Dynamic Bidding in Second Price Auction

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Abstract

We consider equilibrium bidding behavior in a dynamic second price auction where agents have the option to increase bids at random times and values follow a Markov process. We prove that equilibrium exists and is unique and give an algorithm to solve for bids as a function of time and values. Equilibrium bids equal the expected final value conditional on the bid placed being the final one, meaning that either the agent doesn't get another opportunity to rebid or chooses not to increase this bid if given the option. This results in adverse selection with respect to a bidder's own future strategy, and as a result bids are shaded relative to the bidder's expected value. This is true in spite of values being independent across bidders. Under mild conditions, desired bids increase as time increases and the close of the auction is approached. Our results are consistent with repeated bidding and sniping, two puzzling observations in eBay auctions.

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...if something went over my limit early on, I might re-evaluate my budget and make a higher bid because I want the product. If it was the last few minutes I wouldn't have the time to consider if I can afford it.

...work + college = does not give you the right time you need to baby an auction.

The wife wanted to go to lunch right at the same time as the end of the auction, so I decided to drop an early bid an hour before the close.

1 Introduction

Various auction mechanisms happen in a dynamic setting while most of the theoretical models of auctions are static. For example, on eBay, auction listings usually last between three and seven days. Bidders can bid at anytime during the active time of the auction and also increase their bids at any other time prior to the end. As the first quote suggests[1], information might arrive during the auction that can change a bidder's value. The second and third quotes suggest bidders might not have full control of the timing of future bids.[2] In this paper we develop a model of a dynamic auction that captures these two features.

In our model, bidders arrive randomly and possibly at multiple times in a second price auction set in continuous time for a fixed interval. At each time of arrival bidders observe a new signal and choose whether to place a bid or increase a previous one. At the end of the auction, the winner is the highest bidder and the price equal to the second highest bid. We characterize the equilibrium of this dynamic auction.

Our paper is motivated by bidding behavior in eBay auctions, which are close to second price auctions, where bidders can place bids at any point in time up until the end of the auction. Several authors have emphasized what seem to be anomalies in bidding behavior, like submission of multiple bids throughout the auction and a higher concentration of bids towards the end of the auction, as we document below. This type of behavior cannot be

[1] The first two quotes were obtained from a blog http://www.neowin.net/forum/topic/587154-why-do-people-bid-so-early-on-ebay/. The second one is from http://www.kenrockwell.com/tech/ebay/early-bidding.htm.

[2] Sniping programs are an incomplete solution as they do not condition bids on possible changes of valuation that could arise from new information.

rationalized through the lens of a static model, where bidders should bid their valuation only once and at the time of arrival to the auction. Our model provides a way to rationalize this behavior combining two ingredients: new information can change the optimal bid throughout the auction and the possible lack of a future opportunity to bid provides a rationale for early bidding.

In most of the paper, we consider the case of independent private values, where signals and final values are independent across bidders, and where bidding times are exogenous.[3] We model this through a–otherwise unrestricted–joint Markov process of signals and bidding times that are also sufficient statistics for the final expected value. We prove equilibrium is unique and provide a dynamic programming algorithm to derive the optimal bids as functions of the state variable–signal and time.

Bidding behavior has an intuitive analog to a standard second price auctions' behavior. Consider first the case where bidders have a single opportunity to bid prior to the end of the auction. In that case, they bid their expected final value as in a standard second price auction. In contrast, when bidders can return to the auction prior to its end with positive probability and rebid, the expected value is modified taking into account that the current bid will only apply if the bidder chooses not to exercise that option. The optimal bid takes this into account and as we showat each bidding time bidders bid their expected value conditional on that bid being the final one.

The possibility of rebidding is a source of *adverse selection* as the current bid only applies when the bidder chooses not to exercise future options of rebidding and this is correlated with lower future value.[4] As a consequence, bidders shade bids below their unconditional expected final value. In particular, as the probability of bidding at the very end of the auction converges to one, any prior bid goes to zero as would happen with sniping.

To derive further properties, we consider the specialized case where bidding times are Poisson, with possibly time dependent arrival rates, while signals and final values are independent of arrivals. Assuming the expected final value conditional on a signal is non-

[3]Section 7 analyzes a simplified model with endogenous bidding and Section 8 analyzes a simplified model with correlated values among bidders.

[4]An analogue result is found in Harris and Holmstrom [1982], where initially worker's wages are shaded below marginal products, as the wage is effective in the future only if it is less or equal than the realized marginal product of the worker.

decreasing in time, we show that the bidding function increases over time. This provides a rationale why bidders might increase their bids over time independently of competitive pressures. The intuition for this result is straightforward and goes back to the incentives for shading: as the end of the auction approaches and the option of rebidding becomes less likely, the adverse selection problem mentioned above and the incentives for shading tend to disappear.

The problem of solving for the optimal bidding function is not a simple one, as current bids depend on all future bidding behavior. The dynamic programming problem we define provides an indirect method for deriving the optimal bidding functions, defined implicitly as zeros of the value function. We use this method to derive explicitly a closed form solution to the bidding function in a special case we call the *bad news* model, where bidders initial value can become zero if a Poisson shock arrives. This scenario can be motivated by exogenous events, e.g., the impossibility of attending a concert or bidders changing their mind or finding a better alternative for their purchase. In addition to this case, we also specialize the model to signals that follow a Brownian motion with drift, independently of Poisson arrivals for bidding times.[5]

The paper is organized as follows. Section 2 discusses the evidence on eBay bidding behavior and reviews related literature. Section 3 describes the model and defines an equilibrium. Section 4proves existence and uniqueness of equilibrium, characterizes bidding behavior and provides the dynamic programming algorithm. Section 5 gives properties for the case where values are independent of Poisson arrivals for bidding times and solves some special cases. Section 6 shows that the bidding function derived before is still applicable under arbitrary assumptions about the information a bidder observes on the past bids of other bidders. Sections 7 and 8 consider, respectively, the cases of endogenous bidding and correlated signals.

[5]We conjecture there exist differential equations for optimal bids as a function of time and current signals that we have not yet derived.

2 Evidence and Related Literature

The model presented in this paper closely mimics eBay auctions. eBay is an online auction and shipping website introduced in 2005. Sellers can sell their items either through an auction or by setting a fixed price for their item, an option called "Buy it Now." The auction mechanism is similar to a second price or Vickery auction. A seller sets the starting bid of an auction and bidders can bid repeatedly for the item until the end of the auction. Each bidder observes all previous bids, except for the current highest bid. A bidder should bid an amount higher than the current second highest bid, plus some minimum increment.[6] If this value is higher than the current highest bid, the bidder becomes the new highest bidder. Otherwise, he becomes the second highest bidder. The winner has to pay the second highest bid, plus the increment or his/her own bid, whichever is smaller. Auctions last for one to ten days and they have a pre-determined and fixed ending time that cannot be changed once the auction is active.

On eBay, bidders' bid-placement, as noted in the literature, does not follow the prediction of a static model of auction, for instance a disproportional share of bids are placed in the last few seconds of an auction. As a recent paper by Backus et al. [2013] shows, about third of winning bids are placed in the last ten seconds of an auction. Hayne et al. [2003] report that about 15 percent of bids placed are within the last 60 minutes of an auction and also about 61 percent of bids placed by bidders who submitted more than one bid. Gonzalez et al. [2009] similarly report 11 percent of bids placed within the last 60 minutes of auction and 77 percent of bidders placed more than one bids. Hayne et al. [2003] further note that on average there are 6.78 bids placed and about 3.98 unique bidders per auction. Moreover, they show that late bidding has a much higher success rate, about 75.17 percent, much more than early bidding, 7.33 percent, or bidding in between, 40.50 percent. Using a static model of auctions, strategic late bidding or bidding multiple times is not rational.

Several papers have tried to rationalize sniping behavior and incremental bidding. Bajari and Hortacsu [2003] incorporate a model of common value auction to explain this phenomenon. Having informed and uninformed participants, the informed bidders do not bid

[6] The increment is a function of the second highest bid, fixed for all auctions, and is set by eBay.

before the last period since that would reveal their private information to other potential bidders. Hence in the equilibrium, all bidders only place a bid at the very last period of the auction, leading to sniping by everyone. In another set of papers, to explain sniping Ockenfels and Roth [2002] and Ockenfels and Roth [2006] compare auctions mechanisms at eBay and Amazon which have hard ending and soft ending, respectively. Hard ending refers to a fixed ending time for an auction which cannot be extended by the seller or the marketplace; whereas soft ending refers to a tentative ending time: placing a bid within the last few minutes of the auction extends the duration of the auction for another ten minutes. They argue that along with the hard-ending rule at eBay, there are incremental bidders on eBay whom increase their maximum bid as they get outbid by other bidders. Therefore, a strategic bidder facing incremental bidders places bids in the last possible moment not to give incremental bidders time to react and place higher bids, giving rise to sniping.

Gray and Reiley [2004] and Ely and Hossain [2009] independently run experiments evaluating the benefit of sniping. The former does not find a statistically significant effect while reconfirming results in Ockenfels and Roth [2002] and Ockenfels and Roth [2006] regarding the prevalence of sniping; in their dataset 50 percent of bids are placed within the last minute of an auction. On the other hand, the latter find a significant value to sniping about $1 per auction for various DVD listings. Backus et al. [2013] in a more recent paper consider the effect of sniping on the return rate of new buyers to the marketplace. In all of these papers, the concentration is mainly on rationalizing sniping, and neither early bidding nor bidding multiple time is rationalized.

In this paper, we concentrate mainly on dynamics within the span of an auction. However, we are borrowing the result from Zeithammer [2006], Said [2011], Hendricks and Sorensen [2014] and Backus and Lewis [2012] implicitly. These papers model the dynamic value of not winning the current auction and the opportunity cost of participating in the next available auction. There are usually many closely substitutable items simultaneously available for auction on eBay and if bidders do not win a particular auction they have a chance of participating in the next available one. This will lead to a reservation price for the bidders below the static valuation of the item for them. Changes in the available alternative items can change the reservation price for bidders over time, and as they get closer to the end of the auction their valuation of the item at that instant becomes closer to their valuation at the end of the auction. While our paper misses some of the interesting features arising from the link between these auctions, it provides a very tractable reduced form.

3 The Model

There are i=1,...,N potential bidders in an auction. The auction is sealed bid second price and takes place in time interval $[0, T]$ where bids are submitted. As shown below in Section 6 equilibrium bidding functions are also applicable to sequential auctions -such as in eBay- with publicly available information on past bides.

Each bidder has the option of submitting bids only at random times $\tau_1, \tau_2, ...$ Bids can only be increased at any of these random bidding times and cannot be retracted. The valuation of the bidder is modeled as a stochastic process $v_i(t)$ of signals where without loss of generality $v(T)$ corresponds to the final value. Since these signals are only relevant at bidding times τ_n we restrict attention to the corresponding signals v_n at these dates. Assume $\{v_n, \tau_n\}$ follows a joint Markov process with transition function $P(\tau_{n+1}, v_{n+1}|\tau_n, v_n)$. Define the *state* of a bidder as the pair (τ, v) at the last bidding time. [7]We assume this process is independent across bidders.[8] Note that the model allows for random entry and rebidding, as well as a random number of bidders.

A bidding function for bidder i specifies at each possible bidding time τ_n and given signal v_n a desired bid $B_i(\tau_n, v_n)$. Given that bids can only be increased, $b_i(t) = \max\{B_i(\tau_m, v_m) | \tau_m \leq t\}$ is the bid that prevails at time t and in particular $b_i(T)$ is the final bid. Let $b_{-i}(T)$ denote the maximum bid over the remaining bidders at time T and $F_{-i}(b)$ its distribution. Utility for bidder i is given by:

$$U(v(T), b(T)) = \int^{b(T)} (v(T) - u)\, dF_{-i}(u).$$

An optimal bidding function B_i for bidder i is the one that maximizes expected utility at

[7] The process for the value $v_i(t)$ can be considered a continuous Markov process sampled at random stopping times τ_n that are Markov with respect to last stopping time and valuation at that time.

[8] Section 8 considers a simplified model with correlation.

the time the bidder enters the auction, i.e.,

$$\max_{B(.)} E\left(U\left(v\left(T\right), b\left(T\right)\right)|\tau_1, v_1\right) \tag{1}$$

where τ_1 is the time at which the bidder enters the auction and v_1 the initial value.

Definition. An *equilibrium* for the auction is a vector of bidding functions B_i and final distributions F_{-i} for each player i, such that for every bidder i the bidding function B_i is the best responses to F_{-i} and bidding functions are consistent with the final distribution of bids.

4 Equilibrium Bidding

In this section, we characterize the equilibrium bidding functions and provide a dynamic programming problem that can be used to solve them. We show that the optimal bid $B\left(\tau_n, v_n\right) = E\left[v\left(T\right)|\tau_n, v_n, b\left(T\right) = B\left(\tau_n, v_n\right)\right]$, namely the expected final value for bidder i conditional on the current state and the event that no higher bids are placed by the same bidder at a later instance so that the final bid $b\left(T\right)$ equals the current bid. Intuitively, this mimics the notion that in a second price auction it is weakly dominant strategy to bid the valuation or and when it is random, the expected valuation.

Proposition 1. *The optimal bid* $B\left(\tau_n, v_n\right) = E\left[v\left(T\right)|\tau_n, v_n, b\left(T\right) = B\left(\tau_n, v_n\right)\right].$

Proof. (sketch) We provide here a variational argument to characterize bids. Take a candidate optimal bidding function B_i for bidder i and consider bid b in state τ_n, v_n. Let $H\left(b\right)$ denote all paths $\omega = \{\tau_{m,} v_m\}_{m>n}$ starting from the current history where $B\left(\tau_m, v_m\right) \leq b$ including $\tau_{n+1} > T$ (i.e. no rebidding opportunity.) The expected value of bidding b equals:

$$V\left(\tau_n, v_n, b\right) = \int_{H(b)} \int^{b} \left(v\left(T\right) - u\right) dF_{-i}\left(u\right) dP\left(\omega\right) + \int_{H(b)^c} \int^{b(T,\omega)} \left(v\left(T\right) - u\right) dF_{-i}\left(u\right) dP\left(\omega\right)$$

We claim that the optimal bid is $b = B\left(\tau_n, v_n\right) = E_{H(b)} v\left(T\right)$. First, note that the boundary of the set $H\left(b\right)$ consists of all those paths starting from $\left(\tau_n, v_n\right)$ for which the final bid $b\left(T\right)$ is equal to b. So when considering the derivative of the above we can ignore the effect of the change in the supports of the two integrals. The first order condition is then $\partial V/\partial b = \int_{H(b)} (v(T) - b)\, dF_{-i}(b)\, dP(\omega) = 0$ which is equivalent to the statement that $b = E_{H(b)} v(T)$. □

Notice that there is *adverse selection* induced by the possibility of the bidder placing future bids, while the current one will only apply when the bidder chooses not to do so. As a consequence, bidders shade bids below their unconditional expected final value. To see this more clearly, consider the case where a bidder can currently submit a bid and will be able to bid once more only with probability p at the end of the auction. Given a current bid b, the final bid prevailing at the end of the auction is then b with probability $(1-p)$ and $\max(b, v(T))$ with probability p. The initial bid b binds in two cases: 1) there is no opportunity to rebid and 2) there is an opportunity to rebid but $v(T) \leq b$ therefore the bidder decides not to bid. Letting F denote the distribution of $v(T)$ conditional on current information,

$$b = \frac{(1-p)\, E(v(T)) + pF(b) \left(\int_0^b v dF(v) \right) / F(b)}{1 - p + pF(b)},$$

where $E(v(T))$ is the unconditional expectation. The second term in the above equation captures this adverse selection effect. It follows that $b < E(v(T))$ as $\int_0^b v dF(v) / F(b) < E(v(T))$. Indeed, as $p \to 1$ it is easy to see that b decreases monotonically to zero. This example also suggests that bids might increase as the end of the auction gets closer when the probability of getting a second chance to rebid decreases. We will examine this question in detail in Section 5.

While Proposition 1 characterizes bidding at a given history, it also shows that current bidding behavior depends on the whole strategy for future bidding, making the problem of calculating equilibrium bids potentially very complicated. However, there is a natural recursive structure to this problem which we exploit to define a dynamic programming problem that will help derive the optimal bidding function and establish uniqueness.

Define recursively the following function[9]:

$$W(b, v, t) = \int_t^T \min\left(W\left(b, v', \tau\right), 0\right) dP\left(v', \tau | v, t\right) + P(\tau > T | v, t) \left(E\left[v_T | v, t, \tau > T\right] - b\right) \quad (2)$$

[9] When conditioning with respect to $\tau = t$ and $v(\tau) = v$ we will write for short $P[.|v, t]$.

where τ denotes the following bidding time.

Assumption 2. *Assume* $P(\tau > T|v,t) > \delta > 0$ *for all* (v,t)*. In addition, assume all conditional probabilities and expectations are continuous in the state.*

By Assumption 2 and using standard dynamic programming arguments, it follows that there is a unique function satisfying functional equation (2), that it is strictly decreasing in b and continuous. Moreover, it is greater or equal to zero when $b = 0$ and negative for large b. It follows, by the intermediate value theorem, that there is a unique value $B(t,v)$ such that $W(B(t,v), v, t) = 0$. Given a new bidding time τ, value $v(\tau)$ and outstanding bid, this function also defines the rebidding region $\{(b, v(\tau), \tau) | W(b, v(\tau), \tau) > 0\}$ We will next show that this bidding function maximizes the agents expected utility.

Proposition 3. *The function* $B(\tau, v)$ *defined implicitly by* $W(B(\tau,v), v, \tau) = 0$ *satisfies* $E[v(T)|\tau, v, b(T) = B(\tau, v)]$ *and is thus the optimal bidding function.*

Proof. Let $Q(b,v,t)$denote the probability that $b(T) = b$ conditional on $\tau = t$ and $v(\tau) = v$. This can be also expressed as the compound lottery

$$Q(b,v,t) = \int_t^T \chi_{W(b,v',\tau)\leq 0} Q(b,v',\tau)\, dP(v',\tau|v,t) + P(\tau > T|v,t)$$

We now show recursively that

$$W(b,v,t)/Q(b,v,t) = E[v(T)|\tau = t, v(\tau) = v, b(T) = b] - b. \tag{3}$$

Substituting in (2)

$$\begin{aligned}
\frac{W(b,v,t)}{Q(b,v,t)} &= \left\{\int_t^T \min(W(b,v',\tau), 0)\, dP(v',\tau|v,t) + P(\tau > T|v,t)(E[v_T|v,t] - b)\right\} \\
&\quad * Q^{-1}(b,v,t) \\
&= \left\{\int_t^T Q(b,v',\tau) \min\left(\frac{W(b,v',\tau)}{Q(b,v',\tau)}, 0\right) dP(v',\tau|v,t) + P(\tau > T|v,t)(E[v_T|v,t] - b)\right\} \\
&\quad * Q^{-1}(b,v,t) \\
&= \left\{\int Q(b,v',\tau) \min(E[v(T)|\tau, v', b(T) = b] - b, 0)\, dP(v',\tau|v,t)\right. \\
&\quad \left. + P(\tau > T|v,t)(E[v_T|v,t] - b)\right\} * Q^{-1}(b,v,t) \\
&= \left\{\int_t^T \chi_{\{W(b,v',\tau)\leq 0\}} Q(b,v',\tau)(E[v(T)|\tau, v', b(T) = b] - b)\, dP(v',\tau|v,t)\right. \\
&\quad \left. + P(\tau > T|v,t)(E[v_T|v,t] - b)\right\} * Q^{-1}(b,v,t) \\
&= E(v(T)|\tau = t, v(\tau) = v, b(T) = b) - b.
\end{aligned}$$

□

Bid Shading

The example given above suggests that bidders will shade bids as a consequence of adverse selection given by the option of future rebidding. In this section we prove this is true for an arbitrary process. We first establish the following result:

Lemma 4. *$W(b,v,t) \leq E[v_T|v,t] - b$ with strict inequality if $P(\tau \leq T|v,t) > 0$.*

Proof. First note that by the law of iterated expectation, $E(E([v_T|v',\tau] - b|v',\tau)|v,t) = E[v_T|v,t] - b$ and that

$$\begin{aligned} E(E([v_T|v',\tau] - b|v',\tau)|v,t) &= \int_t E([v_T|v',\tau] - b|v',\tau)\,dP(v',\tau|v,t) \\ &= \int_t^T E([v_T|v',\tau] - b|v',\tau)\,dP(v',\tau|v,t) \\ &\quad + \int_T E([v_T|v',\tau] - b|v',\tau)\,dP(v',\tau|v,t) \\ &= \int_t^T E([v_T|v',\tau] - b|v',\tau)\,dP(v',\tau|v,t) \\ &\quad + P(\tau > T|v,t)\,E([v_T|v',\tau] - b|v,t,\tau > T)\,. \end{aligned}$$

We now prove the Lemma by induction. So suppose that $W(b,v',\tau) \leq E(v(T)|v',\tau)$. Then

$$\begin{aligned} W(b,v,t) &= \int_t^T \min(W(b,v',\tau),0)\,dP(v',\tau|v,t) + P(\tau > T|v,t)(E[v_T|v,t,\tau > T] - b) \\ &\leq \int_t^T W(b,v',\tau)\,dP(v',\tau|v,t) + P(\tau > T|v,t)(E[v_T|v,t,\tau > T] - b) \\ &\leq \int_t^T E([v_T|v',\tau] - b|v',\tau)\,dP(v',\tau|v,t) + P(\tau > T|v,t)(E[v_T|v,t,\tau > T] - b)\,. \\ &= E[v_T|v,t] - b \end{aligned}$$

where the first inequality is strict if $P(\tau \leq T|v,t) > 0$, thus completing the proof. □

Since $W(b,v,t)$ is decreasing in b, it follows that:

Proposition 5. *$B(v,t) \leq E[v_T|v,t]$ with strict inequality if and only if rebidding occurs with positive probability.*

5 Properties

This section derives general properties on equilibrium bidding behavior and considers some special cases. We specialize the model here to a case where the process for rebidding times τ_n is Poisson with intensity $\rho(t)$ that is independent of the signals and value $v(T)$. The main result in this section is that the bidding function $B(t,v)$ is increasing in t.

Proposition. *Assume $E[v_T|v,t]$ is weakly increasing in t. Then $\partial W(b,v,t)/\partial t \geq 0$ and bid $B(t,v)$ increases with t.*

Proof. Let $S(t,\tau) = \exp\left(-\int_t^\tau \rho(s)\,ds\right)$ be the probability of no arrival of bidding time in the interval $[t, t+\tau]$. For the specific process considered here, the value function (2) specializes to:

$$
\begin{aligned}
W(b,v,t) &= \int_t^T \rho(\tau) S(t,\tau) \left[\int \min\left(W(b,v',\tau),0\right) dP(v'|v)\right] d\tau \\
&\quad + S(t,T)\left[E(v_T|v,t) - b\right]
\end{aligned}
$$

We prove inductively that $W(b,v,t)$ is weakly increasing in t.

$$
\begin{aligned}
\partial W(b,v,t)/\partial t &= -\rho(t) \int \min\left(W(b,v',t),0\right) dP(v'|v) && (4) \\
&\quad + \int_t^T \rho(\tau)\rho(t) \left[\int \min\left(W(b,v',\tau),0\right) dP(v'|v)\right] d\tau \\
&\quad + \rho(t)\left[E(v_T|v,t) - b\right] + S(t,T) \frac{\partial}{\partial t} E(v_T|v,t) && (5) \\
&> -\rho(t) \int \min\left(W(b,v',t),0\right) dP(v'|v) \\
&\quad + \rho(t)\left[E(v_T|v,t) - b\right] + S(t,T) \frac{\partial}{\partial t} E(v_T|v,t) && (6) \\
&\geq S(t,T) \frac{\partial}{\partial t} E(v_T|v,t) \geq 0
\end{aligned}
$$

where the last inequality follows from Lemma 4. This completes the inductive proof. The second claim of the Proposition follows from the first one, the fact that $W\left(B\left(t,v\right),v,t\right)=0$ and that W is decreasing in b. $\square$

The above result implies that bids increase over time even in the absence of competitive pressure.

We now consider two special cases that satisfy the conditions of the above Proposition: a Poisson "bad news" model and the case of Brownian motion.

5.1 The Bad News Model

The agent starts with a valuation $v\left(0\right)>0$. There is a Poisson process that can turn that valuation to zero forever, with arrival rate λ; otherwise it remains unchanged. The random bidding times $\tau_1,\tau_2,...$ are determined by a Poisson process with arrival ρ. This special case captures the idea that a bidder might receive information during the auction that makes the object auctioned unattractive, e.g. the bidder might find there is another show she wishes to attend that day or engages in other commitments. It immediately follows that $W\left(b,0,t\right)=-b$ for all t. Moreover, the unconditional expectation at T equals $\exp\left(-\lambda\left(T-t\right)\right)v-b$

Substituting in functional equation (2)

$$\begin{aligned}
W\left(b,v,t\right) &= \rho\int_0^{T-t}\exp\left(-\rho\tau\right)\left[\exp\left(-\lambda\tau\right)\min\left(W\left(b,v,t+\tau\right),0\right)-\left(1-\exp\left(-\lambda\tau\right)\right)b\right]d\tau \\
&\quad +\exp\left(-\rho\left(T-t\right)\right)\left(\exp\left(-\lambda\left(T-t\right)\right)v-b\right) \\
&= \rho\int_0^{T-t}\exp\left(-\rho\tau\right)\exp\left(-\lambda\tau\right)\min\left(W\left(b,v,t+\tau\right),0\right)d\tau \\
&\quad -\rho\int_0^{T-t}\exp\left(-\rho\tau\right)\left(1-\exp\left(-\lambda\tau\right)\right)bd\tau \\
&\quad +\exp\left(-\rho\left(T-t\right)\right)\left(\exp\left(-\lambda\left(T-t\right)\right)v-b\right) \\
&= \rho\int_0^{T-t}\exp\left(-\rho\tau\right)\left[\exp\left(-\lambda\tau\right)\min\left(W\left(b,v,t+\tau\right),0\right)-\left(1-\exp\left(-\lambda\tau\right)\right)b\right]d\tau \\
&\quad -\frac{b}{\rho+\lambda}\left[\lambda+\exp\left(-\left(\rho+\lambda\right)\left(T-t\right)\right)\right]+\exp\left(-\left(\rho+\lambda\right)\left(T-t\right)\right)v
\end{aligned}$$

Now make the guess that $W(b,v,t)$ is increasing in t. Given that $W(B(v,t),v,t)=0$ it follows that $W(B(v,t),v,t+\tau)>0$ for all $\tau>0$. This implies that:

$$\begin{aligned} 0 &= W(B(v,t),v,t) \\ &= -\frac{B(v,t)}{\rho+\lambda}\left[\lambda+\rho\exp\left(-(\rho+\lambda)(T-t)\right)\right]+\exp\left(-(\rho+\lambda)(T-t)\right)v \end{aligned}$$

Solving for $B(v,t)$ this gives

$$B(v,t) = \frac{(\rho+\lambda)\exp\left(-(\rho+\lambda)(T-t)\right)v}{\lambda+\rho\exp\left(-(\rho+\lambda)(T-t)\right)}.$$

Note that when $\lambda=0$ or $t=T$ this gives $B(t)=v$ and when $\rho=0$ this gives the expected value $\exp\left(-(\lambda)(T-t)\right)v$, as should be. To connect to our intuition that bids are the expectation conditional on no rebidding, the two events to consider are: (1) no rebidding occurred and the value did not go to zero, that has probability $\exp\left(-(\rho+\lambda)(T-t)\right)$ and value v. The other event is that the value went to zero before the next rebid. This has probability $\lambda\int_t^T\exp\left(-(\rho+\lambda)s\right)ds=\frac{\lambda}{\rho+\lambda}\left(1-\exp\left(-(\rho+\lambda)(T-t)\right)\right)$ and value zero. Note that the sum of these two events:

$$\exp\left(-(\rho+\lambda)(T-t)\right)+\frac{\lambda}{\rho+\lambda}\left(1-\exp\left(-(\rho+\lambda)(T-t)\right)\right)=\frac{\lambda+\rho\exp\left(-(\rho+\lambda)(T-t)\right)}{\rho+\lambda}$$

Dividing the weighted value $\exp\left(-(\rho+\lambda)(T-t)\right)v$ by this sum of probabilities gives the bidding function above.

5.2 Brownian Motion

Suppose that $v(t)$ follows a Brownian motion with drift μ and variance σ^2 and the process for rebidding is Poisson with arrival ρ, as in the previous case. Equation (2) is now:

$$\begin{aligned} W(b,v,t) &= \rho\int_0^{T-t}\exp(-\rho\tau)\int\min\left(W(b,v',t+\tau),0\right)d\Phi\left(v'|v+\mu\tau,\tau\sigma^2\right) \\ &\quad +\exp\left(-\rho(T-t)\right)\int(v'-b)\,d\Phi\left(v'|v+\mu(T-t),(T-t)\sigma^2\right) \\ &= \rho\int_0^{T-t}\exp(-\rho\tau)\int\min\left(W(b,v',t+\tau),0\right)d\Phi\left(v'|v+\mu\tau,\tau\sigma^2\right) \\ &\quad +\exp\left(-\rho(T-t)\right)\left(v+\mu(T-t)-b\right) \end{aligned}$$

Note that for this case, $E[v(T)|v,t] = v + \mu(T-t)$ so if the drift $\mu \leq 0$ then $E[v(T)|v,t]$ is increasing with t, so by Proposition 5, $B(v,t)$ is increasing in t.

6 Bidding with Partially Observed Competing Bids and Censoring

The above assumed that the auction is sealed bid, so there is no information on competing bids. In this section, we show that this assumption is really immaterial and the bidding function derived above still hold, with a slightly different interpretation. Throughout this section we maintain our assumption of independent values across bidders. Let $B_i(v,t)$ denote the maximum bid that bidder i is willing to place at time t. The information on other bids during the auction has an effect on the expected utility of the agent as given by (1) as it affects the distribution of the highest competing final bid. But as we have seen, the optimal bid is independent of this distribution. Therefore, it remains an equilibrium for bidders to "ignore" other bidders bids in choosing their own bid $B(v,t)$.[10]

However, when taking the model to the data one must take into account that when a bidder observes an outstanding bid that is greater than $B(v,t)$, he will choose not to bid. Hence, the process for observed bids is censored and this censoring depends on the available information. In eBay auctions, bidders can see the outstanding second highest value thus defining the threshold below which bids are censored.

Timing of Bids

The model has implications for the observed timing of bids. If at time t a bidder has a high probability of returning to the auction right before its end, its desired bid $B(v,t)$ will be very low and thus it is likely to be censored by an existing higher second bid. As a consequence, it is unlikely to see any bids from this bidder until the end of the auction, consistent with the observed sniping behavior.

There is likely to be asymmetry in the frequency of bidding times for different bidders.

[10] We do not know if there are other equilibria.

The observation of sniping–defined as bids that are overtaken in the last few minutes of the auction–is usually interpreted as indication that many bidders follow this kind of strategy. As the following example shows, this might not be true. Suppose there are n bidders. One of these bidders can bid with probability one at the end of the auction, while the other $n-1$ bidders only bid at the beginning of the auction. To make things extreme, suppose the final value is uniform between $[0,1]$ and all the $n-1$ initial bidders have no information and thus approximately the same expectation equal to $1/2$ and so the initial winning bid is also around $1/2$. It follows that the probability that the remaining "sniping" bidder wins the auction is $1/2$. So in this auction, $1/2$ of the times the auction will be *sniped* while the share of snipers is only $1/n$. Note also that given the information structure sniping is still efficient as in absence of correlated information the expected value of the sniped bidder is $1/2$ and thus lower than the value for the sniper.

7 Endogenous Rebidding

We consider here a very simple model. There are $N = 2$ bidders and 2 periods. Let v_i denote the value for bidder i in the first period drawn from distribution $G(v)$ and v_i' the value in the second period, drawn from conditional distribution $F(v'|v)$. Both bidders can bid freely the first period but must pay a cost $c > 0$ to rebid in the second period. We assume bids are sealed.

Strategies for the bidders can be defined as follows: a bidding function $B_i(v)$ for the first period and a rebidding set $R_i(v)$ for the second period. Let $N_i(v)$denote the complement of $R_i(v)$. Given the strategy for the other player, player $i's$ expected utility is given by:

$$\begin{aligned} U(v) &= \int_{N_i(v)} Q_i(B_1(v))(v' - P_i(B_i(v)))\,dF(v'|v) \\ &+ \int_{R_i(v)} -c + Q_i(v')(v' - P_i(v'))\,dF(v'|v) \end{aligned}$$

where Q_i is the probability of winning function and P_i the expected payment conditional on winning. A symmetric equilibrium $(Q, R)$is a Nash equilibrium in these strategies.

Example 6. Both bidders draw independently their initial value $v \epsilon [0,1]$ from distribution

G and with probability $1-\rho$ get zero value next period and with probability ρ the value remains equal to v. Conjecture a threshold v^* so that:

$B_1(v) = \rho v$ if $v < v^*$and zero otherwise

$R(v) = \{\}$ for $v < v^*$and $R(v) = \{v\}$ otherwise

Consider the player with $v = v^*$. Bidding first or second period doesn't change his probability of winning since for $\rho v^* \leq b \leq v^*$, $Q(b) = G(v^*) + (1-\rho)(1-G(v^*))$. The expected payment is also the same in both cases equal to:

$$\frac{\rho \int_0^{v^*} v dG(v)}{G(v^*) + (1-\rho)(1-G(v^*))}.$$

The difference is that if he chooses not to rebid, he pays this expected value for sure while if he chooses to rebid he pays it with probability ρ. The difference in expected payment is then:

$$(1-\rho)\frac{\rho \int_0^{v^*} v dG(v)}{G(v^*) + (1-\rho)(1-G(v^*))}$$

to which we need to add that he pays an expected cost ρc. So v^* must be such that:

$$c = \frac{(1-\rho)\int_0^{v^*} v dG(v)}{\rho G(v^*) + (1-\rho)}. \tag{7}$$

The derivative of the right hand side with respect to v^* equals in sign to

$$v^*\left[\rho G(v^*) + (1-\rho)\right] - \rho \int_0^{v^*} v dG(v) > 0.$$

The last step is to show that for all $v > v^*$it is optimal to rebid and conversely for those not in this set. The difference between rebidding and not is

$$\begin{aligned} & \rho \int_0^{v} (v-x)\, dF_{-i}(x) - \int_0^{\rho v} (\rho v - x)\, dF_{-i}(x) \\ = \quad & \int_0^{v} (\rho v - \rho x)\, dF_{-i}(x) - \int_0^{\rho v} (\rho v - x)\, dF_{-i} \end{aligned}$$

Using the envelope condition and taking derivatives with respect to v (keeping bids fixed)

$$\rho F_{-i}(v) - \rho F_{-i}(\rho v) > 0.$$

This establishes that the gains from rebidding are increasing in v so the thresholdv^*defined by (7) is an equilibrium and it is unique.

One might expect that similar results will hold with more bidders. Moreover, it is natural to conjecture that the threshold for rebidding increases with the number of bidders, as expected payoffs decrease. Following this conjecture, with endogenous bidding one might expect that the number of bids per player decreases too.

8 Correlated Information

In many cases, it is likely that information or signals observed are correlated across bidders. For example, the arrival of a competing auction with a similar product is an event that creates an opportunity cost and is likely to affect in a correlated way the value of all bidders that keep track of that information. We have not been able yet to address this question at a general level and in this section, we consider a simplified scenario. Suppose time is discrete and there are two periods $\{1, 2\}$ where 2 corresponds to the end of the auction. Each bidder receives an independent initial value v_i, drawn from distribution $F(v)$, that we interpret as the unconditional expected final value in absence of further information. They simultaneously bid in the first period. With probability ρ, they have a chance to bid in the second period. Second period valuations are drawn from distribution $G(v'|v, \theta)$ where θ is a common observable shock that is independent of the initial values. Here v (resp. v') denotes the initial (resp. final) value for bidder one and v_2 (v_2') the corresponding values for bidder two.

An agent that bids in the second period will choose $b_2(v') = v'$. Let $B_1(v)$ denote the bid in the first period. This bid should equal the expected value conditional on the union of the following events: (1) none of the two agents get to rebid and $B_1(v) > B_1(v_2)$; (2) agent one gets to rebid but $v' \leq B_1(v)$and $v_2 \leq v$; (3) other agent rebids but $v_2' < B_1(v)$

and $B_1(v_2) < B_1(v)$, and (4) both get to rebid but $v', v_2' < B_1(v)$ and $B_1(v_2) < B_1(v)$. In a symmetric equilibrium with monotone bidding functions, the condition $B_1(v) \geq B_1(v_2)$ can be substituted by $v \geq v_2$.

Consider a symmetric equilibrium Given a bid b for the first player, we claim the following:

Proposition 7. *$B_1(v)$ is an equilibrium if and only if $E(v'|v, B_1(v)) = B_1(v)$, where*

$$E(v'|v,b) = \frac{F(v)(1-\rho)^2 v_1(b,v) + F(v)\rho(1-\rho)v_2(b,v) + \rho(1-\rho)v_3(b,v) + F(v)\rho^2 v_4(b,v)}{\pi(v,b)} \tag{8}$$

where:

$$\begin{aligned}\pi(v,b) &= F(v)(1-\rho)^2 + F(v)\rho(1-\rho)G(b|v) + \rho(1-\rho)P(v_2 \leq v, v_2' \leq b) \\ &\quad + F(v)\rho^2 \int \chi_{v' \leq b, v_2' \leq b} dP(v', v_2'|v, v_2 \leq v)\end{aligned}$$

$$v_1(b,v) = E[v'|v]$$

$$v_2(b,v) = E[v'|v' \leq b, v]$$

$$v_3(b,v) = \int v' dG(v'|v,\theta)\, dP(\theta|v_2' \leq b \& B_1(v_2) < b)$$

$$v_4(b,v) = \int \chi_{v' \leq b, v_2' \leq b} v' dP$$

and $H(v_2', v_2)$ is the joint distribution that is assumed to be independent of v.

Proof. Note that the denominator is also $\pi(v,b)$ the probability of winning with a bid equal to b. Suppose the bidding function satisfies this condition for all v. By symmetry and monotonicity of the bidding function, $B_1(v_2) \leq B_1(v)$ if and only if $v_2 \leq v$. Hence conditioning on $v_2 \leq v$ is equivalent to conditioning on $B_1(v_2) \leq b$ for $b = B_1(v)$. It is now easy to verify the expression given in equation (8) for $b = B_1(v)$ is precisely the conditional expectation described above. □

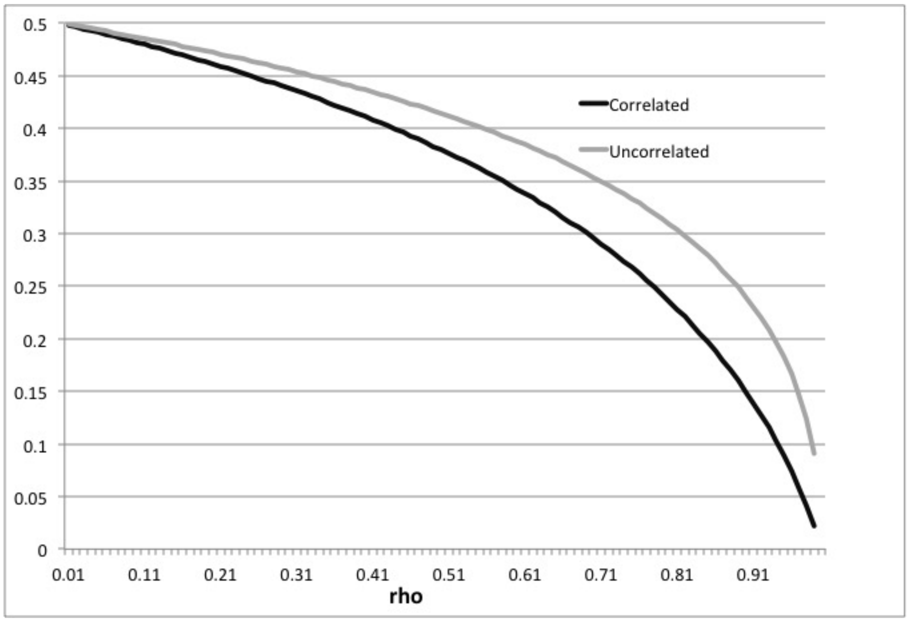

Figure 1: Bidding functions: effect of correlation

Suppose $v_i' = \theta v_i$ where θ is the common component. Rewriting equation (8),

$$E\left(v'|v,b\right) = \frac{F\left(v\right)\left(1-\rho\right)^2 v \int \theta dG\left(\theta\right) + F\left(v\right)\rho\left(1-\rho\right) v \int^{b/v} \theta dG\left(\theta\right) + \rho\left(1-\rho\right) v \int \chi_{\{v_2 \leq v, \theta \leq b/v_2\}} \theta dG\left(\theta\right) dF\left(v_2\right)}{F\left(v\right)\left(1-\rho\right)^2 + F\left(v\right)\rho\left(1-\rho\right) G\left(b/v\right) + \rho\left(1-\rho\right)\int \chi_{\{v_2 \leq v, \theta \leq b/v_2\}} dG\left(\theta\right) dF\left(v_2\right)} \tag{9}$$

Suppose in addition that v and θ are both uniform $[0,1]$. Closed form expressions can be found when θ is uniform $[0,1]$ and v_2 is also uniform.

$$E\left(v'|v,b\right) = \frac{v^2\left(1-\rho\right)^2/2 + \rho\left(1-\rho\right)b^2/2 + \rho\left(1-\rho\right)v\int^v \min\left(\frac{1}{2}, \frac{b^2}{2v_2^2}\right)dv_2 + \frac{\rho^2 v b^2}{2}}{v\left(1-\rho\right)^2 + \rho\left(1-\rho\right)b + \rho\left(1-\rho\right)\int^v \min\left(1, b/v_2\right)dv_2 + \rho^2 b} \tag{10}$$

$$= \frac{v^2\left(1-\rho\right)^2/2 + \rho\left(1-\rho\right)b^2/2 + \rho\left(1-\rho\right)v\left(b - \frac{b^2}{2v}\right) + \frac{\rho^2 v b^2}{2}}{v\left(1-\rho\right)^2 + \rho\left(1-\rho\right)b + \rho\left(1-\rho\right)\left(b + b\ln\left(v/b\right)\right) + \rho^2 b} \tag{11}$$

$$= \frac{v^2\left(1-\rho\right)/2 + \rho v b + \frac{\rho^2 v b^2}{2(1-\rho)}}{v\left(1-\rho\right) + \rho\left(2b + b\ln\left(v/b\right)\right) + \frac{\rho^2 b}{(1-\rho)}} \tag{12}$$

We solve for the fixed point above numerically and compare the bidding function with the one obtained for the uncorrelated case, where θ is independently drawn for the two players from a uniform distribution. Figure 1 plots bidding functions in both scenarios for an initial value $v = 1$. The x axis shows different probabilities of rebidding $\rho$and the y axis the corresponding bids. Consistent with our findings, as $\rho \to 1$ bids go to zero in both cases and at the other extreme, when $\rho = 0$ bids equal the unconditional mean of $\theta = 1/2$. More importantly, when information is correlated bidders shade their bids even more.

References

Matt Backus, Tom Blake, D Masterov, and Steve Tadelis. Is sniping a problem for online auction markets. Technical report, Working Paper, 2013. 2

Matthew Backus and Gregory Lewis. A demand system for a dynamic auction market with directed search. *Harvard University, October*, 2012. 2

P. Bajari and A. Hortacsu. The winner's curse, reserve prices, and endogenous entry: empirical insights from eBay auctions. *RAND Journal of Economics*, pages 329–355, 2003. 2

Jeffrey C Ely and Tanjim Hossain. Sniping and squatting in auction markets. *American Economic Journal: Microeconomics*, pages 68–94, 2009. 2

Raul Gonzalez, Kevin Hasker, and Robin C Sickles. An analysis of strategic behavior in ebay auctions. *The Singapore Economic Review*, 54(03):441–472, 2009. 2

Sean Gray and David Reiley. Measuring the benefits to sniping on ebay: Evidence from a field experiment. *Preliminary and Incomplete Draft*, pages 1–18, 2004. 2

Milton Harris and Bengt Holmstrom. A theory of wage dynamics. *The Review of Economic Studies*, 49(3):315–333, 1982. 4

Stephen C Hayne, CAP Smith, and Leo R Vijayasarathy. Who wins on ebay: An analysis of bidders and their bid behaviours. *Electronic Markets*, 13(4):282–293, 2003. 2

Kenneth Hendricks and Alan Sorensen. The role of intermediaries in dynamic auction markets. Technical report, Working Paper, 2014. 2

Axel Ockenfels and Alvin E Roth. Last-minute bidding and the rules for ending second-price auctions: Evidence from ebay and amazon auctions on the internet. *American Economic Review*, 92(4), 2002. 2

Axel Ockenfels and Alvin E Roth. Late and multiple bidding in second price internet auctions: Theory and evidence concerning different rules for ending an auction. *Games and Economic behavior*, 55(2):297–320, 2006. 2

Maher Said. Sequential auctions with randomly arriving buyers. *Games and Economic Behavior*, 73(1):236–243, 2011. 2

Robert Zeithammer. Forward-looking bidding in online auctions. *Journal of Marketing Research*, 43(3):462–476, 2006. 2