\underline{x}) = -1$. This implies that $\underline{x} = x^*$.

Hence, the support of $F(x)$ and $F_e(x)$ is $[x^*, x_0]$ and the support of $H(p)$ is $[r_S(x^*), r_S(x_0)]$.

2. $H(p)$ has no point mass.

   We first show $Pr(\tilde{p} = r_S(x_0)) = 0$. Suppose $Pr(\tilde{p} = r_S(x_0)) = q > 0$, then $U'(x_0-) = \frac{-q}{1-\beta+\beta q} < c'(x_0)$. This is a contradiction.

   Next, we solve $H(p)$. Notice that $U(x)$ is differentiable for any x on the support, because the support is convex, $c(x)$ is differentiable and $U(x) - c(x) = 0$. This also implies that $r_S(x)$ and $\hat{x}(p)$ are differentiable for any x and p on the support. Hence,

$H(p)$ can be solved from the equilibrium condition that $U'(x) = c'(x)$,

$$H(r_S(x)) = \frac{1 + c'(x)}{1 + \beta c'(x)} \Rightarrow H(p) = \frac{1 + c'(\hat{x}(p))}{1 + \beta c'(\hat{x}(p))} \tag{22}$$

It is straightforward to verify that $H(p)$ has no atom.

3. Point mass at x^* and no other point mass.

   We already know that $r_S(x^*)$ is on the support of $H(p)$. For a buyer who offers this price, he would get zero payoff if $F(x^*) = 0$. In this case, the buyer would find it profitable to deviate to $r_S(x_0) = x_0$ to get strictly positive payoff.

   Suppose that there is a point mass at $x \in (x^*, x_0]$, then there exist ϵ, such that buyers would find it profitable not to offer price $p \in (r_S(x - \epsilon), r_S(x))$. This contradicts the convexity property of the support.

Appendix A.C Proof of Proposition 3

To prove $F(x)$ has first order stochastic dominance over $F_e(x)$, we need to show $F_e(x) - F(x) \geq 0$ for all x and the inequality is strict for some x.

$$\begin{aligned} F_e(x) - F(x) &= \frac{F(x) - \int_{x^*}^{x} H(r_S(\tilde{x}))dF(\tilde{x})}{1 - \int_{x^*}^{x_0} H(r_S(\tilde{x}))dF(\tilde{x})} - F(x) \\ &= F(x)\frac{\int_{x^*}^{x_0} H(r_S(\tilde{x}))dF(\tilde{x}) - \int_{x^*}^{x} H(r_S(\tilde{x}))d\frac{F(\tilde{x})}{F(x)}}{1 - \int_{x^*}^{x_0} H(r_S(\tilde{x}))dF(\tilde{x})} \end{aligned}$$

Therefore, $F_e(x) - F(x) \geq 0$ for any $x \in [x^*, x_0]$ and the inequality is strict except for $x = x_0$.

Appendix A.D Proof of Theorem 2

1. We have shown the first two parts of the theorem.

2. By equation (6), the proportion of type x^* entrants equals,

$$\begin{aligned}F_e(x^*) &= \frac{F(x^*)}{1-\int_{x^*}^{x_0} H(r_S(x))f(x)dx} \\ &= [1+\frac{1-F(x^*)}{F(x^*)}\int_{x^*}^{x_0}(1-H(r_S(x)))\frac{f(x)}{1-F(x^*)}dx]^{-1} \\ &= [1+A]^{-1}\end{aligned}$$

By Median Value Theorem, there exist a $\tilde{x} \in (x^*, x_0)$ such that,

$$\begin{aligned}A =&\frac{1-F(x^*)}{F(x^*)}\int_{x^*}^{x_0}(1-H(r_S(x)))\frac{f(x)}{1-F(x^*)}dx \\ =&\frac{1-F(x^*)}{F(x^*)}[1-H(r_S(\tilde{x}))] \\ =&\frac{-c'(\tilde{x})(x_0-x^*-\beta c(x^*))}{(y_0-x_0)(1+\beta c'(\tilde{x}))}\end{aligned}$$

Take derivative with respect to β,

$$\frac{\partial A}{\partial \beta} = \frac{-c''(\tilde{x})\frac{\partial \tilde{x}}{\partial \beta}(x_0-x^*-\beta c(x^*)) + c'(\tilde{x})[c(x^*)+c'(\tilde{x})(x_0-x^*)]}{(y_0-x_0)(1+\beta c'(\tilde{x}))^2}$$

Therefore, the sufficient conditions for the derivative to be negative are

$$\frac{\partial \tilde{x}}{\partial \beta} > 0 \text{ and } c(x^*) + c'(\tilde{x})(x_0 - x^*) > 0$$

We first show that the first condition is satisfied. $\tilde{x}$ is defined by,

$$\int_{x^*}^{x_0} \frac{1-H(r_S(x))}{1-\beta} d\frac{F(x)}{1-F(x^*)} = \frac{-c'(\tilde{x})}{1+\beta c'(\tilde{x})} \tag{23}$$

From the previous analysis, as β increases, $\frac{1-H(r_S(x))}{1-\beta}$ strictly decreases for any $x \in (x^*, x_0)$. In addition, the mass of the conditional distribution shifts to higher x, where $\frac{1-H(r_S(x))}{1-\beta}$ is lower. As a result, the left hand side of (23) strictly decreases in β. On the other hand,

$$\frac{\partial \frac{-c'(\tilde{x})}{1+\beta c'(\tilde{x})}}{\partial \beta} = \frac{-c''(\tilde{x})\frac{\partial \tilde{x}}{\partial \beta} + [c'(\tilde{x})]^2}{(1+\beta c'(\tilde{x}))^2}$$

Since the derivative has to be negative so that (23) holds, $\frac{\partial \tilde{x}}{\partial \beta} > 0$ must be satisfied.

This in turn implies we only need to check the second sufficient condition for $\beta = 0$, because $c'(\tilde{x})$ is the smallest when $\beta = 0$. It is straight forward to check that the density of the stationary type distribution strictly increases in x, since,

$$f(x) = \frac{(y_0 - x_0 - \beta\pi)(1 + \beta c'(x))}{[y_0 - \beta\pi - x - \beta c(x)]^2}$$

When $\beta = 0$, $1 - H(r_S(x)) = -c'(x)$. Therefore, the condition for $\tilde{x}$ becomes,

$$\begin{aligned} -c'(\tilde{x}) &= \int_{x^*}^{x_0} -c'(x)\frac{f(x)}{1 - F(x^*)}dx < \int_{x^*}^{x_0} -c'(x)\frac{1}{x_0 - x^*}dx = \frac{c(x^*)}{x_0 - x^*} \\ &\Rightarrow c(x^*) + c'(\tilde{x})(x_0 - x^*) > 0 \end{aligned}$$

We have proved that those two sufficient conditions both hold and thus

$$\frac{\partial F_e(x^*)}{\partial \beta} > 0$$

Next, we will prove that in the limit $F_e(x^*) \to 1$. We know that $\tilde{x}$ is defined by,

$$\int_{x^*}^{x_0} (1 - H(r_S(x)))f(x)dx = [1 - H(r_S(\tilde{x}))][1 - F(x^*)] \tag{24}$$

For any $\epsilon \in (0, x_0 - x^*)$, we can rewrite the left hand side of (24) as

$$\begin{aligned} &\int_{x^*}^{x_0-\epsilon} (1 - H(r_S(x)))f(x)dx + \int_{x_0-\epsilon}^{x_0} (1 - H(r_S(x)))f(x)dx \\ =&[1 - H(r_S(x_1))][F(x_0 - \epsilon) - F(x^*)] + [1 - H(r_S(x_2))][1 - F(x_0 - \epsilon)] \end{aligned} \tag{25}$$

where $x_1 \in (x^*, x_0 - \epsilon)$ and $x_2 \in (x_0 - \epsilon, x_0)$.

Combining (24) and (25) we have the following equation,

$$\begin{aligned} [H(r_S(\tilde{x})) - H(r_S(x_2))][1 - F(x^*)] &= [H(r_S(x_1)) - H(r_S(x_2))][F(x_0 - \epsilon) - F(x^*)] \\ \Rightarrow \frac{c'(\tilde{x}) - c'(x_2)}{1 + \beta c'(\tilde{x})}[1 - F(x^*)] &= \frac{c'(x_1) - c'(x_2)}{1 + \beta c'(x_1)}[F(x_0 - \epsilon) - F(x^*)] \end{aligned} \tag{26}$$

Here $F(x_0 - \epsilon) - F(x^*)$ can be rewritten as,

$$(1 - \beta)(y_0 - x_0)\frac{-x^* - \beta c(x^*) + (x_0 - \epsilon) + \beta c(x_0 - \epsilon)}{(y_0 - \beta\pi - x_0 + \epsilon - \beta c(x_0 - \epsilon))(y_0 - \beta\pi - x^* - \beta c(x^*))}$$

We can verify that for any ϵ and ϕ, there exist an $\eta > 0$, such that when $1 - \beta < \eta$, $F(x_0 - \epsilon) - F(x^*) < \phi$. Therefore, the right hand side of (26) converges to 0 in the limit. This implies $\tilde{x}$ converges to x_2 in the limit. Since $x_2 \in (x_0 - \epsilon, x_0)$, for small enough ϵ, $\tilde{x}$ converges to x_0.

Appendix A.E Proof of Proposition 4

Based on the construction, the following relationship between the measures of exits of two consecutive periods must hold: for any $t \geq 1$,

$$e_{t+1} = e_t + e_1(1 - e_t)$$

First of all, we can verify that $e_t < 1$ for any t. e_1 is strictly less than 1 by construction. For any $t > 1$, the above conditions shows that e_t is a convex combination of 1 and e_1 and hence is also strictly less than 1.

This then implies that the sequence $\{e_t\}$ is an increasing sequence, since $e_{t+1} - e_t = e_1(1 - e_t) > 0$.

Hence, there must exist a limit of the bounded sequence. We can solve the limit e_∞ from

$$e_\infty = e_\infty + e_1(1 - e_\infty) \Rightarrow e_\infty = 1.$$

The rest of the proposition is already proved in the main body of the paper.

B Proofs for the Two-Sided Investment Case

Appendix B.A Proof of Proposition 5

1. **$p(y)$ increasing**

   If the support of $G(y)$ is degenerate, then we have nothing to prove.

   Otherwise, consider any y_1 and y_2 on the support with $y_1 > y_2$. Denote $p_1 = p(y_1)$ and $p_2 = p(y_2)$. Since p_1 (p_2) solves the maximization problem of type y_1 (y_2), the following inequality must hold,

$$(y_1 - p_1)F(\hat{x}(p_1)) + [1 - F(\hat{x}(p_1))]\beta\Pi(y_1) \geq (y_1 - p_2)F(\hat{x}(p_2)) + [1 - F(\hat{x}(p_2))]\beta\Pi(y_1)$$
$$(y_2 - p_2)F(\hat{x}(p_2)) + [1 - F(\hat{x}(p_2))]\beta\Pi(y_2) \geq (y_2 - p_1)F(\hat{x}(p_1)) + [1 - F(\hat{x}(p_1))]\beta\Pi(y_2)$$

Adding two equations, we have

$$(y_1 - y_2)[F(\hat{x}(p_1)) - F(\hat{x}(p_2))] \geq [F(\hat{x}(p_1)) - F(\hat{x}(p_2))][\beta\Pi(y_1) - \beta\Pi(y_2)]$$

Since $y_1 - y_2 > \beta[\Pi(y_1) - \Pi(y_2)]$, the above inequality implies $F(\hat{x}(p_1)) \geq F(\hat{x}(p_2))$. This proves $p(y)$ increases in y.

2. $P(y)$ **single-valued**

   We can apply the same proof to show that the support of $F(x)$ is $[x^*, x_0]$. Therefore, $F(x)$ is a strictly increasing function of x.

   Suppose there exist y that is on the support of $G_e(y)$, such that $p_1 > p_2$ are both optimal price offers of a type y buyer. His search stage payoff can be rewritten as,

$$\Pi_i(y) = (y - p_i)F(\hat{x}(p_i)) + [1 - F(\hat{x}(p_i))]\beta\Pi_i(y),\ i = 1, 2$$

   It is easy to verify that $\Pi_1'(y) > \Pi_2'(y)$. This contradicts the equilibrium constraint $\Pi(y) - c(y) = \pi$.

Appendix B.B Proof of Proposition 6

1. We can apply the same proof to show that the support of $F_e(x)$ and $F(x)$ is $[x^*, x_0]$, and that x^* is the unique point mass.

2. The compactness of the support of $G_e(y)$ and $G(y)$ comes from the fact that y^* is finite and that all agents choose to trade when indifferent.

   To see convexity, suppose there exist y_1, y_2 on the support and any $y \in (y_1, y_2)$ is not. Then $p(y_1) < p(y_2)$, otherwise $\Pi'(y)$ will be constant in the interval and the indifference condition cannot be satisfied. However, combined with the monotonicity of $p(y)$, this implies any $p \in (p(y_1), p(y_2))$ is not on the support of $H(p)$. We have a contradiction.

Because the support of $G(y)$ is convex, we can use the envelope condition and indifference condition to determine $\bar{y}$ and $\underline{y}$. That is,

$$e'(\bar{y}) = \Pi'(\bar{y}) = 1 \text{ and } e'(\underline{y}) = \Pi'(\underline{y}) = \frac{F(x^*)}{1-\beta(1-F(x^*))}$$

Therefore $\bar{y} = y^*$ and $\pi = y^* - x_0 - e(y^*)$. Using indifference condition, $\underline{y}$ is pinned down by

$$y^* - x_0 - e(y^*) = (\underline{y} - x^* - \beta c(x^*))e'(\underline{y}) - e(\underline{y})$$

It is straightforward to verify that $\underline{y} < y^*$. The right hand side strictly increases in $\underline{y}$ and equals zero when $\underline{y} = y_0$. In addition, it is strictly larger than the left hand side when $\underline{y} = y^*$. Therefore, $\underline{y}$ is uniquely determined by equation (12) and $\underline{y} < y^*$.

Finally, no point mass result comes from the fact that there is no point mass on the support of $H(p)$.

Appendix B.C Proof of Proposition 7

Combining the envelope condition of $\Pi(y)$ and the indifference condition that $\Pi'(y) = e'(y)$ for any y on the support,

$$F(\hat{x}(p(y))) = \frac{(1-\beta)e'(y)}{1-\beta e'(y)} \text{ for any } y \text{ on the support}$$

We can use the equilibrium condition $\Pi(y) - e(y) = \pi$ to solve $p(y)$.

$$(y - p(y))e'(y) - e(y) = y^* - x_0 - e(y^*)$$
$$\Rightarrow p(y) = y - \frac{e(y) + y^* - x_0 - e(y^*)}{e'(y)}$$

We can verify that $p(y)$ is continuous and strictly increases in y. Therefore, the inverse function $y(p)$ exists. We can then define $F(x)$ and $G(y)$.

From the previous discussion, we see that $F(x)$, $F_e(x)$, $G(y)$ and $G_e(y)$ defined above are the only distributions that satisfies equilibrium restrictions. Hence, the steady state equilibrium is unique.

Appendix B.D Proof of Proposition 8

1. Take derivative of equation (12) with respect to β, we get

$$[\underline{y} - x^* - \beta c(x^*)]e''(\underline{y})\frac{\partial \underline{y}}{\partial \beta} = c(x^*)e'(\underline{y})$$

   Therefore, $\frac{\partial \underline{y}}{\partial \beta}$ is strictly positive.

   Denote the $t * 100$th percentile of $G(y)$ with β as $y_{t,\beta}$, i.e.,

$$G(y_{t,\beta}) = \frac{1 + c'(\hat{x}(p(y_{t,\beta})))}{1 + \beta c'(\hat{x}(p(y_{t,\beta})))} = t$$
$$\Rightarrow 1 - t = -(1 - t\beta)c'(\hat{x}(p(y_{t,\beta})))$$

   Therefore, when β increases, $\hat{x}(p(y_{t,\beta}))$ strictly decreases.

2. $\lim_{\beta \to 1} \underline{y} < y^*$ can be shown by plugging in $\beta = 1$ and $\underline{y} = y^*$ into equation (12). It is easy to check that the left hand side is strictly smaller the right hand side.

   Consider any $y > \underline{y}$. Since $\hat{x}(p(y)) > x^*$, $1 + c'(\hat{x}(p(y))) > 0$. Then it is straightforward to verify that as $\beta \to 1$, $G(y) \to 1$ for any $y > \underline{y}$.

   For any $y > \underline{y}$, there exist $\check{y}$ and $\check{\check{y}}$, such that

$$G_e(y) = \frac{\int_{\underline{y}}^{y} F(\hat{x}(p(\tilde{y})))dG(\tilde{y})}{\int_{\underline{y}}^{y^*} F(\hat{x}(p(\tilde{y})))dG(\tilde{y})} = \frac{F(\hat{x}(p(\check{y})))G(y)}{F(\hat{x}(p(\check{\check{y}})))}$$
$$= \frac{e'(\check{y})(1 - \beta e'(\check{\check{y}}))}{e'(\check{\check{y}})(1 - \beta e'(\check{y}))}G(y)$$

   When $\beta \to 1$, both $\check{y}$ and $\check{\check{y}}$ approaches $\underline{y}$ following the same argument as in the proof of proposition 2, and $G(y)$ approaches 1 for any $y > \underline{y}$. Therefore, $G_e(y) \to 1$ for any $y > \underline{y}$.

C Proofs for the Two-Sided Offer Case

Appendix C.A Proof of Proposition 9

Suppose there are more than one optimal investment levels in a stationary equilibrium. From the condition (17) we can see that there can be at most one x with non-negative net surplus.

Next, consider the scenario where there exist some $\hat{x}$ on the support that generates negative net surplus. Because the net surplus is negative, these sellers never trade. As a result, $v = -c(\hat{x})$. So the only possible $\hat{x}$ is x_0. Because these sellers never leave the market, in a stationary equilibrium the only entrant's type is the invested type.

Appendix C.B Proof of Proposition 10

Equilibrium 1.

Assume $\alpha(y_0 - \bar{x}) \geq c(\bar{x})$.

All seller entrants invest to reduce their production cost to $\bar{x}$. Agents' ex ante payoffs can be solved: $v = \alpha(y_0 - \bar{x}) - c(\bar{x})$ and $\pi = (1-\alpha)(y_0 - \bar{x})$. By the assumption, both are positive.

Equilibrium 2.

Assume $\alpha(y_0 - \bar{x}) \leq c(\bar{x})$.

If a seller invests, he will invest to reduce his production cost to $\bar{x}$. Given π, his ex ante payoff $v = \frac{\alpha}{1+\alpha\beta-\beta}(y_0 - \bar{x} - \beta\pi) - c(\bar{x})$. To make sure that sellers are indifferent between $\bar{x}$ and x_0, v must equal 0. Therefore,

$$\pi = \frac{\alpha(y_0 - \bar{x}) - (1 + \alpha\beta - \beta)c(\bar{x})}{\alpha\beta}$$

If $F(\bar{x}) = q$, we can also solve π as follows,

$$\pi = \alpha\beta\pi + (1-\alpha)[q(y_0 - \bar{x} - \beta c(\bar{x})) + (1-q)\beta\pi]$$

Equating two π, q can be solved,

$$q = \frac{(1-\beta)\pi}{(1-\alpha)(y_0 - \bar{x} - \beta c(\bar{x}) - \beta\pi)}$$

q is always positive. The condition that q is less than 1 is equivalent to $\alpha(y_0 - \bar{x}) \leq c(\bar{x})$.

The last equilibrium condition we need to verify is that $y_0 - x_0 - \beta\pi \leq 0$. After plugging in π, this condition is equivalent to $c(\bar{x}) \leq \frac{\alpha}{1+\alpha\beta-\beta}(x_0 - \bar{x})$. This condition is satisfied since $c'(\bar{x}) = \frac{-\alpha}{1+\alpha\beta-\beta}$ and $c(x)$ is strictly convex.

Appendix C.C Proof of Proposition 11

As $\beta \to 1$, the right hand side of equation (17) converges to -1, which implies $\bar{x} \to x^*$. Since $F_e(x)$ is a point mass at $\bar{x}$ in both forms of equilibrium, the limiting distribution converges to a point mass at x^*.

In equilibrium 1, the social welfare equals $y_0 - \bar{x} - c(\bar{x})$. Hence the social welfare converges to the first best. In equilibrium 2, the social welfare equals π, which also converges to $y_0 - x^* - c(x^*)$, given that α is bounded above zero.

D Proofs for Robustness and Other Extensions

Appendix D.A Proof of Proposition 13

Denote the probability of death shock by $\delta = e^{-r_2 t}$. Then

$$\begin{aligned} F_e(x^*) =& \frac{F(x^*)}{1 - \delta \int_{x^*}^{x_0} H(r_S(x)) f(x) dx} \\ =& [1 + \frac{1 - F(x^*)}{F(x^*)} \int_{x^*}^{x_0} [1 - \delta H(r_S(x))] \frac{f(x)}{1 - F(x^*)} dx]^{-1} \\ =& [1 + A]^{-1} \end{aligned}$$

Plug in $F(x^*)$ and $H(r_S(x))$, and use the Median Value Theorem, there exists $\tilde{x} \in (x^*, x_0)$ such that

$$A = \frac{x_0 - x^* - \hat{\beta} c(x^*)}{(1 - \hat{\beta})(y_0 - x_0)} \frac{1 - \delta + (\hat{\beta} - \delta) c'(\tilde{x})}{1 + \hat{\beta} c'(\tilde{x})}$$

Here $\hat{\beta} = \beta\delta$. Take $t \to 0$,

$$\lim_{t \to 0} A = \frac{x_0 - x^* - c(x^*)}{(y_0 - x_0)} \lim_{t \to 0} \frac{1 - e^{-r_2 t} - (1 - e^{-r_1 t}) e^{-r_2 t} c'(\tilde{x}_t)}{[1 + e^{-(r_2 + r_1) t} c'(\tilde{x}_t)][1 - e^{-(r_2 + r_1) t}]}$$

By the same argument in the proof of Theorem 2, the limit of $\tilde{x}_t$ is x_0. Using the L'Hopital's rule and the fact that $\frac{\partial \tilde{x}_t}{\partial t}$ is finite,

$$\lim_{t \to 0} A = \frac{x_0 - x^* - c(x^*)}{y_0 - x_0} \frac{r_2}{r_1 + r_2}$$

Appendix D.B Proof of Proposition 14

x_0 is on the support, hence the first part of the proposition holds.

As $\beta \to 1$, $F(x) \to 0$ for any $x < x_0$. Therefore, the right-hand side of (20) converges to $x_0 + A$, which is strictly larger than the left-hand side x_0. That is, a buyer will not verify if β is close enough to 1.

Moreover, the right-hand side shifts up with a larger β. It follows from 1) $F(x)$ increases in β, which is equivalent to putting higher weights on larger reserve prices and 2) reserve prices $x + \beta c(x)$ increases in β.

Combining the above two facts, there exists a $\hat{\beta}$ for any given A, such that buyers will choose not to verify when $\beta > \hat{\beta}$.

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