Head Starts And Doomed Losers: Contest Via Search

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Abstract

We study a two-player stochastic contest model in which the arrival of (innovation) candidates to each player is a privately observed Poisson process. Each player's process continues as long as he exerts costly effort until a predetermined deadline. The player whose best candidate is better by the deadline wins a prize. We first solve Nash equilibria for the game with symmetric players. Then we study the effects of head starts on players' equilibrium strategies and payoffs. when the head start is high, the head-starter wins the contest, regardless of the deadline. When the head start is in a middle range, it may either benefit (hurt) the head-starter or his opponent, and the head-starter is doomed to loss if the deadline is long. When the types of both firms' initial candidates are in a middle range, the head starter could improve his expected payoff by discarding his initial candidate and committing to search if the deadline is sufficiently long. Cost (valuation) asymmetries is discussed as well.

Keywords: contest, research tournament, head start, search

JEL classication: C72, C73, O32

1 Introduction

Dynamic contests, such as research tournaments, in which players spend resource to compete against each other in order to win some prize and each player can stop exerting efforts privately at any time are widely observed in the real world. For instance, prior to World War II, the U.S. Army Air Corps regularly sponsored prototype tournaments to award a production contract to the winning manufacturers; the Federal Communication Commission (FCC) held a tournament to decide on the American broadcast standard for high-definition television; Dow and IBM sponsor annual tournaments in which the winning contestants receive grants to develop their projects for commercial use; in professional sports, clubs or franchises search for competent free agents before new seasons start.[1] Despite of its prevalence, the relevant literature is very scarce, leaving alone the study by Taylor (1995) as an important exception.

To the best of our knowledge, most if not all existing works on dynamic contests focus on symmetric players. However, this setup fails to correspond to the reality. In many contest

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[1]See Taylor (1995) for more examples.

situations, players are heterogeneous in starting points. That is, one player may have a higher or lower initial state or position than his rivals. For instance, in the pharmaceutical industry, when a popular prescribed drug is close to its patent expiration date, the patent owner invests in its over-the-counter version, expecting to maintain its market share. Meanwhile, other drug producers intend to have their generic versions of the drug ready when the patent expires. In this case, the patent owner has already a head start in this competition. In this kind of contests, what observed often is that companies with a head start fail in competitions in the end. The failure of Nokia, the former global mobile communication giant, with the rise of iphones is one example. As addressed in the following excerpts:

"Nokia's initial reaction to the iPhone is the most embarrassing example of what went wrong. When Steve Jobs unveiled the device in January 2007, "it was widely disregarded," says former manager Dave Grannan..." (BloombergBusinessweek, June 01, 2011)

" And this was, in retrospect, a classic case of a company being enthralled (and, in a way, imprisoned) by its past success. In that sense, Nokia's failure resulted at least in part from an institutional reluctance to transition into a new era." (New Yorker Times, September 3, 2013)

"Stiff competition from Samsung and Apple, and lack on focus on innovation was the second big reason of collapse. " (NXTInsights, 14 September, 2013)

It seems that sometimes having head starts means being trapped. Why did Nokia refuse to spend resources to innovations (breakthroughs), and was it rational to do so? This motivates us to study the effects of head starts on players strategies and payoffs in dynamic contests .

In this paper, we construct a continuous time stochastic contest model with a preset deadline, and systematically analyze participants' competition strategies, and study the effects of head starts on firms' strategies and payoffs. In the contest, two firms compete for a fixed prize and each has to decide whether to participate in searching for (innovation) candidates before the contest starts. Each participating firm has to have a candidate in order to compete. The arrival of candidates to each firm is a Poisson process. The quality of each candidate is drawn from a fixed distribution. During the contest, one can decide when to stop his own process which continues as long as he expands costly search efforts until the preset deadline. Each firm only observes his own process and has no information about whether the opponent firm is actively searching. At the deadline, each firm compete against the opponent with the best candidate among all he has discovered, and the winner receives all the prize.

First, we study the equilibrium behaviors in a model with symmetric firms. We show that the domain the search cost can be divided into three regions, and firms' equilibrium strategies depends on which region search cost lies in. (1) When search cost is high, these are two pure strategy equilibrium in which one firm searches until he discovers a candidate and the other firm does not search, and a symmetric mixed strategy equilibrium in which each firm assigns a probability p to participating searching until he discovers a candidate and a probability $1 - p$ to not participating searching; (2) when search cost is in a middle range, each firm searches until he discovers a candidate; (3) when search cost is low, each firm continues searching until he discovers a candidate with a quality above certain positive cut-off value before the deadline arrives, and the cut-off strictly increases with the length of deadline and the arrival rate of candidates and strictly decreases with the marginal cost of search.

Next, we extend the benchmark symmetric model to include head starts: one firm (head-starter) has a better initial candidate than that of the other firm (runner-up). In this model, the domain of the head start can be divided into three regions, and equilibrium strategies depend on which region the head start lies in. (1) When the head start is high, neither firm will conduct searching. In this scenario, no innovation or technological progress is created and the head-starter wins the contest directly regardless of the deadline. (2) When the head start is low, both firms play the same equilibrium strategy as they do when neither firm has an initial candidate. In both cases, the head start benefits the head-starter and hurts the runner-up.

Our main findings concern about the cases in which the head start lies in the middle range, when the head start may hurt the head-starter and benefit the runner-up. In this range, the head-starter does not search, whereas the runner-up becomes more active in searching. When the head start is above certain value, the head-starter is doomed to loss when the deadline is very long, because his dominant strategy is still not to search. When the deadline is short the head-starter gets a higher payoff than the runner-up does. When the deadline is long the runner-up gets a higher payoff, in which case he highly benefits from it's initial disadvantage. Particularly, when the deadline goes to be very long, all benefit of the head start goes to the runner-up, whereas the head-starter gets a payoff of zero. For search cost being low and the deadline being long, if both firms' initial candidates are relatively high, the head-starter can improve his payoff by discarding his initial candidate and committing to search. By doing so, he actually makes the runner-up to be the head-starter, because the new head-starter is not willing to discard his initial candidate as shown in the incumbent-entrant game. Moreover, it is possible that by discarding the head start, the (former) head-starter could make both firms better off.

The typical scenario of the incumbent-entrant game, which commonly happens in real-world innovation contests where the incumbent has an existing technology or product whereas the potential entrant begins from nothing, is fully analyzed. It is shown that for search cost being low, a head start always benefits the incumbent, but it may also benefit the potential entrant as well when the deadline is long.

In addition, we discuss at the end a model with asymmetric search costs or valuations. Unlike the effects of head starts, the effects of a cost advantage are straightforward.

This paper is related to two strands of literature. The first strand is on stochastic dynamic contests.[2] The most related papers are Taylor (1995) , Seel and Strack (2013), and Lang et al. (2014). Taylor (1995) studies a T-period contest model. At each period each player decides whether to take an additional draw, the realization of which is a random variable following a commonly known distribution. The player who has the best draw by the deadline wins. In equilibrium, each player continues taking additional draws until having a draw above a certain cut-off. Seel and Strack (2013) consider a contest where each player privately observes a Brownian motion and choses the optimal stopping rule. There is no deadline and no search cost, but each player has to stop if the process reaches zero. In equilibrium, players would not stop immediately even when facing a negative drift of the process. Lang et al. (2014) study a multi-period points accumulation model. They find that in equilibrium the distribution over successes converges to the symmetric equilibrium distribution of an all-pay auction when the deadline is long enough. In addition, there is also a very recent working paper by Bimpikis et al. (2014) on dynamic contest design.

The second strand is on deterministic contests with head starts. Konrad (2002, 2004) studies two-player all-pay auctions with head starts in which the prizes are endogenously determined by players' pre-contest actions (investments and information disclosure). Kaplan et al. (2003) study two-player all-pay auctions with head starts and with time-dependent rewards and sunk costs. Kirkegard (2012) studies the effects of head starts and handicaps on the revenue and welfare of all-pay auctions with heterogeneous players. Seel (2014) studies a two-player all-pay auction model in which one player has a head start with the size being imperfectly known to his rival. Seel and Wasser (2014) consider the problem of optimal head starts in all-pay auctions. Casas-Arce and Martinez-Jerez (2011) solve a model of two-stage all-pay auction. Siegel (2014b) studies both single-prize and multiple-prize contests allowing for a wide range of asymmetries among any number of players.

The rest of the paper is organized as follows. In the following section we introduce the nota-

[2] There is a large amount of research in contest theory focusing on examining equilibrium and individual strategies in reduced-from models. For example, Hillman and Reiley (1989), Che and Gale (1998,2003), Cohen and Sela (2007), Bos (2012), Siegel (2009, 2010, 2014a).

tion. In Section 3, we give some preliminary results. The benchmark model is introduced after that. Section 5 presents the model with head starts. The model with asymmetric normalized search costs is put into the discussion section. Most proofs are relegated to the Appendix.

2 Model And Notation

There are two risk neutral firms, Firm 1 and Firm 2, in the following contest. Each firm decides whether to compete for a pre-specified prize, normalized to be 1. There is no entrance fee (fixed cost). At the beginning of the contest $t = 0$, each firm decides whether to start searching for (innovation or delegation) candidates. At any time point $t \in (0, T)$ each firm decides whether to stop searching. The cost of conducting search is $c > 0$ per unit of time for Firm i.

The arrival of candidates to each firm follows a Poisson process with an arrival rate of λ. That is, the probability of discovering m candidates in an interval of length Δ is $\frac{e^{-\lambda\Delta}(\lambda\Delta)^m}{m!}$. The types of candidates are drawn independently from distribution F, defined on $(0, 1]$ with $F(0) := \lim_{a\to 0} F(a) = 0$ and F being continuous and strictly increasing over the domain. At time T, the firm with the best candidate wins the prize. If no firm has discovered any candidate, the prize is reserved; If there is a tie between the two firms, the prize is randomly allocated to either of them with equal probability.

The searching processes of the two firms are independent and with recall. It is unobservable that whether the opponent firm is actively searching. Whether a firm has discovered any candidate and the types of discovered candidates are both private information until time T.

For convenience, we say a firm is in **state** $a \in (0, 1] \cup \{-1\}$ at time t if the type of the best candidate he has discovered by time t, where $a = -1$ means that the firm has discovered no candidate. The **initial states** of Firm 1 and Firm 2 are denoted by a_1^I and a_2^I, respectively.

We are going to study Nash equilibrium in two models, a benchmark model with symmetric players and no initial candidates and a model with head starts. Before that, let us have some preliminary results and make two assumption.

3 Preliminaries

The main objective of this part is to show the pattern of a firm's best responses to his opponent's strategies, based on which we restrict each firm's strategy space. First, we say $\hat{a}_i^t \in [0, 1]$ is a **cut-off rule** of Firm i if he stops searching when he is in a state above $\hat{a}_i^t$ at time t and continues searching when he is in a state below $\hat{a}_i^t$. In the following we define another two concepts.

Definition 3.1. *For a given strategy played by Firm j, we say $\underline{a}_i^t$ is Firm i's* ***lower optimal cut-off*** *at time t if*

$$\underline{a}_i^t = \inf\{\tilde{a} \geq a_i^I |\ \textit{in state } \tilde{a}, \textit{Firm } i \textit{ weakly prefers stopping searching to continuing searching at } t\};$$

$\bar{a}_i^t$ is Firm i's ***upper optimal cut-off*** *at time t if*

$$\bar{a}_i^t = \inf\{\tilde{a} \geq a_i^I |\ \textit{in state } \tilde{a}, \textit{Firm } i \textit{ strictly prefers stopping searching to continuing searching at } t\}.$$

Then we apply apply the above concepts in the following two lemmas.

Lemma 3.1. *Suppose $a_1^I = a_2^I = -1$.*

1. *if $\lambda < c$, then not to search is Firm i's dominant strategy: $\bar{a}_i^t = \underline{a}_i^t = -1$ for all $t \in [0, T]$;*
2. *if $\lambda > c$, then if Firm j does not search, Firm i's best response is to continue searching if he is in state -1 and stop searching once he is in a state higher than 0: $\bar{a}_i^t = \underline{a}_i^t = 0$ for all $t \in [0, T]$;*

3. *if $\lambda = c$, then if Firm j does not search, at any time point Firm i is indifferent between continuing searching and stopping searching if he is in state -1 and strictly prefers stopping searching if he is in a state higher than 0: $\bar{a}_i^t = 0$ and $\underline{a}_i^t = -1$ for all $t \in [0,T]$; if Firm j participates searching with a positive probability, Firm i strictly prefers not to search: $\bar{a}_i^t = \underline{a}_i^t = -1$ for all $t \in [0,T]$.*

Proof. See Appendix A.1. □

This lemma directly implies that for $c < \lambda$ there is no firm going to search. To rule out this case, we make the following assumption for the whole paper.

Assumption 3.1. $c_i \in (0, \lambda)$.

Lemma 3.2. *Suppose $a_1^I = a_2^I = -1$. If $\lambda > c$, for any fixed strategy played by Firm j, Firm i's best response is either*

1. *not to search: $\bar{a}_i^t = \underline{a}_i^t = -1$ for all $t \in [0,T]$,*
2. *or to search with a constant cut-off rule $\hat{a}_i \geq 0$: $\bar{a}_i^t = \underline{a}_i^t = \hat{a}_i \geq 0$ for all $t \in [0,T]$,*
3. *or with any mixed action between stopping searching and continuing searching at any time point: $\bar{a}_i^t = 0$ and $\underline{a}_i^t = -1$ for all $t \in [0,T]$.*

Proof. See Appendix A.1. □

In the first and second scenarios of Lemma 3.1*and*3.2, the best responses are singleton. In the third scenario of Lemma 3.2 and the first part of the third scenario of Lemma 3.1, the set of best responding strategies is large, and it is not convenient to write out a general strategy in this set in a clear mathematical form. We therefore select some strategies and take a subset. Specifically, we only look at strategies in which Firm i choose a probability p_i to participate searching at $t = 0$ and $1 - p_i$ to not to search. That is, we only allow firm play mixed actions before searching starts. Thus, we make the following assumption for Section 4 and **??**.

Assumption 3.2. *Each Firm's strategy space is $\mathcal{S}^F := ([0,1] \times \{0\}) \cup (\{1\} \times [0,1])$.*

We can therefore write a general strategy of Firm i as $[p_i, \hat{a}_i] \in \mathcal{S}^F$ in which p_i is Firm i's probability to participate searching and $\hat{a}_i$ is his cut-off rule once he started searching. We also just write strategy $\hat{a}_i$ as $[p_i, \hat{a}_i]$ for short when $p_i = 1$. A general strategy profile is written as $([p_i, \hat{a}_i], [p_j, \hat{a}_j])$, and $(\hat{a}_i, \hat{a}_j)$ for short when $p_i = 1$ and $p_j = 1$.

A remark is that a firm would delay the acceptance of a candidate if and only if the type of the candidate is below the cut-off. Once a firm meets a type above the cut-off, he will accept the player immediately.

Lemma 3.3. *Suppose a firm's initial state is -1, and he searches with a cut-off $\hat{a} \geq 0$. Then, his probability of ending up in a state **lower** than $a \in (0,1] \cup \{-1\}$ at time T is*

$$Z(a|\hat{a},T) = \begin{cases} 0 & if\ \ a = -1 \\ e^{-\lambda T[1-F(a)]} & if\ \ 0 < a \leq \hat{a} \\ e^{-\lambda T[1-F(\hat{a})]} + \left[1 - e^{-\lambda T[1-F(\hat{a})]}\right] \frac{F(a)-F(\hat{a})}{1-F(\hat{a})} & if\ \ a > \hat{a}. \end{cases}$$

Proof. See Appendix A.1. □

$1 - e^{-\lambda T[1-F(\hat{a})]}$ is the probability that searching stops before time T, and $\frac{F(a)-F(\hat{a})}{1-F(\hat{a})}$ is the conditional probability that the player above the threshold the firm meets is in between $\hat{a}$ and a. An important property of Z, which we will use quite often, is stated below.

Lemma 3.4. *Given $a > a'$, $Z(a|\tilde{a},T) - Z(a'|\tilde{a},T)$*

1. *is constant in $\tilde{a}$ for $\tilde{a} \geq a$;*
2. *strictly decreases in $\tilde{a}$ for $\tilde{a} \in (a', a)$;*
3. *strictly increases in $\tilde{a}$ for $\tilde{a} \leq a'$.*

Proof. See Appendix A.1. □

4 Symmetric Players: $a_1^I = a_2^I = -1$

In this section we study the benchmark model, in which search costs of both firms are the same and both firms start with no candidate. We show that the existence of pure strategy equilibrium is ensured. The domain of searching cost c and deadline T can be partitioned into three regions as show in Figure 1, and the types of the equilibria depends on which region (c, T) lie in. The first main result is stated as below.

Theorem 4.1.

i. If $c \in (\lambda, \frac{1}{2}\lambda(1+e^{-\lambda T}))$, there are two pure strategy equilibrium, $(0, [0, 0])$ and $([0, 0], 0)$, in each of which one firm searches with cut-off 0 and the other firm does not, and there is a unique symmetric equilibrium, $([p, 0], [p, 0])$ where $p = \frac{2}{1-e^{\lambda T}} \cdot \left(1 - \frac{c}{\lambda}\right)$, in which each firm assigns a probability p to search with cut-off 0 and probability $1 - p$ to not to participate.

i^+. If $c = \frac{1}{2}\lambda(1 + e^{-\lambda T})$, there are two pure strategy equilibrium, $(0, [0, 0])$ and $([0, 0], 0)$.

ii. If $c \in [\frac{1}{2}\lambda(1 - e^{-\lambda T}), \frac{1}{2}\lambda(1 + e^{-\lambda T}))$, there is a unique equilibrium, $(0, 0)$, in which both firms search with cut-off 0.

iii. If $c \in (0, \frac{1}{2}\lambda(1 - e^{-\lambda T}))$, there is a unique equilibrium, $(\hat{a}, \hat{a})$ in which $\hat{a} > 0$ is the unique value that satisfies

$$\frac{1}{2}[1 - F(a^*)]\left[1 - e^{-\lambda T[1-F(a^*)]}\right] = \frac{c}{\lambda}, \tag{1}$$

in which both firms search with cut-off $\hat{a}$.

Proof. See the proofs of Proposition A.1, A.2, and A.3 in Appendix A.2. □

The result is illustrated by Figure 1. Region 1 is for *Case* [*i*], in which there is a mixed strategy equilibrium, holds. Region 2, which is partitioned into two sub-regions, Region 2_u and Region 2_d, is for *Case* [*ii*], in which both firms search with cut-off 0. Region 3 is for *Case* [*iii*], in which both firms search with a positive cut-off.

It is clear from Figure 1 that, as T turns long, the interval for *Case* [*ii*] diminishes and that for Case [*i*] (*Case* [*iii*]) expends if $c > (<)\frac{\lambda}{2}$. As T goes to infinity, only *Case* [*i*] and *Case* [*iii*] remains, and *Case* [*ii*] totally disappear. Whereas, as c increases from 0 (decreases from λ) to $\frac{\lambda}{2}$, the interval for *Case* [*iii*] (*Case* (*i*)) diminishes and that for Case [*ii*] (*Case* [*iii*]) expends. As c goes to $\frac{\lambda}{2}$, only *Case* [*ii*] remains, and both *Case* [*i*] and *Case* [*iii*] disappear.

Corollary 4.1. *When $\sqrt{2cT}$ is small, $a^* \approx F^{-1}\left(1 - \sqrt{\frac{2c}{\lambda^2 T}}\right)$.*

Proof. First, we assume that $\lambda T[1-F(a^*)]$ is small, and we come back to check that it is implied by that $\sqrt{2cT}$ is small. Applying equation (1), we have

$$\begin{aligned}
&\frac{c}{\lambda} = \frac{1}{2}[1 - F(a^*)]\left[1 - e^{-\lambda T[1-F(a^*)]}\right] \approx \frac{1}{2}\lambda T[1 - F(a^*)]^2 \\
\Leftrightarrow &[1 - F(a^*)]^2 \approx \frac{2c}{\lambda^2 T} \\
\Leftrightarrow &a^* \approx F^{-1}\left(1 - \sqrt{\frac{2c}{\lambda^2 T}}\right) \quad \text{and} \quad \lambda T[1 - F(a^*)] \approx \sqrt{2cT}.
\end{aligned}$$

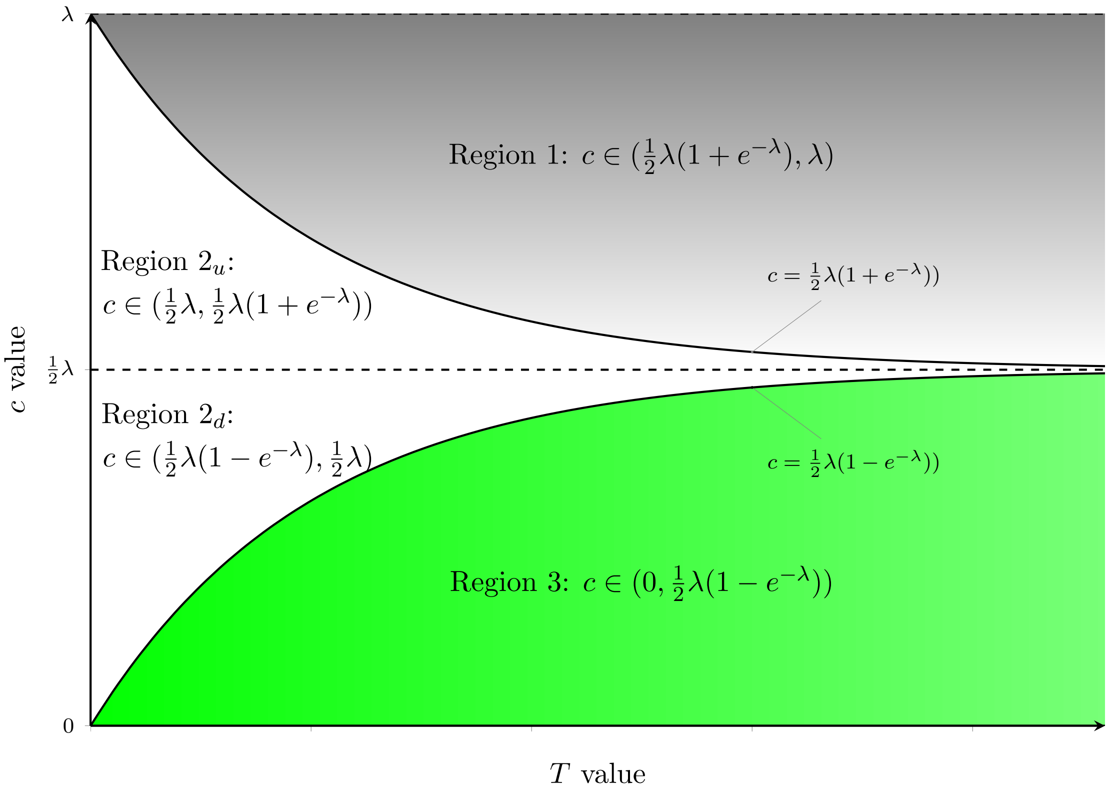

Figure 1: *Regions*

□

For later reference we, based on the previous theorem, define a function $a^* : [0, \lambda]\times[0, +\infty) \to [0, 1]$ where

$$a^*(c, T) = \begin{cases} 0 & \text{for the } c \in [\frac{1}{2}\lambda(1-e^{-\lambda T}), \lambda) \\ \text{the } a^* \text{ that solves (1)} & \text{for the } c \in (0, \frac{1}{2}\lambda(1-e^{-\lambda T})), \end{cases}$$

and we state some simple properties which will be used in the later sections are stated as below.

Lemma 4.1. *In Region 3, $a^*(c, T)$ is strictly increasing in T and strictly decreasing in c (it is also strictly increasing in λ).*

Lemma 4.2.

1. *If $c \geq \frac{\lambda}{2}$ (in Region 1 and 2_u), $a^*(c, T) = 0$.*
2. *If $c < \frac{\lambda}{2}$ (in Region 2_d and 3), $\lim_{T\to+\infty} a^*(c, T) = F^{-1}(1 - \frac{2c}{\lambda})$.*

Proof. For part 2, take the limit of T of equation (1). □

We end this section by computing the expected payoffs to the firms. First, for a firm searching with a cut-off $a \geq 0$, the expected total cost is

$$\begin{aligned} & c\left[\int_0^T \frac{\partial(1 - Z(a|a, t))}{\partial t} t dt + T Z(a|a, T)\right] \\ =& \frac{c}{\lambda[1 - F(a)]}(1 - e^{-\lambda T[1-F(a)]}), \end{aligned} \tag{2}$$

which is strictly increasing in a. In Region 2 and 3, in equilibrium the expected probability of winning for each firm is

$$\begin{aligned} &\frac{1 - Z^2(0|a^*(c,T),T)}{2} \\ =&\frac{1 - e^{-2\lambda T}}{2}, \end{aligned} \tag{3}$$

and thus the expected payoff for each player in these two regions, is just the difference between the expected probability of winning (3) and the the expected search cost (2), setting a to be $a^*(a,T)$. A simple result follows.

Lemma 4.3. *If $c < \frac{\lambda}{2}$, each firm's expected payoff in equilibrium goes to 0 as the deadline T goes to infinity.*

Proof. Applying Lemma 4.1 and 4.2, we have

$$\lim_{T\to+\infty} (3) - (2)|_{a=a^*(c,T)} = \frac{1}{2} - \frac{1}{2} = 0.$$

□

Though the equilibrium expected payoff goes to 0 at the limit, it is not monotonely decreasing to 0 as T goes to 0. One can verify that by taking the first order condition of the expected payoff with respect to T.

5 Model II: Head-start $(a_1^I > a_2^I)$

In this section, we study the model in which Firm 1 has a head start before the competition begins. In particular, the scenario in which that $a_1^I > 0 > a_2^I = -1$ is a typical incumbent-entrant game. We will derive the equilibrium strategies for these two firms in this model. But before that, we make an assumption, similar to Assumption 3.2, for this section.

Assumption 5.1. *Each firm's strategy space is $\mathcal{S}^H := ([0,1] \times \{a_1^I\}) \cup (\{1\} \times [a_1^I, 1])$.*

The justification for this assumption is similar to Lemma 3.2, and we do not repeat it again here. In the following part, we derive the equilibrium strategies for the game, and then we explore equilibrium properties.

5.1 Equilibrium Strategies

First, we state the two crucial lemmas for the whole section.

Lemma 5.1.

1. *For $a_1^I > a^*(c,T)$, not to search is Firm 1's strictly dominant strategy.*
2. *For $a_1^I = a^*(c,T)$, not to search is Firm 1's weakly dominant strategy, in which he is indifferent between searching with cut-off a_1^I and no searching if Firm 2 searches with cut-off a_1^I and strictly prefers not to search otherwise.*

Proof. See Appendix A.3 □

Lemma 5.2.

1. *For $a_1^I > F^{-1}(1 - \frac{c}{\lambda})$, not to search is Firm 2's strictly dominant strategy.*

2. *For* $a_1^I = F^{-1}(1-\frac{c}{\lambda})$, *not to search is Firm* 2*'s weakly dominant strategy, in which he is indifferent between searching with cut-off* a_1^I *and no searching if Firm* 1 *does not search and strictly prefers not to search otherwise.*

3. *For* $a_1^I = F^{-1}(1-\frac{c}{\lambda})$, *Firm* 2*'s strictly prefers to search with cut-off* a_1^I *if Firm* 1 *does not search.*

Proof. See Appendix A.3 □

The intuition for Lemma 5.1 is as follows. Firm 1 does no need to search if Firm 2 does not search. If Firm 2 searches, Firm 1's marginal gain in probability of winning from searching does not compensate the cost for searching, because he is already in a high state. For Lemma 5.2, $F^{-1}(1-\frac{c}{\lambda})$ is simply the state value at which Firm 1 is indifferent between searching and not searching when Firm 1 does no search. The second main result is as below.

Theorem 5.1. *Suppose* $\lambda > c$ *and* $a_1^I > a_2^I$.

1. *For* $a_1^I > F^{-1}(1-\frac{c}{\lambda})$, *there is a unique equilibrium, in which no firm searches, and thus Firm* 1 *wins the prize;*

2. *for* $a_1^I = F^{-1}(1-\frac{c}{\lambda})$, *any strategy profile in which Firm* 1 *does not search and Firm* 2 *assigns any probability to participating searching with cut-off* a_1^I *is an equilibrium.*

3. *for* $a_1^I \in (a^*, F^{-1}(1-\frac{c}{\lambda}))$, *there is a unique equilibrium, in which Firm* 1 *does not search and Firm* 2 *searches with cut-off* a_1^I.

4. *for* $a_1^I = a^*(c,T)$, *any strategy profile in which Firm* 2 *searches with cut-off* a_1^I *and Firm* 1 *assigns probability* $p \in [0,1]$ *to searching with cut-off* a_1^I *and* $1-p$ *to no searching is an equilibrium;*

5. *for* $a_1^I \in (0, a^*(c,T))$, *the unique equilibrium is that both Firm* 1 *and Firm* 2 *search with the same cut-off* $a^*(c,T)$.

Proof. [1],[2], and [3] directly follow from Lemma 5.1&5.2. See Appendix A.3 for the proof of [4] and [5]. □

Remark. *Case [4] and [5] exist only if* $c \geq \frac{1}{2}\lambda[1-e^{-\lambda T}]$ *(Region* 1 *and* 2*).*

This theorem says that, comparing the equilibrium strategies of Model II to that of Model I, a head start of Firm 1 does not alter his own equilibrium strategy but Firm 2's, and the initial state of Firm 2, the runner-up, is irrelevant to equilibrium strategies.

Specifically, in *Case* [5], in which Firm 1's head start is low, the head start has no effect on either firm's equilibrium strategy, and both firms search with the same cut-off $a^*(c,T)$ as in Model I.

In *Case* [3], in which Firm 1's head start is high, Firm 2 is deterred away and Firm 1 wins the prize without conducting searching. This is the case "head starts and doomed winners". Moreover, it is independent of the deadline T, hence it has some insights on the cases in which search processes are publicly observable. We leave it for later discussion.

In *Case* [1], in which Firm 1's head start is in a middle range, Firm 1's dominant strategy is still not to search, whereas Firm 2 is more active in searching (searches with a cut-off higher than in Model I). This leads to the case of "head start and doomed losers", which is formally stated as below.

Corollary 5.1. *For* $a_1^I \in (\max\{0, F^{-1}(1-\frac{2c}{\lambda})\}, F^{-1}(1-\frac{2c}{\lambda}))$, *Firm* 2*'s probability of winning is higher than Firm* 1*'s for any* $T > \frac{\lambda \ln 2}{1-F(a_1^I)}$, *and it increases to* 1 *as* T *goes to infinity.*

Proof. Notice that following from Lemma 4.2 $a^*(c,T)$ is 0 if $c \geq \frac{\lambda}{2}$ and is less than $F^{-1}(1-\frac{2c}{\lambda})$ if $c < \frac{\lambda}{2}$. For a_1^I in the prescribed range of the above corollary, Firm 1's strictly dominant strategy is not to search and Firm 2 searches with cut-off a_1^I. Thus, Firm 2 wins if he discovers a candidate with a type higher than a_1^I, and the probability of winning is thus $1-e^{-\lambda T}$, which increases to 1 as T turns to be infinity. □

As Firm 1 wins with a probability close to 1 if the deadline is very short and losses with a probability close to 1 if the deadline is excessively long and the head start is in the above range.

The above theorem also implies the following property.

Lemma 5.3. *As the deadline increases to infinity,*

1. *Firm 1's equilibrium payoff goes to be* 0 *for* $a_1^I \in (0, F^{-1}(1-\frac{\lambda}{c}))$; 1 *for* $a_1^I \in (F^{-1}(1-\frac{\lambda}{c}), 1]$
2. *Firm 2's equilibrium payoff goes to be* 0 *for* $a_1^I \in (\max\{0, F^{-1}(1-\frac{2c}{\lambda})\}, F^{-1}(1-\frac{2c}{\lambda})) \cup (F^{-1}(1-\frac{c}{\lambda}), 1)$; $1 - \frac{c}{\lambda[1-F(a_1^I)]} \in (0, \frac{1}{2})$ *for* $a_1^I \in (F^{-1}(1-\frac{2c}{\lambda}), F^{-1}(1-\frac{c}{\lambda}))$.

1. and the first half of [2] directly follows from Lemma 4.3. For the second half of [2], take the limit of equation (5). □

The natural questions follow are that who benefits (hurts) from a head start and under what conditions. We answer these questions in the following section.

5.2 Comparisons of Equilibrium Payoffs

In order to answer these questions, we first study the incumbent-entrant game, and then we show a result on the cases with two incumbent firms.

5.3 Model III: Incumbent-entrant Game ($a_1^I > 0, a_2^I = -1$)

We first discuss payoffs to Firm 1, the head starter, and then payoffs to Firm 2.

5.3.1 Payoffs To Firm 1: Model III vs. Model I

Proposition 5.1. *In Region* 2 *and* 3*, in which* $c < \frac{1}{2}\lambda(1+e^{-\lambda T})$*, a head start* $a_1^I > 0$ *always benefits Firm* 1*, comparing to the equilibrium payoff he gets in Model I.*

Proof. We apply Theorem 5.1 here for the analysis. If the head start $a_1^I < a^*(c,T)$, which happens only when $a^*(c,T) > 0$ (Region 3), both firms search with cut-off $a^*(c,T)$, which is the same as in Model I. Hence, the mere effect of such a head start only increases (decreases) Firm 1's (Firm 2's) probability of winning, and thus Firm 1 (Firm 2) is better (worse) off. Suppose Firm 1 has a head start $a_1^I = a^*(c,T)$, he is indifferent between searching with cut-off a^* and not searching, hence his payoff from both strategies is $e^{-\lambda T[1-F(a^*)]}$, the probability of Firm 2 finding no candidate with a type higher than a^*.[3] If he searches, the effect of a head start merely increases his probability of winning, and he is better off. If $a^I > a^*(c,T)$, Firm 1's probability of winning is even higher, and thus he can only be further better off. □

The above result itself is somehow corresponds to expectations. What is unexpected is the mechanism through which Firm 1 gets better off. As a head start gives Firm 1 a higher position, we would expect that he is better off by (1) having a better chance to win and (2) spending less on searching. The together with Theorem 5.1, the above proposition shows that in Region 1 and 2, Firm 1 is better off purely from an increased probability of winning when $a_1^I < a^*(c,T)$; purely from spending nothing on searching when $a_1^I \in (a^*(c,T), F{-}1(1-\frac{c}{\lambda}))$

[3]For the cases in which $a^* = 0$, simple regard $a_1^I = a^*$ as the limiting case.

(though there could be a payoff loss from a reduced probability of winning); from the increase in the probability of winning and reduction in the cost of searching when $a_1^I > F^{-1}(1-\frac{c}{\lambda})$.

The interesting property is that a head start may not be beneficial when search cost is relatively high.

Proposition 5.2. *In Region 1, in which $c > \frac{1}{2}\lambda(1+e^{-\lambda T})$ and $a^*(c,T) = 0$,*

1. *for*

$$(1-e^{-\lambda T})(1-\frac{c}{\lambda}) > e^{-\lambda T[1-F(a_1^I)]}, \tag{4}$$

if Firm 1 can take the first move to search with cut-off 0 so as to deter Firm 2 away from competition, Firm 1 can improve his expected payoff even if he totally discards the head start (if he is able to do so);

2. *there is a $\hat{a}_1^I > 0$ such that (4) holds for $a_1^I \in (0, \hat{a}_1^I)$ and it holds in the opposite direction for $a_1^I \in (\hat{a}_1^I, F^{-1}(1-\frac{c}{\lambda}))$.*

Proof. [1] is straightforward. The term on the left side if inequality (4) is Firm 1's expected payoff in Model I and the right side is his expected payoff in Model III. [2] is due to that the right side is increasing in a_1^I. □

According to this proposition, by committing search and deterring firm 2 away, Firm1 can get a higher payoff. However, if Firm 2 is deterred away, Firm 1 would no longer have any incentive to search. This could simply not be an equilibrium. To make a credible threat to scare Firm 2 away, Firm 1 has to discard the head start (giving up the initial candidate) before the competition starts. By doing so, Firm 1 could then increase his expected payoff.

Moreover, The left side of (4) increases in T, and it goes to -1 when T goes to 0 and it increases to $1-\frac{c}{\lambda}$ when T goes to infinity. The intermediate value theorem insures the following result.

Corollary 5.2. *For any $a_1^I \in (0, F^{-1}(1-\frac{c}{\lambda}))$, there is a $\hat{T}(a_1^I)$ such that (4) holds for $T > \hat{T}(a_1^I)$ and it holds in the opposite direction for $T < \hat{T}(a_1^I)$.*

Since $\frac{1}{2}\lambda(1+e^{-\lambda T})$ decreases to $\frac{1}{2}$ as T increases to infinity. This corollary implies that:

Corollary 5.3. *For $c \geq \frac{\lambda}{2}$, for any head start $a_1^I \in (0, F^{-1}(1-\frac{c}{\lambda}))$, as long as the deadline is sufficiently long there is always such a chance that Firm 1 could improve his payoff.*

5.3.2 Payoffs To Firm 2: Model III vs. Model I

Next, we compare the payoffs to Firm 2, the runner-up, in these two models. First, In Region 1, when search cost is sufficiently high and there are multiple equilirbia in Model I, whether a head start of Firm 1 benefits or hurts Firm 2 depends on which equilirbrium in Model I we compare to.

Proposition 5.3. *In Region 1, in which $c > \frac{1}{2}\lambda(1+e^{-\lambda T})$, the equilibrium payoff Firm 2 gets in Model III is*

- *strictly lower than his payoff in the equilibrium in which Firm 1 does not search and Firm 2 searches with cut-off 0 in Model I, and*
- *strictly higher than (equal to) any other equilibrium payoff in Model I if $a_1^I < (\geq) F^{-1}(1-\frac{c}{\lambda})$.*

Proof. In Model III, if $a_1^I < F^{-1}(1-\frac{c}{\lambda})$, Firm 1 does not search, Firm 2's equilibrium payoff is the difference between the probability of Firm 2 discovering a candidate with a type higher than a_1^I and expected cost of searching:

$$(1-e^{-\lambda T[1-F(a^I)]})\left(1-\frac{c}{\lambda[1-F(a_1^I)]}\right), \tag{5}$$

which is strictly decreasing in a_1^I, and it decreases to 0 as in a_1^I increases to $F^{-1}(1-\frac{c}{\lambda})$; if $a_1^I \geq F^{-1}(1-\frac{c}{\lambda})$, Firm 2 gets a payoff of 0.

In Model I, in the equilibrium in which only Firm 2 searches, Firm 2's expected payoff is

$$(1-e^{-\lambda T})\left(1-\frac{c}{\lambda T}\right),$$

strictly higher than (5) for $a_1^I > 0$; in any other equilibrium, Firm 2 gets 0 payoff, strictly lower than (5) for $a_1^I < F^{-1}(1-\frac{c}{\lambda})$. □

Second, the effect of a head start in the lower range and the upper range is straightforward. If $a_1^I < a^*(c,T)$, which can happen only in Region 3, in which $c < \frac{1}{2}\lambda(1-e^{-\lambda T})$, both firms' equilibrium strategies are the same as in Model I, but Firm 2's probability of winning decreases. If $a_1^I \geq F^{-1}(1-\frac{c}{\lambda})$, Firm 2 does not search and thus ends up with a payoff of 0. Hence, in Region 2 and 3, such a head start of Firm 1 hurts Firm 2. Formally, we state it as below.

Proposition 5.4. *In Region 2 and 3, if $0 < a_1^I < a^*(c,T)$ or $a_1^I \geq F^{-1}(1-\frac{c}{\lambda})$, Firm 2's equilibrium payoff is lower than that in Model I.*

What then remains to compare is when $a_1^I \in (a^*(c,T), F^{-1}(1-\frac{c}{\lambda}))$ in Region 2 and 3. We show that this middle range of a_1^I could be potentially divided into a lower beneficial region and upper hurtful region for Firm 2.

Proposition 5.5. *In Region 2 and 3, in which $c < \frac{1}{2}\lambda(1+e^{-\lambda T})$,*

1. *if*

$$(1-e^{-\lambda T[1-F(a^*(c,T))]}) - \frac{1}{2}(1-e^{-2\lambda T}) > 0, \tag{6}$$

there exists a $\tilde{a}_1^I \in (a^*(c,T), F^{-1}(1-\frac{c}{\lambda}))$ such that the head start a_1^I hurts Firm 2 if $a_1^I \in (\tilde{a}_1^I, F^{-1}(1-\frac{c}{\lambda}))$ and benefits Firm 2 if $a_1^I \in (a^*(c,T), \tilde{a}_1^I)$;

2. *if (6) holds in the opposite direction, any head start $a_1^I \in (a^*(c,T), F^{-1}(1-\frac{c}{\lambda}))$ hurts Firm 2.*

Proof. Firm 2's equilibrium payoff in the model with a head start for Firm 1 is (5), and his equilibrium payoff in Model I is

$$\frac{1}{2}(1-e^{-2\lambda T}) - (1-e^{-\lambda T[1-F(a^*(c,T))]})\frac{c}{\lambda[1-F(a^*(c,T))]}. \tag{7}$$

The difference between these two payoffs, (5) − (7), is strictly decreasing in a_1^I, and it goes to −(7) < 0 as a_1^I goes to $F^{-1}(1-\frac{c}{\lambda})$ and goes to

$$(1-e^{-\lambda T[1-F(a^*(c,T))]}) - \frac{1}{2}(1-e^{-2\lambda T}) \tag{8}$$

as a_1^I goes to $a^*(c,T)$. Hence, if (8) > 0, the intermediate value theorem yields the desired result for *Case* 1; if (8) < 0, *Case* 2 holds. □

The following lemma shows that there is always a beneficial region of a_1^I when search cost is large or the deadline is sufficiently long if search cost is small.

Corollary 5.4.

1. *For $c \in [\frac{1}{2}\lambda(1-e^{-\lambda T}), \frac{1}{2}\lambda(1+e^{-\lambda T}))$, in which case $a^*(c,T)=0$, (7) holds.*

2. *For $c \in [\frac{\lambda}{4}, \frac{1}{2}\lambda(1-e^{-\lambda T}))$, in which case $a^*(c,T)>0$, (7) also holds.*

3. *For $c \in (0, \frac{\lambda}{4})$, (7) holds for T being sufficiently large (or equivalently, $a^*(c,T)$ being sufficiently close to $F^{-1}(1-\frac{2c}{\lambda})$); (7) holds in the opposite direction for T being sufficiently close to 0 (or equivalently, a^* being sufficiently close to 0).*

Proof. For [1],

$$(8) = \frac{1}{2}(1-e^{-\lambda T})^2 > 0.$$

For [2] and [3], we have $(1-F(a^*(c,T))) \in (\frac{2c}{\lambda}, 1)$ and

$$(1) \Rightarrow T = -\frac{1}{\lambda[1-F(a^*(c,T))]} \ln\left(1 - \frac{2c}{\lambda[1-F(a^*(c,T))]}\right).$$

Substituting it into (8), we have

$$\begin{aligned}(8) &= \frac{2c}{\lambda[1-F(a^*(c,T))]} - \frac{1}{2}\left[1 - \left(1 - \frac{2c}{\lambda[1-F(a^*(c,T))]}\right)^{\frac{2}{1-F(a^*(c,T))}}\right] \\ &= \frac{1}{2}\left(1 - \frac{2c}{\lambda[1-F(a^*(c,T))]}\right)^{\frac{2}{1-F(a^*(c,T))}} - \frac{1}{2}\left(1 - \frac{4c}{\lambda[1-F(a^*(c,T))]}\right). \qquad (9)\end{aligned}$$

Hence, for [2], (9) > 0 for $(1-F(a^*(c,T))) \in (\frac{2c}{\lambda}, 1)$, because the term in the first bracket of (9) is positive and the term in the second bracket is negative. For [3], when T goes to infinity, $1-F(a^*(c,T))$ goes to $\frac{2c}{\lambda}$, and thus (9) goes to 1; when T goes to 0, $1-F(a^*(c,T))$ goes to 1, and thus (9) goes to $\frac{1}{2}(1-e^{-\lambda T})^2 > 0$. □

Even though Firm 1 does not search when the head start $a_1^I > a^*(c,T)$, it seems that a low search cost may benefit Firm 2. On the contrary, a head start of Firm 1 would always hurt Firm 2 when search cost is sufficiently small.

Corollary 5.5. *For any fixed deadline T, if search cost is sufficiently small, (6) holds in the opposite direction.*

That is because when c is close to 0, $a^*(c,T)$ is close to 1, and the interval in which Firm 1 does not search while Firm 2 searches is very small, and thus the chance for Firm 2 to win is too low when $a_1^I > a^*(c,T)$, even though the expected search cost is low as well.

Last, a long deadline may either be good or bad for Firm 2.

Corollary 5.6. *Suppose $c < \frac{\lambda}{2}$.*

1. *For any $a_1^I \in (0, F^{-1}(1-\frac{2c}{\lambda}))$, there is a $\tilde{T}(a_1^I)$ such that for the head start hurts Firm 2 if $T > \tilde{T}(a_1^I)$.*

2. *For any $a_1^I \in [F^{-1}(1-\frac{2c}{\lambda}), F^{-1}(1-\frac{c}{\lambda}))$, $\exists \tilde{T}(a_1^I)$ such that the head start benefits Firm 2 $\forall T > \tilde{T}(a_1^I)$.*

Proof. For [1], $a^*(c,T)$ increases to $F^{-1}(1-\frac{2c}{\lambda})$ as T goes to infinity, and then Proposition 5.4 applies. For [2], the difference between Firm 2's equilibrium payoffs in Model III and Model I, (5) − (7), goes to $\frac{1}{2}$ as T goes to infinity. □

5.3.3 Which Firm Does A Head Start Benefit More?

The next question is, between Firm 1 and Firm 2 which firm benefits more from a head start? That is, whether Firm 1 gets a higher payoff or Firm 2 get a higher payoff in the model with a head start. It is summarized as below.

Proposition 5.6. *In Model II,*

1. *when $a_1^I \leq a^*(c,T)$ or $a_1^I \geq F^{-1}(1-\frac{c}{\lambda})$, Firm 1 gets a higher expected payoff than Firm 2 does;*
2. *when $a_1^I \in (a^*(c,T), F^{-1}(1-\frac{c}{\lambda}))$, Firm 1 gets a higher expected payoff if*

$$e^{-\lambda T[1-F(a_1^I)]} - (1-e^{-\lambda T[1-F(a_1^I)]})(1-\frac{c}{\lambda[1-F(a_1^I)]}) > 0, \tag{10}$$

whereas Firm 2 gets a higher expected payoff if (10) holds in the opposite direction.

In particular, because the term on the left side of (10) is increasing in a_1^I, if (10) holds when $a_1^I = a^*(c,T)$, then it holds for any $a_1^I \in (a^*(c,T), F^{-1}(1-\frac{2c}{\lambda}))$. In the following we give some sufficient conditions for (10) to hold (or hold in the opposite direction) for $a_1^I \in (\max\{0, F^{-1}(1-\frac{2c}{\lambda})\}, F^{-1}(1-\frac{c}{\lambda}))$.

Corollary 5.7. *Suppose $c \geq \frac{\lambda}{2}$, in which case $a^*(c,T) = 0$ for all $T > 0$.*

1. *If $T \geq \frac{1}{\lambda}\ln\frac{2-\frac{c}{\lambda}}{1-\frac{c}{\lambda}}$, then (10) holds for all $a_1^I \in (0, F^{-1}(1-\frac{c}{\lambda}))$.*
2. *if $T < \frac{1}{\lambda}\ln\frac{2-\frac{c}{\lambda}}{1-\frac{c}{\lambda}}$, there is a $\breve{a}_1^I \in (0, F^{-1}(1-\frac{c}{\lambda}))$ such that (10) holds for all $a_1^I \in (\breve{a}_1^I, F^{-1}(1-\frac{c}{\lambda}))$ and it hold in the opposite direction for all $a_1^I \in (0, \breve{a}_1^I)$. In particular, this condition holds if $c \leq (1-\frac{\sqrt{2}}{2})\lambda$.*

Proof. See Corollary A.1 and A.2 in Appendix A.3. □

Corollary 5.8. *Suppose $c < \frac{\lambda}{2}$, in which case $\sup_{T>0} a^*(c,T) = F^{-1}(1-\frac{2c}{\lambda})$.*

1. *If $T \leq \frac{\ln 3}{2c}$, then (10) holds for all $a_1^I \in (F^{-1}(1-\frac{2c}{\lambda}), F^{-1}(1-\frac{c}{\lambda}))$.*
2. *If $T > \frac{\ln 3}{2c}$, there is a $\breve{a}_1^I \in (F^{-1}(1-\frac{2c}{\lambda}), F^{-1}(1-\frac{c}{\lambda}))$ such that (10) holds for all $a_1^I \in (\breve{a}_1^I, F^{-1}(1-\frac{c}{\lambda}))$ and it holds in the opposite direction for all $a_1^I \in (F^{-1}(1-\frac{2c}{\lambda}), \breve{a}_1^I)$.*

Proof. See Corollary A.3 and A.4 in Appendix A.3. □

A comparison shows that the effects of deadline when search cost is low are opposite of that when search cost is high. For a given head start $a_1^I > F^{-1}(1-\frac{2c}{\lambda}))$, when search cost is high, Firm 1 is (not) ensured to obtains a higher payoff when deadline is long (short), whereas when search cost is low, he is (not) ensured to obtain a higher payoff when deadline is short (long).

We end this part by showing a asymptotic property of the effects of a head start on firm's payoffs.

Proposition 5.7. *Suppose $c < \frac{\lambda}{2}$. As the deadline increases to infinity,*

1. *the benefit of the head start to Firm 1 converges to 0 for $a_1^I \in (0, F^{-1}(1-\frac{2c}{\lambda}))$; 1 for $a_1^I \in (F^{-1}(1-\frac{2c}{\lambda}), 1]$*
2. *the difference between Firm 2's equilibrium payoffs in Model III and Model I converges to*
   - *0 for any $a_1^I \in (0, F^{-1}(1-\frac{2c}{\lambda})) \cup (F^{-1}(1-\frac{c}{\lambda}), 1]$ and*

- $1 - \frac{c}{\lambda[1-F(a_1^I)]} \in (0, \frac{1}{2})$ *for* $a_1^I \in (F^{-1}(1 - \frac{2c}{\lambda}), F^{-1}(1 - \frac{c}{\lambda}))$

Proof. Follow from Lemma 4.3 and 5.3. □

This property suggests that the effects of a head start on firms' payoffs do not vanish as the deadline goes to infinity. For $a_1^I \in (F^{-1}(1 - \frac{2c}{\lambda}), F^{-1}(1 - \frac{c}{\lambda}))$, the benefit totally goes to Firm 2 , whereas Firm 1's benefit vanishes.

5.4 Compete To Be A Runner-up

Previously, we have shown that in the incumbent-entrant game, when search cost is low and deadline is long, the incumbent could improve his expected payoff if he could scare the potential entrant away by discarding the initial candidate and committing to search. In this part we show that in an incumbent-incumbent game there can be a unique subgame perfect equilibrium, in which the head-starter would discard his initial candidate, if the head-starter can take the first move. To make it more clear, we state procedures as below.

Model IV:

- stage 1: Firm 1 decides whether to discard his initial candidate,
- stage 2: Firm 2 decides whether to discard his initial candidate,
- stage 3: both firm simultaneously start playing the game as of Model II.

The third main result is as follows.

Proposition 5.8. *Suppose* $c < \frac{\lambda}{2}$ *and* $a_1^I, a_2^I \in (F^{-1}(1 - \frac{2c}{\lambda}), F^{-1}(1 - \frac{c}{\lambda}))$*. In Model IV, there is a* $\breve{T}(a_1^I, a_2^I)$ *such that*

- *for* $T > \breve{T}(a_1^I, a_2^I)$*, there is a unique subgame perfect equilibrium, in which Firm* 1 *discards his initial candidate and searches with cut-off* a_2^I *and Firm* 2 *keeps the initial candidate and does not search;*
- *for* $T < \breve{T}(a_1^I, a_2^I)$*, subgame perfect equilibria exist, and in each equilibrium Firm* 1 *keeps his initial candidate and does not search and Firm* 2 *searches with cut-off* a_1^I*.*

Proof. If Firm 1 discard his initial candidate, Then Firm 2 becomes the head-starter, and by Proposition 5.1, Firm 2 strictly prefers retain the initial candidate. In this case, Firm 1's payoff would be

$$(1 - e^{-\lambda T[1-F(a_2^I)]})(1 - \frac{c}{\lambda[1 - F(a_2^I)]}). \tag{11}$$

If Firm 1 retains the initial candidate, then Firm 2 is indifferent between discarding the initial innovation or not. In this case, Firm 1's payoff would be

$$e^{-\lambda T[1-F(a_1^I)]}. \tag{12}$$

The difference between these two payoffs, (11) − (12), is increasing in T, and it equals −1 when $T = 0$ and goes to $1 - \frac{c}{\lambda[1-F(a_2^I)]} > 0$ as T goes to infinity. Hence, the desired result derives, following from the intermediate value theorem. □

Remark.

- *When $c \geq \frac{\lambda}{2}$, for T being large, there are three subgame perfect equilibria, including the one in which Firm 1 discards his initial candidate and make Firm 1 the head-starter so as to improve his own payoff.*
- *When T is large, ,by giving up the initial candidate, Firm 1 makes himself better off, but it hurts Firm 2. However, it is possible that by doing so, Firm 1 can make both firms better off as shown in the following example.*

Example 5.1. *Suppose F is the uniform distribution, $c = \frac{1}{3}$, $\lambda = 1$, $a_1^I = \frac{1}{2}$, and $a_2^I = \frac{1}{3}$. If Firm 1 discards his initial candidate, then his payoff would be $\frac{1}{2}(1 - e^{-\frac{T}{3}})$, and Firm 2's payoff would be $e^{-\frac{T}{3}}$; if Firm 1 does not discard his initial candidate, then his payoff would be $e^{-\frac{T}{2}}$, and Firm 2's payoff would be $\frac{1}{3}(1 - e^{-\frac{T}{2}})$.*

Firm 1 would be better off by discarding his initial candidate if $T > 2.52$. If $T \in (2.52, 3.78)$, by discarding the initial candidate, Firm 1 makes both firms better off. If T is larger, then doing so would only make Firm 2 worse off.

6 Discussion Asymmetric Search Costs ($a_1^I = a_2^I = -1$, $c_1 < c_2$)

Previously, we showed that the effects of head starts are ambiguous, in this section, we show that the effects of costs advantages are quite clear. The firm with a cost advantage is encouraged to search more actively, whereas the opponent firm is discouraged.

We now asumme that the pre-specified prize worths V_i to Firm $i = 1, 2$ and that the search cost is C_i per unit of time for Firm i. However, at each time point Firm i only makes a binary decision on whether to stop searching or to continue searching, what matters is whether it is profitable to continue searching, and thus the scale of V_i and C_i does not matter. Instead, what matters is the ratio $\frac{C_i}{V_i}$. Thus, we can normalize the valuation of each player to be 1 and the searching cost to be $\frac{C_i}{V_i} =: c_i$. W.l.o.g, we assume $c_1 < c_2$. For convenience, we define a function

$$I(a_i|a_j, c_i) := \lambda \int_{a_i}^{\bar{a}} [Z(a|a_j) - Z(a_i|a_j)]dF(a) - c_i.$$

We emphasize on the cases in which the search costs of both firms are low, which is the most important case.

Proposition 6.1. *For $0 < c_1 < c_2 < \frac{1}{2}\lambda(1 - e^{-\lambda T})$, there must exist a unique equilibrium (a_1^*, a_2^*), in which $a_1^* > a_2^* \geq 0$. Specifically,*

1. *if $I\left(0 \middle| F^{-1}\left(1 - \sqrt{\frac{2c_1}{\lambda(1-e^{-\lambda T})}}\right), c_2\right) > 0$, the unique equilibrium is a pair of cut-off rules (a_1^*, a_2^*), $a_1^* > a_2^* > 0$, that satisfies*

$$\lambda \int_{a_i^*}^{\bar{a}} \left[Z(a|a_j^*, T) - Z(a_i^*|a_j^*, T)\right] dF(a) = c_i;$$

2. *if $I\left(0 \middle| F^{-1}\left(1 - \sqrt{\frac{2c_1}{\lambda(1-e^{-\lambda T})}}\right), c_2\right) \leq 0$, the unique equilibrium is a pair of cut-off rules $\left(F^{-1}\left(1 - \sqrt{\frac{2c_1}{\lambda(1-e^{-\lambda T})}}\right), 0\right)$.*

Proof. See Appendix A.4 □

The existence of equilibrium is proved by using Brouwer's fixed point theorem. As expected, a cost (valuation) advantage of a firm would drive the firm searching more actively than the opponent firm. The following statement show that while an increase on cost advantage of the firm in advantage would make the firm more active in searching and make the opponent firm less actively in searching , an further cost disadvantage of the firm in disadvantage would make both firms less active in searching.

Proposition 6.2. *For* $0 < c_1 < c_2 < \frac{1}{2}\lambda(1 - e^{-\lambda T})$ *and* $I\left(0|F^{-1}\left(1 - \sqrt{\frac{2c}{\lambda(1-e^{-\lambda T})}}\right), c_2\right) > 0$*, in which case there is a unique equilibrium* (a_1^*, a_2^*)*,* $a_1^*, a_2^* > 0$*,*

1. *for fixed* c_2*,* $\frac{\partial a_1^*}{\partial c_1} < 0$ *and* $\frac{\partial a_2^*}{\partial c_1} > 0$*;*
2. *for fixed* c_1*,* $\frac{\partial a_1^*}{\partial c_2} < 0$ *and* $\frac{\partial a_2^*}{\partial c_2} < 0$*.*

Proof. See Appendix A.4 □

The intuition is simple, when the cost of the firm in advantage decreases, this firm would be more willing to search, while the opponent firm would be discouraged because the marginal benefit from searching in a higher state is decreased, hence he would lower his cut-off; when the cost of the firm in disadvantage increases, the firm would be less willing to search, and the opponent firm would consider it less necessary to search actively because the probability of winning has increased.

Corollary 6.1. *For* $0 \le c_1 < c_2 < \frac{1}{2}\lambda(1 - e^{-\lambda T})$ *and* $I\left(0|F^{-1}\left(1 - \sqrt{\frac{2c}{\lambda(1-e^{-\lambda T})}}\right), c_2\right) > 0$*, in which case there is a unique equilibrium* (a_1^*, a_2^*)*,* $a_1^*, a_2^* > 0$*,*

1. $a_1^* < a^*(c, T)$ *for the corresponding* $c = c_1 > c_2$*;*
2. $a_2^* < a^*(c, T)$ *for the corresponding* $c = c_2 < c_1$*.*

The other important case is that the searching costs of both firms are relatively high. When both costs are high, there are equiibria in which the firm with a lower cost has lower chance to participate searching than the firm with higher cost does. Because of the crowd out effect, the market is not large and profitable enough for both firms to survive.

Proposition 6.3. *If* $\frac{1}{2}\lambda(1 + e^{-\lambda T}) < c_1 < c_2 < \lambda$*, then*

1. *there exists a unique mixed strategy Nash equilibrium, in which Firm i participate the game with probability*

$$p_i = \frac{2}{1 - e^{\lambda T}} \cdot \left(1 - \frac{c_j}{\lambda}\right)$$

and stops searching once he meets an agent with a type higher than 0*. Hence,* $p_1 < p_2$*.*

2. *there are two pure strategy equilibrium, in which one firm searches with cut-off* 0 *and the other firm doe not search.*

Proof. Same as the proof of Proposition A.1 □

The equilibrium for the other non-marginal cases are stated in the following proposition without proof. Basically, it shows that Firm 1 is more active in searching than Firm 2 does.

Proposition 6.4.

1. *For* $c_1 < \frac{1}{2}\lambda(1 - e^{-\lambda T})$ *and* $c_2 \in (\frac{1}{2}\lambda(1 - e^{-\lambda T}), \frac{1}{2}\lambda(1 + e^{-\lambda T}))$*, the pair of cut-off rules,* $\left(F^{-1}\left(1 - \sqrt{\frac{2c_1}{\lambda(1-e^{-\lambda T})}}\right), 0\right)$*, is the unique equilibrium.*
2. *For* $c_1 < \frac{1}{2}\lambda(1 + e^{-\lambda T})$ *and* $c_2 > \frac{1}{2}\lambda(1 + e^{-\lambda T})$*, there is a unique equilibrium, in which Firm* 1 *searches with cut-off* 0 *and Firm* 2 *does not search.*
3. *For* $c_1, c_2 \in (\frac{1}{2}\lambda(1 - e^{-\lambda T}), \frac{1}{2}\lambda(1 + e^{-\lambda T}))$*, there is a unique equilibrium, in which both Firms search with cut-off* 0*.*

Remark. *We could find similar results for a model with the same costs but with different arrival rates of candidates for two firms. It is just more computational tedious.*

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A Appendix

A.1 Proofs of The Lemmas

Proof of Lemma 3.1. If Firm j does not search, the **instantaneous gain** from continuing searching for Firm i in state -1 at any time point is

$$\lim_{\delta\to 0}\frac{\lambda\delta e^{-\lambda\delta}+o(\delta)-c_i\delta}{\delta}=\lambda-c_i,$$

where $\lambda\delta e^{-\lambda\delta}+o(\delta)$ is the probability of winning the prize in a time interval of δ, and the instantaneous gain is even lower if firm j searches. The results then follow from the sign of $\lambda - c_i$. □

Proof of Lemma 3.2. Fix a strategy of Firm j. Let $P(a)$ denote the probability of Firm j ending up in a state **below** a at time T. $P(a)$ is either constant in a or strictly increasing in a. It is a constant if and only if Firm j does not search.[4] If this is the case, by Lemma 3.1 Firm i's best response is to continue searching with a fixed cut-off $\hat{a}_i^t=\bar{a}^{t_i}=\underline{a}^{t_i}=0$ for all t. In the following, we study the case that $P(a)$ strictly increases in a.

Step 1. We argue that, given a fixed strategy played by Firm j, Firm i's best response is a (potentially time-dependent and state-dependent) cut-off rule. Suppose at time t Firm i is in a state $\tilde{a}\in(0,1)\cup\{-1\}$. If he is strictly marginally profitable to stop (continue) searching at t, then it is also strictly marginally profitable to continue searching if he is in a state higher (lower) than $\tilde{a}$. Let the upper and lower optimal cut-offs at time t be $\bar{a}_i^t$ and $\underline{a}_i^t$, respectively, as defined previously.

Step 2. We show that $\{\bar{a}_i^t\}_{t=0}^T$ and $\{\underline{a}_i^t\}_{t=0}^T$ should be time-invariant and state-independent. We use a discrete version to approximate the continuous version. Take any $\tilde{t}\in[0,T)$. Let $\{t_l\}_{l=0}^k$, where $t_l-t_{l-1}=\frac{T-\tilde{t}}{k}=:\delta$ for $l=1,...,k$, be a partition of the interval $[\tilde{t},T]$. Suppose Firm i can only make decisions at $\{t_l\}_{l=0}^k$ in the interval $[\tilde{t},T]$. Let $\{\bar{a}^{t_l}\}_{l=0}^{k-1}$ and $\{\underline{a}^{t_l}\}_{l=0}^{k-1}$ be the corresponding upper and lower optimal cut-offs, respectively, and $G^\delta(a)$ be Firm i's probability of discovering **no** candidate with a type **above** a in an interval δ.

At t_{k-1}, for Firm i in a state a, if he stops searching, his expected payoff is $P(a)$; if he continues searching, his expected payoff is

$$\begin{aligned}&G^\delta(a)P(a)+\int_a^1 P(\tilde{a})dG^\delta(\tilde{a})-\delta c_i\\=&P(a)+\int_a^1[P(\tilde{a})-P(a)]dG^\delta(\tilde{a})-\delta c_i.\end{aligned}$$

The firm strictly prefers continuing searching if and only if searching in the last period strictly increases his expected payoff:

$$e^\delta(a):=\int_a^1[P(\tilde{a})-P(a)]dG^\delta(\tilde{a})-\delta c_i>0.$$

Since $e^\delta(a)$ strictly decreases in a and $e^\delta(1)\le 0$, while $e^\delta(0)$ can be either negative or positive. Therefore, there are several cases.

Case 1. If $e^\delta(-1)<0$, Firm i is strictly better off stopping searching at any state $a\in(0,1]\cup\{-1\}$. Thus, $\bar{a}^{t_{k-1}}=\underline{a}^{t_{k-1}}=-1$.

Case 2. If $e^\delta(-1)=0$, Firm i is indifferent between not participating searching and searching with cut-off 0, if he is in state -1; strictly prefers stopping searching, if he is in any state above 0. Then $\bar{a}^{t_{k-1}}=0$ and $\underline{a}^{t_{k-1}}=-1$.

[4] More generally, it is constant if and only if the opponent firm conducts search with a measure 0 over $[0,T]$.

Case 3. If $e^{\delta}(-1) > 0 \geq \lim_{a \to 0} e^{\delta}(a)$, Firm i is strictly better off continuing searching in state $a = -1$, but stopping searching once he's in a state $a > 0$. Thus, $\bar{a}^{t_{k-1}} = \underline{a}^{t_{k-1}} = 0$.
Case 4. If $\lim_{a \to 0} e^{\delta}(a) > 0$, then Firm i's is strictly better off stopping searching if he is in a state above $\hat{a}^{t_{k-1}}$ and continuing searching if he is in a state below $\hat{a}^{t_{k-1}}$, where the optimal cut-off $\hat{a}^{t_{k-1}} > 0$ is the unique value that solves, w.r.t. a,

$$\int_a^1 [P(\tilde{a}) - P(a)] dG^{\delta}(\tilde{a}) - \delta c = 0.$$

Thus, in this case $\bar{a}^{t_{k-1}} = \underline{a}^{t_{k-1}} = \hat{a}^{t_{k-1}}$.

Hence, the continuation payoff at $t_{k-1} \geq 0$ for Firm i in state $a \in (0, 1] \cup \{-1\}$ is

$$\omega(a) = \begin{cases} P(a) + \int_a^1 [P(\tilde{a}) - P(a)] dG^{\delta}(\tilde{a}) - \delta c & for\ a < \underline{a}^{t_{k-1}} \\ P(a) & for\ a \geq \underline{a}^{t_{k-1}}. \end{cases}$$

Then, we look at the time point t_{k-2}. In the following, we argue that $\bar{a}^{t_{k-2}} = \bar{a}^{t_{k-1}}$. The proof for $\underline{a}^{t_{k-2}} = \underline{a}^{t_{k-1}}$ is very similar and thus is omitted.

First, we show that $\bar{a}^{t_{k-2}} \leq \bar{a}^{t_{k-1}}$. Suppose $\bar{a}^{t_{k-2}} > \bar{a}^{t_{k-1}}$. Suppose Firm i is at state $\bar{a}^{t_{k-2}}$ at time t_{k-2}. Suppose Firm i searches between t_{k-2} and t_{k-1}. If he does not discover any candidate with a type higher than $\bar{a}^{t_{k-2}}$, then at the end of this period he stops searching and takes $\bar{a}^{t_{k-2}}$. However, $\bar{a}^{t_{k-2}} > \bar{a}^{t_{k-1}}$ implies

$$\begin{aligned} 0 &= \int_{\bar{a}^{t_{k-2}}}^1 [P(\tilde{a}) - P(\bar{a}^{t_{k-2}})] dG^{\delta}(\tilde{a}) - \delta c_i \\ &< \int_{\bar{a}^{t_{k-1}}}^1 [P(\tilde{a}) - P(\bar{a}^{t_{k-1}})] dG^{\delta}(\tilde{a}) - \delta c \leq 0. \end{aligned}$$

The search cost is not compensated by the increase in the probability of winning from searching between t_{k-2} and t_{k-1}, and thus the firm must strictly prefer stopping searching to continuing searching at time t_{k-2}, which contradicts the assumption that $\bar{a}^{t_{k-2}}$ is the upper optimal cut-off. Hence, it must be the case that $\bar{a}^{t_{k-2}} \leq \bar{a}^{t_{k-1}}$.

Next, we show that $\bar{a}^{t_{k-2}} = \bar{a}^{t_{k-1}}$.

In *Case 1*, it is straightforward that Firm i strictly prefers stopping searching at t_{k-2}, since he is for sure not going to search between t_{k-1} and t_k. Hence, Firm i stops searching before t_{k-1}, and $\bar{a}^{t_{k-2}} = \bar{a}^{t_{k-1}} = \underline{a}^{t_{k-2}} = \underline{a}^{t_{k-1}} = -1$.

For $\bar{a}^{t_{k-1}} \geq 0$, we prove by contradiction that $\bar{a}^{t_{k-2}} < \bar{a}^{t_{k-1}}$ is not possible. Suppose the inequality holds. If Firm i stops searching at t_{k-2}, he would choose to continue searching at t_{k-1}, and his expected continuation payoff at t_{k-2} is $\omega(\bar{a}^{t_{k-2}})$. If the firm continues searching, his expected continuation payoff is

$$\omega(\bar{a}^{t_{k-2}}) + \int_{\bar{a}^{t_{k-2}}}^1 [\omega(a) - \omega(\bar{a}^{t_{k-2}})] dG^{\delta}(a) - \delta c. \tag{13}$$

In *Case 2 and 3*, $\bar{a}^{t_{k-1}} = 0$ implies $\bar{a}^{t_{k-2}} = -1$. Then,

$$\begin{aligned} &\int_{\bar{a}^{t_{k-2}}}^1 [\omega(a) - \omega(\bar{a}^{t_{k-2}})] dG^{\delta}(a) - \delta c_i \\ =& \int_{-1}^1 [P(a)] dG^{\delta}(a) - \delta c_i \\ =& e^{\delta}(-1) \\ \geq& 0 \end{aligned}$$

which means that that Firm i in state -1 is weakly better off continuing searching between t_{k-2} and t_{k-1}, and thus it implies that $\bar{a}^{t_{k-2}} \geq 0$, resulting in a contradiction.

For *Case 4*, in which $\bar{a}^{t_{k-1}} > 0$, we have in (13)

$$
\begin{aligned}
&\int_{\bar{a}^{t_{k-2}}}^{1} [\omega(a) - \omega(\bar{a}^{t_{k-2}})] dG^\delta(a) \\
=& \int_{\bar{a}^{t_{k-2}}}^{\bar{a}^{t_{k-1}}} \left[\left(P(a) + \int_a^1 [P(\tilde{a}) - P(a)] dG^\delta(\tilde{a}) \right) - \left(P(\bar{a}^{t_{k-2}}) + \int_{\bar{a}^{t_{k-2}}}^1 [P(\tilde{a}) - P(\bar{a}^{t_{k-2}})] dG^\delta(\tilde{a}) \right) \right] dG^\delta(a) \\
&+ \int_{\bar{a}^{t_{k-1}}}^1 \left[P(a) - \left(P(\bar{a}^{t_{k-2}}) + \int_{\bar{a}^{t_{k-2}}}^1 [P(\tilde{a}) - P(\bar{a}^{t_{k-2}})] dG^\delta(\tilde{a}) \right) \right] dG^\delta(a) \\
=& \int_{\bar{a}^{t_{k-2}}}^1 \left[P(a) - P(\bar{a}^{t_{k-2}}) \right] dG^\delta(a) + \int_{\bar{a}^{t_{k-2}}}^{\bar{a}^{t_{k-1}}} \left[\int_a^1 [P(\tilde{a}) - P(a)] dG^\delta(\tilde{a}) \right] dG^\delta(a) \\
&- \int_{\bar{a}^{t_{k-2}}}^1 \left[\int_{\bar{a}^{t_{k-2}}}^1 [P(\tilde{a}) - P(\bar{a}^{t_{k-2}})] dG^\delta(\tilde{a}) \right] dG^\delta(a) \\
=& G^\delta(\bar{a}^{t_{k-2}}) \int_{\bar{a}^{t_{k-2}}}^1 \left[P(a) - P(\bar{a}^{t_{k-2}}) \right] dG^\delta(a) + \int_{\bar{a}^{t_{k-2}}}^{\bar{a}^{t_{k-1}}} \left[\int_a^1 [P(\tilde{a}) - P(a)] dG^\delta(\tilde{a}) \right] dG^\delta(a) \\
>& 0.
\end{aligned}
$$

Hence, at t_{k-2} Firm i would strictly prefer continuing searching, which again contradicts the assumption that $\bar{a}^{t_{k-2}}$ is the upper optimal cut-off. Consequently, $\bar{a}^{t_{k-2}} = \bar{a}^{t_{k-1}}$.

By backward induction from t_{k-1} to t_0, we have $\bar{a}^{t_0} = \bar{a}^{t_{k-1}}$. Taking the limit we get

$$\bar{a}^t = \lim_{\delta \to 0} \bar{a}^{T-\delta} =: \bar{a} \quad \text{for all } t \in [0, T).$$

Similarly,

$$\underline{a}^t = \lim_{\delta \to 0} \underline{a}^{T-\delta} =: \underline{a} \quad \text{for all } t \in [0, T).$$

In addition, $\bar{a} \neq \underline{a}$ when and only when $\bar{a} = 0$ and $\underline{a} = -1$.

As a consequence, Firm i's best response is not to search, if $\bar{a} = \underline{a} = -1$; to continue searching if he is in a state below $\bar{a}$ and stopping searching once he's in a state above $\bar{a}$, if $\bar{a} = \underline{a} \geq 0$; any strategy that assigns, at any time point, a probability to continuing searching with cut-off 0 and the remaining probability to stopping searching, if $\bar{a} = 0$ and $\underline{a} = -1$. □

Proof of Lemma 3.3. For $a = -1$, it is clear that $Z(a|\hat{a}, T) = 0$.

For $0 < a \leq \hat{a}$,

$$Z(a|\hat{a}, T) = \sum_{j=0}^{\infty} \frac{e^{-\lambda T} (\lambda T)^l}{l!} F^j(a) = e^{-\lambda T [1 - F(a)]}.$$

For $a > \hat{a}$, we approximate it by discrete time model. Let $\{t_l\}_{l=0}^k$, where $0 = t_0 < t_1 < ... < t_k = T$, be a partition of the interval $[0, T]$, and let $\delta_l = t_l - t_{l-1}$ for $l = 1, 2, ..., k$. Define π as

$$\| \pi \| = \max_{1 \leq l \leq k} |\delta_l|.$$

Then,

$$
\begin{aligned}
Z(a|\hat{a},T) =& Z(\hat{a}|\hat{a},T) + \lim_{\|\pi\|\to 0}\sum_{l=1}^{k} Z(a|\hat{a},t_{l-1})\left[\sum_{n=1}^{\infty}\frac{e^{-\lambda\delta_l}(\lambda\delta_l)^n}{n!}[F^n(a)-F^n(\hat{a})]\right] \\
=& Z(\hat{a}|\hat{a},T) + \lim_{\|\pi\|\to 0}\sum_{l=1}^{k} e^{-\lambda t_{l-1}[1-F(\hat{a})]}\lambda e^{-\lambda\delta_l}\big([F(a)-F(\hat{a})]+O(\delta_l)\big)\delta_l \\
=& Z(\hat{a}|\hat{a},T) + \int_0^T \lambda e^{-\lambda t[1-F(\hat{a})]}[F(a)-F(\hat{a})]dt \\
=& Z(\hat{a}|\hat{a},T) + \left[1-e^{-\lambda T[1-F(\hat{a})]}\right]\frac{F(a)-F(\hat{a})}{1-F(\hat{a})}
\end{aligned}
$$

where the second term on the right side of each equality is the firm's probability of ending up in a state between $\hat{a}$ and a. The term $Z(\hat{a}|\hat{a},t_n)$ used here is a convenient approximation when δ_l is small. The second equality is derived from the fact that

$$
\sum_{n=2}^{\infty}\frac{(\lambda\delta_l)^n}{n!}[F^n(a)-F^n(a^*)] < \frac{\lambda^2\delta_l^2}{2(1-\lambda\delta_l)} = O(\delta_l^2).
$$

□

Proof of Lemma 3.4. First, we show that $\frac{1-e^{-\lambda Tx}}{x}$ strictly decreases in x over $(0,1]$ as follows. Define $s := \lambda T$ and take x_1, x_2, $0 < x_1 < x_2 \le 1$, we have

$$
\frac{1-e^{-sx_1}}{x_1} > \frac{1-e^{-sx_2}}{x_2},
$$

implied by

$$
\begin{aligned}
&\frac{\partial(1-e^{-sx_1})x_2-(1-e^{-sx_2})x_1}{\partial l} = x_1x_2(e^{-sx_1}-e^{-sx_2}) \ge 0 \quad (=0 \ \text{ iff } \ s=0) \quad \text{and} \\
&(1-e^{-sx_1})x_2-(1-e^{-sx_2})x_1 = 0 \quad \text{for} \quad s=0.
\end{aligned}
$$

Next, define $x := 1-F(a)$, $x' := 1-F(a')$, and $\tilde{x} := 1-F(\tilde{a})$. We have

$$
Z(a|\tilde{a},T)-Z(a'|\tilde{a},T) = \begin{cases} e^{-\lambda Tx}-e^{-\lambda Tx'} & \text{for } \tilde{a}\ge a \\ (1-e^{-\lambda Tx'})-(1-e^{-\lambda T\tilde{x}})\frac{x}{\tilde{x}} & \text{for } \tilde{a}\in(a',a) \\ (1-e^{-\lambda T\tilde{x}})\frac{x'-x}{\tilde{x}} & \text{for } \tilde{a}\le a'. \end{cases}
$$

It is independent of $\tilde{a}$ for $\tilde{a}\ge a$, strictly increases in $\tilde{x}$ and thus strictly decreases in $\tilde{a}$ for $\tilde{a}\le a'$, and strictly decreases in $\tilde{x}$ and thus strictly increases in $\tilde{a}$ for $\tilde{a}\le a'$. □

Lemma A.1. *Suppose Firm j with initial state -1 plays a strategy $[p_j,\hat{a}_j]$. Then, the* ***instantaneous gain*** *on payoff for Firm i from searching is*

$$
\lambda\int_{a_i}^{1} p_j[Z(a|\hat{a}_j,T)-Z(a_i|\hat{a}_j,T)]dF(a)-c
$$

if he is in a state $a_i>0$, and

$$
\lambda\int_0^1 [p_jZ(a|\hat{a}_j,T)+(1-p_j)]dF(a)-c
$$

if he is in state -1.

Proof of Lemma A.1. For convenience, let $p_j Z(a|\hat{a}_j, T) + (1 - p_j)$ be denoted as $H(a)$. The instantaneous gain from searching for Firm i in a state $a_i > 0$ is

$$
\begin{aligned}
& \lim_{\delta \to 0} \frac{\left(e^{-\lambda\delta} H(a_i) + \lambda\delta e^{-\lambda\delta}\left[\int_{a_i}^1 H(a)dF(a) + F(a_i)H(a_i)\right] + o(\delta) - \delta c\right) - H(a_i)}{\delta} \\
\Leftrightarrow & \lim_{\delta \to 0} \frac{-(1 - e^{-\lambda\delta})H(a_i) + \lambda\delta e^{-\lambda\delta}\left[\int_{a_i}^1 H(a)dF(a) + F(a_i)H(a_i)\right] + o(\delta) - \delta c}{\delta} \\
\Leftrightarrow & \lim_{\delta \to 0} \frac{-\lambda e^{-\lambda\delta} H(a_i) + \lambda\delta e^{-\lambda\delta}\left[\int_{a_i}^1 H(a)dF(a) + F(a_i)H(a_i)\right] + o(\delta) - \delta c}{\delta} \\
\Leftrightarrow & - \lambda H(a_i) + \lambda \left[\int_{a_i}^1 H(a)dF(a) + F(a_i)H(a_i)\right] - c \\
\Leftrightarrow & \lambda \int_{a_i}^1 p_j \left[Z(a|\hat{a}_j, T) - Z(a_i|\hat{a}_j, T)\right] dF(a) - c.
\end{aligned}
$$

The proof for the case of $a_i = -1$ is similar and thus is omitted. □

Lemma A.2. $\int_0^1 Z(a|1, T)dF(a) > \frac{1}{2}(1 - e^{-\lambda T})$.

Proof of Lemma A.2. Let us define a function

$$D(x) := \left(\frac{1}{2} + \frac{1}{x}\right)(1 - e^{-x}) \text{ for } x > 0.$$

Next, take the derivative of the function, we have

$$
\begin{aligned}
\frac{\partial D(x)}{\partial x} &= - \frac{1}{x^2}(1 - e^{-x}) + \left(\frac{1}{x} + \frac{1}{2}\right) e^{-x} \\
&= \frac{e^{-x}}{x^2}(1 + x + \frac{1}{2}x^2 - e^x) \\
&= - \frac{e^{-x}}{x^2} \sum_{n=3}^{\infty} \frac{x^n}{n!} \\
&< 0.
\end{aligned}
$$

Then, we have the desired result that

$$
\begin{aligned}
& \int_0^1 Z(a|1, T)dF(a) - \frac{1}{2}(1 - e^{-\lambda T}) \\
= & \int_0^1 (1 - e^{-\lambda T[1 - F(a)]})dF(a) - \frac{1}{2}(1 - e^{-\lambda T}) \\
= & 1 - \left(\frac{1}{2} + \frac{1}{\lambda T}\right)(1 - e^{-\lambda T}) \\
= & 1 - D(\lambda T) \\
> & \lim_{x \to 0} 1 - D(x) \\
= & 1 - \frac{0}{2} - \lim_{x \to 0} \frac{e^{-x}}{1} = 0,
\end{aligned}
$$

where the second last equality follows from the L'Hospital's rule. □

A.2 Proofs for The Benchmark Model

Proposition A.1.

1. *For $\frac{1}{2}\lambda(1+e^{-\lambda T}) < c < \lambda$, there exists a non-degenerate mixed strategy equilibrium, in which each firm assigns probability p to searching with cut-off 0, where*
$$p = \frac{2}{1-e^{\lambda T}} \cdot \left(1 - \frac{c}{\lambda}\right),$$
and $1-p$ to not participating searching, and it is the unique non-degenerate mixed strategy equilibrium.

2. *For $\frac{1}{2}\lambda(1+e^{-\lambda T}) < c < \lambda$, there exist two pure strategy equilibria, In each pure strategy equilibrium, one firm does not search and the other searches with cut-off 0.*

3. *For $\frac{1}{2}\lambda(1+e^{-\lambda T}) = c$, any strategy profile in which one firm searches with cut-off 0 and the other firm participates searching with any probability $p \in [0,1]$ and searches with cut-off 0 with a probability $1-p$ is an equilibrium.*

Proof of Proposition A.1. First, we prove [2] and [3]. For $\lambda > c$, when Firm i does not search, Firm j's best response is to search with cut-off 0. For $\frac{1}{2}\lambda(1+e^{-\lambda T}) \leq c$, when Firm j searches with any cut-off $a_j \geq 0$, Firm i's best response is not to search, since the instantaneous gain from searching for Firm i in state -1 is

$$\begin{aligned}
&\lambda \int_0^1 [Z(a|a_j,T) - Z(0|a_j,T)]dF(a) - c \\
\leq&\lambda \int_0^1 [Z(a|0,T) - Z(0|0,T)]dF(a) - c \\
=&\frac{1}{2}\lambda(1+e^{-\lambda T}) - c \\
\leq&0 \quad (= 0 \text{ iff } \frac{1}{2}\lambda(1+e^{-\lambda T}) = c),
\end{aligned}$$

where the first inequality follows from Lemma 3.4. Hence, for [2] there are two pure strategy equilibria, in each of which one firm does not search and the other firm searches with cut-off 0, and for [3] any strategy profile in which one firm searches with cut-off 0 and the other firm participates searching with any probability is an equilibrium.

Second, we prove [1]. From the above analysis, we know that if there is a mixed strategy equilibrium, it must be the case that in this equilibrium both firms play mixed strategies. Suppose Firm i participates searching with a probability p_i, Firm j should be indifferent between non-participating and participating with cut-off 0. Hence, the instantaneous gain from participating for Firm j in state -1 is 0. That is

$$\begin{aligned}
0 =&\lambda \int_0^1 [p_i Z(a|0,T) + (1-p_i)]dF(a) - c \\
=&\lambda \int_0^1 [p_i e^{-\lambda T} + p_i(1-e^{-\lambda T})F(a) + 1 - p_i]dF(a) - c \\
=&\lambda \left[p_i e^{-\lambda T} + \frac{1-e^{-\lambda T}}{2} \cdot p_i + (1-p_i)\right] - c \\
\Leftrightarrow p_i =&\frac{2}{1-e^{\lambda T}}\left[1 - \frac{c}{\lambda}\right],
\end{aligned}$$

while $p_i \in (0,1)$ iff

$$\frac{1}{2}\lambda(1+e^{-\lambda T}) < c < \lambda.$$

□

Proposition A.2. *If* $\frac{1}{2}\lambda(1-e^{-\lambda T}) \le c < \frac{1}{2}\lambda(1+e^{-\lambda T})$*, then there is a unique equilibrium,* $(0,0)$ *, in which each firm searches with cut-off* 0*.*

Proof of Proposition A.2. First, we show there can be no equilibrium in which one firm searches with a cut-off higher than 0. Suppose Firm j searches with cut-off $\hat{a}_j > 0$. The instantaneous gain from searching for Firm i in a state $a_i > 0$ is

$$\begin{aligned}
&\lambda\int_{a_i}^{1}[Z(a|\hat{a}_j,T)-Z(a_i|\hat{a}_j,T)]dF(a)-c\\
\le&\lambda\int_{a_i}^{1}[Z(a|a_i,T)-Z(a_i|a_i,T)]dF(a)-c\\
=&\frac{1}{2}\lambda(1-e^{-\lambda T})[1-F(a_i)]^2-c\\
<&0,
\end{aligned}$$

where the first inequality follows from Lemma 3.4. The instantaneous gain from searching for Firm i in state -1 is

$$\begin{aligned}
&\lambda\int_{0}^{1}Z(a|\hat{a}_j,T)dF(a)-c\\
<&\lambda\int_{0}^{1}Z(a|0,T)dF(a)-c\\
=&\frac{1}{2}\lambda(1-e^{-\lambda T})-c\\
<&0.
\end{aligned}$$

Therefore, if Firm j searches with a cut-off higher than 0, Firm i's best response is not to search. Whilst, if Firm i does not search, Firm j's best response is to search with a cut-off 0. Therefore, there is no equilibrium in this case.

Second, we show that the prescribed strategy profile is the unique equilibrium. Suppose Firm j searches with a probability $p_j \in [0,1]$ and cut-off 0, the instantaneous gain for Firm i in a state $\tilde{a} > 0$ from searching is

$$\begin{aligned}
&\lambda\int_{\tilde{a}}^{1}\big(p_j[Z(a|0,T)-Z(\tilde{a}|0,T)]+[1-p_j]\big)dF(a)-c\\
<&\lambda\int_{0}^{1}\big(p_j[Z(a|0,T)-Z(\tilde{a}|0,T)]+[1-p_j]\big)dF(a)-c\\
=&\frac{1}{2}\lambda(1-e^{-\lambda T})p_j+\lambda(1-p_j)-c\\
\le&\frac{1}{2}\lambda(1-e^{-\lambda T})-c\\
\le&0;
\end{aligned}$$

the instantaneous gain for Firm i in state $a=-1$ from searching is

$$\begin{aligned}
&\lambda\int_{0}^{1}[p_jZ(a|0,T)+(1-p_j)]dF(a)-c\\
=&\lambda\int_{0}^{1}\big(p_j[e^{-\lambda T}+(1-e^{-\lambda T})F(a)]+[1-p_j]\big)dF(a)-c\\
=&\lambda\left[e^{-\lambda T}+\frac{1}{2}(1-e^{-\lambda T})\right]p_j+\lambda[1-p_j]-c\\
\ge&\frac{1}{2}\lambda(1+e^{-\lambda T})-c\\
>&0.
\end{aligned}$$

That is, Firm i is strictly better off continuing searching in state -1, and strictly better off stopping searching once he is in a state above 0. Hence, the prescribed strategy profile is the unique equilibrium. □

Proposition A.3. *For $c < \frac{1}{2}\lambda(1-e^{-\lambda T})$, there is a unique equilibrium, in which each firm searches with the same cut-off $a^* > 0$, where a^* satisfies equation (1).*

Proof of Proposition A.3. First, we prove that among the strategy profiles in which each firm searches with a cut-off higher than 0, the prescribed symmetric strategy profile is the unique equilibrium. Suppose a pair of cut-off rules (a_1^*, a_2^*), in which $a_1^*, a_2^* > 0$, is an equilibrium, then Firm i in state a_i^* is indifferent between continuing searching and not. That is, by Lemma A.1, we have

$$\lambda \int_{a_i^*}^{1} \left[Z(a|a_j^*, T) - Z(a_i^*|a_j^*, T) \right] dF(a) - c = 0. \tag{14}$$

Suppose $a_1^* \neq a_2^*$. W.l.o.g., we assume $a_1^* < a_2^*$. Then,

$$\begin{aligned} c =& \lambda \int_{a_1^*}^{1} [Z(a|a_2^*, T) - Z(a_1^*|a_2^*, T)] dF(a) \\ >& \lambda \int_{a_2^*}^{1} [Z(a|a_2^*, T) - Z(a_2^*|a_2^*, T)] dF(a) \\ >& \lambda \int_{a_2^*}^{1} [Z(a|a_1^*, T) - Z(a_2^*|a_1^*, T)] dF(a) = c \end{aligned}$$

resulting in a contradiction. Hence, it must be the cases that $a_1^* = a_2^*$.

Next, we show the existence of equilibrium by deriving the unique equilibrium cut-off value $a^* := a_1^* = a_2^*$ explicitly. Applying Lemma 3.3 to (14), we have

$$\begin{aligned} & \lambda \int_{a^*}^{1} \left[1 - e^{-\lambda T[1-F(a^*)]} \right] \frac{F(a) - F(a^*)}{1 - F(a^*)} dF(a) = c \\ \Leftrightarrow & \frac{1}{2}[1 - F(a^*)] \left[1 - e^{-\lambda T[1-F(a^*)]} \right] = \frac{c}{\lambda}. \end{aligned} \tag{15}$$

The existence of solution is ensured by the intermediate value theorem: when $F(a^*) = 1$, the term on the left side of (20) equals to 0, smaller than $\frac{c}{\lambda}$; when $F(a^*) = 0$, it equal to $\frac{1-e^{-\lambda T}}{2}$, larger than or equals to $\frac{c}{\lambda}$. The uniqueness of the solution is insured by that the term on the left side of the above equality is strictly decreasing in a^*.

Second, there is no equilibrium in which both firms search with cut-off 0. Suppose both search with cut-off 0. Then, we have

$$\begin{aligned} & \lambda \int_{0}^{1} [Z(a|0, T) - Z(0|0, T)] dF(a) - c \leq 0 \\ \Leftrightarrow & \frac{1}{2}\lambda \left[1 - e^{-\lambda T[1-F(a^*)]} \right] \leq c \end{aligned}$$

resulting in a contradiction.

Third, we show that there is no equilibrium in which one firm searches with a cut-off strictly higher than 0 and the other firm with cut-off 0. Take such a pair of cut-off rules $(\hat{a}_i, \hat{a}_j)$ in which $\hat{a}_i = 0$ and $\hat{a}_j > 0$. $(\hat{a}_i, \hat{a}_j)$ is an equilibrium if and only if

$$\lambda \int_{\hat{a}_j}^{1} [Z(a|0, T) - Z(\hat{a}_j|0, T)] dF(a) - c = 0 \quad \text{and} \tag{16}$$

$$\lambda \int_{0}^{1} [Z(a|\hat{a}_j, T) - Z(0|\hat{a}_j, T)] dF(a) - c \leq 0. \tag{17}$$

However,

$$\begin{aligned}0 =& \int_{\hat{a}_j}^1 [Z(a|0,T) - Z(\hat{a}_j|0,T)]dF(a) - c \\ <& \int_{\hat{a}_j}^1 [Z(a|\hat{a}_j,T) - Z(\hat{a}_j|\hat{a}_j,T)]dF(a) - c \\ <& \int_0^1 [Z(a|\hat{a}_j,T) - Z(0|\hat{a}_j,T)]dF(a) - c \leq 0,\end{aligned}$$

resulting in a contradiction.

Last, we show there is no equilibrium in which one firm assigns a positive probability to not to participate searching. Suppose Firm j assigns a probability $p_j \in [0,1)$ to searching with cut-off 0. Firm i's best response is to search with a cut-off higher than or equal to 0, since the instantaneous gain from searching for Firm i in state -1 is

$$\begin{aligned}&\lambda \int_0^1 [p_j Z(a|0,T) + (1-p_j)]dF(a) - c \\ =&\lambda \int_0^1 \big(p_j[e^{-\lambda T} + (1-e^{-\lambda T})F(a)] + [1-p_j]\big)dF(a) - c \\ =&\lambda \left[e^{-\lambda T} + \frac{1}{2}(1-e^{-\lambda T})\right] p_j + \lambda[1-p_j] - c \\ \geq&\frac{1}{2}\lambda(1+e^{-\lambda T}) - c \\ >&0.\end{aligned}$$

Then, we prove by contradiction that if Firm 1 searches with a cut-off $a_i > 0$, Firm j strictly prefers searching with cut-off 0 to not participating searching. The instantaneous gain from searching for Firm i in a state $a_i > 0$ is

$$\lambda \int_{a_i}^1 p_j[Z(a|0,T) - Z(a_i|0,T)]dF(a) - c \tag{18}$$

$$=\lambda \int_{a_i}^1 p_j(1-e^{-\lambda T})[F(a) - F(a_i)]dF(a) - c$$

$$=\frac{1}{2}\lambda(1-e^{-\lambda T})[1-F(a_i)]^2 p_i - c. \tag{19}$$

For $p_j \leq \frac{2c}{\lambda(1-e^{-\lambda T})}$, (19) $\leq [1-F(a_i)]^2 c - c < 0$, and thus Firm i's best response is to search with cut-off 0. For $p_j > \frac{2c}{\lambda(1-e^{-\lambda T})}$, following the intermediate value theorem and monotonicity of (18) w.r.t. to a_i, there must exist a unique $\hat{a}_i \in (0,1)$ such that

$$\lambda \int_{\hat{a}_i}^1 p_j[Z(a|0,T) - Z(\hat{a}_i|0,T)]dF(a) - c = 0, \tag{20}$$

and thus Firm i's best response is to search with with cut-off $\hat{a}_i$.

However, if Firm i searches with cut-off $a_i \in [0,1)$, the instantaneous gain from searching for Firm j in state -1 is

$$\begin{aligned}&\lambda \int_0^1 Z(a|a_i,T)dF(a) - c \\ >&\lambda \int_0^1 Z(a|1,T)dF(a) - c \\ \geq&\frac{1}{2}\lambda(1-e^{\lambda T}) - c \\ >&0,\end{aligned}$$

where the second in equality follows from Lemma A.2. It means that Firm j should strictly prefer searching with cut-off 0 to no searching, resulting in a contradiction. □

A.3 Proof for The Model with Head-start Advantage

Proof of Lemma 5.1. Suppose Firm 2 plays strategy $[p, a_2]$, $p \in [0,1]$ and $a_2 \geq a_1^I$. If Firm 1 searches with cut-off $a_1 \geq a_1^I$, following from Lemma 3.4, the instantaneous gain from searching for Firm 1 at any state $a_1 \geq a_1^I \geq a^*(c,T)$ is

$$\begin{aligned}
&\lambda \int_{a_1}^{\bar{a}} p[Z(a|a_2,T) - Z(a_1|a_2,T)]dF(a) - c \\
\leq&\lambda \int_{a^*(c,T)}^{\bar{a}} p[Z(a|a^*(c,T),T) - Z(a^*(c,T)|a^*(c,T),T)]dF(a) - c, \qquad (21)
\end{aligned}$$

where equality holds if and only if $a_1 = a_2 = a^*(c,T)$.

$$(21) \leq 0 \quad (= 0 \text{ iff } c \geq \frac{1}{2}\lambda[1-e^{-\lambda T}] \text{ and } p = 1).$$

Hence, the desired results follow. □

Proof of Lemma 5.2. If Firm 1 does not search, the instantaneous gain from searching for Firm 2 in a state $a_2 \leq a_1^I$ is

$$\lim_{\delta \to 0} \frac{\lambda \delta e^{-\lambda\delta}[1-F(a_1^I)] + o(\delta) - c\delta}{\delta} = \lambda[1-F(a_1^I)] - c \begin{cases} < 0 & \text{in Case [1]} \\ = 0 & \text{in Case [2]} \\ > 0 & \text{in Case [3].} \end{cases}$$

If Firm 1 searches, Firm 2's instantaneous gain is even lower. Hence, the desired results follow. □

Proof of Theorem 5.1. [4]. Following Lemma 5.1, if Firm 2 searches with cut-off a_1^I, any strategy in which Firm 1 assigns a probability p to searching with cut-off a_1^I and $1-p$ to no searching is Firm 1's best response. When Firm 1 plays $[p, a_1^I]$, the instantaneous gain from searching for Firm 2 is

$$\begin{aligned}
&\lambda \int_{a_1^I}^{1} [pZ(a|a_1^I,T) + (1-p)]dF(a) - c \\
=&(1-p)\lambda[1-F(a_1^I)] + p\lambda \int_{a_1^I}^{1} \left[e^{-\lambda T[1-F(a_1^I)]} + (1-e^{-\lambda T[1-F(a_1^I)]})\frac{F(a)-F(a_1^I)}{1-F(a_1^I)}\right] dF(a) - c \\
=&(1-p)\lambda[1-F(a_1^I)] + \frac{1}{2}\lambda(1+e^{-\lambda T[1-F(a_1^I)]})[1-F(a_1^I)] - c \\
>&\lambda[1-F(a_1^I)] - c \\
>&0
\end{aligned}$$

if he is in a state $a_2 < a_1^I = a^*(c,T)$; it is

$$\lambda \int_{a_2}^{1} [Z(a|a_1^I,T) - Z(a_2|a_1^I,T)]dF(a) - c < 0$$

if he is in a state $a_2 > a_1^I = a^*(c,T)$. Hence, any prescribed strategy profile is an equilibrium.

[5]. We first show that there is no equilibrium in which Firm 2 plays $[p_2, a_1^I]$, $p_2 \in [0,1]$. Suppose such a strategy is played. The instantaneous gain from searching for Firm 1 in state a_1^I is

$$\lambda \int_{a_1^I}^1 p_2[Z(a|a_1^I, T) - Z(a_1^I|a_1^I, T)]dF(a) - c \tag{22}$$

$$= \frac{1}{2}\lambda(1 - e^{-\lambda T[1-F(a_1^I)]})[1 - F(a_1^I)]p_2 - c. \tag{23}$$

(23) > 0 for $p_2 = 1$, since $a_1^I < a^*(c,T)$; (23) < 0 for $p_2 = 0$. Hence, there is a $\tilde{p} \in (0,1)$ such that (23) $= 0$ for $p_2 = \tilde{p}$.

For $p_2 < \tilde{p}$, (23) < 0, and thus Firm 1's best response is not to search. However, if Firm 1 does not search, Firm 2's best response is to search with cut-off a_1^I rather than $[p_2, a_1^I]$, resulting in a contradiction.

For $p_2 \geq \tilde{p}$, Firm 1's best response is to search with a cut-off $\hat{a}_1 \in [a_1^I, a^*(c,T))$. Because, (22) ≥ 0 for $p_2 \geq \tilde{p}$ and by Lemma 3.4

$$\begin{aligned}
&\lambda \int_{a^*}^1 p_2[Z(a|a_1^I, T) - Z(a^*|a_1^I, T)]dF(a) - c \\
<&\lambda \int_{a^*}^1 [Z(a|a^*, T) - Z(a^*|a^*, T)]dF(a) - c \\
=&0,
\end{aligned}$$

and then, by the intermediate value theorem and the strict monotonicity, there is a unique $\hat{a}_1 \in [a_1^I, a^*(c,T))$ such that

$$\lambda \int_{a'}^1 p_2[Z(a|a_1^I, T) - Z(a'|a_1^I, T)]dF(a) - c \begin{cases} < 0 & \text{for } a' > \tilde{a} \\ = 0 & \text{for } a' = \tilde{a} \\ > 0 & \text{for } a' < \tilde{a}. \end{cases}$$

However, if Firm 1 searches with a cut-off $\hat{a}_1 \in [a_1^I, a^*(c,T))$, Firm 2's best response is to search with a cut-off $\hat{a}_2 \in (\hat{a}_1, a^*(c,T))$ rather than $[p_2, a_1^I]$. Because

$$\lambda \int_{a'}^1 [Z(a|\hat{a}_1, T) - Z(a'|\hat{a}_1, T)]dF(a) - c \begin{cases} < 0 & \text{for } a' = a^*(c,T) \\ > 0 & \text{for } a' = \hat{a}_1 \end{cases}$$

and it is monotone w.r.t. a'. This results in another contradiction. Hence, there is no equilibrium in which Firm 2 plays $[p_2, a_1^I]$.

Next, we argue that there is no equilibrium in which Firm 1 plays $[p_1, a_1^I]$, $p_1 \in [0,1]$. Suppose such a strategy is played. The instantaneous gain from searching for Firm 2 in a state a_2 is

$$\lambda \int_{a_1^I}^1 [pZ(a|a_1^I, T) + (1-p)]dF(a) - c > 0$$

if $a_2 < a_1^I$ as in [4], and

$$\lambda \int_{a_2}^1 p[Z(a|a_1^I, T) - Z(a_2|a_1^I, T)]dF(a) - c \tag{24}$$

if $a_2 > a_1^I$.

Then, following the exactly the same logic as in [4], we have for $p_1 < \tilde{p}$, (24) < 0, and thus Firm 2's best response is to search with cut-off a_1^I; for $p_1 \geq \tilde{p}$, Firm 2's best response is to search with a cut-off $\tilde{a} \in [a_1^I, a^*(c,T))$. However, if Firm 2 searches with a cut-off $\hat{a}_2 \in [a_1^I, a^*(c,T))$, Firm 1's best response is to search with a cut-off $\hat{a}_1 \in (\hat{a}_2, a^*(c,T))$ rather than $[p_1, a_1^I]$, resulting in a contradiction. Hence, there is no equilibrium in which Firm 1 plays $[p_1, a_1^I]$.

Lastly, we only need to consider the case in which each firm searches with a cut-off higher than a_1^I, and following the same argument as in the proof of Proposition A.3, we that that $(a^*(c,T), a^*(c,T))$ is the unique equilibrium. □

Proof of Lemma 5.5. First, applying the implicit function theorem on equation (1), we have $\frac{2c}{\lambda[1-F(a^*)]} = 1 - e^{-\lambda T[1-F(a^*)]}$ being increasing in c, and it goes to 0 as c goes to 0. Hence, (9) goes to 0 as c goes to 0. Take the first order derivative of (9) with respect to c we have

$$\begin{aligned}\frac{d(9)}{dc} =& -\frac{1}{2}\frac{2}{1-F(a^*(c,T))}\left(1-\frac{2c}{\lambda[1-F(a^*)]}\right)^{\frac{2}{1-F(a^*(c,T))}-1}\frac{d\frac{2c}{\lambda[1-F(a^*(c,T))]}}{dc}\\ &+\frac{1}{2}\left(1-\frac{2c}{\lambda[1-F(a^*(c,T))]}\right)^{\frac{2}{1-F(a^*(c,T))}}\ln\left(1-\frac{2c}{\lambda[1-F(a^*)(c,T)]}\right)\frac{d\frac{2}{\lambda[1-F(a^*(c,T))]}}{dc}\\ &+\frac{1}{2}\frac{d\frac{4c}{\lambda[1-F(a^*(c,T))]}}{dc}\\ <& \left[[1-F(a^*(c,T))] - \left(1-\frac{2c}{\lambda[1-F(a^*(c,T))]}\right)^{\frac{2}{1-F(a^*(c,T))}-1}\right]\frac{1}{1-F(a^*(c,T))}\frac{d\frac{2c}{\lambda[1-F(a^*(c,T))]}}{dc}.\end{aligned}$$

As the term in the bracket goes to -1 when c goes to 0 and the other two terms are both positive for $c > 0$, this derivative is negative when c is sufficiently close to 0, and so is (9). □

Corollary A.1. *Fix T. In Region 1 and 2 and the boundaries, in which case $a^*(c,T) = 0$,*

1. *if $c \geq \lambda\left[\frac{1-2e^{-\lambda T}}{1-e^{-\lambda T}}\right]$, then (10) holds for all $a_1^I \in (a^*(c,T), F^{-1}(1-\frac{c}{\lambda}))$. In particular, the condition is satisfied for $e^{-\lambda T} \geq \frac{1}{2}$.*

2. *if $c < \lambda\left[\frac{1-2e^{-\lambda T}}{1-e^{-\lambda T}}\right]$, there is a $\breve{a}_1^I \in (a^*(c,T), F^{-1}(1-\frac{c}{\lambda}))$ such that (10) holds for all $a_1^I \in (\breve{a}_1^I, F^{-1}(1-\frac{c}{\lambda}))$ and it hold in the opposite direction for all $a_1^I \in (a^*(c,T), \breve{a}_1^I)$.*

Proof. (10) is strictly increasing in a_1^I and goes to $e^{-Tc} > 0$.

$$(2-\frac{c}{\lambda})e^{-\lambda T} - (1-\frac{c}{\lambda})\begin{cases} > 0 & \text{under the condition in the first part} \\ < 0 & \text{under the condition in the second part.}\end{cases}$$

Hence, the first part results directly; the second part results, following the intermediate value theorem. □

Corollary A.2. *Fix c. In Region 1 and 2 and the boundaries, in which case $a^*(c,T) = 0$,*

1. *if $T \geq \frac{1}{\lambda}\ln\frac{2-\frac{c}{\lambda}}{1-\frac{c}{\lambda}}$, then (10) holds for all $a_1^I \in (a^*(c,T), F^{-1}(1-\frac{c}{\lambda}))$.*

2. *if $T < \frac{1}{\lambda}\ln\frac{2-\frac{c}{\lambda}}{1-\frac{c}{\lambda}}$, there is a $\breve{a}_1^I \in (a^*(c,T), F^{-1}(1-\frac{c}{\lambda}))$ such that (10) holds for all $a_1^I \in (\breve{a}_1^I, F^{-1}(1-\frac{c}{\lambda}))$ and it hold in the opposite direction for all $a_1^I \in (a^*(c,T), \breve{a}_1^I)$. In particular, this condition holds if $c \leq (1-\frac{\sqrt{2}}{2})\lambda$.*

Proof. The proof is the same as that of Corollary A.1. The second condition holds if $c \leq (1 - \frac{\sqrt{2}}{2})\lambda$ because

$$c \leq (1 - \frac{\sqrt{2}}{2})\lambda \Leftrightarrow \frac{2 - \frac{c}{\lambda}}{1 - \frac{c}{\lambda}} \geq \frac{1}{1 - \frac{2c}{\lambda}}.$$

□

Corollary A.3. *Fix T. In Region 3, in which case $a^*(c,T) > 0$,*

1. *if $\frac{c}{1-F(a^*(c,T))} \leq \frac{(2-\sqrt{2})\lambda}{2}$, then (10) holds for all $a_1^I \in (a^*(c,T), F^{-1}(1-\frac{c}{\lambda}))$. In particular, this condition holds if $e^{-\lambda T} \geq \sqrt{2} - 1$.*
2. *if $\frac{c}{1-F(a^*(c,T))} > \frac{(2-\sqrt{2})\lambda}{2}$, there is a $\breve{a}_1^I \in (a^*(c,T), F^{-1}(1 - \frac{c}{\lambda}))$ such that (10) holds for all $a_1^I \in (\breve{a}_1^I, F^{-1}(1 - \frac{c}{\lambda}))$ and it holds in the opposite direction for all $a_1^I \in (a^*(c,T), \breve{a}_1^I)$.*

Proof. Following (1), as a_1^I goes to $a^*(c,T)$, (10) goes to

$$2\left[1 - \frac{c}{\lambda[1 - F(a^*(c,T))]}\right]^2 - 1, \tag{25}$$

where the term in the bracket is positive.

$$(25) \begin{cases} > 0 & \text{under the condition in the first part} \\ < 0 & \text{under the condition in the second part.} \end{cases}$$

Hence, the first part results directly; the second part results, following the intermediate value theorem. By (1),

$$(25) = \frac{1}{2}(1 + e^{-\lambda T[1-F(a_1^I)]})^2 - 1,$$

and it is positive if for any $a_1^I > 0$ if $e^{-\lambda T} \geq \sqrt{2} - 1$. □

Remark. *Notice that because $\frac{c}{1-F(a^*(c,T))}$ is increasing in c, if the condition in the first part holds for some value of c, it holds for any lower value of c.*

Corollary A.4. *Fix c. In Region 3, in which case $a^* > 0$,*

1. *if $T[1 - F(a^*(c,T))] \leq \frac{1}{\lambda} \ln \frac{1}{\sqrt{2}-1}$, then (10) holds for all $a_1^I \in (a^*(c,T), F^{-1}(1 - \frac{c}{\lambda}))$.*
2. *if $T[1-F(a^*(c,T))] > \frac{1}{\lambda} \ln \frac{1}{\sqrt{2}-1}$, there is a $\breve{a}_1^I \in (a^*(c,T), F^{-1}(1-\frac{c}{\lambda}))$ such that (10) holds for all $a_1^I \in (\breve{a}_1^I, F^{-1}(1 - \frac{c}{\lambda}))$, and it holds in the opposite direction for all $a_1^I \in (a^*, \breve{a}_1^I)$.*

Proof. The proof is the same as that for Corollary A.3. □

Remark. *Since $T[1 - F(a^*(c,T))]$ is increasing in T, if the condition in the second part holds for some value of T, it holds for any higher value of T.*

A.4 Proofs for The Model with Asymmetric Costs

Proposition A.4. *If* $0 < c_1 < c_2 < \frac{1}{2}\lambda(1 - e^{-\lambda T})$ *there exists a pure strategy equilibrium* (a_1^*, a_2^*) *with* $a_1^*, a_2^* \geq 0$.

Proof of Proposition A.4. We prove the existence of equilibrium by applying Brouwer's fixed point theorem. First, same as in the previous proofs, if Firm j searches with a cut-off $\hat{a}_j \geq 0$, the instantaneous gain from searching for Firm i in state -1 is

$$\lambda \int_0^{\bar{a}} Z(a|\bar{a}, T) - c_i > 0,$$

and thus he is better off continuing searching in state -1.

Next, let us define for each firm j a critical value

$$\alpha_j = \sup\{a_j \in [0, \bar{a}] \mid I(0|\alpha_j, c_i) = \lambda \int_0^{\bar{a}} [Z(a|\alpha_j) - Z(0|\alpha_j)]dF(a) - c_i > 0\}.$$

Suppose there is a $\alpha_j \in (0, \bar{a})$ such that

$$I(0|\alpha_j, c_i) = \lambda \int_0^{\bar{a}} [Z(a|\alpha_j) - Z(0|\alpha_j)]dF(a) - c_i = 0.$$

For any $\hat{a}_j \in [0, \alpha_j]$

$$I(0|\hat{a}_j, c_i) \geq 0 \text{ and}$$
$$I(\bar{a}|\hat{a}_j, c_i) < 0.$$

By the intermediate value theorem and the strict monotonicity of $Q(a|\hat{a}_j, c_i)$ in a, there must exist a unique $\tilde{a}_i \in [0, \bar{a})$ such that

$$I(\tilde{a}_i|\hat{a}_j, c_i) = 0.$$

That is, if Firm j searches with cut-off $\hat{a}_j$, Firm i's best response is to search with cut-off $\tilde{a}_i$.

For any $\hat{a}_j \in (\alpha_j, \bar{a}]$, if the set is not empty,

$$I(0|\hat{a}_j, c_i) < 0.$$

That is, Firm i's best response is to search with cut-off 0.

Then, we could define two best response functions $BR_i : [0, \bar{a}] \to [0, 1]$ where

$$BR_i(\hat{a}_j) := \begin{cases} 0 & \text{for } \hat{a}_j \in (\alpha_j, \bar{a}] \text{ if it is not empty} \\ \tilde{a}_i & \text{where } I(\tilde{a}_i|\hat{a}_j, c_i) = 0 \text{ for } \hat{a}_j \in [0, \alpha_j]. \end{cases}$$

It is also easy to verify that BR_i is a continuous function over $[0, \bar{a}]$. Hence, we have a continuous self map $BR : [0, 1]^2 \to [0, 1]^2$ where

$$BR = (BR_1, BR_2)$$

on a compact set, and by Brouwer's fixed point theorem, there must exist of a pure strategy equilibrium in which each Firm searches with a cut-off higher then or equals to 0. □

Proof of Proposition 6.1. First, using the same arguments as in the proof of Proposition A.3, we have that if there exists a equilibrium it must be the case that each firm searches with a cut-off higher than equal to 0 with one firm strictly higher than 0.

Next, we show that there can be no equilibrium in which Firm 2 searches with a cut-off $\hat{a}_2 > 0$ and Firm 1 searches with cut-off 0. Such a strategy profile $(0, \hat{a}_2)$ is an equilibrium if and only if

$$\lambda \int_0^{\bar{a}} [Z(a|\hat{a}_2, T) - Z(0|\hat{a}_2, T)]dF(a) - c_1 \leq 0, \quad \text{and}$$
$$\lambda \int_{\hat{a}_2}^{\bar{a}} [Z(a|0, T) - Z(\hat{a}_2|0, T)]dF(a) - c_2 = 0.$$

However,

$$\begin{aligned} 0 &= \lambda \int_{\hat{a}_2}^{\bar{a}} [Z(a|0, T) - Z(\hat{a}_2|0, T)]dF(a) - c_2 \\ &< \lambda \int_{\hat{a}_2}^{\bar{a}} [Z(a|\hat{a}_2, T) - Z(\hat{a}_2|\hat{a}_2, T)]dF(a) - c_2 \\ &< \lambda \int_0^{\bar{a}} [Z(a|\hat{a}_2, T) - Z(0|\hat{a}_2, T)]dF(a) - c_1 \leq 0, \end{aligned}$$

resulting in a contradiction.

Next, we derive the necessary and sufficient conditions for the existence of equilirbium in which Firm 2 searches with a cut-off 0 and Firm 1 searches with a cut-off strictly higher than 0. A pair of cut-off rules $(\hat{a}_1, 0)$, $\hat{a}_1 > 0$, is an equilibrium if and only if

$$\lambda \int_{\hat{a}_1}^{\bar{a}} [Z(a|0, T) - Z(\hat{a}_1|0, T)]dF(a) - c_1 = 0 \quad \text{and} \tag{26}$$
$$\lambda \int_0^{\bar{a}} [Z(a|\hat{a}_1, T) - Z(0|\hat{a}_1, T)]dF(a) - c_2 \leq 0, \tag{27}$$

where

$$(26) \Leftrightarrow \frac{1}{2}\lambda(1 - e^{-\lambda T})[1 - F(\hat{a}_1)]^2 - c = 0 \Leftrightarrow \hat{a}_j = F^{-1}\left(1 - \sqrt{\frac{2c_1}{\lambda(1 - e^{-\lambda T})}}\right). \tag{28}$$

Then, (28) and (27) together imply that $(\hat{a}_i, 0)$ is an equilibrium if and only if

$$I\left(0 \middle| F^{-1}\left(1 - \sqrt{\frac{2c_1}{\lambda(1 - e^{-\lambda T})}}\right), c_2\right) \leq 0. \tag{29}$$

We will see that if (29) holds there is no other equilibrium.

When (29) does not hold, there is a unique equilibrium, in which each firm searches with a cut-off strictly higher than 0. Because by Proposition A.4 there must exists an equilibrium. Let (a_1^*, a_2^*) be such an equilibium, we first show that $a_1^* > a_2^*$ must hold by proof by contradiction, and then we show that it must be a unique equilibrium. Such a pair (a_1^*, a_2^*) is an equilibrium if and only if

$$\lambda \int_{a_i^*}^{\bar{a}} [Z(a|a_j^*, T) - Z(a_i^*|a_j^*, T)]dF(a) = c_i \quad \text{for } i = 1, 2 \text{ and } j \neq i. \tag{30}$$

Suppose $a_1^* \leq a_2^*$. Applying Lemma 3.4, we have

$$\begin{aligned}
c_1 =& \lambda \int_{a_1^*}^{\bar{a}} [Z(a|a_2^*, T) - Z(a_1^*|a_2^*, T)] dF(a) \\
\geq& \lambda \int_{a_2^*}^{\bar{a}} [Z(a|a_2^*, T) - Z(a_2^*|a_2^*, T)] dF(a) \\
\geq& \lambda \int_{a_2^*}^{\bar{a}} [Z(a|a_1^*, T) - Z(a_2^*|a_1^*, T)] dF(a) = c_2,
\end{aligned}$$

resulting in a contradiction.

Then, we show the uniqueness of the equilibrium for *Case* [1] and [2] by contradiction. For *Case* [1] we show that the solution to (30) is unique, and for *Case* [2] and [3] we show that there can be no equilibrium in which each firm searches with a cut-off higher than 0 coexisting with equilibrium $\left(F^{-1}\left(1 - \sqrt{\frac{2c_1}{\lambda(1-e^{-\lambda T})}}\right), 0\right)$. We can prove them together. Suppose there are two equilibria (a_1^*, a_2^*) and $(\tilde{a}_1^*, \tilde{a}_2^*)$, in which (a_1^*, a_2^*) is a solution to (30) and $(\tilde{a}_1^*, \tilde{a}_2^*)$ is either $\left(F^{-1}\left(1 - \sqrt{\frac{2c_1}{\lambda(1-e^{-\lambda T})}}\right), 0\right)$ or a solution to (30). It is sufficient to show that the following two cases are not possible:

1. $\tilde{a}_1^* > a_1^* > a_2^* > \tilde{a}_2^* \geq 0$ and
2. $a_1^* > \tilde{a}_1^* > a_2^* > \tilde{a}_2^* \geq 0$.

Suppose $\tilde{a}_1^* > a_1^* > a_2^* > \tilde{a}_2^* \geq 0$. Applying Lemma 3.4 we have

$$\begin{aligned}
0 =& \lambda \int_{a_1^*}^{\bar{a}} [Z(a|a_2^*, T) - Z(a_1^*|a_2^*, T)] dF(a) - c_1 \\
<& \lambda \int_{\tilde{a}_1^*}^{\bar{a}} [Z(a|a_2^*, T) - Z(\tilde{a}_1^*|a_2^*, T)] dF(a) - c_1 \\
<& \lambda \int_{\tilde{a}_1^*}^{\bar{a}} [Z(a|\tilde{a}_2^*, T) - Z(\tilde{a}_1^*|\tilde{a}_2^*, T)] dF(a) - c_1 = 0,
\end{aligned}$$

resulting in a contradiction.

Suppose $a_1^* > \tilde{a}_1^* > a_2^* > \tilde{a}_2^* \geq 0$. Applying Lemma 3.4 again, we have

$$\begin{aligned}
0 \geq& \lambda \int_{\tilde{a}_2^*}^{\bar{a}} [Z(a|\tilde{a}_1^*, T) - Z(\tilde{a}_2^*|\tilde{a}_1^*, T)] dF(a) - c_2 \\
>& \lambda \int_{a_2^*}^{\bar{a}} [Z(a|\tilde{a}_1^*, T) - Z(a_2^*|\tilde{a}_1^*, T)] dF(a) - c_2 \\
>& \lambda \int_{a_2^*}^{\bar{a}} [Z(a|a_1^*, T) - Z(a_2^*|a_1^*, T)] dF(a) - c_2 = 0,
\end{aligned}$$

resulting in a contradiction.

□

Proof of Proposition 6.2. For fixed c_2 we have

$$\frac{\partial a_2^*}{\partial a_1^*} = -\frac{\frac{\partial \int_{a_2^*}^{\bar{a}} [Z(a|a_1^*, T) - Z(a_2^*|a_1^*, T)] dF(a)}{\partial a_1^*}}{\frac{\partial \int_{a_2^*}^{\bar{a}} [Z(a|a_1^*, T) - Z(a_2^*|a_1^*, T)] dF(a)}{\partial a_2^*}} = \frac{\int_{a_2^*}^{\bar{a}} \frac{\partial [Z(a|a_1^*, T) - Z(a_2^*|a_1^*, T)]}{\partial a_1^*} dF(a)}{\frac{\partial Z(a_2^*|a_1^*, T)}{\partial a_2^*}} < 0.$$

Then,

$$\frac{\partial a_1^*}{\partial c_1} = -\frac{-1}{\lambda \int_{a_1^*}^{\bar{a}} \left[\frac{\partial [Z(a|a_2^*,T) - Z(a_1^*|a_2^*,T)]}{\partial a_2^*} \frac{\partial a_2^*}{\partial a_1^*} - \frac{\partial Z(a_1^*|a_2^*,T)}{a_1^*} \right] dF(a)} < 0 \text{ and}$$

$$\frac{\partial a_2^*}{\partial c_1} = \frac{\partial a_2^*}{\partial a_1^*} \frac{\partial a_1^*}{\partial c_1} > 0.$$

For fixed c_1 we have

$$\frac{\partial a_1^*}{\partial a_2^*} = -\frac{\frac{\partial \int_{a_1^*}^{\bar{a}} [Z(a|a_2^*,T) - Z(a_1^*|a_2^*,T)] dF(a)}{\partial a_2^*}}{\frac{\partial \int_{a_1^*}^{\bar{a}} [Z(a|a_2^*,T) - Z(a_1^*|a_2^*,T)] dF(a)}{\partial a_1^*}} = \frac{\int_{a_1^*}^{\bar{a}} \frac{\partial [Z(a|a_2^*,T) - Z(a_1^*|a_2^*,T)]}{\partial a_2^*} dF(a)}{\frac{\partial Z(a_1^*|a_2^*,T)}{\partial a_1^*}} > 0.$$

Then,

$$\frac{\partial a_2^*}{\partial c_2} = -\frac{-1}{\lambda \int_{a_2^*}^{\bar{a}} \left[\frac{\partial [Z(a|a_1^*,T) - Z(a_2^*|a_1^*,T)]}{\partial a_1^*} \frac{\partial a_1^*}{\partial a_2^*} - \frac{\partial Z(a_2^*|a_1^*,T)}{a_2^*} \right] dF(a)} < 0 \text{ and}$$

$$\frac{\partial a_1^*}{\partial c_1} = \frac{\partial a_1^*}{\partial a_2^*} \frac{\partial a_2^*}{\partial c_2} < 0.$$

□