Competing Mechanism Design with Frictions

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Abstract

This paper considers multiple principals' competing mechanism design problems in directed search markets with frictions. It proposes the notion of a robust DIC-response equilibrium where a principal responds to a competing principal's deviation with a DIC (dominant-strategy incentive compatible) direct mechanism and following his deviation to any arbitrary mechanism a principal cannot gain in every continuation equilibrium. In a robust DIC-response equilibrium, a principal only needs to know the identity of the deviating principal but not the whole market information that agents have in the market. The robustness can be checked with a principal's deviation to only BIC (Bayesian incentive compatible) direct mechanism. This paper provides a sharp characterization of a robust DIC-response equilibrium allocations where the greatest lower bound of a principal's payoff is expressed in terms of incentive compatible direct mechanisms. It also discusses the implications of how those results can be applied to directed search models including the BIC-DIC equivalence.

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Keywords: competing mechanism design, directed search, robust equilibrium, dominant strategy incentive compatibility, Bayesian incentive compatibility.

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1 Introduction

Directed search models generate rich applications with frictions.[1] Game-theoretic analysis with multiple sellers (principals) and multiple buyers (agents) provide micro-foundations for directed search models. For example, buyers may choose sellers with equal probability if their trading mechanisms are the same. This mixed selection strategy reflects the lack of coordination among buyers and it induces frictions with a non-degenerate distribution of the number of participating buyers for each seller. Directed search models based on game-theoretic analysis study various problems ranging from selling private goods or hiring workers through posted prices (Burdett, et al. 2001, Peters 1991, 1997), direct mechanisms under complete information (Coles and Eekhout 2003, Geromichalos 2012) to auctions with incomplete information on bidders' valuations (Burguet and Sákovics 1999, McAfee 1993, Peters and Severinov 1997, Virag 2010).[2]

In a decentralized market, agents typically have information not only on their payoff types, but also what is happening in the market because they actively search for better deals in the market. The latter is called market information, such as competing sellers' trading mechanisms, their terms of trade, etc. Therefore, unlike the Revelation Principle for a single principal, when principals compete in trading mechanisms, the message space (i.e., payoff type space) in a direct mechanism may not be sufficiently large to incorporate the agent's information. This creates two difficult questions that are often ignored in applications. First, can we check whether an equilibrium in a market with a certain set of restrictive trading mechanisms can survive if a principal deviates to a mechanism that is not available in the market? This is a question about the robustness of an equilibrium. Second, what kind of equilibrium outcomes can be sustained in robust equilibrium? This paper studies those problems in game-theoretical directed search models.

In game-theoretic directed search models, Epstein and Peters (1999) study an equilibrium that is robust to the possibility of a principal's deviation to any arbitrary mechanism, not just a mechanism that is allowed in the market. They showed that any robust equilibrium can be reproduced as the payoff-equivalent equilibrium in which no principal can gain in any continuation equilibrium upon his deviation to any arbitrary mechanism given agents' truthful reporting to non-deviating principals who offer universal mechanisms. A universal

[1] There is a fairly large volume of literature on directed search in macro and labor. Among many of them are Acemoglu and Shimer (1999), Burdett and Mortensen (1998), Menzio and Shi (2011), Moen, (1997), Montgomery (1991), Rocheteau and Wright (2005), Shi (2002, 2009), Shimer (2005).

[2] Directed search models also fit public goods with use exclusions (Norman 2004, Hellwig, 2005, Fang and Norman 2010), public goods with congestions (Birulin 2006), tax competition (Black and Hoyt 1989, Burbidge, et al. 2006, Wilson 1999), screening (Dam and Perez-Castillo 2006, Inderst 2001) among others.

mechanism is equipped with the universal language that allows agents to describe any mechanism that a competing principal might deviate to. However, this language itself is quite complex and it does not provide a characterization of robust equilibrium allocations.

This paper relaxes their assumption that an agent can communicate only with the principal who she eventually selects for trading; it is, in fact, often observed that buyers shop around for better deals and they may communicate with many different sellers. We assume that there are three or more agents and that each agent can communicate with every principal even though an agent eventually selects only one principal for trading. Because an agent selects only one principal, market frictions are still induced in equilibrium with agents' mixed selection strategies.

Consider a market with a certain set of mechanisms. In any continuation equilibrium, agents' communication with and their selection decision on principal ℓ, induces a direct mechanism from the mechanism that principal ℓ offers. Importantly, our paper proposes a notion of "*DIC-response equilibrium*" that is tractable for checking its robustness and characterizing equilibrium allocation. In a DIC-response continuation equilibrium following principal j's deviation to any arbitrary mechanism, a dominant strategy incentive compatible (DIC) direct mechanism, say μ_ℓ^j, is always induced from non-deviating principal ℓ's equilibrium mechanism.

We show that any DIC-response equilibrium in a market can be understood as DIC-response equilibrium where a principal offers a deviator-reporting direct mechanism. In principal ℓ's deviator-reporting direct mechanism, an agent is asked to report her payoff type conditional on selecting principal ℓ and the identity of the deviating principal, if any, regardless of selecting principal ℓ. If the majority of agents report j as the identity of the deviating principal, principal ℓ always assigns a DIC direct mechanism μ_ℓ^j. If the majority of agents report no deviation by any principal, principal ℓ assigns a Bayesian incentive compatible (BIC) direct mechanism $\bar{\mu}_\ell$ such that $\bar{\mu} = (\bar{\mu}_1, \ldots, \bar{\mu}_J)$ is BIC given a profile of agents' selection strategies that constitute a (truth-telling) continuation equilibrium.[3]

The beauty of a DIC-response equilibrium is that it is always optimal for an agent to report his payoff type to non-deviating principals on and off the equilibrium path because incentive compatibility is ensured by the BIC property of $\bar{\mu}_\ell$ on the path and by the DIC

[3]The implementation of a deviator-reporting direct mechanisms can be tractable. If principals are all identical, agents only need to reveal whether or not there exists a deviating principal. For example, a seller's website in on-line markets, or equivalently a computer program that implements it, may be viewed as a mechanism. If a seller's website can ensure that a series of buyers' clicking behavior or their viewing history can reveal whether or not there exists a better alternative for them, then he can gather market information on whether or not there is a deviating seller. A seller's website may update his price when there is enough evidence on a competing seller's deviation.

property of μ^j_ℓ off the path following principal j's deviation. The DIC property of μ^j_ℓ is particularly useful for identifying the payoff level that principal j can achieve upon his deviation to any arbitrary mechanism. Because every principal ℓ ($\ell \neq j$) fixes his mechanism to a DIC direct mechanism μ^j_ℓ, any continuation equilibrium upon principal j's deviation to an arbitrary mechanism can be preserved by a payoff-equivalent continuation equilibrium upon his deviation to some direct mechanism that is BIC conditional on μ^j_{-j}, a profile of DIC direct mechanisms that non-deviating principals will assign upon principal j's deviation. Let $\mathcal{M}^B_j(\mu^j_{-j})$ be the set of all possible BIC direct mechanisms available for principal j conditional on μ^j_{-j}. In Proposition 2, we show that a DIC-response equilibrium is then *robust* if and only if each principal j cannot gain in every continuation equilibrium upon his deviation to any BIC direct mechanism in $\mathcal{M}^B_j(\mu^j_{-j})$, given agents' truthful reporting to non-deviating principals. This means that we do not need to worry about principal j's deviation to any other mechanism to check the robustness of a DIC-response equilibrium.

This paper also gives us the tight characterization of the set of all possible robust DIC-response equilibrium allocations where principals employ pure strategies for their mechanism offer. Let $\pi(\bar{\mu})$ be an agent's selection strategy that specifies the probability of selecting a principal in continuation equilibrium given a profile of BIC direct mechanisms. Theorem 1 establishes that any BIC allocation $(\bar{\mu}, \pi(\bar{\mu}))$ can be supported by a robust DIC-response equilibrium if each principal j's payoff associated with it is greater than or equal to his min-max-max payoff (to be precise, inf-sup-sup) that is derived by, going from the most inner max operator: (i) maximization over the set of all possible agents' selection strategies given a profile of direct mechanisms, (ii) maximization over the set of all possible $\mathcal{M}^B_j(\mu^j_{-j})$ conditional on μ^j_{-j}, and (iii) minimization over the set of all possible DIC direct mechanisms $\mu^j_{-j} \in (\Omega^D)^{J-1}$, where Ω^D is the set of all possible direct mechanism for each non-deviating principal and $J-1$ is the number of non-deviating principals.

The result on how to check the robustness of a DIC-response equilibrium (Proposition 2) and the characterization of all robust DIC-response equilibrium allocations (Theorem 1) are derived in the general environment without restricting the nature of principals' allocation decisions or the payoff functional forms. However, it is frequently observed that the applications of the directed search model are studied in the environment with linear payoff functions, with monetary transfers and independent one-dimensional private-value types.[4] In this environment, Gerhskove, et al. (2013) established the BIC-DIC equivalence for a single principal case in the sense that for any BIC mechanism there exists an equivalent DIC

[4]Indeed, all the directed search papers we mentioned earlier belong to this payoff environment (Burdett, et al. 2001, Burguet and Sákovics 1999, Coles and Eekhout 2003, Geromichalos 2012, McAfee 1993, Peters 1991, 1997, Peters and Severinov 1997, Virag 2010).

mechanism that delivers the same interim expected payoffs for all agents and the same ex ante expected social surplus.[5] We show that the BIC-DIC equivalence in Gershkove et al. (2013) is also established in the case with multiple principals, even though the distributions of agents' participation decisions are endogenously determined. Given the BIC-DIC equivalence, instead of $\mathcal{M}_j^B(\mu_{-j}^j)$ conditional on μ_{-j}^j, the DIC direct mechanisms that non-deviating principals will assign upon principal j's deviation, we can focus on only Ω^D, the set of all possible DIC direct mechanisms, for deviating principal j, whether we check the robustness of a DIC-response equilibrium or we characterize the greatest lower bound for principal j's payoff in characterizing the set of all robust DIC-response equilibrium allocations.

We also discuss the generality of our theory in terms of its implications. Our main interest is in a robust DIC-response equilibrium in which principals employ pure strategies for their mechanism offer. In the discussion section, we show that the results on how to check the robustness of an equilibrium and the greatest lower bound of a principal's equilibrium payoff goes through even for robust DIC-response equilibrium in which principals employ mixed strategies for their mechanism offer. We show how to characterize robust DIC-response equilibrium allocations with principals' mixed-strategies. We also note that a directed search model restricts principals to offer mechanisms in a subset of DIC mechanisms such as posted prices or second-price auctions. Offering a DIC direct mechanism is equivalent to offering a deviator-reporting direct mechanism that assigns the same DIC direct mechanism on and off the path. Applying Proposition 1 and its counter part with BIC-DIC equivalence, we can show that the robustness of an equilibrium in a market with a subset of DIC direct mechanisms can be checked whether there exists an agent's continuation equilibrium selection strategy that makes a principal gain upon his deviation to only a BIC direct mechanism or equivalently DIC direct mechanism with the BIC-DIC equivalence.

The DIC property is based on the fact that an agent's payoff type has only private value. We can extend our main results even when agent's payoff type has interdependent values. In this case, we can consider an ex-post incentive compatible (EPIC) direct mechanism for the mechanism that non-deviating principals assign in continuation equilibrium following a principal's deviation. As shown later in the paper, the EPIC property also does not depend on agents' selection strategy or the deviating principals' mechanism. Therefore, an EPIC-response equilibrium under interdependent values provides the same analytical tractability.

[5]Earlier literature on implementation focuses on equivalence between two mechanisms in terms of ex-post allocation. Mookerjee and Reichelstein (1992) show that such a condition for BIC-DIC equivalence is too restrictive. Manelli and Vincent's innovation (2010) was to establish BIC-DIC equivalence in terms of (interim) payoffs in the environments for standard single-unit private value auctions. Gershkov, et al. (2013) extends BIC-DIC equivalence in terms of (interim) payoffs in general social choice problems.

Furthermore, we can establish the result on how to check the robustness of a DIC-response equilibrium and the characterization of all robust DIC-response equilibrium allocations in the same way we did in Proposition 2 and Theorem 1.

1.1 Literature on competing mechanism

Epstein and Peters (1999) propose a solution to the problem associated with the message space for a direct mechanism in game-theoretic directed search models. They show how the universal language can allow agents to describe all the market information that they have. The language, however, is complex and it does not provide the characterization of robust equilibrium allocations.

Since then, most of the progress has been made in competing mechanisms without frictions, where all principals' allocation decisions may affect every agent's payoff so that an agent may trade with all principals.[6] In our paper, agents can communicate with all principals but they eventually select only one principal, as in Epstein and Peters (1999). The notion of robustness our paper adopts is consistent with the (strong) robustness in Epstein and Peters; any robust equilibrium can be reproduced as the payoff-equivalent equilibrium where no principal can gain in any continuation equilibrium following his deviation to any arbitrary mechanism, given agents' truthful report to non-deviating principals with deviator-reporting direct mechanisms. On the other hand, most of the literature on competing mechanisms without frictions is concerned about weak robustness.[7] Let us explain them.

First of all, in a common agency model with a single agent (Martimort and Stole 2002, Peters 2001), a robust equilibrium can be understood as an equilibrium where a principal simply offers a menu of alternatives. Therefore, competition in a market with menus can support robust equilibrium allocations. However, a menu theorem does not provide how to derive an equilibrium.[8] In common agency, Pavan and Calzolari (2010) show that it is useful to use extended direct mechanisms that ask the agent to report only the payoff-relevant market information such as actions taken by competing principals but not the whole market information when it comes to deriving a robust equilibrium.

When there is only one agent, it is not certain whether he tells the truth or not. However,

[6]In this case, we do not need to formulate an agent's mixed selection strategy, which causes frictions in the market and often makes equilibrium analysis quite difficult because a principal faces a non-degenerate probability distribution of the number of participating agents.

[7]An equilibrium in a competing mechanism game without frictions is weakly robust as long as there exists a continuation equilibrium, following a principal's deviation to any arbitrary mechanism, in which the deviating principal cannot gain.

[8]Han (2006) extends the menu theorem for the bilateral contracting environment between multiple principals and multiple agents.

if a principal deals with multiple agents, he can compare agents' reports on the market information. With three or more agents, Yamashita (2010) shows that reporting market information to a principal is equivalent to recommending to the principal a direct mechanism that he should implement. He proposes a recommendation mechanism that assigns the direct mechanism if agents all recommend it, together with the type reported. Then, he shows that any equilibrium in a market with arbitrary set of mechanisms can be supported when principals offer corresponding recommendation mechanisms. However, it may not be an easy task for an agent to recommend a direct mechanism because an agent may need to report the entire mapping of the direct mechanism, i.e. a mapping from the type spaces to the space of allocation decisions. But, more importantly, this approach also does not provide a tight characterization of equilibrium allocation. Yamashita did provide the greatest lower bound for each principal's payoff level, but we cannot specify exactly what it is because it was expressed in terms of the min-max-min value taken over the set of arbitrary mechanisms that are allowed in a market.

Peters and Troncoso-Valverde (2013) generalize Yamashita's idea without distinguishing between principals and agents because everyone can offer mechanisms and send messages (over two rounds). In the two rounds of communication procedure, all players observe anyone who deviates from the equilibrium path, and they can revise their "equilibrium contracts" to "punishing contracts" so as to punish the deviator. They show that any outcome function that is implementable in the sense of Myerson (1979) is also supportable as an equilibrium in this game. However, the communication procedure is complex and the characterization is offered through equilibrium without subgame perfection.[9]

Notably, our paper proposes the notion of a robust DIC-response equilibrium for game-theoretic directed search models, in which principals respond to a competing principal's deviation with DIC direct mechanisms. As explained earlier, it has very tractable equilibrium properties in a decentralized economy with frictions; agents only need to report the identity of the deviating principal, we only need to check a principal's deviation to any incentive compatible direct mechanisms for robustness, and we can provide a sharp characterization for all robust DIC-response direct mechanisms by using only BIC or DIC direct mechanisms in specifying the greatest lower bound for a principal's payoff.

[9]Formally, they adopt Bayesian Nash equilibrium. However, we adopt perfect Bayesian equilibrium as others do. All the papers mentioned in this section assume that principals cannot observe a competing principal's mechanism or even if they do, they cannot offer a mechanism that directly makes their contracts contingent on other principals' contracts. This is why agents' communication on the market information is important. The notable exception is Peters and Szentes (2010) who study the case where mechanisms can make the contract assignment directly contingent on the contract offered by other principals.

2 Model

There are J principals and I agents. Let $\mathcal{J} = \{1,\dots,J\}$ be the set of principals and $\mathcal{I} = \{1,\dots,I\}$ the set of agents. Assume that $J \geq 2$ and $I \geq 3$. Like most of directed search models, we consider ex-ante homogeneous agents.[10] Each agent's type is independently drawn from a probably distribution F with support $X = [\underline{x}, \bar{x}] \subset \mathbb{R}_+$.[11] An agent eventually selects a principal for the principal's allocation decision $a \in A$. Let $u(a, x)$ denote the agent's payoff when x is her type and a is the allocation decision of the principal who she selects. Let α be a (random) allocation decision and let $\mathcal{A} := \Delta(A)$ be the set of all possible allocation decisions.

If an agent does not choose principal j, he treats the agent's type as x°. Let $\bar{X} = X \cup \{x^\circ\}$. Then, $\mathbf{x} \in \bar{X}^I$ can conveniently characterize the type profile of the agents who select principal j. Let $u_j(a, \mathbf{x})$ denote principal j's payoff when $\mathbf{x} \in \bar{X}^I$ is the type profile of the agents who select him and a is his allocation decision.

As agents search for better deals in the market they can freely communicate with principals, even though they eventually have to select only one of them. Consider a market where $\bar{\Gamma}_j$ is the set of mechanisms available for each principal. Let $\bar{\Gamma} := \times_{\ell=1}^J \bar{\Gamma}_\ell$. Because agents are ex-ante homogeneous, we assume that mechanisms are *anonymous* and hence *non-discriminatory*. Let us formulate a mechanism. Let $\bar{M}$ be a message space that includes all possible messages each agent can send. Let $H = \{0, 1\}$ be the set of an agent's selection decisions. If an agent sends $h = 1$ to a principal together with any message in $\bar{M}$, it implies that the agent selects the principal; sending $h = 0$, with any message in $\bar{M}$, implies that an agent does not select him. A mechanism is denoted by $\gamma_j : \bar{M}^I \times H^I \to \mathcal{A}$ and specifies principal j's allocation decision as the function of all agents' messages and selection decisions on whether to select principal j.

A competing mechanism game in a market with $\bar{\Gamma}$ starts when each principal j simultaneously offers a mechanism from $\bar{\Gamma}_j$. After observing a profile of mechanisms $\gamma = (\gamma_1, \dots, \gamma_J)$, each agent simultaneously sends a message and her selection decision to every principal. According to each principal j's mechanism, messages and selection decisions determine his action alternative and monetary transfers. Finally, payoffs are realized.

[10] All the results can go through even when we assume ex-ante heterogenous agents.

[11] This is only necessary for the BIC-DIC equivalence. We impose it at the beginning for the notational simplicity. All the other results go through if we assume that the profile of agents' types follows a symmetric joint probability distribution and the type space X is more generally set up. Since we assume ex-ante homogenous agents, the symmetry of the joint probability distribution is needed.

3 Competition

3.1 Continuation equilibrium of subgames

We first formulate how agents communicate with and select principals at each possible subgame defined by the distribution of mechanisms $(\gamma_1, \ldots, \gamma_J) \in \bar{\Gamma}$ offered by principals. Let $c(\gamma_j, \gamma_{-j}) : X \to \Delta(\bar{M})$ denote the agent's strategy for communicating with principal j when principal j's mechanism is γ_j and (the distribution of) the other principals' mechanisms is γ_{-j}. Therefore, an agent of type x sends to principal j a message that is drawn from a probability distribution $c(\gamma_j, \gamma_{-j})(x) \in \Delta(\bar{M})$

A mapping $\pi(\gamma_j, \gamma_{-j}) : X \to [0, 1]$ is the agent's selection strategy that describes the probability with which the agent selects principal j when principal j's mechanism is γ_j and (the distribution of) the other principals' mechanisms is γ_{-j}. An agent of type x selects principal j with probability $\pi(\gamma_j, \gamma_{-j})(x) \in [0, 1]$

Note that the name of the principal does not matter when an agent communicates with a principal or selects a principal. This imposes the symmetry in the sense that when two principals offer the same mechanism, agents communicate with them in the same way and select one of them with equal probability. This is the key that creates market frictions. The incentive consistency of a continuation equilibrium formulated later includes this symmetry embedded in the agent's strategy.[12]

To formulate payoffs, it is convenient to utilize a direct mechanism. Consider principal j's direct mechanism. It specifies his allocation decision contingent on the types reported by agents who select the principal. Let us fix (the distribution of) mechanisms offered by principals except for principal j to γ_{-j}. From principal j's mechanism γ_j, a communication strategy $c(\gamma_j, \gamma_{-j})$ induces the principal's direct mechanism $\beta_j(\gamma_j, c(\gamma_j, \gamma_{-j}))$ as follows.

Let N denote the number of agents who select principal j (those agents with $h = 1$) Then, for every $N \leq I$ and every $(x_1, \ldots, x_N) \in X^N$, $\beta_j(\gamma_j, c(\gamma_j, \gamma_{-j}))$ is defined as

$$\beta_j(\gamma_j, c(\gamma_j, \gamma_{-j}))(x_1, \ldots, x_N, \mathbf{x}^{\circ}_{-N}) := \tag{1}$$
$$\int_{X^{I-N}} \left(\int_{\bar{M}} \cdots \int_{\bar{M}} \gamma_j(\mathbf{m}, \mathbf{h}) dc(\gamma)(x_1) \ldots dc(\gamma)(x_N) dc(\gamma)(s_{N+1}) \cdots dc(\gamma)(s_I) \right) dF^{I-N},$$

where $\gamma = (\gamma_j, \gamma_{-j})$ and $\mathbf{x}^{\circ}_{-N} = (x^{\circ}, \ldots, x^{\circ})$ is the array of x°'s for $I - N$ agents who do not select principal j (i.e., those agents with $h = 0$). In essence, the agent's communication strategy $c(\gamma_j, \gamma_{-j})$ converts principal j's mechanism γ_j into a direct mechanism $\beta_j(\gamma_j, c(\gamma_j, \gamma_{-j}))$

[12]The terminology of "incentive consistency" is used by Peters (1997) and is adopted in Han (2014a).

given γ_{-j}.

The agent's selection strategy $\pi(\gamma_j, \gamma_{-j})$ induces the probability distribution over $\bar{X}$. Define $z_j(\pi(\gamma_j, \gamma_{-j}))(x)$ as follows

$$z_j(\pi(\gamma_j, \gamma_{-j}))(x) := 1 - \int_x^{\bar{x}} \pi(\gamma_j, \gamma_{-j})(s)dF. \tag{2}$$

$z_j(\pi(\gamma_j, \gamma_{-j}))(x)$ is the probability that an agent either has her type below x or selects a principal other than j. We can now derive an agent's payoff upon selecting principal j given $c(\gamma_j, \gamma_{-j})$ and $\pi(\gamma_j, \gamma_{-j})$. First of all, we can derive the reduced-form direct mechanism $B_j(\gamma_j, c(\gamma_j, \gamma_{-j}))$ that an agent faces when she selects principal j. For all $x \in X$,

$$B_j(\gamma_j, c(\gamma_j, \gamma_{-j}))(x) := \mathbb{E}_{\mathbf{s}}[\beta_j(\gamma_j, c(\gamma_j, \gamma_{-j}))(x, s_2 \ldots, s_I)|z_j(\pi(\gamma_j, \gamma_{-j}))],$$

where $\mathbb{E}_{\mathbf{s}}[\cdot|z_j(\pi(\gamma_j, \gamma_{-j}))]$ is the expectation operator over $\mathbf{s} = (s_2, \ldots, s_I)$ with each s_i being independently drawn from $z_j(\pi(\gamma_j, \gamma_{-j}))$.

Given $\gamma = (\gamma_1, \ldots, \gamma_J)$, let $\beta_{-j}(\gamma_{-j}, c(\gamma_{-j}, \gamma_j)) = \{\beta_\ell(\gamma_\ell, c(\gamma_\ell, \gamma_{-\ell}))\}_{\ell \neq j}$ be an array of direct mechanisms, for all principals except principal j, that the agent's communication strategy $c(\gamma)$ induces. The expected payoff to the agent of type x from selecting principal j is

$$\begin{aligned} v(x, \gamma_j, \gamma_{-j}, c(\gamma_j, \gamma_{-j}), \pi(\gamma_j, \gamma_{-j})) = \\ \hat{v}(x, \beta_j(\gamma_j, c(\gamma_j, \gamma_{-j})), \beta_{-j}(\gamma_{-j}, c(\gamma_{-j}, \gamma_j)), \pi(\gamma_j, \gamma_{-j})) = \\ \int_A u(a, x) dB_j(\gamma_j, c(\gamma_j, \gamma_{-j}))(x), \end{aligned} \tag{3}$$

where v expresses the agent's payoff in terms of mechanisms $\gamma = (\gamma_1, \ldots, \gamma_J)$ offered by principals and $\hat{v}$ in terms of direct mechanisms that γ induces via agents' communication strategies $c(\gamma)$. For any given $c(\gamma)$ and $\pi(\gamma)$, let $c = \bigcup_{\gamma \in \bar{\Gamma}} c(\gamma)$ and $\pi = \bigcup_{\gamma \in \bar{\Gamma}} \pi(\gamma)$.

Definition 1 *A profile of agents' strategies* $(\bar{c}, \bar{\pi})$ *is an* incentive consistent continuation equilibrium *(henceforth simply continuation equilibrium) if for every* $\gamma \in \bar{\Gamma}$ *and almost every* $x \in X$,

1. *it is optimal for each agent to use* $\bar{c}(\gamma)$ *for communicating with a principal given that any other agent uses* $(\bar{c}(\gamma), \bar{\pi}(\gamma))$ *for communicating with and selecting a principal and*

2. $\bar{\pi}(\gamma_j, \gamma_{-j})(x) = 0 \Longrightarrow \exists \ell \neq j$:

$$v(x,\gamma_\ell,\gamma_{-\ell},\bar{c}(\gamma_\ell,\gamma_{-\ell}),\bar{\pi}(\gamma_\ell,\gamma_{-\ell})) \geq v(x,\gamma_j,\gamma_{-j},\bar{c}(\gamma_j,\gamma_{-j}),\bar{\pi}(\gamma_j,\gamma_{-j})),$$
$$\bar{\pi}(\gamma_j,\gamma_{-j})(x) > 0 \Longrightarrow \forall \ell,$$
$$v(x,\gamma_j,\gamma_{-j},\bar{c}(\gamma_j,\gamma_{-j}),\bar{\pi}(\gamma_j,\gamma_{-j})) \geq v(x,\gamma_\ell,\gamma_{-\ell},\bar{c}(\gamma_\ell,\gamma_{-\ell}),\bar{\pi}(\gamma_\ell,\gamma_{-\ell}))$$

3.2 DIC-response equilibrium of the whole game

Because $(\bar{c}(\gamma), \bar{\pi}(\gamma))$ induces a continuation equilibrium at γ, it is clear that the profile of direct mechanisms

$$\beta(\gamma, \bar{c}(\gamma)) = (\beta_1(\gamma_1, \bar{c}(\gamma_1,\gamma_{-1})), \ldots, \beta_J(\gamma_J, \bar{c}(\gamma_J,\gamma_{-J})))$$

is Bayesian incentive compatible (BIC) when agents select principals according to $\bar{\pi}(\gamma)$ given truthful type reporting to each principal. This implies that selecting principals according to $\bar{\pi}(\gamma)$ constitutes a continuation equilibrium together with truthful type reporting when principals directly offer $\beta(\gamma, \bar{c}(\gamma))$.

Consider the payoff to principal j who offers γ_j. Given (the distribution of) the other principals' mechanisms γ_{-j}, the payoff to principal j is

$$\begin{aligned}
\Phi(w_j,\gamma_j,\gamma_{-j},\bar{c}(\gamma_j,\gamma_{-j}),\bar{\pi}(\gamma_j,\gamma_{-j})) = \\
\hat{\Phi}(w_j,\beta_j(\gamma_j,\bar{c}(\gamma_j,\gamma_{-j})),\beta_{-j}(\gamma_{-j},\bar{c}(\gamma_{-j},\gamma_j)),\bar{\pi}(\gamma_j,\gamma_{-j})) = \\
\int_{\bar{X}} \cdots \int_{\bar{X}} \int_A u_j(a,\mathbf{s}) d\beta_j(\gamma_j,\bar{c}(\gamma_j,\gamma_{-j}))(\mathbf{s}) dz_j(\bar{\pi}(\gamma_j,\gamma_{-j}))(s_1) \cdots dz_j(\bar{\pi}(\gamma_j,\gamma_{-j}))(s_I), \quad (4)
\end{aligned}$$

where $\mathbf{s} = (s_1, \ldots, s_I)$ and Φ expresses the agent's payoff in terms of mechanisms γ offered by principals and $\hat{\Phi}$ in terms of direct mechanisms that γ induces via agents' communication strategies.

The BIC property of a direct mechanism is based on an agent's interim payoff, which depends on the other agents' decisions on selecting the principal. An agent's selection decision in turn depends on what mechanisms other principals offer. Therefore, the BIC property is endogenous. The 'dominant strategy incentive compatible' (DIC) property, however, is based on an agent's ex-post payoff; a direct mechanism μ_j is DIC if, for all $x, x' \in X$ and all $\mathbf{x} \in \bar{X}^{I-1}$

$$\int_A u(a,x) d\mu_j(x,\mathbf{x}) \geq \int_A u(a,x) d\mu_j(x',\mathbf{x}).$$

Therefore, the DIC property does not depend on the other agents' selection decisions nor on the mechanisms offered by the other principals. Let Ω^D be the set of all DIC direct mechanisms.

The analysis in our paper is based on a DIC-response equilibrium where principals employ pure strategies for their mechanism offer and its definition is provided in Definition 2 below. Most of the analysis can also be applied to a DIC-response equilibrium, where principals employ mixed strategies for their mechanism offer. The discussion of this can be found in subsection 6.1.

Definition 2 *A strategy profile $\{\bar{\gamma}, \bar{c}, \bar{\pi}\}$ is a DIC-response equilibrium in a market with $\bar{\Gamma}$ if*

1. *$(\bar{c}, \bar{\pi})$ is a continuation equilibrium and*

2. *for all $j \in \mathcal{J}$ and all $\gamma_j \in \bar{\Gamma}_j$*

$$\hat{\Phi}(w_j, \beta_j(\bar{\gamma}_j, \bar{c}(\bar{\gamma}_j, \bar{\gamma}_{-j})), \beta_{-j}(\bar{\gamma}_{-j}, \bar{c}(\bar{\gamma}_{-j}, \bar{\gamma}_j)), \bar{\pi}(\bar{\gamma}_{-j}, \bar{\gamma}_j))) \geq \hat{\Phi}(w_j, \beta_j(\gamma_j, \bar{c}(\gamma_j, \bar{\gamma}_{-j})), \beta_{-j}(\gamma_{-j}, \bar{c}(\bar{\gamma}_{-j}, \gamma_j)), \bar{\pi}(\gamma_j, \bar{\gamma}_{-j})).$$

3. *DIC-response: for all $j \in \mathcal{J}$ and all $\gamma_j \neq \bar{\gamma}_j$,*

$$\beta_{-j}(\bar{\gamma}_{-j}, \bar{c}(\bar{\gamma}_{-j}, \gamma_j)) = \mu^j_{-j} \in (\Omega^D)^{J-1}$$

Conditions 1 and 2 are required for $\{\bar{\gamma}, \bar{c}, \bar{\pi}\}$ to be an equilibrium in a market with $\bar{\Gamma}$. Condition 3 imposes the DIC-response property. It requires that the direct mechanism the agent's communication induces from non-deviating principal ℓ's equilibrium mechanism $\bar{\gamma}_\ell$ in continuation equilibrium upon principal j's deviation be always a DIC direct mechanism μ^j_ℓ regardless of the mechanism that principal j offers upon deviation. Therefore, when principal j deviates to any mechanism γ_j, the array of non-deviating principals' direct mechanisms induced by agents' communication is $\mu^j_{-j} = (\mu^j_1, \ldots \mu^j_{j-1}, \mu^j_{j+1}, \ldots \mu^j_J)$.

For example, in an auction environment, principals implicitly agree to sell their products through second price auctions with reserve prices that are sufficiently higher than their costs. Then, principals' equilibrium mechanisms may require agents to report not only their types but also their market information. If principal j deviates from his equilibrium mechanism to attract more buyers, a buyer's communication with other principals in continuation equilibrium always induces an array of DIC direct mechanisms; for example, second price auctions with certain reserve prices, possibly very close to zero.

Subsequently we will show that the property of DIC-response makes it very tractable to check the robustness of DIC-response equilibrium and to characterize the set of robust

DIC-response equilibrium allocations. Equilibrium analysis in the next section first starts with formulating the notion of robustness.

4 Robust DIC-Response Equilibrium

4.1 Robustness

We fix an equilibrium $(\bar{\gamma}, \bar{c}, \bar{\pi})$ in a market with $\bar{\Gamma}$. A principal may want to offer a mechanism that is not available in the market if he can gain by doing so. For a principal's payoff upon deviation to such a mechanism, it is important to know how agents would communicate with and select other principals upon his deviation. Suppose that given $\bar{\gamma}_{-j}$, principal j deviates to an arbitrary mechanism γ_j in Γ_j. Let M be the set of messages available for an agent in those mechanisms in Γ_j. Therefore, Γ_j includes mechanisms with $M^I \times H^I$ as the domain. A set of mechanisms Γ_j may include mechanisms that are different from those available in $\bar{\Gamma}_j$ because M may be different from $\bar{M}$. In order to not restrict the nature of communication in deviating principal j's mechanism in Γ_j, we assume that principal j can deviate to a mechanism in Γ_j, which is bigger than the set of all direct mechanisms, denoted by Ω. Formally, Γ_j is bigger than Ω if there exists an embedding $\eta_j : \Omega \to \Gamma_j$. It implies that there are more mechanisms in Γ_j than in Ω.

Given $\bar{\gamma}_{-j} \in \bar{\Gamma}_{-j} := \times_{\ell \neq j} \bar{\Gamma}_\ell$, for simplicity, let an agent's strategy for communicating with deviating principal j be characterized by $c'(\gamma_j) : X \to \Delta(M)$ for any given $\gamma_j \in \Gamma_j$. Therefore, $c'(\gamma_j)(x)$ is the probability distribution that the agent of type x uses for her communication with deviating principal j. Given $\bar{\gamma}_{-j} \in \bar{\Gamma}_{-j}$, let $\pi'(\gamma_j) : X \to [0,1]$ be an agent's strategy for selecting deviating principal j who offers γ_j. Therefore, when the deviating principal's mechanism is γ_j and (the distribution of) mechanisms offered by non-deviating principals is $\bar{\gamma}_{-j}$, the agent of type x sends to the deviating principal a message that is drawn from probability distribution $c'(\gamma_j)(x)$ and eventually selects the deviating principal with probability $\pi'(\gamma_j)(x)$.

Consider an agent's strategy profile for communicating with and selecting a non-deviating principal. $c^\circ(\bar{\gamma}_\ell, \gamma_j, \bar{\gamma}_{-j,\ell}) : X \to \Delta(\bar{M})$ is an agent's strategy for communicating with non-deviating principal ℓ given (the distribution of) other principals' mechanisms $(\gamma_j, \bar{\gamma}_{-j,\ell})$, where γ_j is deviating principal j's mechanism and $\bar{\gamma}_{-j,\ell}$ is (the distribution of) mechanisms offered by principals except for principals j and ℓ. $\pi^\circ(\bar{\gamma}_\ell, \gamma_{-j}, \bar{\gamma}_{-j,\ell}) : X \to [0,1]$ is an agent's strategy for selecting non-deviating principal ℓ. Let $c^\circ(\gamma_j) = \{c^\circ(\bar{\gamma}_\ell, \gamma_j, \bar{\gamma}_{-j,\ell})\}_{\ell \neq j}$ and $\pi^\circ(\gamma_j) = \{\pi^\circ(\bar{\gamma}_\ell, \gamma_j, \bar{\gamma}_{-j,\ell})\}_{\ell \neq j}$. Then, a continuation equilibrium upon principal j's deviation to $\gamma_j \in \Gamma_j$ is denoted by $(c^\circ(\gamma_j), \pi^\circ(\gamma_j), c'(\gamma_j), \pi'(\gamma_j))$. Let $\Lambda(\gamma_j)$ be the set of all continuation equilibria upon deviating to a mechanism γ_j in Γ_j. Then the deviating principal's payoff upon deviating to γ_j is

$$\hat{\Phi}(w_j, \beta_j(\gamma_j, c'(\gamma_j)), \beta_{-j}(\bar{\gamma}_{-j}, c^\circ(\gamma_j)), \pi'(\gamma_j)),$$

where $\beta_{-j}(\bar{\gamma}_{-j}, c^\circ(\gamma_j)) = \{\beta_\ell(\bar{\gamma}_\ell, c^\circ(\bar{\gamma}_\ell, \gamma_j, \bar{\gamma}_{-j,\ell}))\}_{\ell \neq j}$.

Let $\mathcal{C}^\circ(\gamma_j)$ be the set of all strategies $c^\circ(\gamma_j)$ for communicating with a non-deviating principal so that it is defined as

$$\mathcal{C}^\circ(\gamma_j) := \left\{c^\circ(\gamma_j) : \exists(\pi^\circ(\gamma_j), c'(\gamma_j), \pi'(\gamma_j)) \text{ s.t. } (c^\circ(\gamma_j), \pi^\circ(\gamma_j), c'(\gamma_j), \pi'(\gamma_j)) \in \Lambda(\gamma_j)\right\}.$$

For any given $c^\circ(\gamma_j) \in \mathcal{C}^\circ(\gamma_j)$, define

$$\mathcal{C}'(c^\circ(\gamma_j)) := \left\{c'(\gamma_j) : \exists(\pi^\circ(\gamma_j), \pi'(\gamma_j)) \text{ s.t. } (c^\circ(\gamma_j), \pi^\circ(\gamma_j), c'(\gamma_j), \pi'(\gamma_j)) \in \Lambda(\gamma_j)\right\}.$$

Finally, for any given $c^\circ(\gamma_j) \in \mathcal{C}^\circ(\gamma_j)$ and any given $c'(\gamma_j) \in \mathcal{C}'(c^\circ(\gamma_j))$, define

$$\Pi'(c^\circ(\gamma_j), c'(\gamma_j)) := \left\{\pi'(\gamma_j) : \exists \pi^\circ(\gamma_j) \text{ s.t. } (c^\circ(\gamma_j), \pi^\circ(\gamma_j), c'(\gamma_j), \pi'(\gamma_j)) \in \Lambda(\gamma_j)\right\}.$$

Let $\mathcal{C}^\circ := \bigcup_{\gamma_j \in \Gamma} \mathcal{C}^\circ(\gamma_j)$ and $c^\circ = \bigcup_{\gamma_j \in \Gamma} c^\circ(\gamma_j) \in \mathcal{C}^\circ$.

Definition 3 $\{\bar{\gamma}, \bar{c}, \bar{\pi}\}$ *is a robust DIC-response equilibrium if*

1. *it is a DIC-response equilibrium in a market with* $\overline{\Gamma}$,
2. *robustness: for all* $j \in \mathcal{J}$, *and any* Γ_j, *there exists* $c^\circ \in \mathcal{C}^\circ$ *such that for all* $\gamma_j \in \Gamma_j$,

$$\hat{\Phi}(w_j, \beta_j(\bar{\gamma}_j, \bar{c}(\bar{\gamma}_j, \bar{\gamma}_{-j})), \beta_{-j}(\bar{\gamma}_{-j}, \bar{c}(\bar{\gamma}_{-j}, \bar{\gamma}_j)), \bar{\pi}(\bar{\gamma}_{-j}, \bar{\gamma}_j)) \geq \sup_{c'(\gamma_j) \in \mathcal{C}'(c^\circ(\gamma_j))} \left(\sup_{\pi'(\gamma_j) \in \Pi'(c^\circ(\gamma_j), c'(\gamma_j))} \hat{\Phi}(w_j, \beta_j(\gamma_j, c'(\gamma_j)), \beta_{-j}(\bar{\gamma}_{-j}, c^\circ(\gamma_j)), \pi'(\gamma_j)) \right), \quad (5)$$

3. *DIC-response upon principal j's deviation to any mechanism in any* Γ_j*: for all* $j \in \mathcal{J}$, *any* Γ_j, *and all* $\gamma_j \in \Gamma_j$,

$$\beta_{-j}(\bar{\gamma}_{-j}, c^\circ(\gamma_j)) = \mu^j_{-j} \in (\Omega^D)^{J-1}.$$

Condition 3 applies the 'DIC-response' property to the array of direct mechanisms that c°

induces from non-deviating principals' equilibrium mechanisms upon a principal's deviation to any γ_j in any Γ_j.

Condition 2 in Definition 3 states the robustness. It requires that deviating principal j cannot gain in every possible continuation equilibrium upon his deviation to any possible arbitrary mechanism conditional on an agent's strategy profile c° for communicating with non-deviating principals. If condition 2 is satisfied, no principal has incentives to deviate to any arbitrary mechanism regardless of continuation equilibrium that agents might play upon a principal's deviation conditional on an agent's strategy profile c°.

The robustness captured in condition 2 can be rewritten as follows. Given a DIC-response equilibrium $\{\bar{\gamma}, \bar{c}, \bar{\pi}\}$ in a market with $\overline{\Gamma}$, for principal j's deviation to $\gamma_j \in \Gamma_j$, define

$$\begin{aligned}&\mathcal{E}^\circ(\gamma_j, \mu^j_{-j}) := \\ &\left\{(\mu_j, \tau') = (\beta_j(\gamma_j, c'(\gamma_j)), \pi'(\gamma_j)) : \forall c'(\gamma_j) \in \mathcal{C}'(c^\circ(\gamma_j)), \forall \pi'(\gamma_j) \in \Pi'(c'(\gamma_j), c^\circ(\gamma_j))\right\},\end{aligned} \quad (6)$$

where c° is an agent's strategy for communicating with a non-deviating principal that satisfies conditions 2 and 3 in Definition 3.

Lemma 1 *A DIC-response equilibrium $\{\bar{\gamma}, \bar{c}, \bar{\pi}\}$ in a market with $\overline{\Gamma}$ is robust if and only if for all $j \in \mathcal{J}$, and any Γ_j, there exists $c^\circ \in \mathcal{C}^\circ$ such that for all $\gamma_j \in \Gamma_j$,*

$$\begin{aligned}&\hat{\Phi}(w_j, \beta_j(\bar{\gamma}_j, \bar{c}(\bar{\gamma}_j, \bar{\gamma}_{-j})), \beta_{-j}(\bar{\gamma}_{-j}, \bar{c}(\bar{\gamma}_{-j}, \bar{\gamma}_j)), \bar{\pi}(\bar{\gamma}_{-j}, \bar{\gamma}_j)) \geq \\ &\qquad\qquad \sup_{(\mu_j, \hat{\pi}'(\mu_j)) \in \mathcal{E}^\circ(\mu^j_{-j})} \hat{\Phi}(w_j, \mu_j, \mu^j_{-j}, \hat{\pi}'(\mu_j)),\end{aligned} \quad (7)$$

where $\mathcal{E}^\circ(\mu^j_{-j}) = \bigcup_{\gamma_j \in \Gamma_j} \mathcal{E}^\circ(\gamma_j, , \mu^j_{-j})$.

Proof. By its definition, $\mathcal{E}^\circ(\mu^j_{-j})$ includes the set of all possible pairs of BIC direct mechanisms μ_j for principal j, and an agent's strategies $\hat{\pi}'(\mu_j)$ for selecting him that can be induced in all continuation equilibria upon principal j's deviation to a mechanism in Γ_j given $c^\circ \in \mathcal{C}^\circ$ and $\mu^j_{-j} \in (\Omega^D)^{J-1}$ Then, the right-hand side of (5) is equal to

$$\sup_{(\mu_j, \hat{\pi}'(\mu_j)) \in \mathcal{E}^\circ(\mu^j_{-j})} \hat{\Phi}(w_j, \mu_j, \mu^j_{-j}, \hat{\pi}'(\mu_j)). \quad (8)$$

Therefore, we can replace the right-hand side of (5) with (8). ■

We ask two key questions. First, is there a class of mechanisms that principals can use instead of their equilibrium mechanism in robust DIC-response equilibrium? The answer to this first question also provides implications on how to check the robustness of a DIC-response equilibrium. Second, what is the set of BIC allocations that can be supported as an equilibrium allocation in robust DIC-response equilibrium? An equilibrium allocation is characterized by a profile of BIC direct mechanisms $\bar{\mu} = \beta(\gamma, \bar{c}(\gamma)) = (\beta_1(\bar{\gamma}_1, \bar{c}(\bar{\gamma}_1, \bar{\gamma}_{-1})), \ldots, \beta_J(\bar{\gamma}_J, \bar{c}(\bar{\gamma}_J, \bar{\gamma}_{-J})))$ and an agent's selection strategy $\pi(\bar{\mu}) = \bar{\pi}(\bar{\gamma})$. For this purpose, we define a BIC allocation.

Definition 4 *A pair of (i) BIC direct mechanism profile $\bar{\mu}$ and (ii) an agent's selection strategy $\pi(\bar{\mu})$ is a BIC allocation if it is a continuation equilibrium for an agent to report her true type upon selecting a principal according to $\pi(\bar{\mu})$ when principals offer $\bar{\mu}$.*

We address the first question in the next subsection and the second in the subsequent subsection.

4.2 Deviator-reporting direct mechanisms

Suppose that principal ℓ offers a deviator-reporting direct mechanism $\tilde{\gamma}_\ell$ in which a message space for each agent is $\mathcal{J} \times \bar{X}$. Therefore, each agent reports (i) the identity of a deviating principal (if any) and (ii) either a payoff type or x°. Let (d_i, x_i) denote agent i's report. Note that each agent's selection decision is included in her type report (i.e., x° implies that the agent does not select principal j). If $d_i = \ell$ is reported to principal ℓ, it implies that no principal deviated. For any $\mathbf{d} = (d_1, \ldots d_I) \in \mathcal{J}^I$ and any $\mathbf{x} = (x_1, \ldots, x_I) \in \bar{X}^I$, a deviator-reporting direct mechanism $\tilde{\gamma}_\ell$ takes the following form,

$$\tilde{\gamma}_\ell(\mathbf{d}, \mathbf{x}) = \zeta_\ell(\mathbf{d})(\mathbf{x}),$$

where $\zeta_\ell(\mathbf{d})$ is a direct mechanism such that

$$\zeta_\ell(\mathbf{d}) := \begin{cases} \mu_\ell^j \in \Omega^D & \text{if } |\{i : d_i = \ell \text{ for } \ell \neq j\}| > |\mathcal{I}|/2 \\ \bar{\mu}_\ell & \text{otherwise} \end{cases} \tag{9}$$

Given a BIC allocation $(\bar{\mu}, \pi(\bar{\mu}))$, assume that each principal ℓ offers a deviator-reporting direct mechanism $\tilde{\gamma}_\ell$ that satisfies (9). Then, a deviator-reporting direct mechanism changes the principal's direct mechanism contingent on agents' reports on the identity of a deviating principal. If more than half of agents report j to principal ℓ, then principal ℓ assigns μ_ℓ^j. Otherwise, he assigns $\bar{\mu}_\ell$. Given this rule, it is always optimal for each agent to do the same when every other agent reports the true answer to whether or not there is a deviating principal and which principal deviates if any. Suppose that no principal deviates. If every other agent reports ℓ to each principal ℓ, any single agent cannot change principal ℓ's mechanism away from $\bar{\mu}_\ell$ even if she reports something other than ℓ. If principal j deviates and every other agent reports j to principal ℓ, a single agent again cannot change principal ℓ's mechanism away from $\mu_\ell^j \in \Omega^D$. Therefore, we can always fix truth telling on whether or not there is a deviating principal and which principal deviates if any.

If no principal deviates, agents' true reports on the identity of the deviating principal induces $\bar{\mu}_\ell$ for every principal ℓ. Because $(\bar{\mu}, \pi(\bar{\mu}))$ is BIC, it is a continuation equilibrium in which each agent also reports her true type upon selecting principal ℓ given their selection strategy $\tilde{\pi}(\tilde{\gamma}) = \pi(\bar{\mu})$. If principal j deviates, agents' true reporting to principal ℓ on the identity of the deviating principal induces a DIC direct mechanism $\mu_\ell^j \in \Omega^D$. Because of the DIC property, we can also fix an agent's truthful type reporting to each non-deviating principal ℓ in continuation equilibrium. Therefore, we can always fix an agent's truthful reporting to each non-deviating principal on the identity of the deviating principal, and on her type upon selecting him.

Suppose that $\tilde{\gamma} = (\tilde{\gamma}_1, \ldots, \tilde{\gamma}_J)$ is a profile of deviator-reporting direct mechanisms that principals choose in a market with $\tilde{\Gamma} = \times_{j=1}^J \tilde{\Gamma}_j$, where $\tilde{\Gamma}_j$ includes $\tilde{\gamma}_j$ as well as some mechanisms in $\overline{\Gamma}_j$. Let $(\tilde{c}, \tilde{\pi})$ be a profile of agents' strategies for communicating with and selecting principals respectively. $\tilde{c}(\tilde{\gamma}_j, \tilde{\gamma}_{-j})$ and $\tilde{\pi}(\gamma_j, \tilde{\gamma}_{-j})$ denote an agent's strategies for communicating with and selecting principal j respectively when no one deviates. On the other hand, given (the distribution of) other principals' mechanisms, $\tilde{\gamma}_{-j}$, $\tilde{c}(\gamma_j, \tilde{\gamma}_{-j})$ and $\tilde{\pi}(\gamma_j, \tilde{\gamma}_{-j})$ denote an agent's strategies for communicating with and selecting deviating principal j when he deviates to $\gamma_j \in \tilde{\Gamma}_j$.

Let $\tilde{\beta}_j(\tilde{\gamma}_j, k)$ be the direct mechanism that $\tilde{\gamma}_j$ induces when every agent reports k as a deviating principal given truthful reporting. We can define a DIC-response equilibrium $(\tilde{\gamma}, \tilde{c}, \tilde{\pi})$ in a market with $\tilde{\Gamma}$ similar to Defintion 2: $(\tilde{\gamma}, \tilde{c}, \tilde{\pi})$ is a DIC-response equilibrium in a market with $\tilde{\Gamma}$ if $(\tilde{c}, \tilde{\pi})$ is a continuation equilibrium and

$$\hat{\Phi}(w_j, \tilde{\beta}_j(\tilde{\gamma}_j, j), \tilde{\beta}_{-j}(\tilde{\gamma}_{-j}, -j), \tilde{\pi}(\tilde{\gamma}_j, \tilde{\gamma}_{-j})) \geq \hat{\Phi}(w_j, \beta_j(\gamma_j, \tilde{c}(\gamma_j, \tilde{\gamma}_{-j})), \tilde{\beta}_{-j}(\tilde{\gamma}_{-j}, j), \tilde{\pi}(\gamma_j, \tilde{\gamma}_{-j})),$$

where $\tilde{\beta}_{-j}(\tilde{\gamma}_{-j}, -j) = \{\tilde{\beta}_\ell(\tilde{\gamma}_\ell, \ell)\}_{\ell \neq j}$ and $\tilde{\beta}_{-j}(\tilde{\gamma}_{-j}, j) = \{\tilde{\beta}_\ell(\tilde{\gamma}_\ell, j)\}_{\ell \neq j}$.

Now consider the case when principal j deviates to a mechanism γ_j in an alternative set of mechanisms Γ_j, which includes mechanisms with $M^I \times H^I$ as the domain, given other principals' mechanisms $\tilde{\gamma}_{-j}$. Then, let $\tilde{c}(\gamma_j) : X \to \Delta(M)$ be an agent's strategy for communicating with deviating principal j and $\tilde{\pi}'(\gamma_j) : X \to [0,1]$ an agent's strategy for selecting deviating principal j when the deviating principal's mechanism is γ_j given $\tilde{\gamma}_{-j}$. For an agent's strategy profile for selecting a non-deviating principal, we use the notation

$\tilde{\pi}^\circ(\gamma_j)$. Given truthful reporting to non-deviating principals, a continuation equilibrium upon principal j's deviation to $\gamma_j \in \Gamma_j$ is denoted by $(\tilde{c}'(\gamma_j), \tilde{\pi}'(\gamma_j), \tilde{\pi}^\circ(\gamma_j))$. Let $\tilde{\Lambda}(\gamma_j)$ be the set of all continuation equilibria upon principal j's deviation to a mechanism γ_j in Γ_j, given $\tilde{\gamma}_{-j}$. Given truthful reporting to non-deviating principals, define

$$\tilde{\mathcal{C}}'(\gamma_j) := \left\{ \tilde{c}'(\gamma_j) : \exists (\tilde{\pi}'(\gamma_j), \tilde{\pi}^\circ(\gamma_j)) \text{ s.t. } (\tilde{c}'(\gamma_j), \tilde{\pi}'(\gamma_j), \tilde{\pi}^\circ(\gamma_j)) \in \tilde{\Lambda}(\gamma_j) \right\}.$$

For any given $\tilde{c}'(\gamma_j) \in \tilde{\mathcal{C}}'$, define

$$\tilde{\Pi}'(\tilde{c}'(\gamma_j), \mu^j_{-j}) := \left\{ \tilde{\pi}'(\gamma_j) : \exists \tilde{\pi}^\circ(\gamma_j) \text{ s.t. } (\tilde{c}'(\gamma_j), \tilde{\pi}'(\gamma_j), \tilde{\pi}^\circ(\gamma_j)) \in \tilde{\Lambda}(\gamma_j) \right\}.$$

Then, the robustness of a DIC-response equilibrium $(\tilde{\gamma}, \tilde{c}, \tilde{\pi})$ is defined as follow.

Definition 5 *A DIC-response equilibrium $(\tilde{\gamma}, \tilde{c}, \tilde{\pi})$ in a market with $\tilde{\Gamma}$ is robust if, for all $j \in \mathcal{J}$, for any Γ_j, and all $\gamma_j \in \Gamma_j$,*

$$\begin{aligned} &\hat{\Phi}(w_j, \tilde{\beta}_j(\tilde{\gamma}_j, j), \tilde{\beta}_{-j}(\tilde{\gamma}_{-j}, -j), \tilde{\pi}(\tilde{\gamma}_j, \tilde{\gamma}_{-j})) \geq \\ &\qquad \sup_{\tilde{c}'(\gamma_j) \in \tilde{\mathcal{C}}'} \left(\sup_{\tilde{\pi}'(\gamma_j) \in \tilde{\Pi}'(\tilde{c}'(\gamma_j), \mu^j_{-j})} \hat{\Phi}(w_j, \beta_j(\gamma_j, \tilde{c}'(\gamma_j)), \tilde{\beta}_{-j}(\tilde{\gamma}_{-j}, j), \tilde{\pi}'(\gamma_j)) \right). \end{aligned} \tag{10}$$

The robustness of a DIC-response equilibrium examines if there exists a continuation equilibrium where a principal can gain upon his deviation to any arbitrary mechanism given agents' truthful reporting to non-deviating principals who offer deviator-reporting direct mechanisms. If there does not exists such a continuation equilibrium, a DIC-response equilibrium is said to be robust. This is consistent with the notion of (strong) robustness adopted in Epstein and Peters (1999) who also consider the robustness of an equilibrium in the same manner given agents' truthful reporting to non-deviating principals who offer universal mechanisms.[13]

Similar to Lemma 1, we can rewrite the robustness of a DIC-response equilibrium $(\tilde{\gamma}, \tilde{c}, \tilde{\pi})$. Given truthful reporting to non-deviating principals, a continuation equilibrium upon principal j's deviation to $\gamma_j \in \Gamma_j$ is denoted by $(\tilde{c}'(\gamma_j), \tilde{\pi}'(\gamma_j), \tilde{\pi}^\circ(\gamma_j))$. For principal j's deviation

[13]This is a stronger notion of the weak robustness that is mostly adopted in common agency (Peters 2001, Martimort and Stole 2002) or competing mechanism design without frictions (Yamashita 2010). The weak robustness is satisfied as long as there exists one continuation equilibrium where a principal does not gain. An exception is Han (2007) who studies the strongly robust equilibrium, consistent with the robustness notion in Epstein and Peters (1999), in competing mechanism without frictions under complete information.

to $\gamma_j \in \Gamma_j$, define

$$\tilde{\mathcal{E}}(\gamma_j, \mu^j_{-j}) := \left\{(\mu_j, \tau') = (\beta_j(\gamma_j, \tilde{c}'(\gamma_j)), \tilde{\pi}'(\gamma_j)) : \forall c'(\gamma_j) \in \tilde{\mathcal{C}}'(\gamma_j), \forall \tilde{\pi}'(\gamma_j) \in \tilde{\Pi}'(\tilde{c}'(\gamma_j), \mu^j_{-j})\right\}. \quad (11)$$

Lemma 2 *A DIC-response equilibrium $(\tilde{\gamma}, \tilde{c}, \tilde{\pi})$ in a market with $\tilde{\Gamma}$ is robust if and only if, for all $j \in \mathcal{J}$, for any Γ_j, and all $\gamma_j \in \Gamma_j$,*

$$\hat{\Phi}(w_j, \tilde{\beta}_j(\tilde{\gamma}_j, j), \tilde{\beta}_{-j}(\tilde{\gamma}_{-j}, -j), \tilde{\pi}(\tilde{\gamma}_j, \tilde{\gamma}_{-j})) \geq \sup_{(\mu_j, \hat{\pi}'(\mu_j)) \in \tilde{\mathcal{E}}(\mu^j_{-j})} \hat{\Phi}(w_j, \mu_j, \mu^j_{-j}, \hat{\pi}'(\mu_j)), \quad (12)$$

where $\tilde{\mathcal{E}}(\mu^j_{-j}) = \bigcup_{\gamma_j \in \Gamma_j} \tilde{\mathcal{E}}(\gamma_j, , \mu^j_{-j})$.

Proof. $\tilde{\mathcal{E}}(\mu^j_{-j})$ includes the set of all possible pairs of BIC direct mechanisms μ_j for principal j and an agent's strategies $\hat{\pi}'(\mu_j)$ for selecting him that can be induced in all continuation equilibria given $\mu^j_{-j} \in (\Omega^D)^{J-1}$, the profile of non-deviating principals' DIC direct mechanisms that $\tilde{\gamma}_{-j}$ assigns when agents are truthfully reporting the identity of the deviating principal. Then, the right-hand side of (10) is equivalent to $\sup_{(\mu_j, \hat{\pi}'(\mu_j)) \in \tilde{\mathcal{E}}(\mu^j_{-j})} \hat{\Phi}(w_j, \mu_j, \mu^j_{-j}, \hat{\pi}'(\mu_j))$. ■

We establish the equivalence between robust DIC-response equilibrium allocations in any market and robust DIC-response equilibrium allocations that are supported by deviator-reporting direct mechanisms.

Proposition 1 *For any robust DIC-response equilibrium $\{\bar{\gamma}, \bar{c}, \bar{\pi}\}$ in a market with $\overline{\Gamma}$, there exists a corresponding robust DIC-response equilibrium $\{\tilde{\gamma}, \tilde{c}, \tilde{\pi}\}$ in a market with $\tilde{\Gamma}$ in which each principal offers a deviator-reporting direct mechanism and the equilibrium allocation is preserved.*

Proof. Consider a robust DIC-response equilibrium $\{\bar{\gamma}, \bar{c}, \bar{\pi}\}$ in a market with $\overline{\Gamma}$, where $\bar{\gamma}_\ell$ always induces a DIC direct mechanism μ^j_ℓ $(j \neq \ell)$ in continuation equilibrium off the equilibrium path following principal j's deviation. On the equilibrium path, let

$$\bar{\mu} = (\bar{\mu}_1, \ldots, \bar{\mu}_J) = (\beta_1(\bar{\gamma}_1, \bar{c}\left(\bar{\gamma}_1, \bar{\gamma}_{-1}\right)), \ldots, \beta_J(\bar{\gamma}_J, \bar{c}\left(\bar{\gamma}_J, \bar{\gamma}_{-J}\right))) \quad (13)$$

be the equilibrium profile of BIC direct mechanisms. Then, we can use the notation $(\bar{\mu}, \pi(\bar{\mu}))$ to denote the equilibrium allocation with $\pi(\bar{\mu}) = \bar{\pi}(\bar{\gamma})$.

Now let $\tilde{\gamma}_\ell$ be principal ℓ's deviator-reporting direct mechanism that satisfies (9), and $\tilde{\Gamma}_j$ be the set of mechanisms available for principal j that is constructed by replacing $\bar{\gamma}_j$ in

$\overline{\Gamma}_j$ with $\tilde{\gamma}_j$ for all $j \in \mathcal{J}$. Let $\tilde{\Gamma} = \times_{j=1}^{J} \tilde{\Gamma}_j$. Consider a competition in a market with $\tilde{\Gamma}$. Let each principal ℓ's strategy be to offer the deviator-reporting mechanism $\tilde{\gamma}_\ell$. Let $\tilde{\pi}$ be a profile of an agent's strategies for selecting principals on and off the path following principal j's deviation to a mechanism in $\tilde{\Gamma}_j$. We fix truthful reporting as an agent's strategy for communicating with non-deviating principals on and off the path. Let $\tilde{c}$ be a profile of an agent's strategies for communicating with a deviating principal. Let agents choose the profile of strategies $(\tilde{c}, \tilde{\pi})$ that satisfies

$$\tilde{\pi}(\tilde{\gamma}_j, \tilde{\gamma}_{-j}) = \bar{\pi}(\bar{\gamma}_j, \bar{\gamma}_{-j}), \tag{14}$$

$$\tilde{\pi}(\gamma_j, \tilde{\gamma}_{-j}) = \bar{\pi}(\gamma_j, \bar{\gamma}_{-j}) \text{ for any } \gamma_j \in \tilde{\Gamma} \text{ with } \gamma_j \neq \tilde{\gamma}_j, \tag{15}$$

$$\tilde{c}(\gamma_j, \tilde{\gamma}_{-j}) = \bar{c}(\gamma_j, \bar{\gamma}_{-j}) \text{ for any } \gamma_j \in \tilde{\Gamma} \text{ with } \gamma_j \neq \tilde{\gamma}_j. \tag{16}$$

Given an agent's truthful reporting to each principal j, $\tilde{\gamma}_j$ induces $\tilde{\beta}_j(\tilde{\gamma}_j, j) = \bar{\mu}_j$ without any principal's deviation. $\bar{\mu}_j$ is the same as the one $\beta_j(\bar{\gamma}_j, \bar{c}\left(\bar{\gamma}_j, \bar{\gamma}_{-j}\right)) = \bar{\mu}_j$ derived from $\bar{\gamma}_j$. Because (14) implies that $\tilde{\pi}(\tilde{\gamma}) = \bar{\pi}(\bar{\gamma}) = \pi(\bar{\mu})$, the equilibrium allocation $(\bar{\mu}, \pi(\bar{\mu}))$ is preserved. Furthermore, it is straightforward to show that $(\tilde{\gamma}, \tilde{c}, \tilde{\pi})$, satisfying (9), (14), (15), and (16), is a DIC-response equilibrium in a market with $\tilde{\Gamma}$ because an agent's selection behavior and her communication behavior with a deviating principal are preserved.

To complete the proof, we now show the robustness of the DIC-response equilibrium $(\tilde{\gamma}, \tilde{c}, \tilde{\pi})$ in a market with $\tilde{\Gamma}$. In equilibrium $\{\bar{\gamma}, \bar{c}, \bar{\pi}\}$ in a market with $\overline{\Gamma}$, non-deviating principals' mechanisms $\bar{\gamma}_{-j}$ always induce μ^j_{-j} on and off the equilibrium path following principal j's deviation. Because we fix μ^j_{-j}, we have that $\mathcal{E}^\circ(\gamma_j, , \mu^j_{-j})$ in (6) is equal to $\tilde{\mathcal{E}}(\gamma_j, , \mu^j_{-j})$ in (11) for all $\gamma_j \in \Gamma_j$. Therefore, we have $\mathcal{E}^\circ(\mu^j_{-j}) = \tilde{\mathcal{E}}(\mu^j_{-j})$. Because the equilibrium allocation is preserved,

$$\hat{\Phi}(w_j, \beta_j(\bar{\gamma}_j, \bar{c}(\bar{\gamma}_j, \bar{\gamma}_{-j})), \beta_{-j}(\bar{\gamma}_{-j}, \bar{c}(\bar{\gamma}_{-j}, \bar{\gamma}_j)), \bar{\pi}(\bar{\gamma})) = \hat{\Phi}(w_j, \tilde{\beta}_j(\tilde{\gamma}_j, j), \tilde{\beta}_{-j}(\tilde{\gamma}_{-j}, -j), \tilde{\pi}(\tilde{\gamma}_j, \tilde{\gamma}_{-j})). \tag{17}$$

Combining (17) with $\mathcal{E}^\circ(\mu^j_{-j}) = \tilde{\mathcal{E}}(\mu^j_{-j})$, we can show that (7) in Lemma 1 implies (12) in Lemma 2. Therefore the robustness of the DIC-response equilibrium $(\tilde{\gamma}, \tilde{c}, \tilde{\pi})$ is ensured. ■

Concerning the robust DIC-response equilibrium allocation, principals do not need to consider any complex mechanisms but only deviator-reporting direct mechanisms. It implies that agents only need to convey market information on which principal deviates, if any. If principals are all identical, this can be more simplified, i.e., agents only need to reveal whether or not there exists a deviating principal. This makes the implementation of a deviator-reporting direct mechanisms very tractable.

4.3 Equilibrium characterization

In this subsection, we study the set of all robust DIC-response equilibrium allocations. Because of Proposition 1 in the previous subsection, we consider only robust DIC-response equilibria where principals offer deviator-reporting direct mechanisms.

For the characterization, we first introduce some notations. For any profile of incentive compatible direct mechanisms (μ_j, μ_{-j}), let $\hat{\Pi}'(\mu_j, \mu_{-j})$ be the set of all possible strategies for selecting principal j given the agent's truthful reporting when principals offer (μ_j, μ_{-j}). Then, define $\mathcal{M}_j^B(\mu_{-j}^j)$ as follows: for $\mu_{-j}^j \in (\Omega^D)^{J-1}$,

$$\mathcal{M}_j^B(\mu_{-j}^j) := \left\{\mu_j \in \Omega \text{ with } \hat{\Pi}'(\mu_j, \mu_{-j}^j) \neq \varnothing\right\}$$

In words, $\mathcal{M}^B(\mu_{-j}^j)$ includes all possible BIC direct mechanisms μ_j for principal j that can be combined with μ_{-j}^j to induce a continuation equilibrium where agents report truthfully their types upon selecting a principal. Now we establish the necessary and sufficient condition for robust DIC-response equilibrium allocations.

Proposition 2 *$(\bar{\mu}, \pi(\bar{\mu}))$ can be supported as an equilibrium allocation in a robust DIC-response equilibrium $(\tilde{\gamma}, \tilde{c}, \tilde{\pi})$ in a market with $\tilde{\Gamma}$ if and only if*

1. *$(\bar{\mu}, \pi(\bar{\mu}))$ is a BIC allocation and*

2. *for every $j \in \mathcal{J}$, there exists $\mu_{-j}^j \in (\Omega^D)^{J-1}$ such that*

$$\hat{\Phi}(w_j, \bar{\mu}_j, \bar{\mu}_{-j}, \pi(\bar{\mu})) \geq \sup_{\mu_j \in \mathcal{M}_j^B(\mu_{-j}^j)} \left(\sup_{\hat{\pi}'(\mu_j) \in \hat{\Pi}'(\mu_j, \mu_{-j}^j)} \hat{\Phi}(w_j, \mu_j, \mu_{-j}^j, \hat{\pi}'(\mu_j)) \right). \tag{18}$$

Proof. "only if" part: Suppose that $(\bar{\mu}, \pi(\bar{\mu}))$ is supported as an equilibrium allocation in robust DIC-response equilibrium $(\tilde{\gamma}, \tilde{c}, \tilde{\pi})$ in a market with $\tilde{\Gamma}$. When there is no-deviating principal, each principal j's deviator-reporting direct mechanism induces $\tilde{\beta}_j(\tilde{\gamma}_j, j) = \bar{\mu}_j$ given agents' truthful reporting, and each agent's selection strategy follows $\tilde{\pi}(\tilde{\gamma}) = \pi(\bar{\mu})$ in order to support $(\bar{\mu}, \pi(\bar{\mu}))$ as an equilibrium allocation. Because truthful reporting is part of continuation equilibrium on the equilibrium path, $(\bar{\mu}, \pi(\bar{\mu}))$ must be a BIC allocation, which is condition 1. Suppose that condition 2 is not satisfied: for some j and all $\mu_{-j}^j \in (\Omega^D)^{J-1}$, we have that

$$\hat{\Phi}(w_j, \bar{\mu}_j, \bar{\mu}_{-j}, \pi(\bar{\mu})) < \sup_{\mu_j \in \mathcal{M}_j^B(\mu_{-j}^j)} \left(\sup_{\hat{\pi}'(\mu_j) \in \hat{\Pi}'(\mu_j, \mu_{-j}^j)} \hat{\Phi}(w_j, \mu_j, \mu_{-j}^j, \hat{\pi}'(\mu_j)) \right) \tag{19}$$

The left-hand side is the equilibrium payoff to principal j in (19). Let μ_{-j}^j be the profile of DIC direct mechanisms that non-deviating principals' deviator-reporting direct mechanisms $\tilde{\gamma}$ assign upon principal j's deviation. Then, $\mathcal{M}_j^B(\mu_{-j}^j)$ is a possible set of mechanisms that principal j can consider for his deviation. (19) implies that there exists a BIC direct mechanism $\mu_j \in \mathcal{M}_j^B(\mu_{-j}^j)$ that induces a continuation equilibrium in which principal j can gain given agents' truthful reporting to non-deviating principals. Then, it contradicts the robustness of the DIC-response equilibrium $(\tilde{\gamma}, \tilde{c}, \tilde{\pi})$. Therefore, condition 2 must be satisfied. This concludes the proof of the "only if" part.

"if" part: Suppose that conditions 1 and 2 are satisfied. Let $\tilde{\gamma}_\ell$ be principal ℓ's deviator-reporting direct mechanism that is constructed according to (9) given μ_ℓ^j for each j from condition 2 and $\bar{\mu}_\ell$ from $\bar{\mu}$ in condition 1. Then, $\tilde{\pi}(\tilde{\gamma}) = \pi(\bar{\mu})$, together with agents' truthful reporting to each principal, constitutes a continuation equilibrium on the equilibrium path and we have

$$\hat{\Phi}(w_j, \tilde{\beta}_j(\tilde{\gamma}_j, j), \tilde{\beta}_{-j}(\tilde{\gamma}_{-j}, -j), \tilde{\pi}(\tilde{\gamma})) = \hat{\Phi}(w_j, \bar{\mu}_j, \bar{\mu}_{-j}, \pi(\bar{\mu})).$$

On the other hand, for any Γ_j and any $(\mu_j, \hat{\pi}'(\mu_j)) \in \tilde{\mathcal{E}}(\mu_{-j}^j)$, it is clear that $\mu_j \in \mathcal{M}_j^B(\mu_{-j}^j)$ and $\hat{\pi}'(\mu_j) \in \hat{\Pi}'(\mu_j, \mu_{-j}^j)$. Therefore, for $\tilde{\mathcal{E}}(\mu_{-j}^j)$ associated with any Γ_j, we have that

$$\sup_{\mu_j \in \mathcal{M}_j^B(\mu_{-j}^j)} \left(\sup_{\hat{\pi}'(\mu_j) \in \hat{\Pi}'(\mu_j, \mu_{-j}^j)} \hat{\Phi}(w_j, \mu_j, \mu_{-j}^j, \hat{\pi}'(\mu_j)) \right) \geq \sup_{(\mu_j, \hat{\pi}'(\mu_j)) \in \tilde{\mathcal{E}}(\mu_{-j}^j)} \hat{\Phi}(w_j, \mu_j, \mu_{-j}^j, \hat{\pi}'(\mu_j)) \tag{20}$$

Because principal j can directly deviate to a BIC direct mechanism in $\mathcal{M}_j^B(\mu_{-j}^j)$, the left-hand side of (20) is the lowest upper bound that principal j can expect from any possible deviation across all possible Γ_j. Applying (18) to (20) yields

$$\hat{\Phi}(w_j, \tilde{\beta}_j(\tilde{\gamma}_j, j), \tilde{\beta}_{-j}(\tilde{\gamma}_{-j}, -j), \tilde{\pi}(\tilde{\gamma})) \geq \sup_{(\mu_j, \hat{\pi}'(\mu_j)) \in \tilde{\mathcal{E}}(\mu_{-j}^j)} \hat{\Phi}(w_j, \mu_j, \mu_{-j}^j, \hat{\pi}'(\mu_j)). \tag{21}$$

Lemma 2 shows that (21) is equivalent to (10); if (10) is satisfied, there is no continuation equilibrium in which principal j cannot gain upon deviating to any arbitrary mechanism in any Γ_j including $\tilde{\Gamma}_j$. Therefore, $(\bar{\mu}, \pi(\bar{\mu}))$ can be supported as an equilibrium allocation in a robust DIC-response equilibrium $(\tilde{\gamma}, \tilde{c}, \tilde{\pi})$ in a market with $\tilde{\Gamma}$ if conditions 1 and 2 are satisfied. ■

Condition 2 in Proposition 2 provides an important implication. It shows that, given $\mu^j_{-j} \in (\Omega^D)^{J-1}$ that non-deviating principals will assign upon principal j's deviation, principal j only needs to consider deviation to a BIC direct mechanism in $\mathcal{M}^B_j(\mu^j_{-j})$ given the agent's selection strategy that is best for principal j, in order to examine whether there exists a profitable deviation. It is possible to derive this tight necessary and sufficient condition for robust DIC-response equilibrium because of the notion of (strong) robustness.

Given three or more agents and the strict majority rule specified in the deviator-reporting direct mechanism, no single agent can change the non-deviating principal's DIC direct mechanism when every agent reports the true identity of the deviating principal. It means that the DIC direct mechanism assigned upon a principal's deviation does not affect the agent's incentive to report the true identity of the deviating principal. This is why non-deviating principals can assign any DIC direct mechanisms in continuation equilibrium following a principal's deviation. Therefore, non-deviating principals can consider any μ^j_{-j} in $(\Omega^D)^{J-1}$ to punish deviating principal j, as Condition 2 asserts.

If there are only two agents and their reports on the identity of a deviating principal are inconsistent with each other, a non-deviating principal does not know which agent lied. If monetary transfers are part of a principal's allocation decision, even with two agents, non-deviating principals can assign any DIC direct mechanisms that they want upon a principal's deviation, so Proposition 2 and Theorem 1 (and also Proposition 1) are valid with two agents. The reason is that the non-deviating principal can always set very low monetary transfer to each participating agent (i.e., very high payment from each participating agent) upon agents' inconsistent reports. This will prevent lying on the identity of a deviating principal.

Proposition 2 also allows us to characterize the set of allocations that can be supported in a robust DIC-response equilibrium. For the characterization, first let Ψ_B be the set of BIC allocations such that

$$\Psi_B := \{(\bar{\mu}, \pi(\bar{\mu})) : (\bar{\mu}, \pi(\bar{\mu})) \text{ is a BIC allocation}\}. \tag{22}$$

We then define the greatest lower bound for the payoff to principal j:

$$\Phi^*_j := \inf_{\mu^j_{-j} \in (\Omega^D)^{J-1}} \left[\sup_{\mu_j \in \mathcal{M}^B_j(\mu^j_{-j})} \left(\sup_{\hat{\pi}'(\mu_j) \in \hat{\Pi}'(\mu_j, \mu^j_{-j})} \hat{\Phi}(w_j, \mu_j, \mu^j_{-j}, \hat{\pi}'(\mu_j)) \right) \right].$$

Now we present the characterization of the equilibrium allocations in robust DIC-response equilibrium in Theorem 1 below.

Theorem 1 *The set of robust DIC-response equilibrium allocations is*

$$\Psi_B^* := \{(\bar{\mu}, \pi(\bar{\mu})) \in \Psi^B : \hat{\Phi}(w_j, \bar{\mu}_j, \bar{\mu}_{-j}, \pi(\bar{\mu})) \geq \Phi_j^* \ \forall j\}.$$

Proof. Due to Proposition 2, we only need to prove that Ψ_B^* is the set of all allocations that satisfy conditions 1 and 2 in Proposition 2.

First of all, for $(\bar{\mu}, \pi(\bar{\mu}))$ to be an equilibrium allocation in robust DIC-response equilibrium, it is necessary that $(\bar{\mu}, \pi(\bar{\mu})) \in \Psi^B$ (i.e., condition 1 in Proposition 2) and that $\hat{\Phi}(w_j, \bar{\mu}_j, \bar{\mu}_{-j}, \pi(\bar{\mu}))$ is no less than Φ_j^*; if $\hat{\Phi}(w_j, \bar{\mu}_j, \bar{\mu}_{-j}, \pi(\bar{\mu})) < \Phi_j^*$, there is no μ_{-j}^j satisfying condition 2 in Proposition 2. Therefore, any equilibrium allocation $(\bar{\mu}, \pi(\bar{\mu}))$ that is supported in robust DIC-response equilibrium belongs to Ψ_B^*.

To complete the proof, we need to show that any $(\bar{\mu}, \pi(\bar{\mu})) \in \Psi_B^*$ satisfies conditions 1 and 2 in Proposition 2. Any $(\bar{\mu}, \pi(\bar{\mu})) \in \Psi_B^*$ satisfies condition 1 because $(\bar{\mu}, \pi(\bar{\mu}))$ is in Ψ_B. Therefore, we only need to examine condition 2. Suppose that $\hat{\Phi}(w_j, \bar{\mu}_j, \bar{\mu}_{-j}, \pi(\bar{\mu})) > \Phi_j^*$ for some j. Then, clearly there exists μ_{-j}^j such that

$$\hat{\Phi}(w_j, \bar{\mu}_j, \bar{\mu}_{-j}, \pi(\bar{\mu})) \geq \sup_{\mu_j \in \mathcal{M}_j^B(\mu_{-j}^j)} \left(\sup_{\hat{\pi}'(\mu_j) \in \hat{\Pi}'(\mu_j, \mu_{-j}^j)} \hat{\Phi}(w_j, \mu_j, \mu_{-j}^j, \hat{\pi}'(\mu_j)) \right) \geq \Phi_j^*$$

The first inequality implies condition 2 in Proposition 2.

Suppose that $\hat{\Phi}(w_j, \bar{\mu}_j, \bar{\mu}_{-j}, \pi(\bar{\mu})) = \Phi_j^*$ for some j. It means that $(\bar{\mu}_j, \bar{\mu}_{-j}, \pi(\bar{\mu}))$ satisfies

$$\begin{aligned}
\pi(\bar{\mu}) &\in \arg\max_{\hat{\pi}'(\bar{\mu}_j) \in \hat{\Pi}'(\bar{\mu}_j, \bar{\mu}_{-j})} \hat{\Phi}(w_j, \bar{\mu}_j, \bar{\mu}_{-j}, \hat{\pi}'(\bar{\mu}_j)), \\
\bar{\mu}_j &\in \arg\max_{\mu_j \in \mathcal{M}_j^B(\bar{\mu}_{-j})} \left(\sup_{\hat{\pi}'(\mu_j) \in \hat{\Pi}'(\mu_j, \bar{\mu}_{-j})} \hat{\Phi}(w_j, \mu_j, \bar{\mu}_{-j}, \hat{\pi}'(\mu_j)) \right), \\
\bar{\mu}_{-j} &\in \arg\min_{\mu_{-j}^j \in (\Omega^D)^{J-1}} \left[\sup_{\mathcal{M}_j^B(\mu_{-j}^j)} \left(\sup_{\hat{\pi}'(\mu_j) \in \hat{\Pi}'(\mu_j, \mu_{-j}^j)} \hat{\Phi}(w_j, \mu_j, \mu_{-j}^j, \hat{\pi}'(\mu_j)) \right) \right].
\end{aligned}$$

Because $\hat{\Phi}(w_j, \bar{\mu}_j, \bar{\mu}_{-j}, \pi(\bar{\mu})) = \Phi_j^*$, $\mu_{-j}^j = \bar{\mu}_{-j}$ satisfies condition 2 in Proposition 2 with equality. This completes the proof. ■

Non-deviating principals' equilibrium mechanisms punish a deviating principal j by responding with the same array of DIC direct mechanisms, say μ_{-j}^j, regardless of deviating principal j's mechanism. Because the robustness considers the situation where agents play the best continuation equilibrium for deviating principal j, given their truthful reporting to non-deviating principals, the supremum of deviating principal j's payoff is

$$\sup_{\mu_j \in \mathcal{M}_j^B(\mu_{-j}^j)} \left(\sup_{\hat{\pi}'(\mu_j) \in \hat{\Pi}'(\mu_j, \mu_{-j}^j)} \hat{\Phi}(w_j, \mu_j, \mu_{-j}^j, \hat{\pi}'(\mu_j)) \right). \tag{23}$$

The worst punishment can be then made by choosing μ_{-j}^j that minimizes (23). This is why the greatest lower bound for principal j's equilibrium payoff is equal to Φ_j^*.

5 DIC Equilibrium Allocations

The linear payoff environment with independent one-dimensional private-value types is required in Gershokov, et al (2013) to establish the BIC-DIC equivalence in the environment with a single mechanism designer and multiple agents. They focus on the (interim) payoff equivalence. They show that for any BIC direct mechanism, there exists a corresponding DIC direct mechanism that delivers the same interim payoffs for all agents and the same ex ante expected social surplus given the probability distribution over agents' types.

Our framework is different because agents can select principals. The probability distribution over an agent' message space in a principal's direct mechanism is endogenously determined whereas in Gershkov et al (2013), it is the same as the exogenously given probability distribution over agents' types. It turns out that the underlying logic in their BIC-DIC equivalence is sufficiently general for it to be extended to the case with multiple principals and multiple agents given the linear payoff environment with independent one-dimensional private-value types.

5.1 Payoff environment

Following the environment in Gershkov, et al (2013), we assume that each principal j's allocation decision is to choose one of the action alternatives from the finite set, $\mathcal{K} = \{1, \dots, K\}$, and to choose monetary transfers to agents who select him. An agent's payoff depends on the action taken by the principal she eventually selects and the monetary transfer. An agent's payoff from choosing principal j who takes an action alternative k is $b^k x + g^k + t$, where $b^k \in \mathbb{R}_+$ and $g^k \in \mathbb{R}$ for all $k \in \mathcal{K}$ and $t \leq 0$ is a monetary transfer to the agent.[14]

[14]We consider the cases where an agent (buyer) pays a positive amount of money to a principal (seller) because t is a monetary transfer to the agent and t is assumed to be non-positive. The assumption of $t \leq 0$

Principal j's payoff depends on his choice of action alternative k and monetary transfer t'; $a^k w + y^k + t'$,where $a^k, y^k \in \mathbb{R}$ for all $k \in \mathcal{K}$, $w \in \mathbb{R}_+$. We impose the *budget balance* for each principal j so that the sum of t' and monetary transfers to agents who choose him is equal to zero. Therefore, $t' \geq 0$ because monetary transfer to an agent is assumed to be non-positive.

A mechanism γ_j in the set of mechanisms available for each principal j is now denoted by $\gamma_j = \{q_j^1, \ldots, q_j^K, t_j\}$, where $q_j^k : \bar{M}^I \times H^I \to [0,1]$ specifies the probability for action alternative $k \in \mathcal{K}$ as the function of all agents' messages and selection decisions and $t_j : \bar{M}^I \times H^I \to \mathbb{R}_-$ specifies principal j's monetary transfer to an agent as the function of all agents' messages and selection decisions. We assume that principal j's monetary transfer to an agent who does not select him (i.e., an agent with $h = 0$) is always equal to zero.

To specify payoffs, we now reformulate a direct mechanism. Consider principal j's direct mechanism. It specifies his action alternative and monetary transfers contingent on the types of agents who select him. For example, an auction specifies the probabilities of winning the object and monetary transfers as the function of bids submitted by participating bidders. A direct mechanism $\mu_j = \{q_j^1, \ldots, q_j^K, t_j\}$ with $q_j^k : \bar{X}^I \to [0,1]$ for all k and $t_j : \bar{X}^I \to \mathbb{R}_-$ for all i. Let us fix a profile of mechanisms offered by principals except for principal j to γ_{-j}. From principal j's mechanism γ_j, a communication strategy $c(\gamma_j, \gamma_{-j})$ induces the principal's direct mechanism,

$$\beta_j(\gamma_j, c(\gamma_j, \gamma_{-j})) = \{\rho_j^1(\gamma_j, c(\gamma_j, \gamma_{-j})), \ldots, \rho_j^K(\gamma_j, c(\gamma_j, \gamma_{-j})), \tau_j(\gamma_j, c(\gamma_j, \gamma_{-j}))\},$$

as follows.

Let N denote the number of agents who select principal j (those agents with $h = 1$) Then, for every $N \leq I$ and every $(x_1, \ldots, x_N) \in X^N$ and every k, $\rho_j^k(\gamma_j, c(\gamma_j, \gamma_{-j}))$ is defined as

$$\rho_j^k(\gamma_j, c(\gamma_j, \gamma_{-j}))(x_1, \ldots, x_N, \mathbf{x}_{-N}^\circ) :=$$
$$\int_{X^{I-N}} \left(\int_{\bar{M}} \cdots \int_{\bar{M}} q_j^k(\mathbf{m}, \mathbf{h}) dc(\gamma)(x_1) \ldots dc(\gamma)(x_N) dc(\gamma)(s_{N+1}) \cdots dc(\gamma)(s_I) \right) dF^{I-N},$$

where $\gamma = (\gamma_j, \gamma_{-j})$ and $\mathbf{x}_{-N}^\circ = (x^\circ, \ldots, x^\circ)$ is the array of x°'s for $I - N$ agents who do not select principal j (i.e., those agents with $h = 0$). In the same way, we can derive $\tau_j(\gamma_j, c(\gamma_j, \gamma_{-j}))$ from $t_j : \bar{M}^I \times S^I \to \mathbb{R}_-$ based on a communication strategy c.

essentially implies that a seller (principal) cannot make a positive amount of monetary transfer to the agent. This is what we usually observe in practice. We can assume $t \geq 0$ for the cases where principals are buyers and agents are sellers.

The agent's selection strategy $\pi(\gamma_j, \gamma_{-j})$ induces $z_j(\pi(\gamma_j, \gamma_{-j}))$, the probability distribution over $\bar{X}$, as specified in (2). Then, we can derive the expected probability for action alternative k that an agent of type x faces when she selects principal j:

$$Q_j^k(\gamma_j, c(\gamma_j, \gamma_{-j}))(x) := \mathbb{E}_{\mathbf{s}}[\rho_j^k(\gamma_j, c(\gamma_j, \gamma_{-j}))(x, s_2 \ldots, s_I) | z_j(\pi(\gamma_j, \gamma_{-j}))],$$

where $\mathbb{E}_{\mathbf{s}}[\cdot | z_j(\pi(\gamma_j, \gamma_{-j}))]$ is the expectation operator over $\mathbf{s} = (s_2, \ldots, s_I)$ with each s_i being independently drawn from $z_j(\pi(\gamma_j, \gamma_{-j}))$.

We can also derive the expected payment to an agent of type x when she selects principal j:

$$T_j(\gamma_j, c(\gamma_j, \gamma_{-j}))(x) := \mathbb{E}_{\mathbf{s}}[\tau_j(\gamma_j, c(\gamma_j, \gamma_{-j}))(x, s_2, \ldots, s_I) | z_j(\pi(\gamma_j, \gamma_{-j}))].$$

Given $\gamma = (\gamma_1, \ldots, \gamma_J)$, let $\beta_{-j}(\gamma_{-j}, c(\gamma_{-j}, \gamma_j)) = \{\beta_\ell(\gamma_\ell, c(\gamma_\ell, \gamma_{-\ell}))\}_{\ell \neq j}$ be an array of direct mechanisms for principals except for principal j that the agent's communication strategy $c(\gamma)$ induces. The expected payoff to the agent of type x from selecting principal j is

$$\begin{aligned} v(x, \gamma_j, \gamma_{-j}, c(\gamma_j, \gamma_{-j}), \pi(\gamma_j, \gamma_{-j})) = \\ \hat{v}(x, \beta_j(\gamma_j, c(\gamma_j, \gamma_{-j})), \beta_{-j}(\gamma_{-j}, c(\gamma_{-j}, \gamma_j)), \pi(\gamma_j, \gamma_{-j})) = \\ \sum_{k \in \mathcal{K}} (b^k x + g^k) Q_j^k(\gamma_j, c(\gamma_j, \gamma_{-j}))(x) + T_j(\gamma_j, c(\gamma_j, \gamma_{-j}))(x), \end{aligned} \quad (24)$$

where v expresses the agent's payoff in terms of mechanisms $\gamma = (\gamma_1, \ldots, \gamma_J)$ offered by principals and $\hat{v}$ in terms of direct mechanisms that γ induces via agents' communication strategies $c(\gamma)$.

Given (the distribution of) the other principals' mechanisms γ_{-j}, the payoff to principal j is

$$\begin{aligned} \Phi(w_j, \gamma_j, \gamma_{-j}, \bar{c}(\gamma_j, \gamma_{-j}), \bar{\pi}(\gamma_j, \gamma_{-j})) = \\ \hat{\Phi}(w_j, \beta_j(\gamma_j, \bar{c}(\gamma_j, \gamma_{-j})), \beta_{-j}(\gamma_{-j}, \bar{c}(\gamma_{-j}, \gamma_j)), \bar{\pi}(\gamma_j, \gamma_{-j})) = \\ \sum_{k \in \mathcal{K}} (a^k w_j + y^k) \int_{\underline{x}}^{\bar{x}} Q_j^k(\gamma_j, \bar{c}(\gamma_j, \gamma_{-j}))(s) dz_j(\bar{\pi}(\gamma_j, \gamma_{-j}))(s) \\ - I \times \int_{\underline{x}}^{\bar{x}} T_j(\gamma_j, \bar{c}(\gamma_j, \gamma_{-j}))(x) dz_j(\bar{\pi}(\gamma_j, \gamma_{-j}))(s), \end{aligned} \quad (25)$$

where Φ expresses the agent's payoff in terms of mechanisms γ offered by principals and $\hat{\Phi}$ in terms of direct mechanisms that γ induces via agents' communication strategies.

5.2 BIC-DIC equivalence

We first establish the BIC-DIC equivalence in the case with multiple principals and multiple agents and then derive its implication in the following subsections. Let $\mu = (\mu_1, \ldots, \mu_J)$ be a profile of direct mechanisms that principals offer. Suppose that an agent's strategy for selecting principal j is $\pi'(\mu)$. Then, we can define $z_j(\pi'(\mu))(x)$ similar to (2). Given $\mu_j = \{q_j^1, \ldots, q_j^K, t_j\}$, we can induce the reduced-form direct mechanism $\{Q_j^1, \ldots, Q_j^K, T_j\}$ such that, for all x,

$$Q_j^k(x) = \mathbb{E}_{\mathbf{s}}[q_j^k(x, s_2, \ldots, s_I)|z_j(\pi'(\mu))], \tag{26}$$

$$T_j(x) = \mathbb{E}_{\mathbf{s}}[t_j(x, s_2, \ldots, s_I)|z_j(\pi'(\mu))]. \tag{27}$$

Let $\hat{v}(x, \mu_j, \mu_{-j}, \pi'(\mu))$ be the payoff to the agent of type x associated with reporting the true type upon selecting principal j:

$$\hat{v}(x, \mu_j, \mu_{-j}, \pi'(\mu)) = \sum_{k \in \mathcal{K}} b^k x Q_j^k(x) + T_j(x).$$

Given a profile of direct mechanisms μ and π', principal j's payoff is

$$\hat{\Phi}(w_j, \mu_j, \mu_{-j}, \pi'(\mu)) = a^k w_j \mathbb{E}_s[Q_j^k(s)|z_j(\pi'(\mu))] - I \times \mathbb{E}_s[T_j(s)|z_j(\pi'(\mu))]$$

The BIC property of μ_j is satisfied if, all $x, x' \in X$, and all j,

$$\sum_{k \in \mathcal{K}} b^k x Q_j^k(x) + T_j(x) \geq \sum_{k \in \mathcal{K}} b^k x Q_j^k(x') + T_j(x')$$

As shown above, the BIC property of μ_j depends on the probability distribution of the set of participating agents and their types because the reduced-form mechanism (and therefore an agent's (interim) payoff) depend on $z_j(\pi'(\mu))$. On the other hand, a direct mechanism, $\mu_j = \{q_j^1, \ldots, q_j^K, t_j\}$, is DIC if, for all $x, x' \in X$, all $\mathbf{x} \in \bar{X}^{I-1}$,

$$\sum_{k \in \mathcal{K}} b^k x q_j^k(x, \mathbf{x}) + t_j(x, \mathbf{x}) \geq \sum_{k \in \mathcal{K}} b^k x q_j^k(x', \mathbf{x}) + t_j(x', \mathbf{x})$$

As one can see, the DIC property is independent of the probability distribution of the other agents' types and of the (endogenous) probabilities with which agents select the principal. Therefore, an agent's truthful type reporting can be always fixed, regardless of the other agents' selection decision and their type reporting.

Proposition 3 (BIC-DIC equivalence) *Suppose that given μ_{-j}, μ_j is BIC when agents select principal j according to $\pi'(\mu_j, \mu_{-j})$. Then, there exists a DIC direct mechanism μ'_j and π'' such that (i) $\pi''(\mu'_j, \mu_{-j}) = \pi'(\mu_j, \mu_{-j})$ and (ii)*

$$\hat{v}(x, \mu'_j, \mu_{-j}, \pi''(\mu'_j, \mu_{-j})) = \hat{v}(x, \mu_j, \mu_{-j}, \pi'(\mu_j, \mu_{-j})) \text{ ,} \forall x \in X \tag{28}$$
$$\hat{\Phi}(w_j, \mu'_j, \mu_{-j}, \pi''(\mu'_j, \mu_{-j})) = \hat{\Phi}(w_j, \mu_j, \mu_{-j}, \pi'(\mu_j, \mu_{-j})). \tag{29}$$

Proof. Let Q_j^k be the reduced form of q_j^k in $\mu_j = \{q_j^1, \ldots, q_j^K, t_j\}$ that is derived with $z_j(\pi'(\mu_j, \mu_{-j}))$ according to (27). Let $\acute{Q}_j^k$ be the reduced form of $\acute{q}_j^k$ in $\mu'_j = \{\acute{q}_j^1, \ldots, \acute{q}_j^K, \acute{t}_j\}$ that is derived similarly with $z_j(\pi''(\mu'_j, \mu_{-j}))$. Then, we define $V'_j(x) := \sum_{k\in\mathcal{K}} b^k \acute{Q}_j^k(x)$ and $V_j(x) := \sum_{k\in\mathcal{K}} b^k Q_j^k(x)$.

If $\pi''(\mu'_j, \mu_{-j}) = \pi'(\mu_j, \mu_{-j})$, then

$$z_j(\pi''(\mu'_j, \mu_{-j})) = z_j(\mu_j, \pi'(\mu_j, \mu_{-j})). \tag{30}$$

Because $z_j(\pi''(\mu'_j, \mu_{-j}))$ is the probability distribution over $\bar{X}$ and $\bar{X}$ is the message space for an agent in a direct mechanism, it implies that the probability distribution over each agent's $\bar{X}$ for a direct mechanism is preserved and that it is independent of each other. Given the linear payoff structure, we can apply Theorem 1 and Lemma 3 in Gershkov, et al. (2013) for the case of anonymous (and hence non-discriminatory) mechanisms to show the existence of a DIC direct mechanism μ'_j such that

$$V'_j(x) = V_j(x) \text{ for all } x, \tag{31}$$
$$\mathbb{E}_s[\acute{Q}_j^k(s)|z_j(\pi''(\mu'_j, \mu_{-j}))] = \mathbb{E}_s[Q_j^k(s)|z_j(\pi'(\mu_j, \mu_{-j}))] \text{ for all } k \in \mathcal{K} \tag{32}$$

with the transfers $\acute{t}_j$, which preserves each agent i's payoff upon selecting j, that is (28). Taking the expected value of each side of (28) over x_i and applying (31) yields

$$\begin{aligned} &\mathbb{E}_s[\acute{T}_j(s)|z_j(\pi''(\mu'_j, \mu_{-j}))] \\ &= \mathbb{E}_s[T_j(s)|z_j(\pi'(\mu_j, \mu_{-j}))] \\ &+ g^k \left(\sum_{k\in\mathcal{K}} \mathbb{E}_s[Q_j^k(s)|z_j(\pi'(\mu_j, \mu_{-j}))] - \sum_{k\in\mathcal{K}} \mathbb{E}_s[\acute{Q}_j^k(s)|z_j(\pi''(\mu'_j, \mu_{-j}))] \right) \\ &= \mathbb{E}_s[T_j(s)|z_j(\pi'(\mu_j, \mu_{-j}))] \end{aligned} \tag{33}$$

The first and the second equality in (33) hold because of (31) and (32) respectively. On the other hand, principal j's payoff associated with μ_j satisfies

$$\begin{aligned}
&\hat{\Phi}(w_j, \mu'_j, \mu_{-j}, \pi''(\mu'_j, \mu_{-j})) \\
&= \sum_{k \in \mathcal{K}} (b^k w_j + y^k) \mathbb{E}_s[\acute{Q}^k_j(s) | z_j(\pi''(\mu'_j, \mu_{-j}))] - I \times \mathbb{E}_s[\acute{T}_j(s) | z_j(\pi''(\mu'_j, \mu_{-j}))] \\
&= \sum_{k \in \mathcal{K}} (b^k w_j + y^k) \mathbb{E}_{\mathbf{x}}[q^k_j(\mathbf{x}) | z_j(\pi'(\mu_j, \mu_{-j}))] - I \times \mathbb{E}_s[T_j(s) | z_j(\pi'(\mu_j, \mu_{-j}))] \\
&= \hat{\Phi}(w_j, \mu_j, \mu_{-j}, \pi'(\mu_j, \mu_{-j})).
\end{aligned} \tag{34}$$

where the second equality holds because of (32) and (33). Therefore, principal j's payoff is preserved as well.

Because a monetary transfer to a participating agent is restricted to be non-positive, we need to show whether the DIC direct mechanism μ'_j also has the property of the "non-positive" monetary transfer to a participating agent given that μ_j has that property. Note that given $\mu_j = \{q^1_j, \ldots, q^K_j, t_{1j}, \ldots t_{Ij}\}$, we have

$$T_j(\underline{x}) = \mathbb{E}_{\mathbf{s}}[t_j(\underline{x}, s_2, \ldots, s_I) | z_j(\pi'(\mu_j, \mu_{-j}))] \leq 0,$$

because monetary transfers to a participating agent are non-positive and that

$$V_j(\underline{x}) = \sum_{k \in \mathcal{K}} b^k Q^k_j(\underline{x}) \geq 0$$

because $b^k \geq 0$ and $Q^k_j(\underline{x}) \geq 0$ for all k. Applying (30), as the underlying probability distribution over the message space in the direct mechanism, to Theorem 1 in Gershkov, et al. (2013), monetary transfers in $\mu'_j = \{\acute{q}^1_j, \ldots, \acute{q}^K_j, \acute{t}_j\}$ are determined by

$$\acute{t}_j(x, \mathbf{x}) = \frac{\acute{v}_j(\underline{x}, \mathbf{x})}{V_j(\underline{x})} T_j(\underline{x}) + \acute{v}_j(\underline{x}, \mathbf{x})\underline{x} - \acute{v}_j(x, \mathbf{x})x + \int_{\underline{x}}^{x} \acute{v}_j(s, \mathbf{x}) ds,$$

where $\acute{v}_j$ is defined as $\acute{v}_j(s, \mathbf{x}) = \sum_{k \in \mathcal{K}} b^k \acute{q}^k_j(s, \mathbf{x}) \geq 0$ for all $s \in X = [\underline{x}, \bar{x}]$. Because μ'_j is DIC, $\acute{v}_j(s, \mathbf{x})$ is non-decreasing in s.

First of all, we have

$$\acute{t}_j(\underline{x}, \mathbf{x}) = \frac{\acute{v}_j(\underline{x}, \mathbf{x})}{V_j(\underline{x})} T_j(\underline{x}) \leq 0. \tag{35}$$

because $T_j(\underline{x}) \leq 0$, $V_j(\underline{x}) \geq 0$ and $\acute{v}_j(\underline{x}, \mathbf{x}) \geq 0$.[15] Secondly, consider $\acute{t}_j(x, \mathbf{x}) - \acute{t}_j(x', \mathbf{x})$ for

[15] As in Gershkov, et al (2013), 0/0 is interpreted as 1.

any $x, x' \in X$ with $x > x'$:

$$\acute{t}_j(x, \mathbf{x}) - \acute{t}_j(x', \mathbf{x}) = -\acute{\upsilon}_j(x, \mathbf{x})x + \acute{\upsilon}_j(x', \mathbf{x})x' + \int_{x'}^{x} \acute{\upsilon}_j(s, \mathbf{x})ds \leq 0 \tag{36}$$

The inequality holds because we have

$$\acute{\upsilon}_j(x, \mathbf{x})x - \acute{\upsilon}_j(x', \mathbf{x})x' \geq \int_{x'}^{x} \acute{\upsilon}_j(s, \mathbf{x})ds$$

given that $\acute{\upsilon}_j(s, \mathbf{x})$ is non-decreasing in s with $\acute{\upsilon}_j(s, \mathbf{x}) \geq 0$ for all $s \in X \subset \mathbb{R}_+$. (35) and (36) imply that $\acute{t}_j(x, \mathbf{x}) \leq 0$ for all $x \in X$ given any $\mathbf{x}$. ■

Consider a BIC allocation $\mu, \pi'(\mu)$, where $\mu = (\mu_j, \mu_{-j})$. We can use Proposition 3 to show that we can preserve all players' continuation equilibrium payoffs with another BIC allocation $(\mu', \pi''(\mu'))$, where $\mu' = (\mu'_j, \mu_{-j})$ with a DIC direct mechanism $\pi''(\cdot) = \pi'(\cdot)$, which keeps an agent's principal selecting probabilities. Let us explain its implications.

5.3 Robust DIC-response equilibrium

First of all, Proposition 2 provides the necessary and sufficient condition for $(\bar{\mu}, \pi(\bar{\mu}))$ to be supported in robust DIC-response equilibrium in a market with $\tilde{\Gamma}$. The second condition in the proposition requires the existence of $\mu^j_{-j} \in (\Omega^D)^{J-1}$ for all j such that

$$\hat{\Phi}(w_j, \bar{\mu}_j, \bar{\mu}_{-j}, \pi(\bar{\mu})) \geq \sup_{\mu_j \in \mathcal{M}^B_j(\mu^j_{-j})} \left(\sup_{\hat{\pi}'(\mu_j) \in \hat{\Pi}'(\mu_j, \mu^j_{-j})} \hat{\Phi}(w_j, \mu_j, \mu^j_{-j}, \hat{\pi}'(\mu_j)) \right)$$

It shows that principal j only needs to consider deviation to a direct mechanism μ_j in $\mathcal{M}^B_j(\mu^j_{-j})$, which is the set of all possible BIC direct mechanisms for principal j that can be induced in continuation equilibrium upon his deviation. Because of the BIC-DIC equivalence we can replace this set with Ω^D. Therefore, we can rewrite Proposition 2 as follows:

Proposition 4 $(\bar{\mu}, \pi(\bar{\mu}))$ *can be supported as an equilibrium allocation in a robust DIC-response equilibrium* $(\tilde{\gamma}, \tilde{c}, \tilde{\pi})$ *in a market with* $\tilde{\Gamma}$ *if and only if*

1. $(\bar{\mu}, \pi(\bar{\mu}))$ *is a BIC allocation and*

2. for every $j \in \mathcal{J}$, there exists $\mu^j_{-j} \in (\Omega^D)^{J-1}$ such that

$$\hat{\Phi}(w_j, \bar{\mu}_j, \bar{\mu}_{-j}, \pi(\bar{\mu})) \geq \sup_{\mu_j \in \Omega^D} \left(\sup_{\hat{\pi}'(\mu_j) \in \hat{\Pi}'(\mu_j, \mu^j_{-j})} \hat{\Phi}(w_j, \mu_j, \mu^j_{-j}, \hat{\pi}'(\mu_j)) \right)$$

Proof. Condition 1 is the same as condition 1 in Proposition 2. We only need to show that $\mathcal{M}^B_j(\mu^j_{-j})$ in condition 2 in Proposition 2 can be replace with Ω^D. Recall that $\mathcal{M}^B(\mu^j_{-j})$ includes all possible BIC direct mechanisms for principal j that can be induced in continuation equilibrium upon his deviation. Therefore, $\Omega^D \subset \mathcal{M}^B(\mu^j_{-j})$. On the other hand, consider a BIC allocation associated with (μ_j, μ^j_{-j}) given any $\mu^j_{-j} \in (\Omega^D)^{J-1}$ and any $\mu_j \in \mathcal{M}^B_j(\mu^j_{-j})$. Because of Proposition 3, there exists a DIC allocation associated with (μ'_j, μ^j_{-j}) such that (i) μ'_j is a DIC direct mechanism, (ii) agent's selection strategy preserves the same selection probabilities, and (iii) the payoffs for all principals and agents are preserved. It implies that for a given $\mu^j_{-j} \in (\Omega^D)^{J-1}$

$$\sup_{\mu_j \in \Omega^D} \left(\sup_{\hat{\pi}'(\mu_j) \in \hat{\Pi}'(\mu_j, \mu^j_{-j})} \hat{\Phi}(w_j, \mu_j, \mu^j_{-j}, \hat{\pi}'(\mu_j)) \right) = \sup_{\mu_j \in \mathcal{M}^B_j(\mu^j_{-j})} \left(\sup_{\hat{\pi}'(\mu_j) \in \hat{\Pi}'(\mu_j, \mu^j_{-j})} \hat{\Phi}(w_j, \mu_j, \mu^j_{-j}, \hat{\pi}'(\mu_j)) \right) \quad (37)$$

■

Therefore, given $\mu^j_{-j} \in (\Omega^D)^{J-1}$ that non-deviating principal will assign upon principal j's deviation, principal j only needs to consider deviation to a DIC direct mechanism.

5.4 DIC lower bound

Proposition 4 allows us to specify the greatest lower bound for deviating principal j's payoff by using only DIC direct mechanisms as follows.

$$\bar{\Phi}^*_j := \inf_{\mu^j_{-j} \in (\Omega^D)^{J-1}} \left[\sup_{\mu_j \in \Omega^D} \left(\sup_{\hat{\pi}'(\mu_j) \in \hat{\Pi}'(\mu_j, \mu^j_{-j})} \hat{\Phi}(w_j, \mu_j, \mu^j_{-j}, \hat{\pi}'(\mu_j)) \right) \right]$$

Now we can establish the characterization theorem with the DIC lower bound $\bar{\Phi}^*_j$.

Theorem 2 *We have* $\Phi_j^* = \bar{\Phi}_j^*$ *for all* j*. Therefore, the set of equilibrium allocations* Ψ_B^* *that can be supported in a robust DIC-response equilibrium is equal to the following set:*

$$\bar{\Psi}_B^* := \{(\bar{\mu}, \pi(\bar{\mu})) \in \Psi^B : \hat{\Phi}(w_j, \bar{\mu}_j, \bar{\mu}_{-j}, \pi(\bar{\mu})) \geq \bar{\Phi}_j^* \ \forall j\}.$$

Proof. Because of (37), $\Phi_j^* = \bar{\Phi}_j^*$ for all j and therefore, $\Psi_B^* = \bar{\Psi}_B^*$. ■

The greatest lower bound Φ_j^* for each principal j's payoff that can be supported in robust DIC-response equilibrium involves the supremum over direct mechanisms in $\mathcal{M}_j^B(\mu_{-j}^j)$. By using Proposition 4 that is established due to the BIC-DIC equivalence, Theorem 2 shows that we can replace $\mathcal{M}_j^B(\mu_{-j}^j)$ with Ω^D. Therefore, we only need to consider DIC mechanisms for the specification of the greatest lower bound for each principal's payoff that can be supported in robust DIC-response equilibrium.

5.5 Competing dominant strategy implementation

To study robust DIC-response equilibrium "allocation", one should still consider a BIC allocation. The reason is that the BIC-DIC equivalence concerns only the payoffs to agents and principals but not the allocation per se, so that the DIC direct mechanism that replaces a BIC direct mechanism may induce different ex-post allocation. However, if our interest is the profile of equilibrium payoffs but not the equilibrium allocation, then we can also focus on a DIC allocation.[16]

Similar to the definition of the set of BIC allocation Ψ^B in (22), define the set of DIC allocations Ψ^D as follows:

$$\Psi_D := \{(\bar{\mu}, \pi(\bar{\mu})) : (\bar{\mu}, \pi(\bar{\mu})) \text{ is a DIC allocation}\}.$$

Now define the set

$$\bar{\Psi}_D^* := \{(\bar{\mu}, \pi(\bar{\mu})) \in \Psi^D : \hat{\Phi}(w_j, \bar{\mu}_j, \bar{\mu}_{-j}, \pi(\bar{\mu})) \geq \bar{\Phi}_j^* \ \forall j\}.$$

The set $\bar{\Psi}_D^*$ includes all DIC allocations that can be supported in a robust DIC-response equilibrium. Consider a BIC allocation $(\bar{\mu}, \pi(\bar{\mu})) \in \bar{\Psi}_B^*$ that can be supported in a robust DIC-response equilibrium, where $\bar{\mu}$ is a profile of BIC direct mechanisms.

[16]A DIC allocation $(\bar{\mu}, \pi(\bar{\mu})$ means that it is a continuation equilibrium that agents report their true types upon selecting a principal according to the selection strategy $\pi(\bar{\mu})$ when principals offer $\bar{\mu}$.

Theorem 3 *The set of all payoff profiles associated with DIC allocations in* $\bar{\Psi}_D^*$ *is the same as the set of all payoff profiles associated with BIC allocations in* $\bar{\Psi}_B^*$.

Proof. Note that $\bar{\Psi}_D^* \subset \bar{\Psi}_B^*$ because both are defined with the same greatest lower bound for each principal j's payoff and Ψ^D is a subset of Ψ^B. Therefore, we only need to consider a BIC allocation $(\bar{\mu}, \pi(\bar{\mu}))$ that is in $\bar{\Psi}_B^*$ but not in $\bar{\Psi}_D^*$. By applying Proposition 3 repeatedly over all principals, we can show that there exists a DIC allocation $(\bar{\mu}', \pi'(\bar{\mu}'))$ such that (i) $\pi'(\bar{\mu}') = \pi(\bar{\mu})$ and (ii) payoffs to all principals and agents are preserved. ■

Consider a DIC allocation $(\bar{\mu}, \pi(\bar{\mu}))$ in $\bar{\Psi}_D^*$. For its implementation, each principal ℓ can use a deviator-reporting direct mechanism that assigns the DIC direct mechanism $\bar{\mu}_\ell$ on the equilibrium path and also a DIC direct mechanism μ_ℓ^j that punishes principal j off the path following his deviation.[17] Because both direct mechanisms that the deviator-reporting direct mechanism induces are DIC, we can always fix agents' truthful type reporting on and off the equilibrium path.

6 Discussion

6.1 Random mechanism offer

So far we studied robust DIC-response equilibrium where principals employ pure strategies. The results can be extended for robust DIC-response equilibrium where principals employ mixed strategies.

Consider a DIC-response equilibrium $\{\tilde{\sigma}, \tilde{c}, \tilde{\pi}\}$ in which the support of principal ℓ's mixed strategy σ_ℓ only includes deviator-reporting direct mechanisms, such that each deviator-reporting direct mechanism in the support of σ_ℓ assigns a DIC direct mechanism on the equilibrium path and all of them in the support assign the same DIC direct mechanism, say $\mu_\ell^j \in \Omega^D$ upon principal j's deviation ($j \neq \ell$). For example, suppose that principal ℓ offers deviator-reporting direct mechanisms $\tilde{\gamma}_\ell$ and $\tilde{\gamma}'_\ell$ with equal probability. Then, $\tilde{\gamma}_\ell$ assigns a DIC direct mechanism μ_ℓ, but $\tilde{\gamma}'_\ell$ assigns a different DIC direct mechanism μ'_ℓ on the equilibrium path. However, they all assign the same DIC direct mechanism $\mu_\ell^j \in \Omega^D$.

Let $\delta = (\delta_1, \ldots, \delta_J)$ be a profile of probability distributions with $\delta_j \in \Delta(\Omega^D)$. Define the

[17] For example, $\bar{\mu}_\ell$ can be a second-price auction with high reserve price (or a very high single posted-price) that cannot be sustained if each principal ℓ directly offer $\bar{\mu}_\ell$ and μ_ℓ^j is a second-price auction with very low reserve price (or very low single posted-price) that can attract enough agents away from deviating principal j's mechanism.

set of random DIC allocations as follows,[18]

$$\tilde{\Psi}_D := \left\{(\delta,\pi) : \delta \in \Delta(\Omega^D)^J \text{ and } (\mu,\pi(\mu)) \text{ is a DIC allocation for all } \mu \in \prod_{j=1}^J supp\ \delta_j\right\}$$

Given any $(\delta,\pi) \in \tilde{\Psi}_D$, principal j's payoff is then specified by

$$\Phi(w_j,\delta_j,\delta_{-j},\pi) := \int_{\Omega^D}\int_{(\Omega^D)^{J-1}} \hat{\Phi}(w_j,\mu_j,\mu_{-j},\pi(\mu))d\delta_{-j}d\delta_j.$$

Applying the logic in Propositions 2 and 4, we can establish the following proposition:

Proposition 5 *Any $(\delta,\pi) \in \tilde{\Psi}_D$ can be supported as an equilibrium allocation in a robust DIC-response equilibrium $(\tilde{\sigma},\tilde{c},\tilde{\pi})$ in a market with $\tilde{\Gamma}$ if and only if, for every $j \in \mathcal{J}$, there exists $\mu^j_{-j} \in (\Omega^D)^{J-1}$ such that*

$$\Phi(w_j,\delta_j,\delta_{-j},\pi) \geq \sup_{\mu_j \in \mathcal{M}^B_j(\mu^j_{-j})} \left(\sup_{\hat{\pi}'(\mu_j)\in\hat{\Pi}'(\mu_j,\mu^j_{-j})} \hat{\Phi}(w_j,\mu_j,\mu^j_{-j},\hat{\pi}'(\mu_j))\right), \tag{38}$$

where $\mathcal{M}^B_j(\mu^j_{-j})$ can be replaced with Ω^D when the BIC-DIC equivalence holds.

The reason why (38) provides the necessary and sufficient condition for $(\delta,\pi) \in \tilde{\Psi}_D$ to be supported by a robust DIC-response equilibrium $(\tilde{\sigma},\tilde{c},\tilde{\pi})$ is that all deviator-reporting direct mechanisms in the support of $\tilde{\sigma}_\ell$ respond to principal j's deviation with the same DIC direct mechanism, say $\mu^j_\ell \in \Omega^D$. Therefore, whether $(\delta,\pi) \in \tilde{\Psi}_D$ can be supported in a robust DIC-response equilibrium $(\tilde{\sigma},\tilde{c},\tilde{\pi})$ hinges on the existence of $\mu^j_{-j} \in (\Omega^D)^{J-1}$ that satisfies (38). If it exists, we can use it to construct deviator-reporting direct mechanisms that are included in the support of principals' mixed strategies. Because non-deviating principals' deviator-reporting mechanisms always assign μ^j_{-j} upon principal j's deviation regardless of the realization of their deviator-reporting direct mechanisms from $\tilde{\sigma}_{-j}$, (38) implies that each principal j only needs to consider deviation to a BIC direct mechanism in $\mathcal{M}^B_j(\mu^j_{-j})$ to see if there is a profitable deviation (or equivalently a DIC direct mechanism if the BIC-DIC equivalence holds).

Since we consider a robust DIC-response equilibrium $(\tilde{\sigma},\tilde{c},\tilde{\pi})$ where principals assign DIC direct mechanisms on the path, we can provide the characterization of a competing DIC implementation by a robust DIC-response equilibrium where principals also employ

[18]A random DIC allocation means not only that agents use mixed strategies for selecting a principal, but the realization of a profile of DIC direct mechanism is random.

mixed strategies. For that, define the set

$$\tilde{\Psi}^*_D := \{(\delta, \pi) \in \tilde{\Psi}^D : \hat{\Phi}(w_j, \delta_j, \delta_{-j}, \pi) \geq L_j \ \forall j\},$$

where

$$L_j = \begin{cases} \Phi^*_j & \text{without the BIC-DIC equivalence,} \\ \bar{\Phi}^*_j (= \Phi^*_j) & \text{with the BIC-DIC equivalence.} \end{cases}$$

Theorem 4 $\tilde{\Psi}^*_D$ *is the set of random DIC allocations that can be supported in a robust DIC-response equilibrium where a principal's mechanism induces a DIC direct mechanism on the path and a DIC direct mechanism off the path following a competing principal's deviation.*

When the BIC-DIC equivalence is satisfied, the lower bound Φ^*_j for each principal j's payoff can be replaced with $\bar{\Phi}^*_j$ as shown in Theorem 2 and hence L_j above reflects it. Theorem 4 can be proved similar to Theorem 1, so the proof is omitted.

6.2 Robust EPIC-response equilibrium

Propositions 1 and 2, and Theorem 1 in section 4 can all be generalized for the case with interdependent-value types. In this case, a non-deviating principal punishes a deviating principal with an ex-post incentive compatible (EPIC) direct mechanism in a robust EPIC-response equilibrium. The EPIC property originates from a single principal's mechanism, where an agent cannot increase her ex-post payoff by reporting a false type with any realization of other agents' types given the other agents' truthful type reporting (Bergemann and Morris 2005, Cremer and McLean 1985, Dasgupta and Maskin 2000, Perry and Reny 2002).

For the generalization of Propositions 1 and 2, and Theorem 1, let an agent's payoff depend on both her own type and the types of agents who select the same principal. Let $U_i(\alpha, x, \mathbf{x})$ be the payoff to agent i of type $x \in X$ given $\alpha \in \mathcal{A}$ and $\mathbf{x} \in \bar{X}^{I-1}$. The payoff to principal j is $U^j(\alpha, \mathbf{x})$ given $\alpha \in \mathcal{A}$ and $\mathbf{x} \in \bar{X}^I$. A direct mechanism $\mu_j : \bar{X}^I \to \mathcal{A}$ is EPIC if, for all $x, x' \in X$, all $\mathbf{x} \in \bar{X}^{I-1}$,

$$U_i(\mu_j(x, \mathbf{x}), x, \mathbf{x}) \geq U_i(\mu_j(x', \mathbf{x}), x, \mathbf{x}).$$

Because the EPIC property is based on the agent's ex-post payoff, similar to the DIC property, one can fix the agent's truthful type report regardless of agents' selection strategies. This induces tractable equilibrium analysis for the market with EPIC direct mechanisms.

Propositions 1 and 2 can be properly modified given that a non-deviating principal punishes a deviating principal j with the same EPIC direct mechanism regardless of his deviation in a robust EPIC-response equilibrium. For the generalization of Theorem 1, let Ω^E be the set of all EPIC direct mechanisms. Define the greatest lower bound for the payoff to principal j in the following way.

$$\Phi_j^E := \inf_{\mu_{-j}^j \in (\Omega^E)^{J-1}} \left[\sup_{\mu_j \in \mathcal{M}_j^B(\mu_{-j}^j)} \left(\sup_{\hat{\pi}'(\mu_j) \in \hat{\Pi}'(\mu_j, \mu_{-j}^j)} \hat{\Phi}(w_j, \mu_j, \mu_{-j}^j, \hat{\pi}'(\mu_j)) \right) \right]$$

Then, Theorem 1 can be modified as follows.

Theorem 5 *The set of equilibrium allocations that can be supported in a robust EPIC-response equilibrium is*

$$\Psi_E^* := \{(\bar{\mu}, \pi(\bar{\mu})) \in \Psi^B : \hat{\Phi}(w_j, \bar{\mu}_j, \bar{\mu}_{-j}, \pi(\bar{\mu})) \geq \Phi_j^E \ \forall j\}.$$

We can also establish the characterization of a random robust EPIC-response equilibrium allocations as we did for a random robust DIC-response equilibrium allocations in Theorem 4 in the previous subsection. However, the BIC-EPIC equivalence generally does not hold (See an example in Gershokov, et al. 2013).

6.3 Competition in DIC direct mechanisms

Consider a market where each principal can offer a direct mechanism in a subset of DIC direct mechanisms $\tilde{\Omega}^D \subset \Omega^D$, e.g., second price auctions. Competition in a market with $(\tilde{\Omega}^D)^J$ is simple because each principal does not change his direct mechanism regardless of a competing principal's deviation and hence it induces an equilibrium that is market-information free.

We can provide the necessary and sufficient condition for a DIC allocation $(\bar{\mu}, \pi(\bar{\mu})) \in \Psi^D$ with $\bar{\mu} \in (\tilde{\Omega}^D)^J$ to be supported in a robust (DIC-response) equilibrium in a market with $(\tilde{\Omega}^D)^J$. A DIC direct mechanism $\bar{\mu}_\ell$ for each principal ℓ can be thought of as a deviator-reporting mechanism that assigns the same DIC direct mechanism regardless of agents' reports on the identity of a deviating principal. Therefore principal j takes $\bar{\mu}_{-j}$ as given when he deviates. Given $\bar{\mu}_{-j}$, principal j only needs to consider either a deviation to a BIC direct mechanism in general, according to Proposition 2, or a deviation to a DIC direct mechanism, according to Proposition 4, when the BIC-DIC equivalence is satisfied. Subsequently, we can establish the following proposition

Proposition 6 *A DIC allocation $(\bar{\mu}, \pi(\bar{\mu}))$ with $\bar{\mu} \in (\tilde{\Omega}^D)^J$ can be supported in a robust*

(DIC-response) equilibrium in a market with $(\tilde{\Omega}^D)^J$ *if and only if for every* $j \in \mathcal{J}$,

$$\hat{\Phi}(w_j, \bar{\mu}_j, \bar{\mu}_{-j}, \pi(\bar{\mu})) \geq \sup_{\mu_j \in Z} \left(\sup_{\hat{\pi}'(\mu_j) \in \hat{\Pi}'(\mu_j, \bar{\mu}_{-j})} \hat{\Phi}(w_j, \mu_j, \bar{\mu}_{-j}, \hat{\pi}'(\mu_j)) \right), \tag{39}$$

where

$$Z = \begin{cases} \Omega^D & \text{if the BIC-DIC equivalence holds,} \\ \mathcal{M}_j^B(\bar{\mu}_{-j}) & \text{otherwise.} \end{cases}$$

Proof. This result comes from Propositions 2 and 4. Consider a DIC allocation $(\bar{\mu}, \pi(\bar{\mu}))$ with $\bar{\mu} \in (\tilde{\Omega}^D)^J$. Condition 1 in Propositions 2 and 4 is redundant because we already specify $(\bar{\mu}, \pi(\bar{\mu}))$ as a DIC allocation. A DIC direct mechanism $\bar{\mu}_\ell$ is equivalent to the deviator-reporting direct mechanism that assigns $\bar{\mu}_\ell$ regardless of agents' reports on the identity of a deviating principal. It implies that $\mu_\ell^j = \bar{\mu}_\ell$ is assigned to each principal ℓ ($\ell \neq j$) upon j's deviation. Replacing μ_ℓ^j with $\bar{\mu}_\ell$ in condition 2 in Propositions 2 and 4 yields (39). It completes the proof. ■

Proposition 6 provides the necessary and sufficient condition for a DIC allocation to be supported in a robust DIC-response equilibrium that is independent of transmission of market information from agents to principals because each principal always sticks to his DIC direct mechanism on and off the path. Proposition 6 is useful in reevaluating the literature on directed search.

Coles and Eekhout (2003) consider a market where buyers have the unit demand and their valuations are all equal to Q and observable. In this case, any direct mechanism, let alone DIC direct mechanism, specifies the price as the function of the number of participating buyers.[19] They consider a market with two sellers and two buyers where sellers are able to offer any direct mechanism. Because there are two buyers, any direct mechanism is a pair of prices (p_1, p_2), where p_i is the price when i is the number of participating buyers. Suppose that (p_1, p_2) is a direct mechanism offered by seller 1 and (p_1', p_2') is offered by seller 2. There may be multiple continuation equilibrium strategies for selecting deviating seller j given his direct mechanism. If that's the case, Coles and Eekhout (2003) select the one that is the best for the deviating seller. Therefore, their equilibrium analysis indeed satisfies (39) that takes the supremum over the set of possible continuation equilibrium strategies for selecting a deviating principal j. It implies that equilibrium in Coles and Eekhout (2003) is a robust equilibrium, independent of transmission of market information from agents. They

[19]Because all buyers are identical, anonymous direct mechanism assigns equal probability of trading with each buyer if multiple buyers visit.

derive the complete set of equilibrium allocations, which is therefore the complete set of equilibrium allocations that are robust and independent of market information transmission. Any equilibrium is characterized as the pair of direct mechanisms (p_1, p_2) and (p'_1, p'_2) with $p_1 = p'_1 = Q/2$ and $p_2 = p'_2 \in (0, Q]$. On the equilibrium path, each buyer chooses each seller with equal probability.

Suppose that buyers' valuation is their private information. Many papers (Peters and Severinov 1997, Peters 1997, Virag 2010, Burguet and Sakovics 1999) consider sellers' competition in second-price auctions with reserve prices. Note that second-price auctions are DIC, but in the finite market with the finite number of sellers, we may not have a pure-strategy equilibrium (Burguet and Sakovics, 1999). If a pure-strategy equilibrium does exists and (39) is satisfied, then it is a robust DIC-response equilibrium, independent of market information transformation. Han (2014a) shows that indeed the notion of equilibrium in Peters (1997), for the large market with the infinite number of sellers and buyers, includes the robustness property. Let $\hat{\Phi}^J$ be a seller's payoff function when the number of sellers is J. Then, the robustness condition can be rewritten as

$$\lim_{J\to\infty} \hat{\Phi}^J(w_j, \bar{\mu}_j, \bar{\mu}_{-j}, \pi(\bar{\mu})) \geq \sup_{\mu_j \in \mathcal{M}_j^B(\bar{\mu}_{-j})} \left(\sup_{\hat{\pi}'(\mu_j) \in \hat{\Pi}'(\mu_j, \bar{\mu}_{-j})} \lim_{J\to\infty} \hat{\Phi}^J(w_j, \mu_j, \bar{\mu}_{-j}, \hat{\pi}'(\mu_j)) \right). \quad (40)$$

The limit version of Proposition 6 can also establish (40) as the robsutenss condition. Therefore, Proposition 6 provides an alternative approach for establishing the robustness of competing second price auction equilibrium in the large market.

6.4 Restrictions in communication

This paper assumes that agents can freely communicate with any principal, even though an agent eventually selects one principal. This reflects the fact that agents or buyers may shop around before making a purchase decision. Even if we restrict the number of principals an agent can communicate with, various allocations can be supported in a robust DIC-response equilibrium.

The most strict restriction in an agent's communication is that the agent can communicate only with the principal he selects. Given a probability that each agent selects a non-deviating principal in a continuation equilibrium, there is a positive probability that a non-deviating principal has only one participating buyer or two. In this case, a non-deviating principal is not certain whether a single participating agent reports the true identity of a deviating principal or which agent reports the truth when their reports are inconsistent; a non-deviating principal may not change his direct mechanism with one or two participating agents. However, there is positive probability that a non-deviating principal has three or more participating buyers. Upon three or more agents' participation, a non-deviating principal can change his direct mechanism when there is a deviating principal. It means that a competing principal's deviation will be detected with some probability less than one. For example, principal ℓ can desgin a deviator-reporting direct mechanism that assigns μ_ℓ^j only when there are three or more participating agents and more than half of them report principal j's deviation. Otherwise, the deviator-reporting direct mechanism always assigns $\bar{\mu}_\ell$.

Principals can still sustain a lot of allocations in a robust DIC-response equilibrium, but the scope of collusion diminishes because a non-deviating principal changes his direct mechanism only when three or more agents select him upon a competing principal's deviation. The probability with which a principal has three or more participating agents depends on the ratio of the number of sellers to the number of buyers. Of course, when monetary transfer is in the model, a non-deviating principal can change his direct mechanism following the participation of two or more agents. Han (2014b) provide comparative statics on the set of robust equilibrium prices as this ratio changes in the model with monetary transfers, where buyers' valuation is public information.

7 Conclusion

Directed search models flourish in applications with frictions that are caused by lack of coordination in a decentralized economy. The purpose of this paper is to provide a tractable theory on robust competing mechanism design for directed search models with frictions.

This paper proposes the notion of a DIC-response equilibrium in which a non-deviating principal responds to a principal's deviation with a DIC direct mechanism. Because the DIC property is independent of the number of participating agents or their type reporting, the set of DIC direct mechanisms is defined independent of those endogenous features of the model. This makes it possible to characterize the set of robust DIC-response equilibrium allocations by specifying the greatest lower bound for a principal's payoff as the min-max-max over incentive compatible direct mechanisms. The paper also shows that the robustness of a DIC-response equilibrium can be checked by considering a principal's deviation only to an incentive compatible direct mechanism.

The paper shows that the most recent result on the BIC-DIC equivalence for a single principal's mechanism design problem is naturally extended to the case with multiple principals. It further simplifies the main results of the paper. Given the prevalence of linear payoff structures in applications, the results due to the BIC-DIC equivalence provide an added tractability to the theory.

As we discussed in Section 6, the theory developed here is sufficiently general to provide rich implications on how to model principals' competition in a market with frictions and how to check if a (DIC-response) equilibrium in a market is robust. This is particularly important in decentralized market design because it is directly related to the stability of a decentralized market institution.

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