No Regret Equilibrium

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Abstract

Subgame Perfect Nash Equilibrium (SPNE) assumes that all players possess the ability to perform backward induction and also that this is common knowledge. Both these assumptions are crucial in generating unintuitive and empirically unjustified equilibrium predictions in some sequential games like the Centipede game (22) and Sequential Bargaining (23 and 24). This paper defines the No Regret Equilibrium (NRE), a limited foresight bounded rationality equilibrium concept for the class of two player[1], perfect information, finite alternate move games, which weakens these two assumptions. The NRE model has generally applicability in this class of games and one does not need to generate game specific behavioral types. It allows for each of the two players to have one of various possible levels of foresight as determined by a common knowledge probability distribution on the levels of foresight. We prove that this equilibrium exists. We show that the equilibrium entails learning about the game and belief updating about opponent's foresight type *within* the play of a game. NRE behavior explains high types using/avoiding reputations, and low and intermediate types trying (often successfully) to spot the high foresight types and optimizing the best they can given their foresight bounds. These features produce passing till the last few stages in a centipede game with more than 4 stages as the NRE outcome for arbitrary probability on bounded rationality. NRE is shown to have the ability to explain delays, near equal splits, disadvantageous counter proposals and subgame inconsistency observed in the experimental data on sequential bargaining.

Keywords: Limited Foresight, Sequential Equilibrium, Centipede Game, Sequential Bargaining.

1 Introduction

The use of Subgame Perfect Nash Equilibrium, henceforth SPNE, to make predictions about outcomes in the class of

*I would like to acknowledge Professors James Peck, and Yaron Azrieli at OSU and Professors Abhijit Banerji and Anirban Kar at DSE for extremely useful and insightful comments.

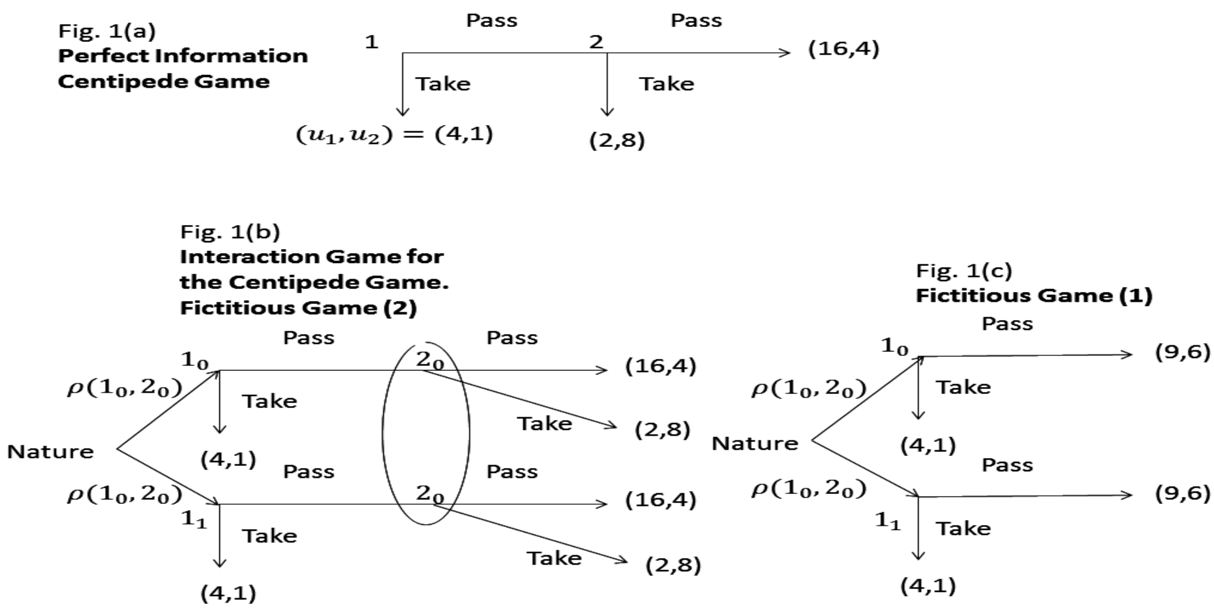
[1]The model is easily extendable to general sequential move, perfect information, finite games. We focus on the two player, alternate move case for ease of illustration and due to the rich set of applications already available for the two player case.

two player, perfect information, alternate move, finite sequential games like chess, tic-tac-toe etcetera, is demanding a lot from the players. SPNE assumes that each player is able to understand the optimal last stage action choice at each possible move in the last stage. Given this understanding, each player is supposed to figure out what the best actions from each second last stage move are. Each player is supposed to backwardly induct in this manner right up to the first stage to understand the best first stage action. SPNE also assumes that both players are assured of each others rationality, in that each player assumes that his opponent will do perfect backward induction. Chess is a good example of how tough this can be. The human race, even with the help of supercomputers has not been able to do it. Thus, when one takes SPNE and tries to predict outcomes of the centipede game (Rosenthal (1981)) or sequential bargaining (Rubinstein (1982) and Stahl (1973)), one finds that the SPNE predictions of "stop immediately" or "settle upon a split of the pie immediately", respectively, are falsified (cf. McKelvey and Palfrey(1992), Ochs and Roth(1988)). This is where the substantial theoretical literature on bounded rationality, that is, a limited understanding of the game and the environment or the error prone nature of strategies of players come in as explanations for the observed and intuitive outcomes of this class of games.

The No Regret Equilibirum (NRE) is one such attempt. NRE is applicable to the class of perfect information games described above. Starting with a game from this class, we construct a richer version of this perfect information game called the *Interaction game*, which nests the original perfect information game as a special case. To construct the Interaction game we relax the assumption of common knowledge of rationality among the players. In the Interaction game, each of the two players of the original perfect information game are replaced by a set of player-types, i.e. each player of the original perfect information game can now be one of several possible types. The probability of each of these player-types is exogenous and a free parameter of the analysis. We assume common knowledge of this prior distribution on the two sets of player types. Any sequence of actions of the Interaction game gives payoffs identical to what that sequence would have fetched in the original perfect information game. This makes the Interaction game a Bayesian game of imperfect information. We would apply a standard solution concept here except the types used here are special. Each type of a player has an associated limited foresight level.

Figure 1 shows how we take a two stage perfect information centipede game (top of figure 1) and construct the Interaction game (bottom left) from it. The original perfect information game has the player set $I_0 = \{1, 2\}$. Player 1 has two possible types, 1_0 (read as player 1-type 0) and 1_1(read as player 1-type 1). Player 2 has only the one type, 2_0. The subscript denotes the foresight level that distinguishes one type from the other. The Interaction game consists of two copies of the centipede game. There are two copies because player 2-type 0 might be playing against player 1- type 0 (1_0), or player 1- type 1 (1_1). Nature decides the probability weight on the two possible pairs $\{(1_0, 2_0), (1_1, 2_0)\}$. The distribution chosen by nature, $\rho(.)$, is assumed to be common knowledge. Player 1's type is his private information. Notice that the payoffs for the same action sequence is the same regardless of the pair chosen by nature.

Figure 1



Following Jehiel (1995), we explore the presence of limited foresight. Each type subscript represents a separate foresight level. Each player-type, at each of his moves, constructs a distorted and curtailed version of the Interaction game, which we call a fictitious game. Jehiel (1995) doesn't use fictitious games per se, but did curtail the underlying game (infinite moves) at the foresight bound. The stage at which the Interaction game is curtailed is determined by the foresight level of that type. In constructing the fictitious game, the player type correctly understands all moves and payoffs upto his foresight. Importantly, for each move at the horizon of his foresight, he takes the mean of the payoff profiles accessible down the game tree from each action at that move and sets this averaged payoff profile as the payoff profile at that action, hence "closing" the fictitious game. Jehiel (2001) takes a different approach for accounting for continuation payoffs. There, a random variable $\tilde{\epsilon}$ captures the continuation payoffs from the foresight horizon; $\tilde{\epsilon}$ can take different values for different moves at the horizon. The rationale for using the mean is that though a player type can't understand how the interaction will pan out beyond his foresight horizon; he does have a sense of what payoffs await him. The mean is a reflection of the payoffs awaiting him down a certain path from his foresight horizon.

In our setup, the fictitious game represents a player-type's distorted understanding of the actual environment of the Interaction game. There are two separate fictitious games in the example in figure 1: figure 1(b) and 1(c). 1(c) shows the fictitious game observed by 1_0 at stage 1. The second fictitious game is the Interaction game itself; it is the fictitious game observed by 1_1 at stage 1, 1_0 at stage 2, and 2_0 at stage 2. Thus, if a player-type i_n is at stage q of the Interaction game, he observes a $(q+n)$ staged curtailed version of the Interaction game with the payoff averaging as shown in the example. If $(q+n) \geq \#$ *Interaction game stages* then i_n observes the whole interaction game and is said to be rational at stage q.

Let i_n be an arbitrary player-type moving at an arbitrary information set, h, which is at stage q. The NRE strategy and belief profile requires the following:

1. The NRE action and a belief for i_n at that information set should be the part of a Sequential Equilibrium (Kreps and Wilson (1982)) of the fictitious game at that move.
2. This Sequential Equilibrium calculation must fix (as nature's move) the actions of all the moves at which the player-type moving there observed a fictitious game that is shorter than the $(q+n)$ staged one observed by i_n at the move in question.

Simply put, NRE requires the following: first, higher foresight types correctly anticipate lower types' moves, for example, a varsity team chess player knows and understands an amateur player's chess moves. Second, a low foresight type discovers that he is playing against some higher type if the latter uses an action different from all types lower than him; if the varsity player uses a complicated action sequence to take your queen, you know you are up against someone better. This feature follows from using Sequential Equilibrium. Third, any foresight type does the best he can within the bound of his foresight given his belief about opponent type. If a low foresight type does find himself at a history of play not possible with a lower foresight type, then he discovers he is playing a higher foresight type and therefore must use his complete foresight to maximize payoffs. If a top varsity player finds out he is playing an International Master, he is going to use all his abilities. Importantly, all this learning and updating happens *within a play of the game.* Using the sequential equilibrium also guarantees us the existence of NRE. In short, at each move, limited foresight types behave in a sequentially rationally manner given the belief about opponent's type. The beliefs at each move reflect the accumulated learning about the game and opponent's type till that point.Though a limited foresight type is allowed to have incorrect beliefs and suboptimal strategies in the context of the entire game, he is optimizing based on his foresight. Thus, no limited foresight type should regret his possibly erroneous strategies and beliefs. Hence the name, "No Regret Equilibrium."

We discuss related literature in section 2. In section 3, we setup the model. Section 4 defines the No Regret

Equilibrium and proves its existence and upper hemicontinuity. In section 5 we show that for any arbitrarily low total probability on limited foresight we get passing till the end stages as the NRE outcome. This is due the averaging of continuation payoffs causing limited foresight types to pass and a powerful reputation effect. Section 5 also shows that NRE has the ability to explain several puzzling qualitative features of the data on the sequential bargaining game.

2 Literature Review

Experimental economics literature has extensively studied the two key assumptions for SPNE. First is the limitations of agents with respect to backward induction, and second is the problem of establishing common knowledge of the ability to backwardly induct. Johnson et al (2002) explored the former. They study three period bargaining to conclude that "results show clearly that subjects do not think as strategically as most game theory applications presume". They monitored subjects using "Mouselab" to observe if, given an option, they look at future pie sizes . They found that the amount of periods subjects look forward was different across subjects, and that it was a good predictor of subjects' actions. The authors also found that if they truncated a game and replaced a subgame with its SPNE payoffs, then subjects are a lot more responsive to the payoffs placed at the node of truncation than to the same payoffs achievable as SPNE after playing the subgame. They conclude that "Backward induction is not computationally complex (subjects can be trained easily); it is simply not natural". They sign off saying that "A complete theory could endogenize which nodes are truncated, and how values are imputed at those nodes, tying these steps to parameters of the game". Clearly, a thought pointing in the direction of this work.

Exploring the importance of common knowledge about the ability of players to perform backward induction, Ignacio Palacios-Huerta and Oscar Volij (2009) found that nearly 70 percent of the professional chess players in their sample stop the centipede game at the first node when matched with other chess players. Grandmasters stop at the very first node if they know their opponent is a capable chess player; where as for the cases where a grandmaster is playing against a student subject, the frequency of stopping at the first node drops. They attribute this result to perfect common knowledge of rationality among capable players, and thus conclude that it is the level of rationality and information about opponent's rationality that determines outcomes rather than altruism or social preference. Levitt, List and Sadoff (2009) test the performance of professional chess players with Elo rating between 1789 and 2367 in the centipede game and two zero sum games requiring 10 steps of backward induction. They find that the chess players play like student subjects in the centipede game, cooperating in the beginning. As we show later, that even an arbitrarily small probability on limited foresight can cause this result. On the other hand, while playing the zero sum games, where cooperation has no benefits, the chess players typically succeed in getting on the SPNE path in the early stages of the game.

The NRE is related to Jehiel (1995) due to the usage of limited foresight. But this paper, which deals with infinite games with a particular structure, ignores the payoffs beyond the horizon which are crucial to our model. Jehiel (2001) includes a random variable draw to capture the payoffs and interaction beyond the foresight horizon, but this still doesn't provide any importance to the actual payoffs beyond the horizon. Also the set of possible foresight levels is not rich enough to generate reputation and updating about the opponent's foresight level. The idea that a player-type averages payoff profiles at the end of his foresight horizon is closely related to the valuation associated with moves in the Valuation Equilibrium by Jehiel and Samet (2007). In Valuation Equilibrium, a player groups his moves into exogenous similarity classes and for each similarity class provides a valuation. The valuation, in equilibrium, must satisfy a consistency condition. Though related, we have obvious differences with this work including their lack of a hierarchical structure, and the fact that the valuation condition there must satisfy an equilibrium consistency condition. Our work has more in common with the sequential level-k model of Ho and Su (2012).

Ho and Su (2012) adapt the Level-K model for sequential move games. They use the approach of Camerer, Ho and Chong (2004), in that a player can be of level 1,2,..,K,.. with a given probability distribution. The level-K type player best responds to the strategies of Level 1 through Level K-1 player types' strategies. Level-1 strategy is a best response to the Level-0 strategy, where Level-0 strategy is to randomize among actions at each move. Thus one can solve for the optimal strategies of each level recursively. They apply it to fit data from experiments on the centipede game and the sequential bargaining game. They allow for *learning across rounds of play* of the same game as opposed to *within* the play of a game as in here. On the other hand, the AQRE model of McKelvey and Palfrey (1998) explains empirical results by fitting the data to a model where the data results are explained through players playing error prone strategies. This approach is effective in Maximizing Likelihood of the observed outcomes but does not capture the interaction dynamics among players with limited understanding of the game.

Thus one way to think of the NRE is that it uses the limited foresight of Jehiel (1995), but uses it in finite games, and provides a hierarchical structure like Ho and Su (2012), but unlike there, in our work lower levels are also trying to optimize against higher types and higher types need to worry about the resulting reputation effects. In the NRE model there is a valuation at the end of the foresight horizon of each player-type, a feature somewhat Similar to Jehiel and Samet (2007) and Jehiel (1999). The reputation effects in our model are similar to the crazy type literature started by Kreps,Wilson, Milgrom and Roberts in 1982, yet there are important difference as their crazy types' behavior was exogenous and their crazy types, whose counterparts in the NRE model would be the player-types with low foresight, had no incentive to discover whom they were playing against.

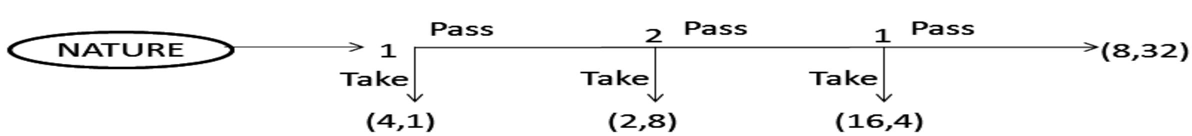
3 Model

This model seeks to pry into the uncertainty and cognitive limitations underlying the interaction of two agents in a seemingly perfect information game. So the underlying game, or the prior, for this analysis is a finite, two player, alternate move, perfect information extensive game : $\Gamma_0 \equiv \{T_0, \prec, A_0, I_0, \iota_0, H_0, u_0, \rho_0\}$. We use Kreps and Wilson (1982) to define the underlying game. Here $T_0 = W_0 \bigcup X_0 \bigcup Z_0$ is the set of all nodes, ordered by a partial order $\prec$, which forms an arborescence. It is the union of initial (W_0), decision (X_0) and terminal nodes (Z_0). The player set is $I_0 = \{1, 2,\ N\}$; it consists of player 1, player 2 and Nature. The player function ι_0 maps non-terminal nodes to the player set, i.e. $\iota_0 : (W_0 \bigcup X_0) \rightarrow I_0$. For the class of alternate move finite games we study, Nature moves in the 0^{th} stage . Subsequently $\iota_0(x_0) = 1$ if $x_0 \in X_0$ is in an odd stage, and $\iota_0(x_0) = 2$ for even stages. $W_0 = \{w_0\}$ is a singleton set with $\iota_0(w_0) = N$,$\forall\ w_0 \in W_0$.

The set of actions available at any node $t_0 \in T_0$ is denoted as $A_0(t_0)$. Without loss of generality assume that actions available at any node are unique, i.e., $A_0(t_0) \bigcap A_0(t_0') = \emptyset$ if $t_0 \neq t_0'$. Action sets are empty for the terminal nodes; $A(z_0) = \emptyset$, $\forall z_0 \in Z_0$.

Exogenously given conditions of the game are described by ρ_0, which tells us the Nature's chosed probability distribution over actions available , whenever it is Nature's chance to move. Mathematically, $\rho_0 : \{t_0 |\ \iota_0(t_0) = N\} \rightarrow \Delta[A(t_0)]$. $A(w_0) = \{1\}$, in the perfect information game. Nature, moving at the initial node, chooses player 1 to move first. H_0 is the set of all information sets, i.e. it is the partition of the set of decision nodes X_0 into information sets. In this perfect information setting, all information sets $h_0 \in H_0$ are Singleton sets, containing exactly one node. Equivalently, $H_0 = X_0$. The payoff function u_0 maps terminal nodes to a real pair of payoffs for the two players; i.e., $u_0 : Z_0 \rightarrow R^2$. As an example, consider a three staged centipede game depicted in figure 3.

Figure 2



The next step is to introduce limited foresight and uncertainty about opponent's ability in Γ_0. Limited Foresight is modeled using multiple types of each player. The uncertainty is about which type one is facing.The **Interaction Game** is a game of **imperfect information** constructed from Γ_0, and reflects the interaction schematic among the players and their types. The underlying game Γ_0 generates a corresponding Interaction game Γ.

3.1 Constructing the Interaction Game from the Underlying Perfect Information Game

The game of imperfect information, said to be the **Interaction Game,** $\Gamma = \{T, \prec; A, \alpha; I, \iota; H, u, \rho\}$, is generated from a two player, perfect information, alternate move game $\Gamma_0 = \{T_0, \prec; A_0, \alpha_0; I_0, \iota_0; H_0, u_0, \rho_0\}$. The Interaction Game nests Γ_0, the underlying game.

3.1.1 Player Set: I

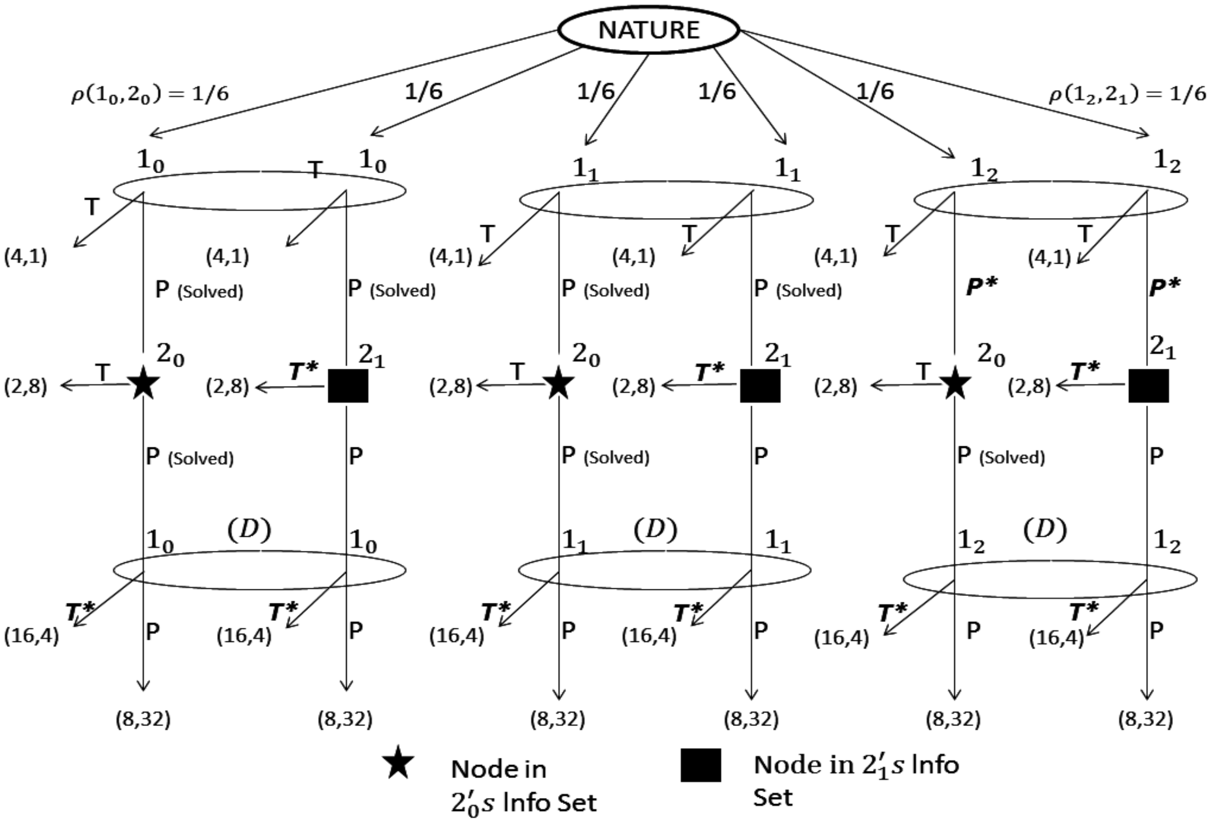
Suppose the underlying game has $N > 1$ stages, not counting Nature's move in the initial (zeroth) stage with the alternate move structure discussed above. Player 1 can be a type that curtails the game 0 stages ahead, i.e. the type that doesn't understand the interaction following his immediate move, as is 1_0 at stage 1 in Figure 1(c), whichever move that may be. Or, player 1 can be of the type who can understand the actions and payoffs accruing 1 stage ahead, as is 1_1 at stage 1 in figure 1(b), and use backward induction from there, but can't understand the interaction following that if the game extends beyond his foresight. Thus, player 1 can be of type 0, 1, and so on upto $(N-1)$ for the interaction game. We stop at type $(N-1)$ because player 1 of type $(N-1)$ can, even from the first time he moves in the first stage, backwardly induct from the very last stage backwards, understanding all the possible actions and payoffs of the whole underlying game, and thus is ex-ante fully rational. Similarly for player 2, we have types $0, 1, ..., (N-2)$. So, we generate the **player-type** set $I = \{1_0, 1_1, ..., 1_{N-1}, 2_0, 2_1, ..., 2_{N-2}, Nature\}$. Let ι be the player-type function for Γ, mapping the non-terminal nodes (described below) to the player-type set.

3.1.2 Nodes and Information Sets: T and H

Let $T = W \bigcup X \bigcup Z$ be the collection of all nodes of Γ. The set of initial nodes in Γ is W. In Γ_0, at w_0, $Nature's$ only action available was "player 1". Instead for Γ, at w (the only initial node), where $\iota(w) = Nature$, $A(w) = \{1_0, 1_1, ..., 1_{N-1}\} \times \{2_0, 2_1, ..., 2_{N-2}\}$, the set of pairs consisting of a player 1 type and player 2 type. $Nature$ chooses the probability distribution over these pairs $\rho \in \Delta[A(w)]$. Now, for each of these pairs in Γ, following nature's move, we exactly copy Γ_0 following nature's initial move at w_0 , and appropriately assign the nodes to information sets. In particular if we set $\rho(1_{N-1}, 2_{N-2}) = 1$, then Γ would just be Γ_0. The interaction game Γ corresponding to the perfect information three staged centipede game in figure 2 is depicted in figure 3. Corresponding to the 3 (so $N = 3$) staged centipede game in figure 2, the player set is $I = \{1_0, 1_1, 1_2; 2_0, 2_1\}$. Thus $A(w) = \{1_0, 1_1, 1_3\} \times \{2_0, 2_1\}$. So, we have 6 pairs of player types possible and hence we make 6 copies, ignoring nature's move in figure 2, of the original 3 staged perfect information centipede game for each of these pairs. In figure 3 we have chosen ρ to be uniform over $A(w) = \{1_0, 1_1, 1_3\} \times \{2_0, 2_1\}$, providing each pair an initial probability of $\frac{1}{6}$.

In particular, for each decision node $x_0 \in X_0$ of Γ_0 we have $N(N-1)$ corresponding decision nodes in X of Γ. This is because each time it was Player 1's (Player 2's) move in Γ_0, he was of only the full foresight type and he knew that Player 2 (Player 1) was also the full foresight type. With multiple foresight types and uncertainty about opponent's type, Player 1 (Player 2) himself can be one of N(respectively $N-1$) types, and thus have N(respectively $N-1$)information sets representing each of those types, and for each of these types, he is ex-ante not aware about which of the $(N-1)$(respectively N) types player 2 is , and hence we have $(N-1)$(respectively N) nodes in each information set. So, if $\iota_0(x_0) = 1$ (respectively 2), then we have N(respectively $N-1$) information sets, each containing $(N-1)$ (resp. N) nodes. This generates all the nodes and information sets in Γ. In figure 3, stage 1 of the interaction game corresponds to the stage 1 of the perfect information game. Player 1 moves in both games' first stage. But, in figure 3, player 1 can be of type $1_0, 1_1$ or 1_2. Each of these player-types is confused about which player 2 type he is facing. Thus there are 3 (N) information sets in stage 1 of the interaction game. As each of these types may be facing either of 2 types ($N-1$ types) of player 2, each information set contains 2 nodes. Similarly, in stage 2 of the interaction game in figure 3, there are two possible types of player 2 causing there to be two information sets and each of these player types is confused about which of the three types of player 1 he is facing leading to three nodes in each of the two information sets.

Figure 3



Let H denote the set of information sets of Γ. Let $\mathcal{T}: \; W_0 \bigcup X_0 \implies W \bigcup X$ as described above.

For each $z_0 \in Z_0$, we again have $N.(N-1)$ corresponding terminal nodes in the set Z in T. So, $z_1, z_2, ..., z_{N(N-1)} \in Z$ correspond to $z_0 \in Z_0$ iff $\forall i = 1, ..., N(N-1)$; $\mathcal{T}^{-1}[h(p_1(z_i))] = p_1(z_0)$. Where $p_1(.)$ function provides the unique immediate predecessor of any node. Thus we broaden $\mathcal{T}: \; T_0 \implies T$. The $\prec$ orders T according to the ordering of the nodes in T_0, i.e., $(t_0 \prec t_0') \iff stage(t) < stage(t') \; \forall \; t \in \mathcal{T}(t_0)$, and $t' \in \mathcal{T}(t_0')$. Also, $(t \prec t') \implies [\mathcal{T}^{-1}(h(t)) \prec \mathcal{T}^{-1}(h(t'))] \; \forall \; t, t' \in T$. For any information set $h \in H$ of Γ, the action set there is the same as the corresponding action set in the underlying game. That is, $A(h) = A_0(\mathcal{T}^{-1}(h)), \;\; \forall h \in H$. As $Z = \mathcal{T}(Z_0)$, for each $z \in Z$, we can obtain the payoffs for the two players by setting $u^i(z) = u^i(\mathcal{T}^{-1}(z))$ for $i = 1, 2$. Thus, we define $u : Z \rightarrow R^2$. This completes our description of the Interaction Game.

3.2 Solving for Strategies and Beliefs: Fictitious Games

The Interaction Game simply reflects the interaction schematic among the player-types but doesn't reflect their various foresight levels which affects what they observe and how they move. We solve for beliefs and strategies using each player-type's particular **Fictitious Game** at each of his moves. Limited Foresight causes each different type to "see" a specific distorted version of the Interaction Game at each move. This is captured in **Fictitious Games**. Each information set $h \in H$ of Γ generates a Fictitious Game (FG), F^h, used by the player-type moving there. The FGs of two information sets, say h, h' are labeled as $F^h, F^{h'}$ respectively. Suppose h is located at stage q of Γ and h' is located at stage r of Γ. Also, suppose $\iota(t) = 1_m$ and $\iota(t) = 2_k$. Then $F^h = F^{h'}$ iff $[q + m = r + k]$ or $[N < q + m \neq r + k > N]$. That is, if the foresight of the player-type moving at an information set plus the information set's stage number is the same as that sum for another information set of Γ, or if both of the player-types at the two information sets can see the whole Γ, then their FGs are the same. Consider figure 5 below. It is the FG of 1_1 at stage 1 as well as 2_0 at stage 2, as both can see a two staged version of the Interaction Game depicted in figure 3. Figure 3 above is the FG of 1_2 at stage 1, 2_1 at stage 2, and all player 1 types at the last stage as they can all see at least a 3 staged game from each of those moves.

Figure 4

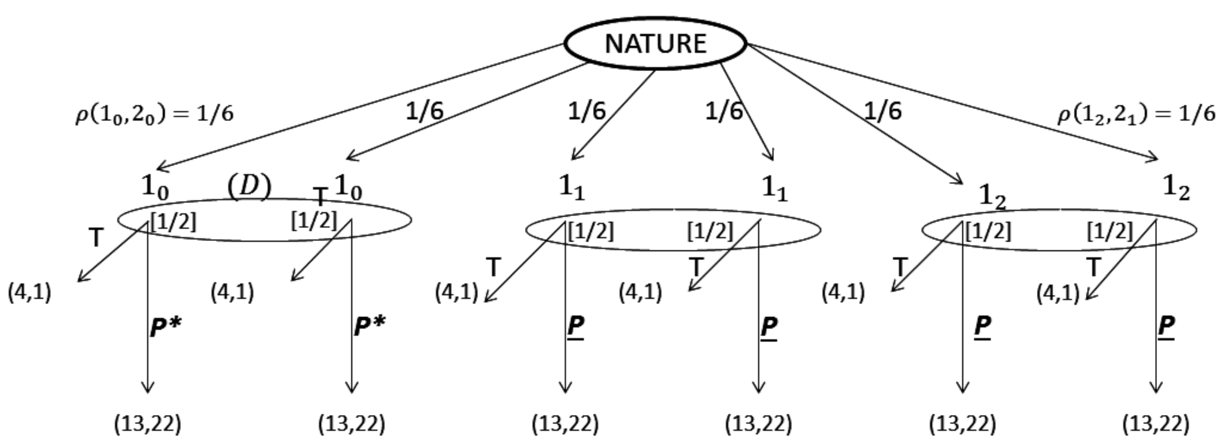
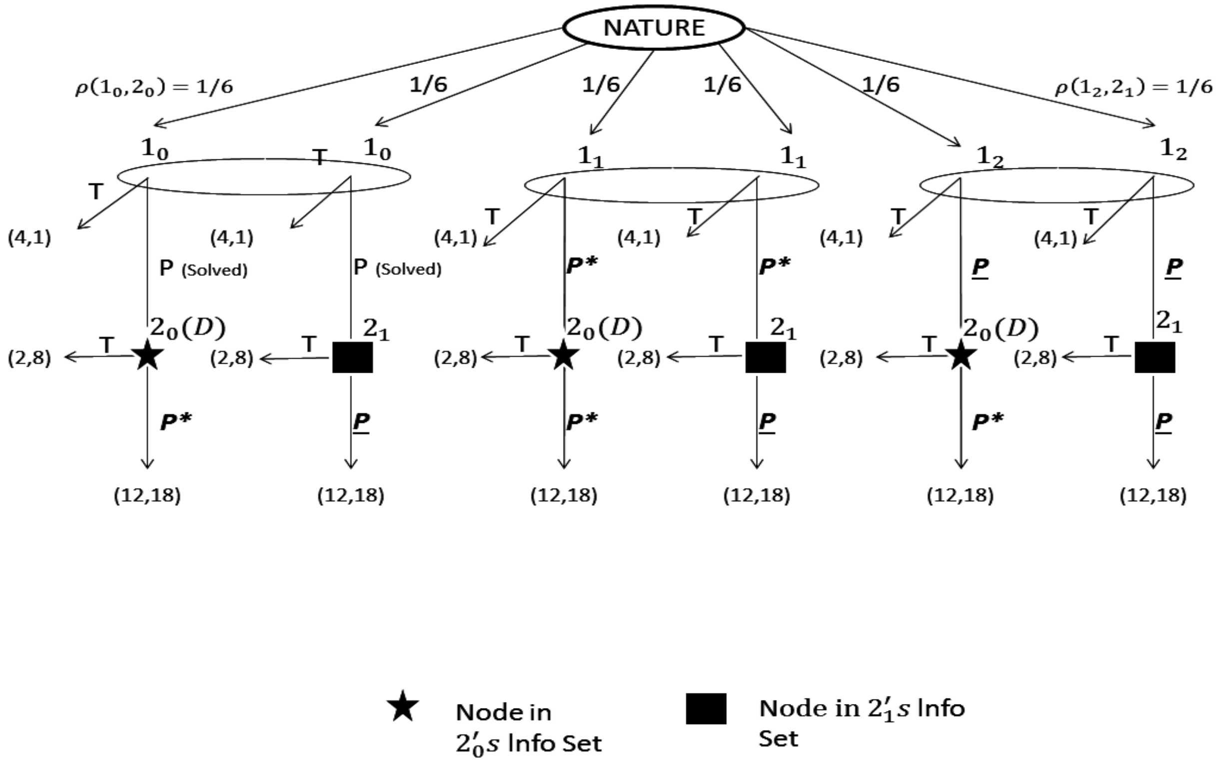


Figure 5



A FG Describes **Order, Choices Available, Player Set, Player Function, Information Sets, Payoffs and Initial Conditions.** Crucially these are often *fictitious constructs* believed by the player-type moving at the information set of Γ where it is nested. A fictitious game is denoted as $F^h \equiv \{T^h, \prec; A^h; I^h, \iota^h; H^h, u^h, \rho\}$, and is a well defined extensive game in itself. Consider the FG seen by 1_1 at stage 1 depicted in figure 5. Notice that it carries over ρ exactly the same as in figure 3, the corresponding Interaction Game to this FG. The FG in figure 5 is a 2 staged game as $1_1's$ foresight is 1 and he is at stage 1 when he sees this FG. Next, observe that all actions sets, information sets and payoffs at terminal nodes are exactly like those in figure 3 till stage 2. Finally note that this FG provides terminal nodes with payoffs after each action at the foresight horizon in stage 2; importantly, these payoffs are the average payoffs of the terminal successors of the node reached in figure 3 by taking that action. For example, in figure 5, consider the case where 2_1 is playing against 1_1 and he chooses P in the second stage. Then, according to the corresponding node in figure 3 we reach the node where 1_1 moves and chooses among actions P or T; both of which lead to terminal nodes with payoff profiles (16,4) and (8,32) respectively. So, in the two staged FG after $2_1's$ move of P, we construct a terminal node with the payoff profile as the average of these two payoff profiles.

In what follows; fix an information set $h \in H$. Let $\mathcal{T}^{-1}(h) = x_0 \in X_0$. We now describe the extensive fictitious game F^h nested there. We identify $\iota(h) = i_m$ (say). Let h be in stage q of Γ; then F^h is an $(q + m)$ staged game.

Where the first $(q+m)$ stages are exactly like Γ curtailed at the $(q+m)^{th}$ stage action. After the $(q+m)^{th}$ stage action, F^h has terminal nodes containing payoffs accruing to the two players. The payoff profile associated with the terminal node, say z^h, of the fictitious game is calculated using the decision node, say x^h, preceding it. To be specific, we look up the counterpart of x^h in Γ, say x. Suppose (x^h, a) leads to the z^h in question, where a is an element of the action set at x^h. Next, we locate the node y in Γ which is reached by taking action a from node x. Then the payoff profile at z^h is the average of the payoff profiles of $y's$ terminal successors[2] $Z(y)$ in Γ.

3.2.1 Fictitious Games : Nodes and Ordering

Fix $h \in H$, arbitrary. Let $\mathcal{T}^{-1}(h) = x_0$ and $\iota(h) = i_m$. Let h be in stage q of Γ. $T^h = Z^h \bigcup X^h \bigcup W^h$ denotes the set of all nodes of F^h. $T^h - Z^h \equiv \{t \in X \bigcup W : t$ belongs to a stage number $\leq q+m$ of $\Gamma\}$. Initial conditions: w^h is exactly like w in Γ. $\iota_0(w^h) = Nature$, $A(w^h) = \{1_0, 1_1, ..., 1_{N-1}\} \times \{2_0, 2_1, ..., 2_{N-2}\}$. Thus the Player-Type set is identical to the Interaction Game, so, $I = I^h$. Nature's choice is common knowledge for all fictitious games, so, $\rho \in \Delta(A(w^h))\ \forall h \in H$. Define $g_h : T \to T^h - Z^h$ an onto and invertible function, such that it preserves the $\prec$ ordering of T. That is, $g_h^{-1}(t) \prec g_h^{-1}(t') \iff t \prec t'\ \forall t, t' \in T^h/Z^h$.

3.2.2 Fictitious Games: Actions, Players and Information

Action sets are identical to the corresponding action sets in the Interaction Game. $A^h(t) = A(g_h^{-1}(t))\ \forall t \in T^h/Z^h$. The player function is also identical to the Interaction Game for corresponding nodes. $\iota^h(t) = \iota(g_h^{-1}(t))\ \forall t \in T^h/Z^h$. Common nodes are partitioned in a manner identical to Γ. $[h_1, ..., h_k \in H,$ is the partition of $T \bigcap (T^h/Z^h)] \iff [\{g_h(h_i)\}_{i=1}^k$ is the partition of T^h/Z^h: $\bigcup_{i=1}^n g_h(h_i) = H^h]$.

3.2.3 Fictitious Games: Payoff

We construct terminal nodes following each action of each $x^h \in X^h \subset (T^h/Z^h)$ such that x^h is in stage $(m+n)$ of F^h *or* the immediate successor of the corresponding node to x^h in Γ is a terminal node, i.e. $S(g_h^{-1}(x^h)) \in Z$. For each such decision node x^h, we start by identifying the node of Γ corresponding to x^h; $x = g_h^{-1}(x^h)$. Next, for each action $a \in A^h(x^h) = A(g_h^{-1}(x^h))$, we identify $t \in T$ of Γ such that x followed by a leads to t. Then in F^h, after $x^h \in (T^h/Z^h)$ followed by a, we add a (x^h, a) specific terminal node , say z^h, and the payoff profile at z^h is a vector where, for $i = 1, 2$, $u_i^h(z^h) = \frac{\sum_{z \in Z(t)} u^i(z)}{|Z(t)|}$ if $|Z(t)| > 0$. However, if $|Z(t)| = 0$ then $u_i^h(z^h) = u^i(z)$ for $i = 1, 2$. For

[2] Fix a node $t \in T$ of a general game Γ. Immediate Predecessor of t is denoted as $p_1(t) \equiv max\{x : x \prec t\}$. n^{th} predecessor of t is denoted as $p_n(t) \equiv p_1(p_{n-1}(t))$ if $p_{n-1}(t) \notin W$. $p_0(t) = t, \forall t$. Number of predecessors of t is denoted as $l(t)$ where $p_{l(t)}(t) \in W$. Immediate successor of t is denoted as $S(t) \equiv p_1^{-1}(t)$. $t \in Z$ implies that $S(t) = \emptyset$. Terminal successors of a node t is the set $\{z \in Z : y \prec z\}$. See Kreps and Wilson (1982) for detailed explanation.

each different (x^h, a) pair s.t. x^h is in stage $(m+n)$ of F^h *or* $S(g_h^{-1}(x^h)) \in Z$, and $a \in A^h(x^h)$, there is a unique successor node $z^h \in Z^h$. Constructing like this, generates the Terminal Node set Z^h of F^h, and $u^h : Z^h \to R^2$as above.

3.3 Learning

Player types learn more about the interaction game as the move forward in it. 1_0 observes only figure 4, a 1 staged FG constructed from Γ at stage 1. But, at stage 3 he observes the complete Γ. Formally, let Γ be an N staged interaction game. Let h, h' be two information sets of Γ at stage q, r respectively. Let $\iota(h) = \iota(h') = i_m$. If $(q+m) < N$ and $(q+m) < (r+m)$, then i_m observes a longer fictitious game F^h at h, compared to $F^{h'}$ at h'. In particular, he knows more about the underlying game Γ. *No Regret Equilibrium* is named so because it requires player-types to choose the best action at each move given their understanding of the Interaction Game at that move. Their understanding is reflected by the fictitious game they observe at that move, which keeps changing at each move as they move further down Γ. As a player-type moves forward, he approximates the Interaction Game better, thus *learning* more about the extensive form during the play of the game.

4 No Regret Equilibrium

The No Regret Equilibrium is an equilibrium of the Interaction Game solved via the Fictitious Games constructed from it. We can't use a solution concept directly on the Interaction Game, as player-types don't observe the complete Interaction Game at most of their information sets, hence can't be optimizing based on it. Thus, we propose the No Regret Equilibrium, where player-types, at each move, optimize based on their understanding of the interaction game at that move, i.e. they use their move specific fictitious game in a bid to optimize.

Consider an interaction game Γ, generated from an underlying Γ_0. Let Γ be an N staged game. Then Γ generates N fictitious games. For example, the Interaction Game in figure 3 generates figures 4,5 and 3 itself as fictitious games. We have one FG for each stage at which Γ can be curtailed to construct a FG. Let the n^{th} fictitious game of Γ, $FG(n)$, be the n staged FG constructed from Γ, where $n \in \{1, 2, ..., N\}$. Note that $FG(n) = F^h = F^{h'}$ if $\iota(h) = i_m$, and $\iota(h') = j_k$, and h is in stage q while h' is in stage r, as long as $(q+m) = (r+k) = n$. The 1^{st} FG is a one stage FG seen by 1_0 at stage 1. 2^{nd} FG is a two staged FG seen by 2_0 at the second stage and 1_1 at the first stage. Similarly, for an even N, the N^{th} FG, $FG(N) = \Gamma$, is seen by 2_0 at stage N, by 2_2 at stage $N-2$,..., by 2_{N-2} at stage 2; it is seen by 1_1 at stage $N-1$, by 1_3 at stage $N-3$,..., and by 1_{N-1} at stage 1. Thus, one FG is seen by multiple player-types at different stages of Γ.

Denote a strategy profile of the N staged Interaction Game as π. Here $\pi = (\pi_{1_0}, ..., \pi_{1_{N-1}}, \pi_{2_0}, ..., \pi_{2_{N-2}})$. For each player-type i_m; $\pi_{i_m} : h \longmapsto \Delta(A(h)),\ \forall\ h \in H\ s.t.\ \iota(h) = i_m$.

Denote a belief system of the Interaction Game as μ. $\mu = (\mu_{1_0}, ..., \mu_{1_{N-1}}, \mu_{2_0}, ..., \mu_{2_{N-2}})$. For each player-type i_m, $\mu_{i_m} : h \longmapsto \Delta(\{x : x \in h\}),\ \forall\ h \in H\ s.t.\ \iota(h) = i_m$.

Let $(\sigma^{FG(n)}, b^{FG(n)})$ be the assessment for fictitious game n, with $\sigma^{FG(n)}$ serving as the strategy profile and $b^{FG(n)}$ serving as the profile of beliefs.

Specifically, $\sigma^{FG(n)} = (\sigma^{FG(n)}_{1_0}, ..., \sigma^{FG(n)}_{1_{N-1}}, \sigma^{FG(n)}_{2_0}, ..., \sigma^{FG(n)}_{2_{N-2}})$. For each player-type i_m, $\sigma_{i_m} : h \longmapsto \Delta(A(h))\ \forall\ h \in H^{FG(n)}\ s.t.\ \iota(h) = i_m$.

Similarly, $b^{FG(n)} = (b^{FG(n)}_{1_0}, ..., b^{FG(n)}_{1_{N-1}}, b^{FG(n)}_{2_0}, ..., b^{FG(n)}_{2_{N-2}})$. For each player-type i_m, $b^{FG(n)}_{i_m} : h \longmapsto \Delta(\{x : x \in h\}),\ \forall\ h \in H^{FG(n)}\ s.t.\ \iota(h) = i_m$.

Let, the set of sequential equilibria of any game G be denoted as $\Psi(G)$.

*Definition 1 : $FG(n)$ is said to be **decisive** for the subset $H(n)$ of H, where $H(n) \equiv \{h : (h \in \text{stage } q) \text{ and } \iota(h) = i_l \text{ and } q + l = n\}$ for $n \in \{1, ..., N-1\}$. $H(N) \equiv H - \{H(n)\}_{n=1}^{N-1}$. Thus $\{H(n)\}_{n=1}^{N}$ forms a partition of H.*

*Definition 2 : (π, μ) is a **No Regret Equilibrium** iff : For each $n = 1, ..., N$, there exists $(\sigma^{\overline{FG(n)}}, b^{\overline{FG(n)}}) \in \Psi(\overline{FG(n)})$ such that $(\pi_{\iota(h)}(h), \mu_{\iota(h)}(h)) = (\sigma^{\overline{FG(n)}}_{\iota(h)}(h), b^{\overline{FG(n)}}_{\iota(h)}(h))$ for all $h \in H(n)$; where $\overline{FG(n)}$ is given by Definition (3).*

Definition 3: $\overline{FG(n)}$ is constructed by taking the NRE strategies for information sets of Γ where the fictitious game is shorter than $FG(n)$ as nature's moves in $FG(n)$. This is done by making changes (3.1) and (3.2) in FG(n), and keeping all other aspects of FG(n) the same.

(3.1) Modifying the player function of FG(n): $\iota^{\overline{FG(n)}}$. For all $h \in \bigcup_{m=1}^{n-1} H(m)$; $\iota^{\overline{FG(n)}}(h) = Nature$. Otherwise $\iota^{\overline{FG(n)}}(.) = \iota(.)$.

(3.2) Modifying the initial conditions of FG(n): For all $h \in \bigcup_{m=1}^{n-1} H(m)$; $\rho^{FG(n)}(h) = \pi_{\iota(h)}(h)$. Where (π, μ) is the NRE for which $\overline{FG(n)}$ is constructed.

That is $(\pi_{\iota(h)}(h), \mu_{\iota(h)}(h))$ of the decisive information sets of $FG(n)$ are taken from sequential equilibrium strategy and belief profile of FG(n), where for solving the SE of FG(n), the NRE strategies for information sets with fictitious games shorter than FG(n) are taken as part of the initial conditions of FG(n) to construct $\overline{FG(n)}$.

4.1 Solving a N staged Interaction Game Γ

Consider as an example the Interaction Game Γ depicted in figure 3. Our solution concept will proceed from the shortest FG, solve for a Sequential Equilibrium (henceforth SE) there. $FG(1)$, depicted in figure 4 provides us with the NRE strategy and belief for 1_0 at stage 1 which is to play P and believe he is facing $(2_0, 2_1)$ with probability $(\frac{1}{2}, \frac{1}{2})$. The (D) marked on $1_0's$ information set tells us that it is a member of the set (Singleton here) of information sets for which $FG(1)$ is decisive. This means that even though we calculate the SE strategies and beliefs of the other two information sets in $FG(1)$ we discard them for the purposes of the NRE. We underline the SE strategies at non-decisive information sets but ignore them. We put a star next to the strategies at the information sets for which $FG(1)$ is decisive and we pluck those out of the SE for collecting as part of the NRE strategy profile.

Next, in $FG(2)$ depicted in figure 5, we consider $1_0's$ 1^{st} stage NRE action (P) as fixed as nature's move and mark it "solved". Adding this to the initial conditions of $FG(2)$ gives us $\overline{FG(2)}$. We solve for $2_0's$ 2^{nd} stage NRE action and $1_1's$ 1^{st} stage NRE action (starred) and their NRE beliefs at these moves, as $FG(2)$ is decisive for them. Both 1_1 and 2_0 choose P . We do this by solving for the SE of $\overline{FG(2)}$. Again, we discard the SE actions (underlined) and beliefs of 2_1 at stage 2 and 1_2 at stage 1. For $FG(3)$, we consider 1_0 and $1_1's$ 1^{st} stage action fixed, and $2_0's$ 2^{nd} stage action as fixed to construct $\overline{FG(3)}$. Then we solve for the 1^{st} and 3^{rd} stage NRE action and NRE belief of 1_2, 2^{nd} stage NRE action and NRE belief of 2_1, and the NRE action and belief of 1_0 and 1_1 at the third stage. We do so by solving for the SE of $\overline{FG(3)}$. Proceeding thus we can solve for the NRE of any Interaction Game $\Gamma = FG(N)$. A strategy profile and belief system constructed thus is a No Regret Equilibrium.

4.2 No Regret Equilibrium Features

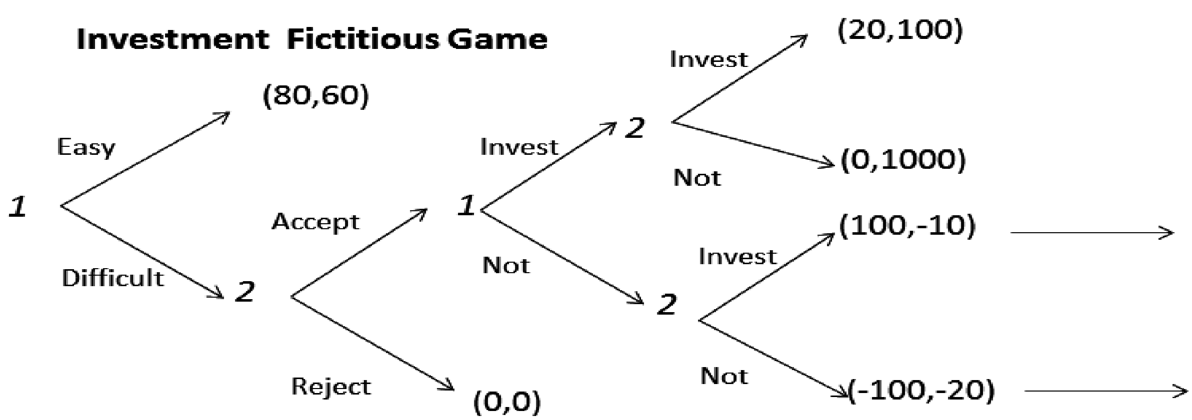
There are four key features that we get from using the above modeling along with sequential equilibrium to define NRE. Consider some player-type i_n at a certain move where he can not observe Γ. First, his double conditional belief distribution on opponent types who observe the same or smaller fictitious games at the previous move must be correct. Second, the conditional belief of Probability (opponent belongs to $\{2_m\}_{m \geq n+1}$; given the current information set) may be incorrect.

For example, suppose, conditional on reaching an information set h, the player-type 1_3 believes with probability $(0.1, 0.2, 0.3, 0.1, 0.1, 0.2)$ that the opponent's type is $(2_0, 2_1, 2_2 2_3, 2_4, 2_5)$, respectively. Then, the conditional probability belief split among foresight types "at least 4" versus "at most 3" is 0.3 and 0.7 respectively; which is allowed to be incorrect. This is because 2_5 moving at the preceding stage, say $s-1$, might have an optimal strategy that takes him away from h. But 1_3 can't observe what 2_5 is observing and can only use his own shorter foresight to guess the optimal action for 2_5 at $s-1$, and the resulting belief vector. Also at s, 1_3 is still solving for the NRE action of 2_4 at $s-1$ using a SE and that can entail an error. In contrast, 1_3 at h can observe a longer FG than what

$2_3, ..., 2_0$ can observe at $s-1$ or stages before than. Hence the double conditional belief probability distribution on $Prob[(2_0, 2_1, 2_2, 2_3)$ given h reached given that opponent 2_m is such that $m \leq n\ ] = (\frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{1}{7})$ must be correct.

Third, a surprising strategy, or being at an off NRE path reveals the higher opponent type. This follows from the first two features. If at h the belief probability is 0 on all opponent types according to NRE actions and initial conditions, then, as we use SE to calculate NRE beliefs, the belief probability must sum to 1 on the set of opponent types whose SE actions are being solved for in $\overline{FG(s+3)}$. In the current example, this set of opponents is $\{2_m\}_{m\geq 4}$. As lower opponent types' actions are part of the initial conditions of $\overline{FG(s+3)}$: $\rho(s+3)$. Note that the opponent types $\{2_m\}_{m\geq 4}$ are identical for 1_3 as against all of them 1_3 must use his complete foresight to calculate their optimal actions not already calculated in $\rho(s+3)$. Thus if a player-type, like 1_3 at h, finds out that he should not have been at h according to the SE path of F^h combined with lower types NRE actions, and therefore is surprised, he must assume that he is facing an opponent type smarter than him. This can be helpful; consider figure 6. We suppress the uncertainty about opponent type in this diagram showing only the curtailed perfect information game. The unmarked arrows Signify that the game continues further. Assume that the continuation payoff structure is well captured by the average displayed in this FG.

Figure 6



Suppose at the beginning of the FG in figure 6, there are only two equiprobable types of player 1: $\{1_0, 1_r\}$. Let there be only one type of player 2: 2_2. 1_0 wants to play "Easy" at first stage to get 80, which is greater than the average of the payoffs leading from the path following "Difficult". In the second stage 2_2 sees figure 6 (we don't show his lack of information about opponent type). If 2_2 believed he was playing 1_0 he would play "Accept" expecting 1_0 to "Invest" at stage 3 getting attracted by higher average payoff achievable from there. But, if at the second stage 2_2 believed he was playing 1_r then the FG from second stage will tell him to "Reject" at stage 2. This is because

1_r would subsequently play "Not" making 2_2 play "Invest" to minimize losses to -10 and providing 1_r with 100. As 2_2 at second stage is assured of $1_0's$ choice of "Easy", he will choose "Reject" following "Difficult", thus making "Difficult" off the NRE path. But as beliefs off NRE path must attribute probability 1 to an opponent-type whose SE strategy in the FG one is still guessing for ; 2_2 puts probability 1 on the opponent type being 1_r conditional on "Difficult" being played, making "Reject" sequentially rational.

Fourth, the believed strategy profile for a fictitious game may be incorrect, but it is optimal in the fictitious game. Following our example, consider 1_3 at h in stage s with an associated fictitious game F^h , an $s+3$ staged fictitious game. According to our algorithm for solving NRE this F^h is $FG(s+3)$. $FG(s+3)$ is decisive for $H(s+3)$, which consists of all h' such that if $(h' \in stage\ q)$ and $\iota(h') = i_l$ then $q+l = s+3$. Solving for the sequential equilibrium will give us the NRE actions and beliefs of all $h' \in H(s+3)$. The fact that we solve for NRE from the smallest fictitious game onwards implies that we have the NRE actions for all $\hat{h} \in \bigcup_{n=1}^{s+2} H(n)$ as part of $\rho(s+3)$. Taking $\rho(s+3)$ as given we solve for the SE of $F^h = FG(s+3)$, and record as NRE moves the SE (select one SE arbitrarily) moves at $H(s+3)$. But notice that $2_5's$ moves at stages $s-1$,$s+1$ and $s+3$ though part of the sequential equilibrium at $FG(s+3)$ are not recorded as part of NRE; neither is $1_2's$ move at stage $s+2$, and in fact nor are any moves from any $\tilde{h} \in H^h - \bigcup_{n=1}^{s+3} H(n)$. Thus the optimal SE actions for all information sets of F^h in which the player moving there can see a FG longer than F^h are ignored after calculating the SE. This is because these actions might not be optimal given the longer foresight for the player-type moving there. 1_3 attributes an optimal action, within the bounds of his own foresight, to those who can see further than him, but recognizes that his guess may be wrong. Thus 1_3 recalculates the SE of a longer FG at his next move.

4.3 NRE Properties

Proposition 1: For every Finite Interaction Game, there exists at least one No Regret Equilibrium.

Proposition 2: Given the Extensive form for an Interaction Game, the correspondence from pairs (p, u) of initial assessments and payoffs to the set of No Regret Equilibriums for the game so defined is upper hemi-continuous.

Both these proofs follow from the proposition 1 and 2 in Kreps Wilson 1982. For proposition 1 notice that one can use the existence of sequential equilibrium to guarantee the existence of NRE strategies and beliefs of $FG(1)$, which are the sequential equilibrium strategies and beliefs of 1_0 . Taking these as part of the initial conditions of FG(2) to get $\overline{FG(2)}$ and then using the existence of SE again we are guaranteed to find the SE and hence NRE strategies and beliefs of 2_0 at stage 2 and 1_1 at stage 1. Taking the first stage NRE actions of 1_0 and 1_1 , the second stage NRE action of 2_0 as part of the initial conditions of $FG(3)$ to construct $\overline{FG(3)}$ we are guaranteed by existence of SE to find the NRE strategies and beliefs of 1_2 at stage 1, 2_1at stage 2, and 1_0 at stage 3. Finally, proceeding thus we are guaranteed a solution for $\overline{FG(N)}$ constructed from $FG(N)$ taking all shorter fictitious games' NRE strategies as given, and hence we are guaranteed to find at least 1 NRE for any Finite Interaction Game.

Proposition 2 can also be reasoned in an inductive manner using the proposition 2 from Kreps Wilson (1982), which says that the correspondence generating SE from (ρ, u) pairs is upper hemicontinuous. We argue this in the Appendix.

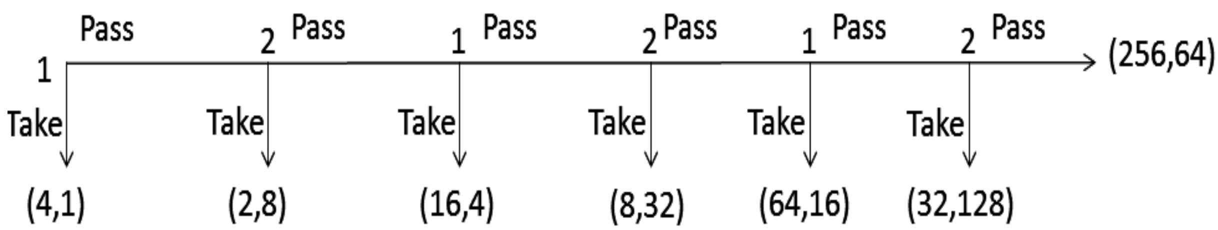
5 Applications

The key aim for developing the general NRE apparatus is to obtain general applicability in solving various existing perfect information game puzzles by modifying the initial distribution on player-types. In this section we apply the NRE model to the Centipede game introduced by Rosenthal (1982) and the Sequential Bargaining game analyzed by Rubinstein (1982) and Ståhl (1973).

5.1 The Centipede Game

The centipede game describes a Situation in which two players alternately decide whether to "take" or "pass" a pile of money. First, player 1 decides whether to take or pass a pile of money; if a player decides to take then he gets the larger share of the existing pile of money, while if that player passes, the pile of money grows. If a player passes, but his opponent takes in the next round, he gets a payoff lower than he would have had he taken; however if his opponent passes too, then the pile grows again and the player has a chance to take again and achieve higher payoff. The unique SPNE prediction is that the first player should take in the very first round, regardless of the number of stages that the pile can be passed and grown for the benefit of both the players. The logic is that in the last round, the player moving there should take and get the higher share of the pile; but given this, one should take in the second last round, and this optimality of take given one's opponent is going to take in the next round continues inexorably backwards, and leads to the SPNE prediction: take in the first round. This is highly unintuitive and various experiments reject the SPNE prediction. We focus on the one using sequential move games (McKelvey and Palfrey (1992)), whose results also soundly reject the SPNE conclusion.

Figure 7



Consider the six staged centipede game used by McKelvey and Palfrey (1992) in figure 7. The averaging of payoffs will give a payoff structure such that any limited foresight type, at any stage from where he can't observe the whole Interaction game generated by the centipede game, will prefer to pass than take. Moreover each limited foresight type will expect all his opponent types to also Pass in all stages leading up to each of his moves. Consider an N staged centipede game as Γ_0. This generates the following $I \in \Gamma$; $I = \{1_0, 1_1, 1_2, ..., 1_{N-1}, 2_0, 2_1, ..., 2_{N-2}\}$. Suppose ρ, the probability distribution on I is such that $\rho(1_n) = \alpha, \forall n < N-1$ and $\rho(1_{N-1}) = 1-(N-1)\alpha$. And independent of this distribution on 1's types, the distribution on 2's types is $\rho(2_n) = \beta, \forall n < N-2$ *and* $\rho(2_{N-2}) = 1-(N-2)\beta$. So, α is the ex-ante probability on each limited foresight type of player 1, and β is the ex-ante probability on each limited foresight type of player 2. The prior probability on the ex-ante rational types of player 1 and player 2 is $[1-(N-1)\alpha]$ and $[1-(N-2)\beta]$ respectively. We say 1_{N-1} is ex-ante rational as other player types of player 1 will also observe Γ as they move forward in the game, but it is only 1_{N-1} who observes the complete Γ from his very first move. Similarly, 2_{N-2} is the ex-ante rational type of player 2. The following lemma holds for the centipede game used by McKelvey and Palfrey (1992).

Lemma 1: (a) For arbitrary α, β such that ρ is a valid probability distribution, in any NRE, at each stage from stage 1 through stage (N-3), the full foresight rational types of player 1 and 2 must pass with strictly positive probability from stage 1 through stage (N-3) (b) The set of NRE is a Singleton, i.e., the NRE is unique.

To see why *lemma 1* holds first note that for all *FG(1) through FG(N-1)* the Sequential Equilibrium consists of beliefs being the same as prior beliefs and all player types playing pass. This is because passing at the last stage of any FG provides both players the highest payoff of the game. Thus for all the decisive information sets of each fictitious game shorter than Γ, the NRE strategy and belief is pass and the prior belief respectively. Put another way, as long as a player type can not see Γ, he passes and expects all his opponent types will also pass and therefore that the prior belief will be maintained. What remains to calculate is the SE of $\overline{FG(N)}$, taking as given that as long as a player type has limited foresight he passes. The payoffs in this game are such that, a player will be indifferent between pass and take at some stage s, given that he plans to take at $s+2$, if he expects that his opponent will play Pass with probability $\frac{1}{7}$.

Consider any arbitrary (α, β) in lemma 1a. Suppose the rational types of, WLOG, player 1 at some stage $s \leq (N-3)$ play Take with probability 1. First, let $s = (N-3)$. Note that this means that in stage $(N-2)$ rational types of player 2 will now face 3 types of player 1 that will play pass in stage $(N-3)$, namely $1_0, 1_1$ and 1_2. Out of these only 1_0 will play Pass again at stage $(N-1)$. This is because at stage $(N-1)$, 1_1 and 1_2, will turn rational and will play take as all player types take in the last round. Thus, if the rational types of player 2 passes at stage $(N-2)$, then with probability $\frac{1}{3} = \frac{\rho(1_0)}{\rho(1_0)+\rho(1_1)+\rho(1_2)} > \frac{1}{7}$ their opponent will pass again. Hence, player 2's rational types pass in stage $(N-2)$, as do player 2's limited foresight types. Thus in response to player 1 rational types taking with probability 1 at $s = (N-3)$, they face an opponent who, regardless of type, will pass with probability 1 in the next stage. Thus player 1 rational types taking with probability 1 at $s = (N-3)$ is not sequentially rational and can not be a SE of $\overline{FG(N)}$, and hence can not be a NRE. Similarly at any stage $s < (N-3)$ if the rational type player moving there, say i, chooses take with probability 1, then in the next stage, the other player, j, faces only $(N-s)$ limited foresight types out of which $(N-s-2)$ will again choose pass. For $s < (N-3)$, $\frac{(N-s)}{(N-s-2)} > \frac{1}{7}$ and thus j will play pass with probability 1, regardless of type, and thus i at s faces pass being played with probability 1; hence, take with probability 1 at stage s can not be sequentially rational.

If $\alpha, \beta > \frac{1}{7}$ then all player types pass with probability 1 from stage 1 through stage (N-2), this is because even the one remaining type in stage $(N-1)$ is enough to strictly incentivize the rational types to Pass with probability 1, and the limited foresight types Pass with probability 1 as argued before. Thus there is a stage $\underline{k}$ till which all player types play Pass with probability 1. This is because if at any stage all player-types pass with probability 1 then in the previous stage pass strictly dominates take for all player types. Next, between stages $\underline{k}$ and $\overline{k}$, the rational player types mix between take and pass to allow their rational opponent to mix. Finally, starting with $\overline{k}$, $(N-1) \geq \overline{k} \geq (N-2)$, the rational player types Take with probability 1.

This analysis is almost parallel to the McKelvey and Palfrey (1992) model without the errors in actions, heterogeneous beliefs and learning components. Both these analysis are in the same vein as the reputation literature of Kreps, Wilson, Milgrom and Roberts (1982). McKelvey and Palfrey say that with probability $(1-q)$ each player in the centipede game is an altruist and thus plays pass at every move. In our analysis, the limited foresight types (altruist for their analysis) slowly transition to being rational as the play moves forward. In both cases the rational type plays pass to pretend to be the type that passes. A noticeable difference from McKelvey and Palfrey (1992) is that unlike there, no type passes in the last round in the NRE. Thus, they have a result that says rational types pass with strictly positive probability till stage $(N-2)$ for any q value, while we can only say so for stage $(N-3)$. Their

$(1-q)$ equals, to begin with, $(N-1)\alpha$ for player 1 and $(N-2)\beta$ for player 2. We can simply borrow their proof for uniqueness (Lemma 1b) in their Appendix A, by treating their last stage as our second last and by adjusting for the fact that $(1-q)$ reduces as stage number increases. One can also verify that the number of stages in the mixing period are even for generic α, β.

5.2 Sequential Bargaining

The SPNE prediction for Rubinstein (1982) and Stahl's (1973) sequential bargaining game has also failed to stand up to experimental tests. The game consists of two player bargaining over a pie of size X. The total pie gets discounted by δ if the two parties disagree over the split of the pie and there is a delay of another period. A period has two stages. Here, we consider the case of the common discount factor. In the first stage, player 1 offers a split $(x_1, X - x_1)$ to player 2, where $x_1 \in [0, X]$. In the next stage of the same period either player 2 accepts this split, in which case the payoff profile $(x_1, X - x_1)$ accrues to player 1 and player 2 respectively. The other option for player 2 is to reject $(x_1, X - x_1)$. With this player 2 decision to accept or reject, the first period ends. If he rejects, the game enters the second period, which means the pie becomes δX. In the third stage, player 2 makes a counter offer of $(y_2, \delta X - y_2) \geq \mathbf{0}$. In stage 4 player 1 can accept $(y_2, \delta X - y_2)$ or reject it and take the game to period 3. If the game enters period 3, the size of the pie becomes $\delta^2 X$. In stage 5 player 1 makes a counteroffer of $(x_3, \delta^2 X - x_3) \geq \mathbf{0}$. This process continues, with the pie getting multiplied by δ at each successive period. Each period has two stages; an offer followed by an acceptance or rejection decision. Rejection leads to the next stage, and the next period. In the last period, say N, the player moving there, player 1 if N is odd and player 2 if N is even, makes an offer of $(x_N, \delta^{N-1} X - x_N) \geq \mathbf{0}$, if N is odd, or $(y_N, \delta^{N-1} X - y_N) \geq \mathbf{0}$ if N is even. In the last stage this last offer can be accepted or rejected by the other player. Both players get 0 upon rejection of the last offer.

The sequential bargaining has been studied extensively in the literature. Binmore et al (1985), using a first round pie of 100 and second round pie of 25, found that SPNE behavior was not observed in the two period game in their first study. Player 1 offered half the pie when the SPNE was to offer to take 75 for himself. However when player 1 did offer to take 75 for himself the offers were predominantly accepted. In the next study, player 2 of the first study was called upon to play as player 1. The authors observed that the erstwhile player 2, now player 1, was offering high x_1 close to the SPNE offer of 75. Thus, they conclude that strategic, rational and selfish play is active in the bargaining game.-

Neelin et al (1988) countered this view by conducting a two part experiment. First, they made subjects play 2, 3 and 5 period bargaining games such that the starting pie and SPNE outcome was fixed. The starting pie was always \$5 and the SPNE outcome was that the first stage offer of (\$3.75,\$1.25) should be immediately accepted. They observed that player 1 offers the second period discounted pie, (\$1.25,\$2.50,\$1.70 respectively for the 2,3, and 5 period games) at the first stage in all these games. Second, they verified the robustness of their results by multiplying the payoffs by 3. Guth and Tietz (1987,1990) also rejected Binmore et al (1985) by studying the two period bargaining game with discount factor 0.9 (SPNE: offer (10%,90%), and immediately accept) and 0.1 (SPNE offer (90%,10%)) , while performing the same reversal of player 2's role post experience. They found that role reversal does not lead to SPNE strategies.

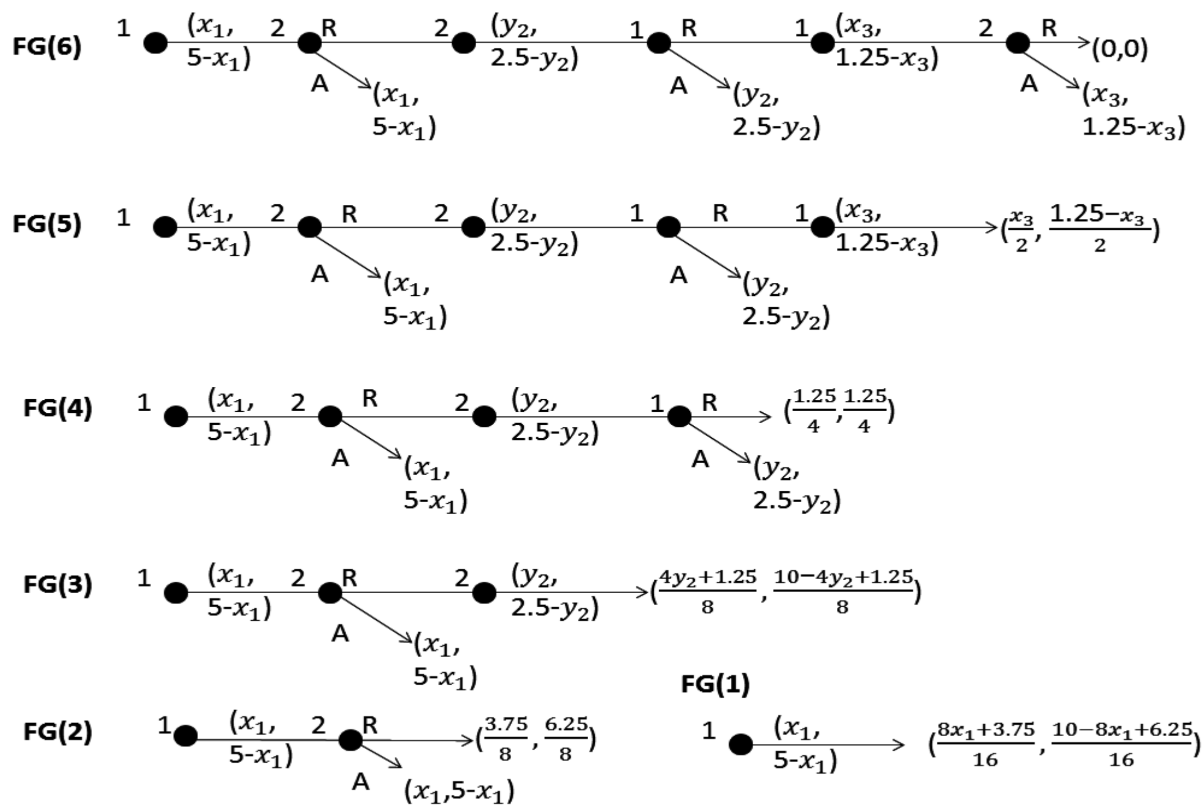
Ochs and Roth (1989) studied 2 and 3 period bargaining games, with both same and different discount factors across the two players. They found SPNE as a poor point predictor of outcomes. They also found that the experimental data does not support the qualitative predictions of SPNE: first, x_1 is larger in a 2 period game compared to the 3 period game; second, increasing 2's discount factor decreases $x_1$even in 2 stage games. They also found that 16% of first round offers are rejected. But following most rejections, player 2 offered a worse for self counter proposal! This led them to conclude that the monetary payoffs of the game do not truly reflect utilities. They confirm this feature of data from the unaggregated data from the above discussed experiments too. They discovered two other pervasive peculiarities in their data: first mover advantage in each subgame; and that "observed mean agreements deviate from the equilibrium predictions in the direction of equal division". They rationalize some aspects of the data by putting in a parameter for disutility from "insultingly" low offers, in other words, a minimum acceptance threshold. This can rationalize first mover advantage, first round offer rejections and the importance of player 1's discount factor. But for the other two oddities of the data, they invoke a discussion of uncontrolled elements affecting the utilities of the agents, for example, the fairness measure of an offer. Bolton (1991) fits a model in which each player's utility depend on the distribution of the monetary payoffs.

Johnson et al (2002) also used a 3 period bargaining game experiment with sizes of pies \$5,\$2.50, and \$1.25 in the three periods, i.e. a discount factor of 0.5. They investigate if the model of limited cognition or the model of equilibrium social preference correspond better to the data. Limited cognition in their paper means that subjects are not examining pie sizes in future periods and that players assume that their opponents will make and accept 40% offers. The figure of 40% is reached due to evidence from ultimatum games. This rule of 40% is said to be level 0 behavior, level 1 behavior is to expect that one's opponent will follow this behavior in the next round, while the fully rational behavior is to expect and respond with sequentially rational behavior. Social preference behavior implies that utility of each player depend on the distribution of payoffs within a payoff vector, and what is considered fair or unfair. They conduct three experiments. First untrained subjects play with each other. They observe that average offers are \$2.11 and low offers, below \$1.80, are rejected in 50% cases. They track if subjects are viewing future pie sizes through "Mouselab". The amount of periods a subject looks ahead explains offer and acceptance behavior well. Second, they make subjects bargain against robots which removes social preference's impact from utility. Though agents make (and accept) lower offers compared to first experiment, averaging $1.84, they are still higher than SPNE offer of $1.25. Experience doesn't improve their performance but training subjects in backward induction does make them follow SPNE behavior. The third study mixes the trained and untrained subjects (5 of each) together. They find that there is a tug of war among SPNE behavior and limited look ahead; subjects offer $1.60 in the first round.

Binmore et al (2002) also test 1 and 2 period bargaining games to investigate if three crucial parts of backward induction theory are verifiable in the lab. These components are: first, rationality, that is, "given a choice between two vectors of payoffs, a player chooses the most preferred" ; second, subgame consistency, the subgame's position within the game should not matter for behavior of the players; third, truncation consistency, that is, replacing a subgame with its SPNE payoff should not affect play at other nodes in the game. They conclude that subgame and truncation consistency do not hold. Players are more responsive to truncation payoffs than to changes in the SPNE payoff achievable after the play of the subgame. Subgame consistency is violated as 2nd period player 2 offers are more equitable than single stage bargaining offers. They conclude by affirming that the utility of a player depends on the vector of payoffs, and that additional elements like the dependence of utility on the extensive game form might be needed to explain the various peculiarities in the data.

We present here the NRE of the bargaining game used by Johnson et al (2002) and Neelin et al (1988). The figure below depicts the fictitious games without accounting for the informational uncertainty. The aim of this exercise is to demonstrate that NRE has the ability to explain some of the qualitative features of the data.

Figure 8



As this is an infinite game, we use a different method for calculating the mean payoff achievable from a certain action. We take the mean of means, ignoring the relative weights of the two means; for example going from FG(4) to FG(3). The reasons for doing so are: first, this is easy to calculate. Secondly, the discretized counterpart to this used in the experiments is computationally taxing but produces similar results as NRE. They key drivers of the NRE is the hierarchical structure and the underestimation of future bargaining power of player 1 as player 2 fails to see that player 1 will get the whole third stage pie.

We assume the prior to be $\rho(1_i) = \frac{1}{6}$ for $i = 0, 1, 2, 3, 4, 5$. Independently $\rho(2_i) = \frac{1}{5}$ for $i = 0, 1, 2, 3, 4$. We report the NRE strategies and beliefs in table 1 and table 2. The threshold numbers have been rounded to tenth decimals. All offers are stated in terms of what they offer player 1. Player 2 is understood to be getting the size of pie minus player 1's share. As discussed above, we start by solving from FG(1). The decisive set consists of only $1_0's$ move at stage I. He chooses the maximum possible $x_1 = 5$. In FG(2) we solve for $2_0's$ move at stage II (accept if $y_2 \leq 4.2$) and $1_1's$ move at I which is to offer the maximum acceptable amount 2_0. In FG(3) we solve for $1_2's$ action at I, $2_1's$

action at II and $2_0's$ action at III, fixing the previous calculations as nature's move. In FG(3) 1_2underestimates his strong last stage bargaining position because of the curtailment and averaging of payoffs at stage 3; Simultaneously 2_0 at III, and 2_1 at II overestimate their position because of the curtailment of foresight at stage 3. Notice that 2_0 at III, and 2_1 at II can tell 1_0 and 1_1 apart based on their x_1 offers. In FG(4), 1_3 at I further underestimates his bargaining position at stage 4, while 2_2 and 2_1 further overestimate it at II and III respectively.

In FG(5) 1_4 has an option to (optimally) offer one of the thresholds facing him: $\{4.2, 3.6, 3.1, 2.8\}$. 1_4 optimally chooses 2.8 to get maximum acceptance from all opponent types, although he is only guessing for the optimal action 2_4 at stage II . 2_3 at II is still underestimating player 1's strong position in the final stage and thus still has maximum acceptance offer at 3.1. 2_2 at III could tell 1_0 apart in stage II, and thus offers him his minimum acceptable $y_2 = 0.3$ for stage IV, for other player 1 types he offers their reservation offer based on his miscalculation due to curtailment and averaging. We construct $\overline{FG(6)}$ taking into account the NRE actions and beliefs solved for in FG(1) through FG(5). In $\overline{FG(6)}$, again 1_5 at stage I offers $(2.8, 2.2)$ as he knows that if he offers 3.6 then 2_2 and 2_3 shall reject, giving him an expected payoff of $(3.6)\frac{3}{5} + (1.25)\frac{2}{5} < 2.8$. 2_3 and 2_4 at stage II can tell apart 1_0 and 1_1 based on their high stage I offers, and offer them their respective (low) reservation offers in stage III. In the final stage all types are rational and thus all player 2 types accept any non-negative offer. In the second last stage all player types except 1_0 know this and demand the third period pie= 1.25. The NRE is detailed in tables 1 and 2. Thus, the following outcomes observed in experiments are observed in the NRE given our assumed ρ:

1. First round offer rejection (Ochs and Roth (1989)): $1_0's$ first round offer is rejected by all player 2 types. $1_1's$ offer is rejected by all but 2_0. $1_2's$ offer is rejected by 2_2 and 2_3.

2. Near equal split offer or second round pie offers (Neelin et al (1988); Guth and Tietz (1987); Ochs and Roth (1989)): $1_3, 1_4$ and even the rational player type, 1_5, offer $(2.8, 2.2)$ to get universal acceptance. This is caused by the high minimum threshold demand by 2_2 and 2_3 who have underestimated player 1's bargaining power come period 3.

3. Subgame consistency violation (Binmore et al (2002)): In the context of the current game, consider the four staged bargaining subgame starting from stage III player 2 offer with pie size of 2.5. Given a fixed ρ of the 6 staged game; if this subgame is solved separately as a game then the NRE will be different than when that subgame is a part of the 3 period game analyzed above. To see this note that 1_0 has probability 1 of reaching this subgame as all player 2 types reject his offer. 1_1 has lower probability than that but his probability of reaching this subgame is higher than 1_2. Among player 2 types, 2_0 has the lowest probability

(highest acceptance threshold) of reaching stage III, $2_4$is next higher, followed by 2_1, 2_3 and finally 2_2, who has the highest probability of reaching the stage III subgame, as 2_2 has the lowest acceptance threshold. Information is markedly different too when a subgame is part of a later game in the NRE setup. Here, 2_1 can tell apart 1_0 and 1_1, while $2_2$and 2_3 know exactly when they are playing against $1_0, 1_1$ and 1_2. Thus the relative probabilities in ρ change by the time play reaches the subgame in a larger game, as opposed to when we start a 4 staged game using ρ.

4. Disadvantageous counter proposals (Ochs and Roth (1989)): If player 2_2 at stage II receives a proposal between $(2.8, 2.2)$ and $(3.1, 1.9)$ he rejects it, given his view of $FG(4)$ and his underestimation of player 1's bargaining power in the last stage. Moving to stage III and observing $FG(5)$, he is better at estimating (though still underestimating) player 1's bargaining power in stage 5 and 6 of period 3 better . Thus 2_2 makes the lower counter proposal of $(0.6, 1.9)$; asking for only 1.9 after rejecting a chance to get a payoff in$(1.9, 2.2)$. The problem is that on the equilibrium path there is no offer between 2.8 and 3.1. Thus, NRE will be hard pressed to justify the quantity of disadvantageous counter proposals observed by Ochs and Roth (1989). But the NRE does justify a player type pursuing a disadvantageous counter proposal policy. Note that this feature disappears if we merge the acceptance/rejection stage with the counteroffer stage. That is, if we tell each player to either accept or make a counteroffer (unless in the last stage), then his acceptance/rejection threshold will exactly match the counteroffer he is going to make as the acceptance/rejection is calculated using the same fictitious game as the one used for making the counteroffer. Therefore this change in extensive form should remove this feature.

Table 1: Player 1-types' NRE strategies and beliefs

Stage	I	IV	V
1_0 Belief	Prior= $(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$	2_0 if $(y_2 = 0)$. Otherwise any distribution on $\{2_i\}_{i\geq 1}$	μ^1_{IV}
1_0 Strategy	$x^0_1 = 5$	*Accept* $\iff y_2 \geq 0.3$	$x_3 = 1.25$
1_1 Belief	Prior	$\mu^{\mathbf{1}}_{\mathbf{IV}} \equiv$[$(2_1, 2_2)$ if $y_2 = (0.3, 0.6)$ resp. O/w, distn on $\{2_i\}_{i\geq 3}$]	μ^5_{IV}
1_1 Strategy	$x^1_1 = 4.2$	*Accept* $\iff y_2 \geq 0.6$	$x_3 = 1.25$
1_2 Belief	Prior	μ^5_{IV}	μ^5_{IV}
1_2 Strategy	$x^2_1 = 3.6$	*Accept* $\iff y_2 \geq 1.25$	$x_3 = 1.25$
1_3 Belief	Prior	μ^5_{IV}	μ^5_{IV}
1_3 Strategy	$x^3_1 = 2.8$	*Accept* $\iff y_2 \geq 1.25$	$x_3 = 1.25$
1_4 Belief	Prior	μ^5_{IV}	μ^5_{IV}
1_4 Strategy	$x^4_1 = 2.8$	*Accept* $\iff y_2 \geq 1.25$	$x_3 = 1.25$
1_5 Belief	Prior	$\mu^{\mathbf{5}}_{\mathbf{IV}} \equiv [(y_2 = 0.6) \implies 2_2$ otherwise $2_3]$	μ^5_{IV}
1_5 Strategy	$x^5_1 = 2.8$	*Accept* $\iff y_2 \geq 1.25$	$x_3 = 1.25$

Table 2: Player 2 types' NRE strategies and beliefs

Stage	II	III	VI
2_0Belief	$(x_1 = 5) \implies 1_0$. Otherwise any distribution on $\{1_i\}_{i\geq 1}$	μ^1_{II}	$(x_1 = 5) \implies 1_0$. O/w 1_5
2_0 Strategy	*Accept* $\iff x_1 \leq 4.2$	$y_2 = 0$	*Accept*
2_1 Belief	$\mu^{\mathbf{1}}_{\mathbf{II}} \equiv [(1_0, 1_1)$ if $x_1 = (5, 4.2)$ resp. O/w distn on $\{1_i\}_{i\geq 2}]$	μ^2_{II}	$(x_1 = 4.2) \implies 1_1$.O/w 1_5
2_1 Strategy	*Accept* $\iff x_1 \leq 3.6$	$y_2 = 0.3$	*Accept*
2_2 Belief	$\mu^{\mathbf{2}}_{\mathbf{II}} \equiv [(1_0, 1_1, 1_2)$ if $x_1 = (5, 4.2, 3.6)$ resp. O/w distn on $\{1_i\}_{i\geq 2}]$	μ^3_{II}	$(\frac{1}{2}, \frac{1}{2})$ on $(1_2, 1_5)$
2_2 Strategy	*Accept* $\iff x_1 \leq 2.8$	$(x_1 = 5) \implies y_2 = 0.3$; O/w $y_2 = 0.6$	*Accept*
2_3 Belief	$\mu^{\mathbf{3}}_{\mathbf{II}} \equiv [(1_0, 1_1, 1_2)$ if $x_1 = (5, 4.2, 3.6)$; $(x_1 = 2.8) \implies (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$; on $(1_3, 1_4, 1_5)$ O/w $1_5]$	μ^4_{II}	1_5
2_3 Strategy	*Accept* $\iff x_1 \leq 3.1$	$(x_1 = 5) \implies y_2 = 0.3$;$(x_1 = 4.2) \implies y_2 = 0.6$; O/w $y_2 = 1.25$	*Accept*
2_4 Belief	$\mu^{\mathbf{4}}_{\mathbf{II}} = \mu^{\mathbf{3}}_{\mathbf{II}}$	μ^4_{II}	1_5
2_4 Strategy	*Accept* $\iff x_1 \leq 3.75$	$(x_1 = 5) \implies y_2 = 0.3$;$(x_1 = 4.2) \implies y_2 = 0.6$; O/w $y_2 = 1.25$	*Accept*

The above analysis shows that the NRE setup provides us several channels to explain several qualitative features of the data on sequential bargaining experiments. Fitting experimental data using this model is left as future work.

6 Conclusion

We define, prove the existence and upperhemicontinuity of the No Regret Equilibrium and end by showing its applicability in two existing puzzles in the class of finite, two player alternate move games, namely, the centipede and the sequential bargaining game. The NRE is defined for general applicability to produce more intuitive or experimentally justifiable results in the class of two player, sequential, alternate move, finite games. The model is set in the environment of self interested behavior within a hierarchical structure of various levels of limited foresight among each of the two players in the game. The environment nests the perfect information case.

The hierarchical structure of foresight levels provides incentives for high foresight types to alter behavior given opponent's foresight level distribution like in the level-k modeling. In the centipede game it unleashes reputation effects a-la Kreps, Wilson, Milgrom and Roberts, leading to cooperative behavior even among rational players. As a unique feature, low types try to understand their opponent's type to do the best they can given foresight bound. As players move forward in the game they get closer to understanding the game in the classical rational sense. In the sequential bargaining application, these features combine to help produce NRE outcomes parallel to qualitative results in the experimental literature.

7 Appendix

7.1 An inductive argument for Proposition 2

Let the correspondence $f : \Delta[A(w)] \times R^2 \implies \Pi \times \mathcal{M}$ be the set valued function, mapping initial conditions and payoffs ,(ρ, u), to the set of NRE strategies and beliefs (π, μ). Consider a sequence $(\rho_k, u_k) \to (\rho, u)$ and an associated sequence $(\pi_k, \mu_k) \in f(\rho_k, u_k)$. To show proposition 2, we need to show that if $(\pi_k, \mu_k) \to (\pi, \mu)$, then $(\pi, \mu) \in f(\rho, u)$.

We can again proceed by forward induction. Consider $FG(1)$. (ρ_k, u_k) generates $(\rho_k, u_k^{FG(1)})$ for $FG(1)$ using the curtail and average method described above. At each $(\rho_k, u_k^{FG(1)})$, the set of NRE strategies and beliefs calculated for 1_0at stage 1, with $(\pi_k(h_0^1), \mu_k(h_0^1))$ as its element, is formed by taking $(\sigma_k^{FG(1)}(h_0^1), b_k^{FG(1)}(h_0^1))$ from each element of the set of sequential equilbria of $FG(1)$ for $(\rho_k, u_k^{FG(1)})$. That is, $(\sigma_k^{FG(1)}(h_0^1), b_k^{FG(1)}(h_0^1))$ is extracted from $(\sigma_k^{FG(1)}, b_k^{FG(1)}) \in \Psi(\rho_k, u_k^{FG(1)})$. We know that $\Psi(.)$ is upper hemi continuous. So if $(\sigma_k^{FG(1)}, b_k^{FG(1)}) \to (\sigma^{FG(1)}, b^{FG(1)})$ we know that $(\sigma^{FG(1)}, b^{FG(1)}) \in \Psi(\rho, u^{FG(1)})$. Thus if $(\sigma_k^{FG(1)}(h_0^1), b_k^{FG(1)}(h_0^1)) = (\pi_k(h_0^1), \mu_k(h_0^1)) \to (\pi(h_0^1), \mu(h_0^1)) \in \Psi(\rho, u^{FG(1)})(h_0^1) = f(\rho, \mu)(h_0^1)$. That is, if a sequence of $1_0's$ first stage NRE action and beliefs, where the NRE set is generated using $(\rho_k, u_k^{FG(1)})$, converges to $(\pi(h_0^1), \mu(h_0^1))$; then as each point, including the limit point, of the sequence of action and beliefs is, by NRE definition, $1_0's$ first stage SE action and beliefs, and SE correspondence is UHC, the limit point belongs to the SE set image of $(\rho, u^{FG(1)})$ at (h_0^1), $\Psi(\rho, u^{FG(1)})(h_0^1)$, and thus to the NRE set image for $(\rho, u^{FG(1)})$ at (h_0^1), i.e. $f(\rho,\mu)(h_0^1)$.

Next, Consider $FG(2)$. (ρ_k, u_k) generates the associates sequence $(\rho_k^{FG(2)}, u_k^{FG(2)})$ for $FG(2)$. $u_k^{FG(2)}$ uses the curtail and average method described above. $\rho_k^{FG(2)}$ also takes into account $1_0's$ NRE strategy at h_0^1, $\pi_k(h_0^1)$, calculated from $FG(1)$ as part of the initial conditions of $FG(2)$. We know that $(\rho_k, u_k) \to (\rho, u)$ and $(\pi_k, \mu_k) \to (\pi, \mu)$ implies that $(\rho_k^{FG(2)}, u_k^{FG(2)}) \to (\rho^{FG(2)}, u^{FG(2)})$. Here we show that $(\pi_k(h_1^1, h_0^2), \mu_k(h_1^1, h_0^2)) \to (\pi(h_1^1, h_0^2), \mu(h_1^1, h_0^2))$ implies $(\pi(h_1^1, h_0^2), \mu(h_1^1, h_0^2)) \in (f(\rho,\mu)(h_1^1, h_0^2))$.

At each $(\rho_k^{FG(2)}, u_k^{FG(2)})$, the set of NRE strategies and beliefs calculated for 1_1 at stage 1, 2_0 at stage 2 with $(\pi_k(h_1^1, h_0^2), \mu_k(h_1^1, h_0^2))$, as its element, is formed by taking $(\sigma_k^{FG(2)}(h_1^1, h_0^2), b_k^{FG(2)}(h_1^1, h_0^2))$ from each element of the set of sequential equilibria of $FG(2)$ for $(\rho_k^{FG(2)}, u_k^{FG(2)})$. That is, $(\sigma_k^{FG(2)}(h_1^1, h_0^2), b_k^{FG(2)}(h_1^1, h_0^2))$ is extracted from $(\sigma_k^{FG(2)}, b_k^{FG(2)}) \in \Psi(\rho_k^{FG(2)}, u_k^{FG(2)})$. We know that $\Psi(.)$ is upper hemi continuous. So if $(\sigma_k^{FG(2)}, b_k^{FG(2)}) \to (\sigma^{FG(2)}, b^{FG(2)})$ we know that $(\sigma^{FG(2)}, b^{FG(2)}) \in \Psi(\rho^{FG(2)}, u^{FG(2)})$. Thus if $(\sigma_k^{FG(2)}(h_1^1, h_0^2), b_k^{FG(2)}(h_1^1, h_0^2)) = (\pi_k(h_1^1, h_0^2), \mu_k(h_1^1, h_0^2)) \to (\pi(h_1^1, h_0^2), \mu(h_1^1, h_0^2)) \in \Psi(\rho^{FG(2)}, u^{FG(2)})(h_1^1, h_0^2) = f(\rho,\mu)(h_1^1, h_0^2)$.

Proceeding thus, and using the UHC of Ψ repeatedly, we can ascertain the validity of proposition 2.

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