Endogenous Stock Price Cycles, Chaos and Sunspot Equilibria with Dynamic Self-Control Preferences

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Abstract

This paper studies the global equilibrium dynamics implied by a Lucas' tree asset pricing model where the representative agent has dynamic self-control preferences, as defined by Gul and Pesendorfer (Econometrica, 2004). It shows that endogenous cycles of period 2 and higher, as well as chaotic dynamics exist provided temptation utility is sufficiently important (with respect to standard commitment utility) and sufficiently convex. For parameterizations leading to complex deterministic dynamics, the model also admits stationary and non-stationary sunspot equilibria.

Keywords: Asset Pricing, Temptation, Self-Control, Endogenous Cycles, Chaotic Dynamics, Sunspot Equilibrium

JEL Classifications: C62, E32, G12

1 Introduction

It is well known that, for any realistic parameterization, representative agent consumption-based asset pricing models with standard constant relative risk aversion preferences - e.g., the Lucas's tree asset pricing model - fail to capture several empirical features of stock price data,

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among which the fact that stock prices appear to be much more volatile than the underlying fundamentals. One reason behind the failure is that, in such framework, under rational expectations, the pricing equation for the risky stock (linking current to future stock prices) is a first order stochastic *linear* difference equation whose unique stable solution features either a constant or a stationary price-dividend ratio.

In the literature, the great majority of attempts to generate excess stock price volatility while preserving the great analytical tractability of the Lucas' tree representative agent framework have, to different extents, involved some departure from the RE hypothesis. Among the many contributions in this direction, two recent ones have produced interesting results. Lansing (2010) derives a general class of rational and near rational bubble solutions in an otherwise standard Lucas' tree environment. He shows that bubbles can be intrinsic (i.e., bubble innovations depend on fundamentals), driftless and recurrent (i.e. they can revert to a zero mean, and show periods where they drop below the underlying fundamentals). The author also shows how a near-rational solution is able to match several unconditional moments of the U.S. price-dividend ratio. Adam, Marcet and Nicolini (2013) assume that the representative agent does not have full knowledge about the pricing function mapping fundamentals (i.e. dividends) into equilibrium prices, but rather holds subjective believes about pay-off relevant variables, namely stock prices and dividends. Beliefs are self-referential: expectations are affected by realized stock prices, while stock prices depend on individual expectations. The authors show that learning from market outcomes leads to momentum and mean reversion in asset prices, similar to what observed in the data.

This paper explores an alternative route, namely the possibility that boom-bust cycles in stock prices could be the result of endogenous belief-driven fluctuations. For that purpose, it considers a deterministic version of a Lucas' tree model (hence, an enviroment with completely stable fundamentals) where the representative agent is characterized by dynamic self-control (henceforth, DSC) preferences as formalized by Gul and Pesendorfer (2002, 2004). In such set-up, the agent faces a temptation to liquidate his entire financial wealth for the purpose of immediate consumption. Resisting temptation involves effort (or self-control) and hence some disutility. Optimal behavior therefore trades off the temptation for immediate satisfaction with long-run optimal consumption smoothing. Gul and Pesendorfer (2004) prove the existence of a recursive representation for dynamic optimization problems with DSC preferences, showing how the pricing kernel for a risky stock becomes a function of future stock prices. While they discuss about the quantitative implication of DSC preferences for the equity premium puzzle, they do not characterize the global equilibrium dynamics generated by the *non-linear* asset pricing equation. This is the objective of my paper.

Using both analytical and numerical techniques, I identify parametric conditions for the existence of both endogenous periodic cycles - i.e., equilibria where the stock price switches deterministically betweeen a finite set of values - and chaotic dynamics - i.e., highly volatile aperiodic (but still deterministic) equilibrium trajectories. I show that the fundamental steady state equilibrium is the unique feasible equilibrium when temptation utility is linear or concave. On the contrary, when temptation utility is convex (and sufficiently so), endogenous cycles of periodicity 2 and higher arise as temptation becomes more important. After the parameter indexing the importance of temptation and self-control passes a certain threshold, the stock price dynamics become chaotic. I also show that, for parameterizations leading to complex deterministic dynamics, it is possible to construct both stationary and non-stationary sunspot equilibria - i.e., equilibrium trajectories that depend on the realizations of a non-fundamental sunspot shock (extrinsic uncertainty) - along the lines of a seminal paper by Azariadis and Guesnerie (1986) and a more recent work by Lagos and Wright (2003).[1]

Few papers have explored the asset pricing implications of DSC preferences. Krusell et al. (2002) show that embedding an otherwise standard economy with a small subset of agents subject to temptation and self-control allows to get a sizable equity premium (together with a lower riskless rate) for degrees of risk aversion which are much lower than what needed in a standard costant relative risk aversion (henceforth, CRRA) set-up without

[1]Azariadis and Guesnerie (1986) study the existence of cycles and sunspot equilibria in a OLG economy. Lagos and Wright (2003) show that these equilibria also exist in a disaggregated model of monetary exchange.

DSC preferences. DeJong and Ripoll (2007) study the ability of DSC preferences to account for the stock price volatility, the risk free rate puzzle and the equity premium puzzle in U.S. data. Their work differs from mine in the fact that their analysis is restricted to a log-linear environment around the model's steady state, and therefore does not explore the global non-linear dynamics implied by DSC preferences.[2] Huang et al. (2013) test for the presence of temptation in portfolio decisions by estimating the consumption Euler equation implied by DSC preferences on U.S. data. Using micro data on IRA and 401(k) accounts, they exploit the fact that, according to the model, invididual with temptation and self-control issues would be more likely to hold (illiquid) commitment assets.

The introduction of DSC preference is certainly not the only way to make the pricing kernel of a risky asset a function of financial wealth. Barberis et al. (2001) show that a Lucas' tree economy where investor are loss averse with respect to fluctuations in their financial wealth can generate risk premia and stock price volatility consistent with the data. Bansal and Yaron (2004) show how Epstein-Zin non-recursive preferences combined with long-run risk can be a solution to a variety of asset pricing puzzles.[3] Bakshi and Chen (1996) and Smith (2001) consider wealth-dependent preferences due to status-seeking or Max Weber's spirit of capitalism hypothesis. They show that wealth-dependent preference introduce an additional risk factor to an otherwise standard Lucas' tree model, and for these reason can generate a higher risk premium. Kamihigashi (2010) shows that wealth-dependent preferences can generate rational bubbles.

The rest of the papaer is organized as follows. Section 2 described the model and derives the related asset pricing equation. Section 3 characterizes the global deterministic equilibrium dynamics of the model. Section 4 discusses about the existence of sunspot equilibria. Section 5 concludes.

[2] They also restrict to a concave temptation utility. In their environment dividends are assumed to grow stochastically around a long-run trend. Concave temptation utility guarantees the existence of a balanced-growth steady state equilibrium.

[3] Backus and Zin (2004) provide an extensive review of non-recursive, as well as other "exotic" preferences used in macroeconomics.

2 The model

The economy is populated by a continuum of identical infinitively-lived households. There is only one asset in this economy, which I call "stock". In every period t, the stock generates exogenous dividends d_t, as in a standard Lucas' tree model. The representative agent buys stock shares a_t at the market price p_t per share. The household uses the financial wealth generated by the stock to finance consumption of a single perishable good and purchases of new shares. His period budget constraint is:[4]

$$c_t + p_t a_t = (p_t + d_t)\, a_{t-1} \tag{1}$$

The households has DSC preferences, as formalized by Gul and Pesendorfer (2002, 2004). DSC preferences imply that, in each period, the household faces the temptation to liquididate all his financial wealth for current consumption purposes, and that in order to resist this temptation he has to incur a self-control effort (disutility). Gul and Pesendorfer (2004) prove the existence of a recursive representation for the household's intertemporal utility maximization problem:

$$\mathcal{W}\,(a_{t-1}, d_t) = \max_{c_t,\, a_t} \{u\,(c_t) + v\,(c_t) + \beta E_t \mathcal{W}\,(a_t, d_{t+1})\} - \max_{\tilde{c}_t,\, \tilde{a}_t} v\,(\tilde{c}_t) \tag{2}$$

where $\beta \in (0, 1)$, and u and v are both Von Neuman-Morgenstern utility functions. While u represents a standard *commmitment* utility, v captures *temptation.* Letting $\tilde{c}_t$ and $\tilde{a}_t$ denote, respectively, the optimal levels of consumption and new stock shares chosen by the household in period t if he falls to temptation, the term $v\,(c_t) - \max_{\tilde{c}_t,\, \tilde{a}_t} v\,(\tilde{c}_t) \leq 0$ corresponds to the disutility due to self-control that the household suffers when he choses c_t instead of $\tilde{c}_t$.

For analytical tractabitliy, I assume the following functional forms for u and v:

[4] All results would equally hold if I introduced exogenous earnings as an additional source of income for the household.

$$u(c_t) = \frac{c_t^{1-\sigma} - 1}{1-\sigma} - \frac{\lambda}{1+\chi} c_t^{1+\chi}, \qquad v(c_t) = \lambda \frac{c_t^{1+\chi} - 1}{1+\chi}. \tag{3}$$

for $\sigma \geq 0$ and $\lambda \geq 0$. This specification has the following features. First, it nests the CRRA utility specification used in benchmark asset-pricing models (for $\lambda = 0$), and the specification used in Section 6.4 in Gul Pesendorfer (2004) (for $\sigma = 0$ and $\chi = 1$) to discuss about the risk premium implications of DSC preferences. Second, temptation utility $v(c_t)$ is strictly increasing in c_t, but can be either linear, strictly concave or strictly convex:

$$\text{sign}\left[v''(c_t)\right] \begin{cases} < 0 & \text{for} \quad \chi < 0\text{: concave temptation utility} \\ = 0 & \text{for} \quad \chi = 0\text{: linear temptation utility} \\ > 0 & \text{for} \quad \chi > 0\text{: convex temptation utility} \end{cases} \tag{4}$$

Third, total momentary utility $u(c_t) + v(c_t)$ is strictly increasing and strictly concave. As in Gul and Pesendorfer (2004), this guarantees that, although temptation utility could be convex (hence, risk loving temptation), temptation will never lead to risk loving behavior.

Due to the strict monotonicity of $v(c_t)$, the dynamic programming problem (2) is equivalent to the following:

$$\mathcal{W}(a_{t-1}, d_t) = \max_{c_t,\, a_t} \{u(c_t) + v(c_t) + \beta E_t \mathcal{W}(a_t, d_{t+1})\} - v[(p_t + d_t) a_{t-1}] \tag{5}$$

subject to the budget constraint (1). The first order condition associated with the choice of optimal holdings of stock shares gives the following asset pricing condition:

$$p_t = E_t \left[m_{t,t+1}^{DSC} (p_{t+1} + d_{t+1})\right] \tag{6}$$

where $m_{t,t+1}^{DSC}$ is the individual stochastic discount factor (henceforth, SDF) - or pricing kernel of the risky stock - defined as

$$m_{t,t+1}^{DSC} \equiv \beta \frac{u'(c_{t+1}) + v'(c_{t+1}) - v'[(p_{t+1} + d_{t+1}) a_t]}{u'(c_t) + v'(c_t)} \tag{7}$$

Since the focus of the analysis will be on the existence of endogenous deterministic dynamics, I eliminate aggregate uncertainty by assuming constant dividend payments: that is, $d_t =$

$d = 1$ in every period t (where without loss of generality, I have normalized them to unity). Using the market clearing conditions $c_t = 1$ and $a_t = 1$ (where, without loss of generality, I have assumed a constant supply of stock shares and normalized it to unity), $m^{DSC}_{t,t+1}$ reduces to

$$m^{DSC}_{t,t+1} = \tilde{m}(p_{t+1}) \equiv \beta \frac{u'(1) + v'(1) - v'[(p_{t+1} + 1)]}{u'(1) + v'(1)}, \tag{8}$$

The SDF is therefore a continuous function $\tilde{m}$ of next period stock price p_{t+1}. Since $v' > 0$, temptation makes the SDF lower with respect to the benchmark CRRA case (where $v' = 0$): i.e., $m^{DSC}_{t,t+1} < m^{CRRA}_{t,t+1} = 1$. This implies that, for a given future stock price and given dividends, under temptation, the risky stock becomes less valuable with respect to the CRRA case as higher financial wealth increases the cost of self-control for the agent. Moreover, by the conditions spelled in (4), the SDF can be either increasing or decreasing in p_{t+1} depending on whether temptation utility is concave or convex: i.e., $\tilde{m}'(p_{t+1}) \gtreqqless 0$ for $\chi \lesseqqgtr 0$. The sign of $\tilde{m}'(p_{t+1})$ will be crucial for the existence of endogenous cycles.

By combining (6) and (8), using the specifications in (3), and letting $w_t \equiv p_t + 1$ denote cum-dividend beginning-of-period wealth, we obtain the following dynamic equation:[5]

$$w_t = 1 + \beta \left[1 - \lambda w^{\chi}_{t+1}\right] w_{t+1} \equiv F(w_{t+1}) \tag{9}$$

Since the model does not feature aggregate uncertainty, I look for non-explosive perfect foresight equilibrium (PFE) sequences $\{w_t\}^{\infty}_{t=0}$ solving the non-linear difference equation (9). Notice that, for $\lambda = 0$ (no DSC preferences), equation (9) reduces to $w_t = 1 + \beta w_{t+1}$, which corresponds to a Lucas's tree asset pricing equation with CRRA utility and constant dividends. It is straightforward to show that, in this case, the unique non-explosive solution to (9) is $w_t = \frac{1}{1-\beta}$ (or equivalently, $p_t = \frac{\beta}{1-\beta}$). With constant fundamentals (i.e., constant dividends), equilibrium wealth and the stock price are constant as well. For $\lambda > 0$ (DSC preferences) instead, the right hand side of (9) becomes non-linear, and there could be time-

[5] Since w_t is a simple scale transformation of p_t, the equilibrium dynamics of the latter will be isomorphic to those of w_t.

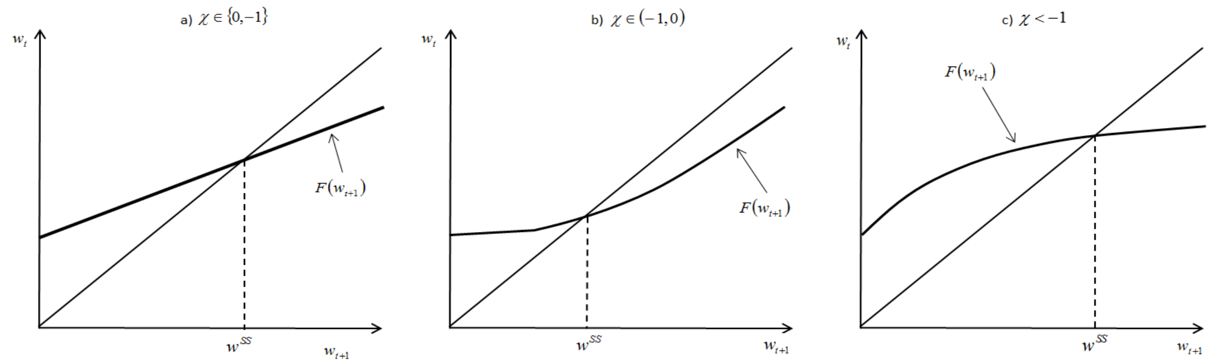


Figure 1: **Mapping F with Linear or Concave Temptation.**

varying PFE trajectories despite the lack of aggregate uncertainty. The purpose of the analysis to follow is to provide a full characterization of those trajectories. In particular, I will assess the possibility of periodic cycles and (topologically) chaotic dynamics.

Definition 1 ***Period-n Cycle*** *A value "q" is a point of a period-n cycle if it is a fixed point of the n-th iterate of a mapping f(.) i.e.* $q = f^n(q)$, *but not a fixed point of an iterate of any lower order. If "q" is such we call the sequence* $\{q, f(q), f^2(q), ..., f^{n-1}(q)\}$ *a period-n cycle.*

Topological Chaos *A mapping* $f(.)$ *is topologically chaotic if there exists a set "S" of uncountable many initial points, belonging to its domain, such that no orbit that starts in "S" will converge to one another or to any existing periodic orbit.*

3 Global Dynamics

This section provides full characterization of the global equilibrium dynamics implied by the non-linear difference equation (9). Since the stock price p_t cannot be negative, any sequence $\{w_t\}_{t=0}^{\infty}$ solving (9) must satisfy the restriction $w_t \geq 1$ for every $t \geq 0$. I start by considering the case where temptation utility is either linear ($\chi = 0$) or strictly concave ($\chi < 0$). As Proposition 1 shows, for $\chi \leq 0$, periodic cycle do not exist.

Proposition 1 *Assume that* $\chi \leq 0$, *i.e. the temptation utillity* v *is linear or strictly concave, and that* $\lambda < 1$. *Then, the dynamic equation (9) possesses a unique steady state equilibrium* $w^{ss} > 1$. *This is also the unique PFE.*

Proof. Consider first the case of a linear temptation utility, i.e. $\chi = 0$. The dynamic difference equation (9) reduces to

$$w_t = 1 + \beta [1 - \lambda] w_{t+1} \tag{10}$$

It is straightforward to show that, as long as $\lambda < 1$, equation (10) has a unique steady state solution $w^{ss} = [1 - \beta(1 - \lambda)]^{-1} > 1$. After iterating (10) forward and imposing the limit $\lim_{J \to +\infty} [\beta (1 - \lambda)]^J w_{t+J} = 0$, it also follows that the steady state $w_t = w^{ss}$ for $t \geq 0$ is also the unique solution that is non-explosive and that does not violate the non-negativity condition on stock prices ($w_t \geq 1$).

Next consider the case of concave temptation, i.e. $\chi < 0$. Here, we have three sub-cases to consider: a) $\chi = -1$; b) $\chi \in (-1, 0)$; c) $\chi < -1$. Sub-case a) is straightforward since it also gives a linear dynamic equation:

$$w_t = 1 + \beta [w_{t+1} - \lambda] \tag{11}$$

Similar to the linear temptation utility case of equation (10), the unique PFE is the steady state $w^{ss} = \frac{1-\beta\lambda}{1-\beta} > 1$. Next, consider sub-case b). Simple calculus shows that the mapping $F(w_{t+1})$ defined in (9) has the following properties: i) $F(1) > 1$; ii) $F'(w_{t+1}) = \beta \left[1 - \lambda (1 + \chi) w_{t+1}^{\chi}\right] > 0$ (since $\lambda \in (0, 1)$, $1 + \chi \in (0, 1)$, $w_{t+1}^{\chi} \in (0, 1)$) with $F'(1) < \beta$; iii) $F''(w_{t+1}) > 0$ (strictly convex) and $\lim_{w_{t+1} \to \infty} F'(w_{t+1}) = \beta$. Hence, there exists a unique steady state solution $w^{ss} > 1$ to the equation $w^{ss} = F(w^{ss})$. Moreover, since $F(w_{t+1}) \gtreqless w_t$ for $w_t \lesseqgtr w^{ss}$, similar to the linear case analyzed above, the steady state is the unique solution to (9) that is non-explosive and does not violate the non-negativity restriction on stock prices. Hence, it is the unique PFE.

To conclude consider sub-case c). The properties of the mapping $F(w_{t+1})$ are similar to those found for sub-case b), except that now $F(.)$ is strictly concave. This is sufficient to establish that, also for this case, there exists a unique steady state solution, which is also the unique PFE. ■

Figure 1 displays the alternative shapes of the mapping F identified in Proposition 1. In all three panels, the map F is monotonically increasing in w_{t+1} over the domain $[1, \infty]$. It is therefore invertible: i.e., $w_{t+1} = F^{-1}(w_t)$. Panel a) considers the case of a liner map F, which obtains for χ equal to either 0 or -1. Panels b) and c) consider the cases of, respectively, $\chi \in (-1, 0)$ and $\chi < -1$. While in both of them temptation utility is strictly concave, the map F is stricly convex for $\chi \in (-1, 0)$ but strictly concave for $\chi < -1$. As displayed in the graphs, when $\chi \leq 0$, the steady state w^{ss} is the unique PFE. Any path starting to the right of w^{ss} will be explosive, while any path starting to the left of w^{ss} will eventually lead to $w_t < 1$, thus violating the non-negativity of stock prices.

Next, I study the case of a strictly convex temptation utility (positive χ). As the analysis will show, depending on the quantitative importance of temptation in preferences (magnitude of λ) and the degree of convexity of v (magnitude of χ), the model can generate endogenous cycles and chaotic dynamics. For this purpose, it is convenient to define the following thresholds for the temptation parameter λ:

$$\lambda^* \equiv \frac{1}{1+\chi}\left[\frac{1+\chi(1-\beta)}{1+\chi}\right]^{\chi}, \qquad \lambda^f \equiv \frac{1}{(1+\chi)} \tag{12}$$

$$\lambda^m \equiv \left[1 - \frac{\chi\beta}{1+\chi}\left(\frac{1}{1+\chi}\right)^{\frac{1}{\chi}}\right]^{\chi}, \qquad \lambda^c \equiv \frac{1+\beta}{(1+\chi)\beta}\left[\frac{2+\chi(1-\beta)}{1+\chi}\right]^{\chi} \tag{13}$$

The next Lemma establishes some useful properties of these thresholds.

Lemma 1 *Let λ^*, λ^c, λ^m and λ^f be defined as in (12)-(13). Assume that $\chi > \chi^*$, where $\chi^* > \beta^{-1}$ is the unique solution to the equation $1 + \chi = \left[\frac{1+\chi(1+\beta)}{1+\chi}\right]^{\chi}$. Then, the following relationships hold:*

$$a)\ \lambda^* < \lambda^c < 1$$

$$b)\ \lambda^* < \lambda^m < \lambda^f < 1$$

Proof. First, consider the following equation:

$$1+\chi=\left[\frac{1+\chi(1+\beta)}{1+\chi}\right]^{\chi} \tag{14}$$

Taking logs of both sides, (14) becomes:

$$\ln(1+\chi)=\chi\ln[1+\chi(1+\beta)]-\chi\ln(1+\chi) \tag{15}$$

Let $LHS(\chi)$ and $RHS(\chi)$ denote, respectively, the left and the right hand sides of (15). Simple calculus gives the following properties for $LHS(\chi)$: (i) $LHS'(\chi)>0$; (ii) $LHS(0)=0$ and $LHS'(0)=1$; (iii) $LHS''(\chi)<0$ with $\lim_{\chi\to\infty}LHS'(\chi)=0$. Hence, $LHS(\chi)$ is a strictly increasing and strictly concave function. For what concerns $RHS(\chi)$, we instead have: (i) $RHS'(\chi)>0$; (ii) $RHS(0)=RHS'(0)=0$; (iii) $RHS''(\chi)>0$. Hence, $RHS(\chi)$ is a strictly increasing and strictly convex function. From these properties it clearly follows that there exists a unique $\chi^*>0$ solving (15), and therefore (14). Moreover, simple but tedious algebra shows that $\chi^*>\beta^{-1}$.

Assume that $\chi>\chi^*$. Consider the thresholds λ^* and λ^f defined in (12). Given that $\chi>\chi^*>0$, and since $\left[\frac{1+\chi(1-\beta)}{1+\chi}\right]^{\chi}\in(0,1)$, it immediately follows that $\lambda^*<\lambda^f<1$. Next, consider the threshold λ^c defined in (13). Simple algebra shows that, since $\chi>\beta^{-1}$, then $\lambda^*<\lambda^c<1$. The last thing to show is that $\lambda^*<\lambda^m<\lambda^f$. Simple algebra shows that $\lambda^*<\lambda^m$, while the inequality $\lambda^m<\lambda^f$ can be written as $1+\chi<\left[\frac{1+\chi(1+\beta)}{1+\chi}\right]^{\chi}$. By the properties of the two sides of equation (15) discussed above, this inequality always holds if $\chi>\chi^*$. ■

The next proposition shows that if temptation utility is sufficiently convex (large enough χ), then there exists a non-empty range for the temptation parameter λ for which the mapping F defined in (9) maps the interval $\left[1,\lambda^{-\frac{1}{\chi}}\right]$ onto itself (i.e., the set is invariant under mapping F, or, globally stable in the sense that all trajectories generated by (9) remain in $\left[1,\lambda^{-\frac{1}{\chi}}\right]$), is single-peaked and displays a unique interior steady state equilibrium.

Proposition 2 *Let $\bar{w}\equiv\lambda^{-\frac{1}{\chi}}$ and assume that $\chi>\chi^*$ (that is, temptation utility is sufficiently convex). Then, for $\lambda\in(\lambda^*,\lambda^m)$, the mapping F defined in equation (9) has the following properties: (i) it possesses a unique steady state $w^{ss}\in(1,\bar{w})$ such that $F(w^{ss})=w^{ss}$;*

(ii) $F(1) = 1 + \beta(1-\lambda) > 1$ *and* $F(\bar{w}) = 1$; *(iii)* F *is single-peaked at* $w^* \equiv [\lambda(1+\chi)]^{-\frac{1}{\chi}} \in (1, w^{ss})$, *with* $F'(w_{t+1}) \gtreqless 0$ *for* $w_{t+1} \lesseqgtr w^*$ *and* $F'(w^{ss}) < 0$; *(iv)* $F : [1, \bar{w}] \to [1, \bar{w}]$.

Proof. Recall the definition of χ^* from Lemma 1, and assume that temptation utility is sufficiently convex, i.e. $\chi > \chi^*$. Then, by the same Lemma, the following inequality holds: $\lambda^* < \lambda^m < \lambda^f < 1$. Assume that the temptation parameter λ belong to the interval (λ^*, λ^m). From equation (9) , we need $F(w_{t+1}) \equiv 1 + \beta\left[1 - \lambda w_{t+1}^{\chi}\right] w_{t+1} \geq 1$ for $w_t \geq 1$ (i.e., for the stock price p_t to be non-negative). In turn, this requires $w_{t+1} \leq \bar{w} \equiv \lambda^{-\frac{1}{\chi}}$. Hence, for $\lambda \in (\lambda^*, \lambda^m)$, the set of feasible values for w_t is $[1, \bar{w}]$. To compute the steady state, set $w_t = w_{t+1} = w$ in (9). After a simple rearrangement of terms, the steady state is the solution to the following equation:

$$\frac{w-1}{w} = \beta(1 - \lambda w^{\chi}) \tag{16}$$

Let $LHS(w)$ and $RHS(w)$ denote, respectively, the left and the right hand sides of (16). Simple calculus shows that $LHS(1) = 0$ and $LHS'(w) > 0$, while $RHS(1) > 0$, $RHS(\bar{w}) = 0$ and $RHS'(w) < 0$. Hence, there exists a unique $w^{ss} \in (1, \bar{w})$ solving (16). By simple differentiation of $F(w_{t+1})$ with respect to w_{t+1} we obtain:

$$F'(w_{t+1}) \gtreqless 0 \qquad \text{for} \qquad w_{t+1} \lesseqgtr w^* \equiv \left[\frac{1}{\lambda(1+\chi)}\right]^{\frac{1}{\chi}}$$

Simple algebra shows that $w^* \in (1, \bar{w})$ given that $\lambda < \lambda^m < \lambda^f \equiv \frac{1}{1+\chi}$. By the properties of $LHS(w)$ and $RHS(w)$ defined above (the two sides of the steady state equation (16)), we have that $w^* \gtreqless w^{ss}$ for $RHS(w^*) \lesseqgtr LHS(w^*)$. After simple algebra this inequality reduces to the following condition:

$$\lambda \lesseqgtr \lambda^* \equiv \frac{1}{1+\chi}\left[\frac{1+\chi(1-\beta)}{1+\chi}\right]^{\chi}$$

Since we are restricting to the case of $\lambda > \lambda^*$, we have that $w^* < w^{ss}$, which also implies that $F'(w^{ss}) < 0$. Simple algebra shows that for $\lambda < \lambda^m \equiv \left[1 - \frac{\chi\beta}{1+\chi}\left(\frac{1}{1+\chi}\right)^{\frac{1}{\chi}}\right]^{\chi}$ we also have $F(w^*) < \bar{w}$. Since $F(w^*)$ is the peak of F, it follows that $F : [1, \bar{w}] \to [1, \bar{w}]$. ■

Proposition 2 shows that for a degree of temptation λ belonging to the interval (λ^*, λ^m), the non-linear difference equation (9) has well-defined backward dynamics, that is, for any

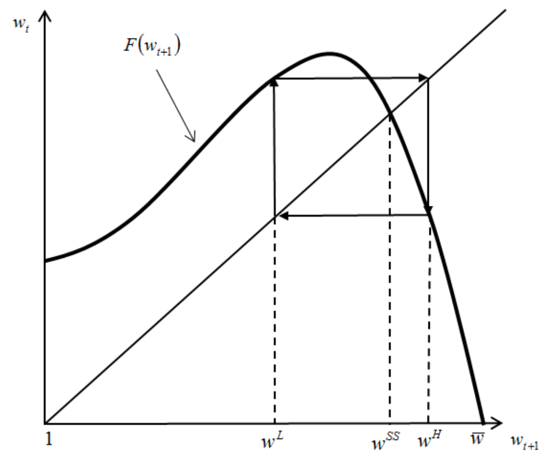


Figure 2: **Two-Period Cycle with Convex Temptation**

given $w_{t+1} \in [1, \bar{w}]$ there exists a unique $w_t \in [1, \bar{w}]$. On the contrary the forward dynamics are not well-defined: since the mapping F is non-invertible over the entire domain $[1, \bar{w}]$, F^{-1} can be a correspondence. This configuration is not new in the literature on economic dynamics. For instance, it often appears in works studying the global equilibrium dynamics of OLG models (Grandmont, 1985; Azariadis and Guesnerie, 1986), money-in-the-utility function models (Fukuda, 1993), cash-in-advance models (Michener and Ravikumar, 1998), and, more recently, search-theoretic models of money and credit (Lagos and Wright, 2003, Rocheteau and Wright, 2013). For the purpose of establishing the existence of periodic orbits, studying the backward equilibrium dynamics is entirely satisfactory. This is indeed the strategy followed by all those works. Interestingly, in a relatively recent contribution, Kennedy and Stockman (2008) show that if one can establish that a backward map F is chaotic, then the forward map F^{-1} is chaotic as well.

Based on these results, I focus the analysis on the backward map F. The single-peakedness and global stability proven by Proposition 2 are necessary but not sufficient conditions for the existence of periodic and chaotic dynamics. The next proposition establishes a sufficient condition for period-2 cycles to occur in this model. This conditions requires the interior steady state to be dynamically unstable.

Proposition 3 *Recall the definition of* λ^c *in (13), and assume that* $\lambda^c \in (\lambda^*, \lambda^m)$. *Then, if* $\lambda \in (\lambda^c, \lambda^m)$ *the model exhibit a period-2 cycle.*

Proof. Recall the definition of λ^c in (13), and assume that $\lambda^c \in (\lambda^*, \lambda^m)$. Then, define the auxiliary function $H(w) \equiv w - F^2(w)$. A period-2 cycle is a solutions to $H(w) = 0$ which is not the steady state w^{ss} (the latter is a trivial period-2 cycle since $F^2(w^{ss}) = w^{ss}$). By the properties of the mapping F in Proposition 2, we have that $H(1) = 1 - F[F(1)] < 0$ (this is because $F(1) \in (1, \bar{w})$ and then $F[F(1)] \in (1, \bar{w})$ as well) and $H(\bar{w}) = \bar{w} - F^2(\bar{w}) = \bar{w} - F(1) > 0$ (this is because $F(1) < F(w^*) < \bar{w}$). Hence, by continuity of the function $H(w)$ on the domain $[1, \bar{w}]$, a sufficient condition for the existence of a period-2 cycle is that $H'(w^{ss}) < 0$, or, equivalently, $F'(w^{ss}) < -1$. More explicitly, we need:

$$\beta \left[1 - \lambda (1+\chi) (w^{ss})^{\chi}\right] < -1$$

Simple rearrangement of terms gives:

$$w^{ss} > \left[\frac{1+\beta}{\beta\lambda(1+\chi)}\right]^{\frac{1}{\chi}} \tag{17}$$

Let $\check{w} \equiv \left[\frac{1+\beta}{\beta\lambda(1+\chi)}\right]^{\frac{1}{\chi}}$. From the proof of Proposition 2, recall that the steady state w^{ss} is the unique solution to the following equation:

$$\frac{w-1}{w} = \beta\left(1 - \lambda w^{\chi}\right) \tag{18}$$

Let $LHS(w)$ and $RHS(w)$ denote the left and the right hand side of (18). It is straightforward to show that the inequality (17) holds if and only if $LHS(\check{w}) < RHS(\check{w})$. As simple algebra shows, the latter can be written as follows:

$$\lambda > \lambda^c \equiv \frac{1+\beta}{(1+\chi)\beta}\left[\frac{2+\chi(1-\beta)}{1+\chi}\right]^{\chi}.$$

■

Figure 2 depicts the case of a period-2 cycle. This is defined as a pair $\{w^L, w^H\}$ for $w^L < w^{ss} < w^H$ such that $w^L = F(w^H)$ and $w^H = F(w^L)$, or, equivalently, $w^i = F^2(w^i)$ for

$i = L, H$. This result implies that there exist a non-empty parametric range of temptation for which the economy switches deterministically between a high stock price value $p^H = w^H - 1$ and a low stock price value $p^L = w^L - 1$ (endogenous boom-bust cycle).

To construct a simple economic intuition for the existence of a period-2 cycle under convex temptation utility consider the pricing equation (6), which, under perfect foresight, reads

$$p = \beta \left[1 - \frac{v'(p' + 1)}{u'(1) + v'(1)} \right] (p' + 1), \tag{19}$$

where I have denoted by p and p', respectively, the current and the future stock price. Suppose the economy is initially at the steady state equilibrium $p^{ss} = w^{ss} - 1$, and that, suddenly, agents believe the stock price will be higher next period: $p' = p^H > p^{ss}$, where again $p^H = w^H - 1$. By the pricing equation (19), the new equilibrium price p is

$$p = \beta \left[1 - \frac{v'\left(p^H + 1\right)}{u'(1) + v'(1)} \right] \left(p^H + 1\right) \tag{20}$$

With v linear or concave, the right hand side of (20) would be strictly increasing in the future price. This would make the current price p also higher then the steady state price ($p > p^{ss}$), which would be inconsistent with a period-2 cycle. On the contrary, with a sufficiently convex temptation utility, the right hand side of (20) becomes strictly decreasing in the future price p'. In this case, a high future price $p' = p^H$ can be consistent with a low current price $p = p^L < p^{ss}$, as well as a low future price can be consistent with a high current price. This is because when temptation utility is convex, the costs of self-control are increasing in future financial wealth, which in turn reduce the equilibrium valuation of wealth itself.

In order to prove the existence of period-2 cycles, Proposition 3 assumes that the temptation threshold λ^c belongs to the interval (λ^*, λ^m). By plotting the temptation thresholds defined in (12)-(13) as functions of χ, Figure 3 provides some perspective on the plausibility of such assumption. To construct these plots, I let one period in the model correspond to a year, and therefore set $\beta = 0.96$ to be consistent with a 4% annual real interest rate. The

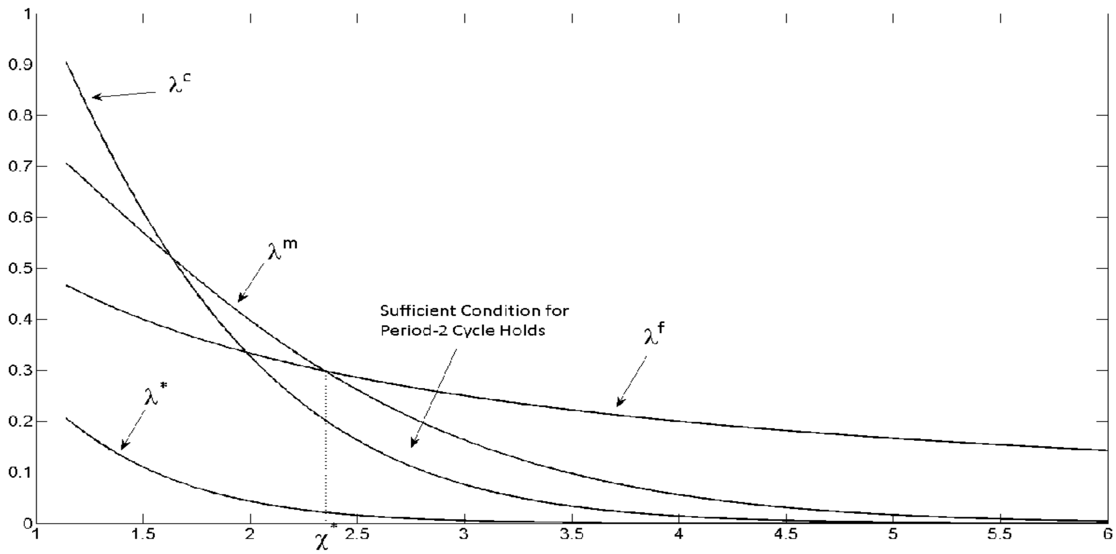


Figure 3: **Temptation Thresholds**

results presented in Propositions 2 and 3 hold for the case of $\lambda^* < \lambda^c < \lambda^m < \lambda^f < 1$. As Figure 3 shows, this requires a sufficiently large χ, that is, a sufficiently large degree of convexity of temptation utility (in the specific example, $\chi > \chi^* \approx 2.3$). For any $\chi > \chi^*$, the range (λ^c, λ^m) - within which the existence of period-2 cycles is guaranteed - is always non-empty. Although this range shrinks as χ increases, by the fact that the bifurcation threshold λ^c is strictly decreasing in χ and quickly goes down to zero, period-2 cycles can obtain for extremely low levels of temptation. In other words, a minimal departure from the benchmark CRRA framework can lead to endogenous stock price fluctuations, provided temptation is sufficiently convex.

Based on the analytical results of Propositions 2 and 3, I construct the orbit bifurcation diagram for the non-linear difference equation (9) with respect to the degree of temptation λ.[6] The panels in Figure 4 presents the results for different degrees of convexity in temptation,

[6]The orbit bifurcation diagram displays all possible long-run (asymptotic) outcomes (steady state(s), periodic equilibria, chaotic attractors, etc..) of a dynamic system as a function of a specific bifurcation parameter. See Lorenz (1993) and Strogatz (1994) for a detailed description.

as indexed by χ. Stronger temptation can lead to periodic cycles as long as temptation utility is sufficiently convex. For instance, if $\chi = 1$ (i.e. quadratic temptation), the steady state is the unique feasible PFE, for any $\lambda \in [0, 1)$, while period-2 cycles exists for λ larger than, about, 0.3 (when $\chi = 2$) and 0.07 (when $\chi = 3$). As we further increase χ, two interesting results emerge. First, period-2 cycles start occurring at much lower lelves of temptation: for instance, for λ around 0.01 if $\chi = 4$, 0.0025 if $\chi = 5$ and 0.0005 if $\chi = 6$. Second, the equilibrium dynamics display also higher order cycles (it is easy to spot period-4 cycles in both panels d. and e., and period-8 cycles in panels e. and f.) and aperiodic trajectories (panels e. and f.).

To assess whether these aperiodic trajectories are indeed chaotic, I compute the Lyapunov exponent associated with the dynamic equation (9). Following Definition 1, to be called chaotic, a dynamic system should display sensitive dependence on initial conditions, in the sense that two obits starting from two arbitrarily close initial conditions should separate explonentially fast. Consider, for instance, the following two intial conditions: w_0 and $\tilde{w}_0 = w_0 + \varepsilon_0$, for ε_0 arbitrarily small. Let $|\varepsilon_n|$ be the distance between the two orbits generated by the difference equation (9) after n iterations. If $|\varepsilon_n| \approx |\varepsilon_0| \, e^{\gamma n}$, then γ is called the Lyapunov exponent. A positive γ is an index of chaotic behavior. Figure 5 plots the Lyapunov exponent for the case of $\chi = 5$ and $\chi = 6$, whose orbit bifurcation diagrams provided some visual evidence of chaos.[7] It clearly appears that a positive Lyaponuv exponent (hence, chaos) start occurring for λ between 0.01 and 0.015 when $\chi = 5$, and for λ larger than just 0.002 for $\chi = 6$.

Both the analytical and the numerical results highlight the fact that the existence of cyclical and chaotic dynamics depends on λ and χ, the two key parameters describing temptation utility. Figure 6 displays a two-dimensional orbit bifurcation digram, with $\lambda \in [0, 0.5]$ on the horizontal axis, and $\chi \in [1.5, 6]$ on the vertical axis. For any pair (λ, χ), it tells whether the PFE trajectories generated by (9) lead to the steady state (white area), a 2-period cycle

[7] The Lyapunov coefficient has been computed using the E&F Chaos software by Dicks et al. (2008). The plots have been done with Matlab.

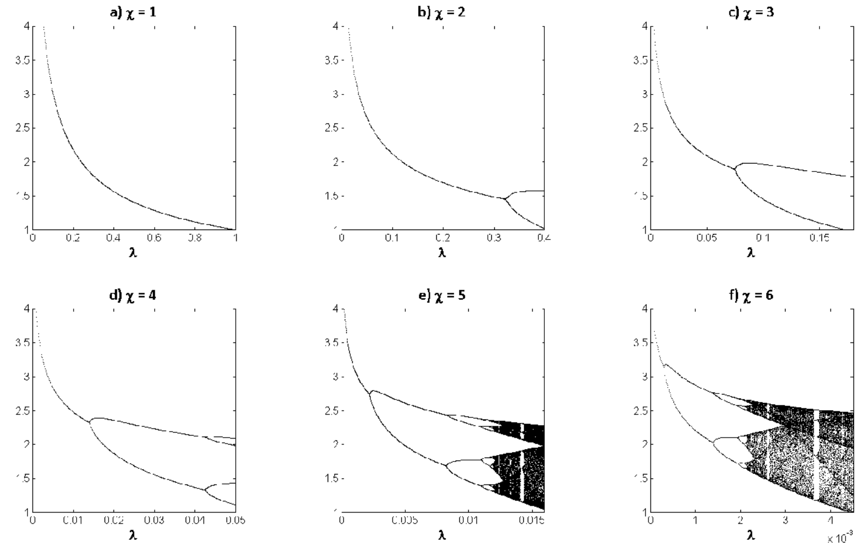


Figure 4: **Orbit Bifurcation Diagrams: Convexity and Strenght of Temptation Utility.**

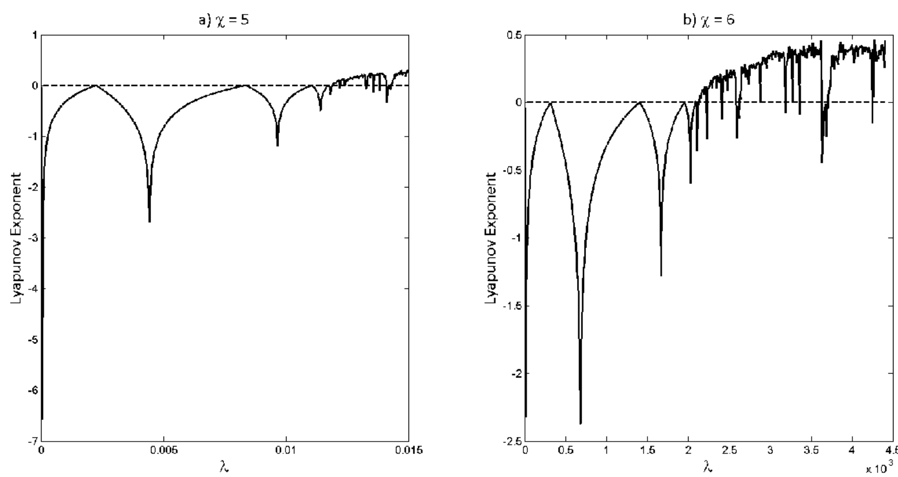


Figure 5: **Lyapunov Coefficient**

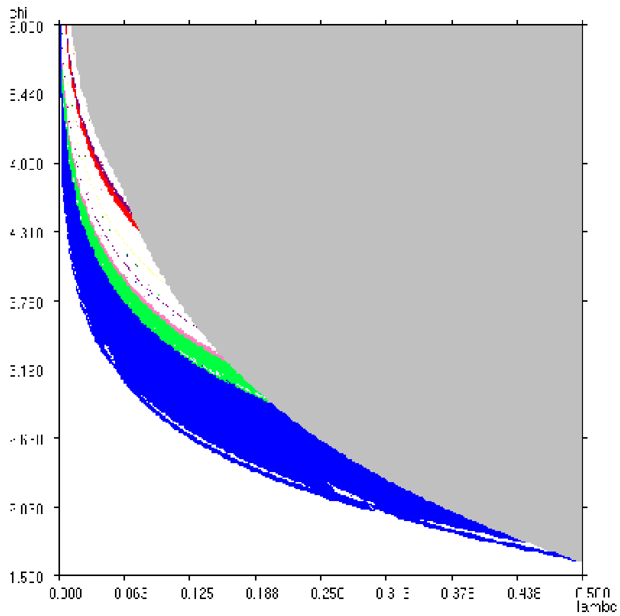


Figure 6: **Two-Dimensional Orbit Bifucation Diagram.**

(blue area), a 4-period cycle (green area), a 8-period cycle (pink area), and a 3-period cycle (red area). For combinations falling within the gray area, the dynamics falls outside the feasible range $[1, \bar{w}]$ and are therefore discarded.

4 Sunspot Equilibria

In this section, I construct both stationary and non-stationary sunspot equilibria, along the lines of Azariadis and Guesnerie (1986) and Lagos and Wright (2003). A sunspot equilibrium is a rational expectation equilibrium where endogenous variables depend on the realizations of an extrinsic (non-fundamental) sunspot shock. I start by studying a 2-state *stationary* sunspot equilibrium (henceforth, 2-SSE). Consider a two-state sunspot $\zeta \in \{\zeta_1, \zeta_2\}$, with transition probabilities $\pi_{12} = prob\left(\zeta_{t+1} = \zeta_2 | \zeta_t = \zeta_1\right)$ and $\pi_{21} = prob\left(\zeta_{t+1} = \zeta_1 | \zeta_t = \zeta_2\right)$. In the context of my model, a 2-SSE is a time-invariant function $w\left(\zeta_t\right)$ such that $w_t = w_1 \equiv w\left(\zeta_1\right)$ if $\zeta_t = \zeta_1$ and $w_t = w_2 \equiv w\left(\zeta_2\right)$ if $\zeta_t = \zeta_2$. More specifically, a 2-SSE is a quadruple $\{w_1, w_2, \pi_{12}, \pi_{21}\}$ solving the following system:

$$w_1 = 1 + \beta \left[\pi_{12} \left(1 - \lambda w_2^{\chi} \right) w_2 + \left(1 - \pi_{12} \right) \left(1 - \lambda w_1^{\chi} \right) w_1 \right] \tag{21}$$

$$w_2 = 1 + \beta \left[\pi_{21} \left(1 - \lambda w_1^{\chi} \right) w_1 + \left(1 - \pi_{21} \right) \left(1 - \lambda w_2^{\chi} \right) w_2 \right] \tag{22}$$

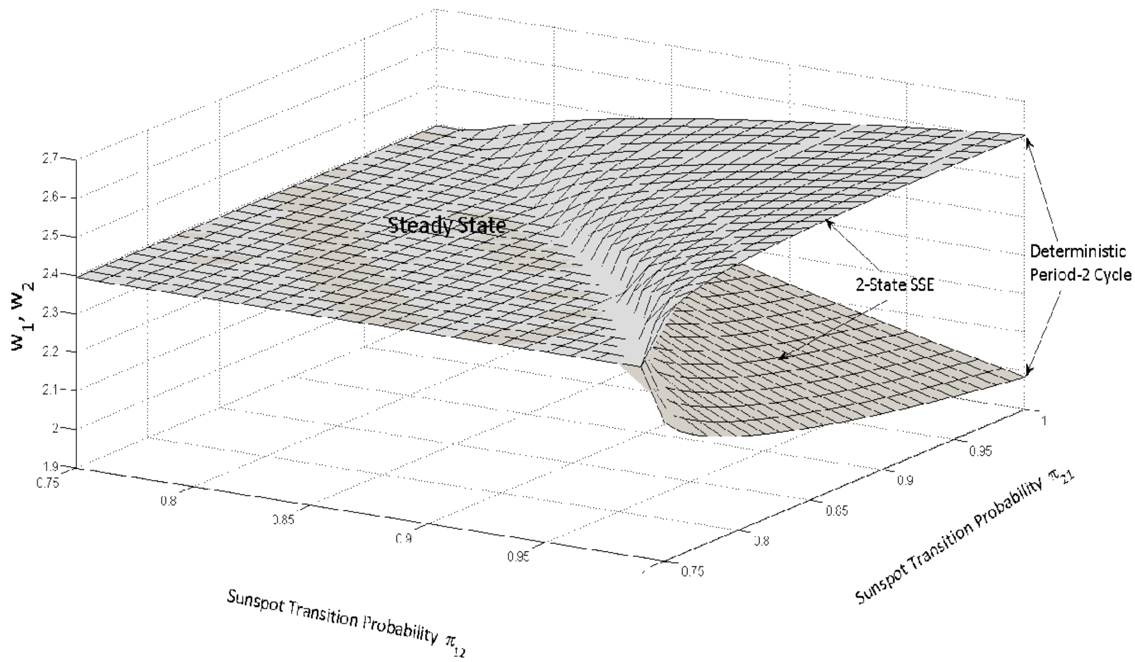
Notice that for $\pi_{12} = \pi_{21} = 1$, this system reduces to

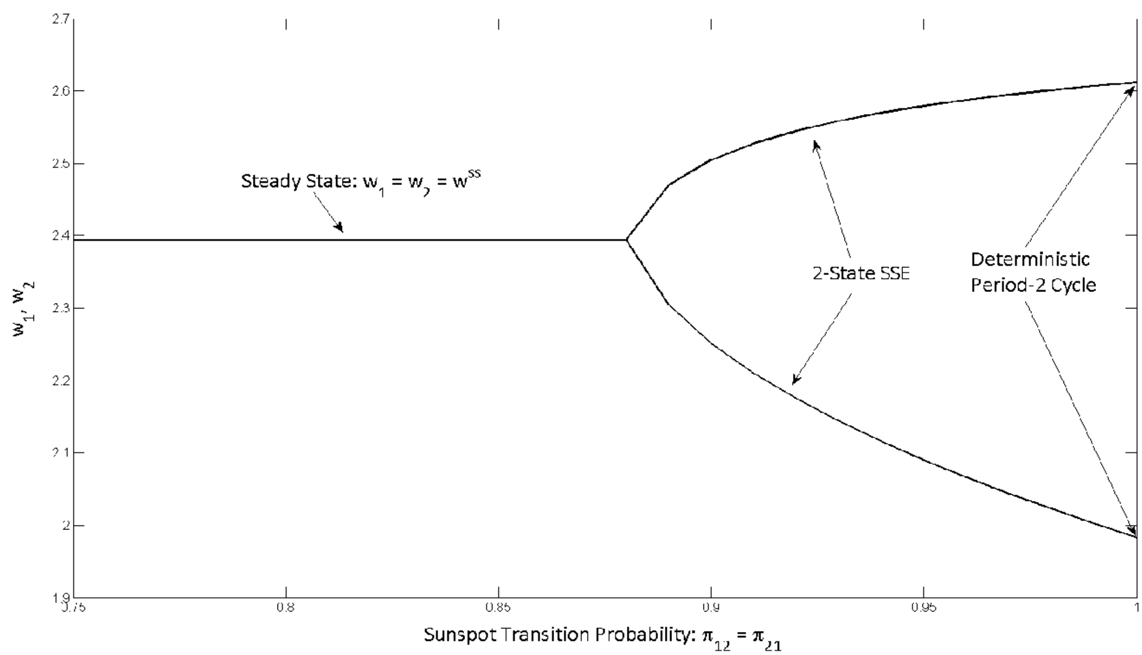
$$w_1 = 1 + \beta \left[\left(1 - \lambda w_2^{\chi} \right) w_2 \right] \qquad \text{and} \qquad w_2 = 1 + \beta \left[\pi_{21} \left(1 - \lambda w_1^{\chi} \right) w_1 \right]. \tag{23}$$

These last two equations define a period-2 cycle (i.e., a trivial 2-SSE), where w_t switches deterministically between the values w_1 and w_2. In this specific case, the conditions for the existence of a 2-SSE coincide with the conditions for the existence of a period-2 cycle, as stated in Proposition 3. It therefore follows that, given a parameterization of the model that satisfies the conditions in Proposition 3, by continuity of the system (21)-(22) in the probabilities π_{12} and π_{21}, there exist a pair w_1, w_2 solving (21)-(22) for some $\pi_{12}, \pi_{21} < 1$.

Figure 4 shows all quadruples $\{w_1, w_2, \pi_{12}, \pi_{21}\}$ solving the system (21)-(22) - hence, the entire set of 2-SSE - for the case of $\chi = 5$ and $\lambda = 0.005$. We know from the analysis of Section 3 that, absent the sunspot shock, this parameterization leads to a period-2 cycle (see panel e. of Figure 4). For combinations $\{\pi_{12}, \pi_{21}\}$ for which the gray-colored surface is flat, we have that $w_1 = w_2 = w^{ss}$: the unique 2-SSE is the steady state (a trivial one). For combinations $\{\pi_{12}, \pi_{21}\}$ for which the surface splits into an upper and a lower part, there exists a 2-SEE given by a pair $w_2 > w_1$ (with w_2 belonging to the upper part, and w_1 belonging to the lower part). This occurs when the sunspot transition probabilities are sufficiently large. The figure also displays the deterministic period-2 cycle, which, as mentioned, occurs for $\pi_{12} = \pi_{21} = 1$. To get an even better sense of how a SSE looks like, I consider the case of a symmetric 2-state sunspot shock: that is, I set $\pi_{12} = \pi_{21} = \pi$. Figure 4 shows that a symmetric 2-state SSE exists for π larger than, about, 0.88. As π approaches unity from below, the SSE converges to the deterministic period-2 cycle.

I conclude the analysis by constructing a *non-stationary* sunspot equibrium (NSSE). For this purpose I set $\chi = 5$ and $\lambda = 0.015$, a parameterization leading to higher order cycles and


Steady State
2-State SSE
Deterministic
Period-2 Cycle
w1, w2
Sunspot Transition Probability π12
Sunspot Transition Probability π21

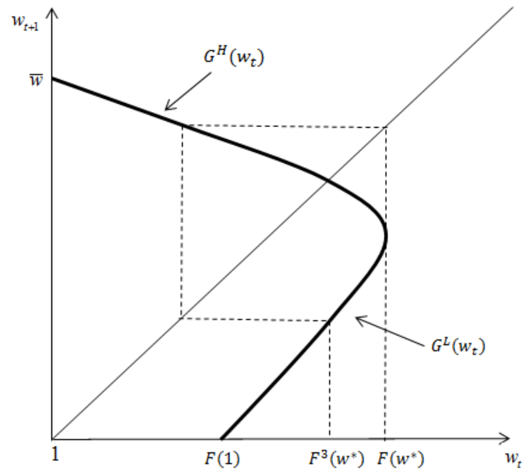

Steady State: w1 = w2 = wss
2-State SSE
Deterministic
Period-2 Cycle
w1, w2
Sunspot Transition Probability: π12 = π21

chaos in the deteministic case (see panel e. of Figure 4 again).[8] From the analysis in Section 3, we know that, under this parameterization, the map F defined in (9) is non-monotonic, hence it is not invertible over the entire admissible range $[1, \bar{w}]$. Letting $G \equiv F^{-1}$, we have in fact that, by the properties of F spelled in Proposition 2, G is single-valued and strictly decreasing for $w_t \in [1, F(1))$, a correspondence for $w_t \in [F(1), F(w^*)]$, and is not defined for $w_t \in (F(w^*), \bar{w}]$. Figure 4 displays $w_{t+1} = G(w_t)$. Notice that for $w_t \in [F(1), F(w^*))$, G displays two branches: a strictly increasing lower branch, G^L, and a strictly decreasing upper branch G^H, which collide for $w_t = F(w^*)$, the point where G starts bending backward. From the figure, it also appears that, for any $w_t \in [F(1), F^3(w^*))$, if $w_{t+1} = G^L(w_t)$, then $w_{t+2} = G^H(w_{t+1}) > F(w^*)$: the sequence falls outside the admissible range. Based on this observation, I define the interval $I \equiv [F^3(w^*), F(w^*))$ and consider a 2-state sunspot $\zeta \in \{\zeta_H, \zeta_L\}$, with arbitrary probabilities $\mathfrak{p}$ and $1-\mathfrak{p}$. I then construct the following NSSE:

$$
\begin{aligned}
w_0 &\in I \\
w_t &= \begin{cases} G^H(w_{t-1}) \ \ (\text{prob. } \mathfrak{p}) \text{ or } G^L(w_{t-1}) \ \ (\text{prob. } 1-\mathfrak{p}) & \text{if } w_{t-1} \in I \\ G^H(w_{t-1}) & \text{if } w_{t-1} \notin I \end{cases} \quad \text{for } t \geq 1
\end{aligned}
$$

Figure 4 displays four alternative NSSE. Each panel assumes a different value for $\mathfrak{p}$ (the probability of being on the higher branch of G), while each trajectory is generated starting from the same initial condition $w_0 \in I$. If the probability of being on the G^H branch is large ($\mathfrak{p} = 0.75$) or if the two branches are equally likely ($\mathfrak{p} = 0.5$), the NSSE converges to the unique steady state equilibrium w^{ss} in finite time. This is because the steady state is determined by the intersection of the upper branch with the 45 degree line, and is stable in forward dynamics. However, once the lower branch G^L becomes more likely ($\mathfrak{p} = 0.25$ or $\mathfrak{p} = 0.05$), the NSSE displays much higher volatility and no tendency to convergence. This is because the branch G^L tends (at least for one iteration) to push the trajectory away from

[8] While a NSSE exists also for $\lambda = 0.005$ - i.e., for the exact same parameterization used for the SSE case - I find that it always converges to the steady state in finite time. The case of $\lambda = 0.015$ delivers more interesting dynamics.

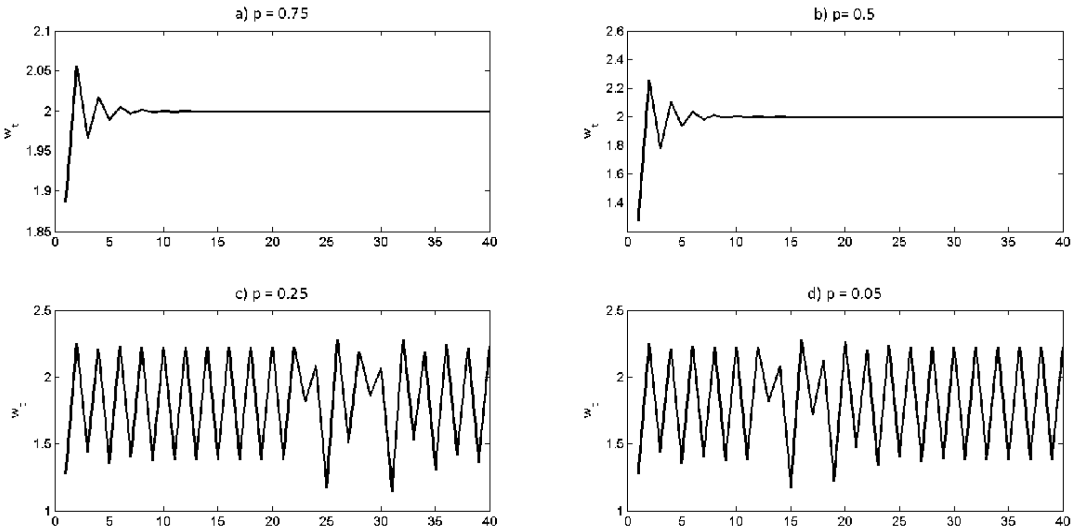


the steady state. When this becomes more frequent (lower $\mathfrak{p}$), convergence to the steady state is less likely, and the NSSE displays volatile cyclical fluctuations around it.

5 Conclusions

This paper studies the global equilibrium stock price dynamics implied by a deterministic Lucas' tree asset pricing model where the representative agent has DSC preferences, as defined by Gul and Pesendorfer (2004). Under this set-up, in every period, the agent faces a temptation to liquidate his entire financial wealth for the purpose of immediate consumption. Resisting temptation involves effort (or self-control) and hence some disutility. Optimal behavior therefore trades off the temptation for immediate satisfaction with long-run optimal consumption smoothing. As shown by Gul and Pesendorfer (2004), DSC preferences imply that the pricing kernel for a risky stock becomes a function of future stock prices, which makes the Lucas' tree asset pricing equation non-linear in future financial wealth.

I show, both analytically and numerically, that DSC preferences can lead to endogenous cycle of period 2 and higher, as well as chaotic dynamics. These complex dynamics attain when the temptation utility is sufficiently convex and sufficiently important with respect to standard commitment utility. For parameterizations leading to complex deterministic



dynamics, the model also admits stationary and non-stationary stochastic sunspot equilibria, that is, equilibrium trajectories where stock price expectations depend on the realization of a non-fundamental sunspot shock (extrinsic uncertainty).

With endogenous cycles being the main objective of the paper, the analysis has been restricted to a set-up with deterministic fundamentals. It would be interesting to assess whether DSC preferences combined with intrinsic uncertainty (e.g. stochastic dividends) could generate recurrent boom-bust stock price cycles similar to what observed in the data, without requiring a departure from rational expectations. I plan to explore this issue in future research.

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