Comparing Optimal Responses in Games with Multi-Dimensional Action Spaces

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Abstract

This paper introduces a single-crossing condition on players' multi-dimensional action spaces which gives rise to a component-wise ordering of optimal responses. Using standard lattice techniques, necessary and sufficient conditions are given on the primitives of the model which characterize when a player will choose to best respond with a higher action in one component relative to another. Furthermore, conditions are given under which this relationship will continue to hold after a positive shock to preferences. These ideas simplify equilibrium analysis in a variety of applications, including multi-product and multi-market oligopolies, as well as equilibria in the corresponding factor markets. Several examples are given.

Preliminary and incomplete

1. Introduction

In many strategic situations, agents have the ability to interact with their environment through a multitude of independent decision making processes. For example, the choice of an optimal bundle for a single consumer can be thought as the utility-maximizing choice of units of individual goods while satisfying a budget constraint. Likewise, the quantity choice of an oligopolist can be seen as a decision of how much labor and capital to employ in the production process. In the latter instance, the optimal decision of how much to choose in one "component" relative to the other has clear economic implications, leading us to questions such as, "How does a firm's technology affect equilibrium in the underlying factor markets?". While classical methods may approach this problem by first deriving and then analyzing the set of equilibria in a game or the optimal responses of an individual decision maker, this paper takes an alternative approach by establishing order properties directly from an agent's preferences which lead to one component being favored over the other. This approach can be applied to strategic as well as non-strategic decision making problems.

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Our analysis draws on the standard lattice techniques developed in Topkis (1998) used in the literature on games of strategic complements (GSC) and substitutes (GSS).[1] This literature allows for multi-dimensional action spaces insofar as they are assumed to be complete lattices in the product order.[2] In the case of GSC, for example, we are able to say that observing a higher action from opponents (in the product order) leads a player to best respond by also choosing a higher action (again in the product order). However, these methods have not yet been extended to compare the component-wise choices that make up such a best response. To this end, we introduce a single-crossing property which gives rise to an order on the components of a player's multi-dimensional action space, so that component k of player i's strategy space is larger than component l if the set of values in the kth component associated with a best response are higher (in the sense of Quah 2007) than those of the lth component. The existence of this order is shown to be not only a sufficient but also a necessary condition for component k to be played higher than component l in an optimal response, and subsequently in equilibrium. In this sense, our method is most closely related to Lazzati (2013), who derives a similar order on the set of players in a game. As in that paper, we provide conditions on differential payoff functions which are sufficient for such an ordering to exist, which lead to various applications including multi-product and multi-market oligopolies, analysis of factor markets, and price discrimiation.

2. Motivating Example

Consider a monopolist which is selling two differentiated products. Suppose that in market $i = 1, 2$, the monopolist charges price p_i, and faces marginal cost and demand given by c_i and $D_i(p_1, p_2)$, respectively. The profit function can then be written as

$$\pi(p_1, p_2) = \pi_1(p_1, p_2) + \pi_2(p_1, p_2)$$

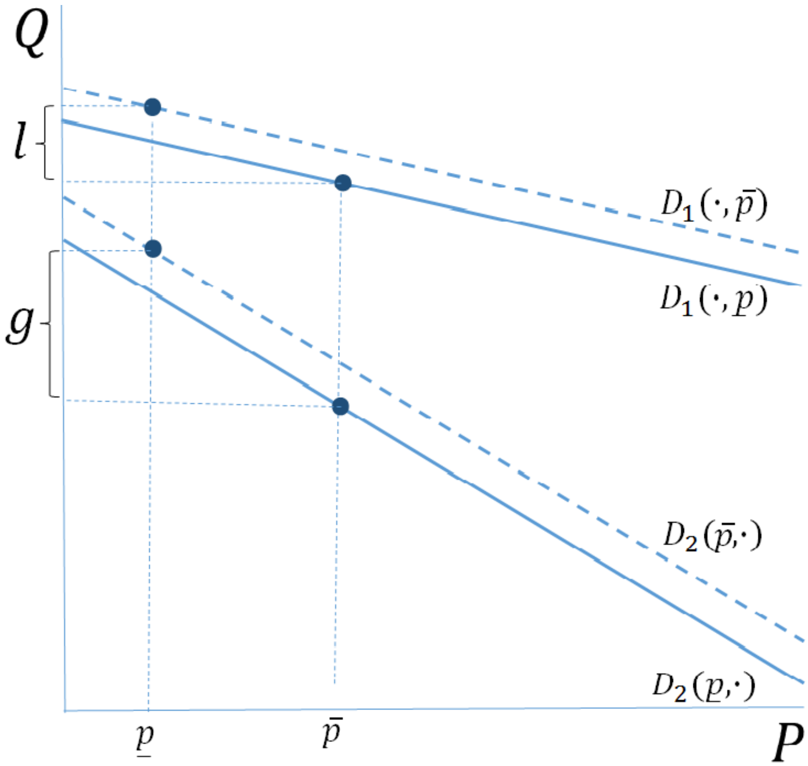
where each $\pi_i(p_1, p_2) = (p_i - c_i) D_i(p_1, p_2)$ represents profits in market i. We will be interested in determining conditions which guarantee a higher price being charged in one market relative to the other. Recall that the goods are substitutes, independent, or complements depending on whether each $\frac{\partial D_i}{\partial p_j} > 0$, $\frac{\partial D_i}{\partial p_j} = 0$, or $\frac{\partial D_i}{\partial p_j} < 0$, respectively. For example, Vives and Singh (1984) consider a linear model in which each

$$D_i(p_1, p_2) = a_i + kp_j - b_i p_i$$

so that each good has the same amount of "substitutability", depending on the sign of k. Let us say that market 1 is larger than market 2 if at equivalent prices, demand in market 1 is larger, and has smaller own price effects $b_1 \leq b_2$, implying a lower elasticity. This is represented below for $c_1 = c_2 = c$.

[1]For a survey article on GSC, see Amir (2005).

[2]Recall that if $(A, \succeq_A)$ and $(B, \succeq_B)$ are two ordered sets, then the product order $\succeq$ on $A \times B$ is defined as, $(a', b') \succeq (a, b)$ if $a' \succeq_A a$ and $b' \succeq_B b$.



Let $(p_1, p_2) = (\underline{p}, \bar{p})$ be a price combination, where $\underline{p} \leq \bar{p}$, and consider the alternative "flipped" price combination $(\bar{p}, \underline{p})$, where the higher price is charged in the larger market 1. We see that when switching to this alternative combination, the relative loss $l = D_1(\underline{p}, \bar{p}) - D_1(\bar{p}, \underline{p})$ in the larger market 1 is less than the relative gain $g = D_2(\bar{p}, \underline{p}) - D_2(\underline{p}, \bar{p})$ in market 2. After a simple rearrangement of terms, this gives us

$$\pi(\bar{p}, \underline{p}) = (\bar{p} - c) D_1(\bar{p}, \underline{p}) + (\underline{p} - c) D_2(\bar{p}, \underline{p}) \geq$$

$$(\underline{p} - c) D_1(\underline{p}, \bar{p}) + (\bar{p} - c) D_2(\underline{p}, \bar{p}) = \pi(\underline{p}, \bar{p})$$

Clearly, if this condition holds for each such price combination $(\underline{p}, \bar{p})$, then any unique optimizer must charge a price in the larger market that is at least as large as the price charged in the smaller market. This paper generalizes and expands on this intuition, and Example 1 gives conditions under which the above analysis can be extended to much more general demand functions, including the possibility that the set of optimizers is not unique.

3. Theoretical Framework

Consider a normal form game $\Gamma \equiv (I, (A_i)_{i \in I}, (\pi_i)_{i \in I})$, where $I = \{1, 2, ..., N\}$ denotes the set of players. For each player i, $A_i = A_i^1 \times A_i^2 \times ... \times A_i^n$ denotes her n-dimensional strategy space, $A_{-i} \equiv \prod_{j \neq i} A_j$ is the strategies of her opponents, and $\pi_i : A_i \times A_{-i} \to \mathbb{R}$ is her payoff function. For any given strategy profile of the other players $a_{-i} \in A_{-i}$, write player i's best-response correspondence as

$$BR_i(a_{-i}) = \{a_i \in A_i \mid a_i \in \underset{a_i' \in A_i}{argmax}\, \pi_i(a_i', a_{-i})\}$$

Moreover, for all a_{-i}, define player i's best response in component k of her action profile as

$$BR_i^k(a_{-i}) = \{a_i^k \in A_i^k \mid \exists a_i^{-k} \in A_i^{-k}, (a_i^k, a_i^{-k}) \in BR_i(a_{-i})\}$$

where $A_i^{-k} = \prod_{l \neq k} A_i^l$ is the strategy space for all components except k.

A strategy profile $a^\star$ is a (pure strategy) Nash equilibrium if

$$a^\star \in BR(a^\star) \equiv (BR_1(a_{-1}^\star), BR_2(a_{-2}^\star), ..., BR_N(a_{-N}^\star))$$

that is $a^\star$ is a fixed point of $BR(\cdot)$.

The objective of this paper is to compare equilibrium choices in any two strategy components by a given player. To order strategy components, we will order the player's best-response correspondences in the respective components and use a set order introduced in Quah (2007).

Definition 1. Let S' and S be non-empty, partially ordered sets. S' is said to be higher than S ($S' \geq_H S$) if for every $x \in S$ there exists $x' \in S'$ such that $x' \geq x$ and for every $x' \in S'$ there exists $x \in S$ such that $x' \geq x$.

Likewise, we say that $S' >_H S$ if for every $x \in S$ there exists $x' \in S'$ such that $x' > x$ and for every $x' \in S'$ there exists $x \in S$ such that $x' > x$.

We will also make use of the following definitions, which are standard in the GSC and GSS literature.

Definition 2. Given a lattice A and a partially ordered set T, if $\pi : A \times T \to \mathbb{R}$ is supermodular in a, then π satisfies increasing (decreasing) differences in (a, t) if $\forall a', a \in A, \forall t', t \in T$ such that $a' \geq a$, and $t' \geq t$, we have that

$$\pi\left(a', t'\right) - \pi\left(a, t'\right) \geq (\leq)\, \pi\left(a', t\right) - \pi\left(a, t\right)$$

Recall that when each π_i satisfies supermodularity in own action and increasing (decreasing) differences between own action and opponents action, or in (a_i, a_{-i}), the game is one of strategic complements (substitutes). Also, we will call a shock parameter t to be positive for player i if π_i has increasing differences between own action and the shock parameter, or in (a_i, t). Recall that from Topkis (1998), if π_i is differentiable, then π_i satisfies increasing differences in (a, t) iff $\frac{\partial \pi_i}{\partial a_i^l}$ is increasing in t for each $1 \leq l \leq n$.

We now define a relation on action components.

Definition 3. Let $i \in I$, and k and m be two components in player i's strategy space such that $A_i^k = A_i^m$. Then

- $k \succeq (\succ)\, m$ at a_{-i} if we have that $BR_i^k(a_{-i}) \geq_H (>_H)\, BR_i^m(a_{-i})$.
- $k \succeq (\succ)\, m$ on $\mathcal{O} \subseteq A_{-i}$ if $k \succeq (\succ)\, m$ at each $a_{-i} \in \mathcal{O}$.

$k \succeq m$ at a_{-i} has the following interpretation: For every best response a_i to a_{-i}, Player i can "cook up" some best response which is at least as large in the kth component as a_i is in the lth, and similarly one that is at least as small in the lth component as a_i is in the kth. Note that if $BR_i(a_{-i})$ is a singleton, then this is equivalent to $BR_i^k(a_{-i})$ being larger in the strong set order than $BR_i^m(a_{-i})$.[3]

4. Main results

We will mainly be interested in situations where, for a player i and a profile of opponents' actions a_{-i}, we have that for each component j of A_i, $BR_i^j(a_{-i})$ is non-empty and contains largest and smallests elements $\vee BR_i^j(a_{-i})$ and $\wedge BR_i^j(a_{-i})$, respectively. Without imposing any specific conditions on payoffs and action spaces in this regard, we list below some popular and widely-used environments in which this condition hold is known to hold:

1. *The set of maximizers* $M = \underset{A_i}{argmax}\,(\pi_i\,(a_i,\, a_{-i}))$ *is compact.* This is true under a variety of conditions, including those of the Berge Maximum Theorem.[4] In this case, by considering the (continuous) jth projection $p_j : M \to A_j$, we are guaranteed the existence of largest and smallest elements $\vee BR_i^j(a_{-i})$, $\wedge BR_i^j(a_{-i}) \in BR_i^j(a_{-i})$.
2. *Games of Strategic Complements (GSC) and Substitutes (GSS).* With increasing/decreasing differences in own action and opponents' action $(a_i,\, a_{-i})$, along with other mild conditions on action spaces and payoffs,[5] this requirement will be satisfied in GSC/GSS. Because both cases guarantee the existence of a largest and smallest best-response $\vee BR_i(a_{-i})$, $\wedge BR_i(a_{-i}) \in BR_i(a_{-i})$, it is an easy fact to check that the jth components of $\vee BR_i(a_{-i})$ and $\wedge BR_i(a_{-i})$ represent the largest and smallest components of $BR_i^j(a_{-i})$, respectively. This is done in Corollary 1.

3. *Action spaces are finite.* This is trivially a special instance of Case 1 above.

The first result says that if that $k \succeq m$ on $\mathcal{O}$, then player i's smallest and largest best response in component k is higher than in component m. In fact, this condition is both necessary and sufficient.

Theorem 1. *Let* $\mathcal{O} \subseteq A_{-i}$. *Then*

[3] Recall that $BR_i^k(a_{-i}) \geq_S BR_i^m(a_{-i})$ if $\forall x \in BR_i^k(a_{-i})$, $\forall y \in BR_i^m(a_{-i})$, we have that $x \vee y \in BR_i^k(a_{-i})$ and $x \wedge y \in BR_i^m(a_{-i})$

[4] See Aliprantis, C. D., & Border, K. C. (1999).

[5] See Milgrom, P., & Shannon, C. (1994).

1. *$k \succeq m$ on $\mathcal{O}$ if and only if for each $a_{-i} \in \mathcal{O}$,*

$$\vee BR_i^k(a_{-i}) \geq \vee BR_i^m(a_{-i}) \, and \, \wedge BR_i^k(a_{-i}) \geq \wedge BR_i^m(a_{-i})$$

2. *$k \succ m$ on $\mathcal{O}$ if and only if for each $a_{-i} \in \mathcal{O}$*

$$\vee BR_i^k(a_{-i}) > \vee BR_i^m(a_{-i}) \, and \, \wedge BR_i^k(a_{-i}) > \wedge BR_i^m(a_{-i})$$

Proof. Fix $a_{-i} \in \mathcal{O}$ arbitrarily, and suppose that $k \succeq m$. To show that $\vee BR_i^k(a_{-i}) \geq \vee BR_i^m(a_{-i})$, we know from Proposition 1 that $\vee BR_i^m(a_{-i}) \in BR_i^m(a_{-i})$. Since $k \succeq m$, there exists $a_i^k \in BR_i^k(a_{-i})$ such that $a_i^k \geq \vee BR_i^m(a_{-i})$. Since $\vee BR_i^k(a_{-i}) \in BR_i^k(a_{-i})$, we have that $\vee BR_i^k(a_{-i}) \geq a_i^k \geq \vee BR_i^m(a_{-i})$, giving the result. The case for $\wedge BR_i^k(a_{-i}) \geq \wedge BR_i^m(a_{-i})$ as well as the strict cases follow similiarly.

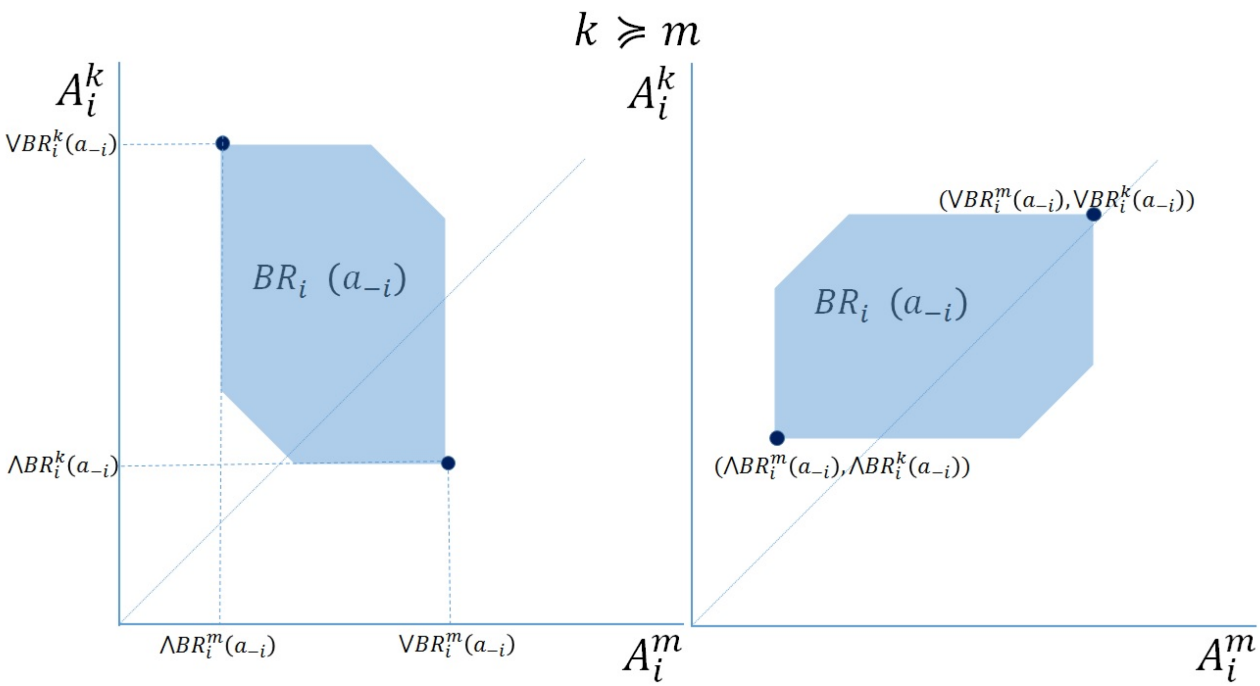
Conversely, suppose that for each $a_{-i} \in \mathcal{O}$, $\vee BR_i^k(a_{-i}) \geq \vee BR_i^m(a_{-i})$ and $\wedge BR_i^k(a_{-i}) \geq \wedge BR_i^m(a_{-i})$. Let $z \in BR_i^m(a_{-i})$ be arbitrary. We then have that

$$z \leq \vee BR_i^m(a_{-i}) \leq \vee BR_i^k(a_{-i}) \in BR_i^k(a_{-i})$$

This satisfies the first requirement of $BR_i^k(a_{-i}) \geq_H BR_i^m(a_{-i})$. The second requirement as well as the strict case follow similarly.

□

Corollary 1 shows that in the case when the largest and smallest best-responses $\vee BR_i(\cdot)$ and $\wedge BR_i(\cdot)$ exist, we can say even more. The pictures below gives two instances in a $2-$dimensional action space, the one on the left is when the extremal best-responses do not exist, the one on the right is when they do. Notice that in the latter case, both points must lie above the “45 degree line”.



Corollary 1. *Suppose $\vee BR_i(\cdot)$ and $\wedge BR_i(\cdot)$ exist, and $k \succeq (\succ)\, m$ on $\mathcal{O} \subseteq A_{-i}$. If $a^\star \in \mathcal{O}$ is a fixed point of either $\vee BR_i(\cdot)$ or $\wedge BR_i(\cdot)$ then $a_i^{*k} \geq (>)\, a_i^{*m}$. In particular, if best responses are unique and $k \succeq (\succ)\, m$, then any Nash equilibrium satisfies this property.*

Proof. We show that $\vee BR_i^k(a_{-i})$ and $\vee BR_i^k(a_{-i})$ exist and are in $BR_i^k(a_{-i})$. Since the case for $\wedge BR_i^k(a_{-i})$ and component m follow similarly, the result then follows from Theorem 1. Write $\vee BR_i(a_{-i}) = (\bar{a}_i^k(a_{-i}))_{k=1}^n \in BR_i(a_{-i})$, and let $a_i^k \in BR_i^k(a_{-i})$ be arbitrary. Then there exists (a_i^{-k}) such that $(a_i^k, a_i^{-k}) \in BR_i(a_{-i}) \leq \vee BR_i(a_{-i}) = (\bar{a}_i^k, \bar{a}_i^{-k})$. Thus $a_i^k \leq \bar{a}_i^k$. Since $\bar{a}_i^k(a_{-i}) \in BR_i^k(a_{-i})$, $\vee BR_i^k(a_{-i})$ exists and equals $\bar{a}_i^k$.

□

In fact, when GSC or GSS are considered, we can say even more, as the next Corollary shows.

Corollary 2. *If each $\vee BR_i(\cdot)$ and $\wedge BR_i(\cdot)$ are continuous and $k \succeq (\succ)\, m$ on A_{-i}, then*

1. *In the case of a GSC, the upper and lower equilibria $\bar{a}$ and $\underline{a}$ exist and satisfy $\bar{a}_i^k \geq (>)\, \bar{a}_i^m$ and $\underline{a}_i^k \geq (>)\, \underline{a}_i^m$. In particular, any unique Nash equilibrium satisfies this property.*
2. *In the case of a GSS, the upper and lower serial undominated strategies $\bar{a}$ and $\underline{a}$ exist and satisfy $\bar{a}_i^k \geq (>)\, \bar{a}_i^m$ and $\underline{a}_i^k \geq (>)\, \underline{a}_i^m$. In particular, any dominance solvable solution satisfies this property.*

Proof. For the first case, the existence of $\bar{a}$ and $\underline{a}$ are given in Milgrom and Roberts (1990). In their construction, $\bar{a} = \underset{k}{lim}\left(\bar{a}^k\right) = \underset{k}{lim}\left(\vee BR(\bar{a}^{k-1})\right)$ and $\underline{a} = \underset{k}{lim}\left(\underline{a}^k\right) = \underset{k}{lim}\left(\wedge BR(\underline{a}^{k-1})\right)$, where $\left(\bar{a}^k\right)_{k=0}^{\infty}$ and $\left(\underline{a}^k\right)_{k=0}^{\infty}$ define their best-response dynamics. Continuity gives the result.

For the second case, the existence of $\bar{a}$ and $\underline{a}$ are given in Roy and Sabarwal (2010). In their construction, $\bar{a} = \underset{k}{lim}\left(\bar{a}^k\right) = \underset{k}{lim}\left(\wedge BR(\underline{a}^{k-1})\right)$ and $\underline{a} = \underset{k}{lim}\left(\underline{a}^k\right) = \underset{k}{lim}\left(\vee BR(\bar{a}^{k-1})\right)$, where $\left(\bar{a}^k\right)_{k=0}^{\infty}$ and $\left(\underline{a}^k\right)_{k=0}^{\infty}$ define their best-response dynamics. Continuity gives the result.

□

5. Sufficient Conditions

We now give two sufficient conditions on the primitives of the model which guarantee that $k \succeq m$. The first, an ordinal property on payoff functions, can be useful when action spaces are finite or payoffs are non-differentiable. The second is a condition on first derivatives when action spaces are continuous.

5.1. A Single-Crossing Condition in Components

In what follows, let the first argument denotes player i's strategy in component k, the second argument in component m, and the third argument all components other than k and m. Moreover, for convenience, we introduce the following notation. For all a_i'', a_i', $a_i \in A_i^k$, a_i'', a_i', $a_i \in A_i^m$, $a_i^{-k,m} \in A_i^{-k,m}$, and $a_{-i} \in A_{-i}$, let

$$\Delta_k \pi_i((a_i'',\, a_i'),\, a_i;\, a_i^{-k,m},\, a_{-i}) := \pi_i(a_i'',\, a_i,\, a_i^{-k,m},\, a_{-i}) - \pi_i(a_i',\, a_i,\, a_i^{-k,m},\, a_{-i})$$

and

$$\Delta_m \pi_i(a_i,\, (a_i'',\, a_i');\, a_i^{-k,m},\, a_{-i}) := \pi_i(a_i,\, a_i'',\, a_i^{-k,m},\, a_{-i}) - \pi_i(a_i,\, a_i',\, a_i^{-k,m},\, a_{-i})$$

where Δ_k denotes that the change occurs in component k with a fixed a_i in component m, and where Δ_m denotes that the change occurs in component m with a fixed a_i in component k. The bracketed term in the delta notation denotes the different actions in that component, while the rest are the action components by player i and the other players that remain unchanged.

We then have the following single-crossing condition.

Definition 4. The payoff function π_i satisfies the single-crossing condition in components k and m if on $\mathcal{O} \subseteq A_{-i}$ if $\forall a_i'' \geq a_i'$, $\forall a_i^{-k,m} \in A_i^{-k,m}$, $\forall a_{-i} \in \mathcal{O}$, $\forall \varepsilon \in \mathbb{R}_+$,

$$\Delta_m \pi_i(a_i',\, (a_i'',\, a_i');\, a_i^{-k,m},\, a_{-i}) \geq \varepsilon \Rightarrow$$

$$\Delta_k \pi_i((a_i'',\, a_i');\, a_i',\, a_i^{-k,m},\, a_{-i}) \geq \varepsilon$$

This condition captures the intuition that when best responding to some a_{-i}, if it is beneficial to increase a response in the mth component, it is at least as beneficial to do so in the kth component. Theorem 2 below shows that the single-crossing condition being satisfied is enough to conclude that $k \succeq m$. This condition then has two useful advantages. First, the single-crossing condition tells us that we need only consider deviations from the "45 degree line" of the $k-m$ axis. Second, as an ordinal condition, we need only compare those deviations which give non-negative benefit in the lower component.

Theorem 2. *If π_i satisfies the single-crossing condition in components k and m on $\mathcal{O} \subseteq A_{-i}$, then $k \succeq m$ on $\mathcal{O}$.*

Proof. Fix $a_{-i} \in A_{-i}$, and let $a_i \in BR_i(a_{-i})$ be arbitrary. Consider the two cases:

Case 1: $a_i^k > a_i^m$.

Case 2: $a_i^m \geq a_i^k$: Let $a_i^m = a_i''$ and $a_i^k = a_i'$ be the largest and the smallest elements according to Definition 3, respectively, and set $\varepsilon = \Delta_m \pi_i(a_i', (a_i'', a_i')\, a_i^{-k,m}, a_{-i})$. Since $(a_i', a_i'', a_i^{-k,m}) \in BR_i(a_{-i})$ by hypothesis, we have that

$$\Delta_m \pi_i(a_i', (a_i'', a_i'), a_i^{-k,m}, a_{-i}) = \varepsilon \geq 0$$

By the single-crossing property, this implies that

$$\Delta_k \pi_i((a_i'', a_i'); a_i', a_i^{-k,m}, a_{-i}) \geq \varepsilon$$

or equivalently

$$\Delta_k \pi_i((a_i'', a_i'); a_i', a_i^{-k,m}, a_{-i}) \geq \Delta_m \pi_i(a_i', (a_i'', a_i')\, a_i^{-k,m}, a_{-i})$$

Therefore, we have that $\pi_i((a_i'', a_i', a_i^{-k,m}), a_{-i}) \geq \pi_i((a_i', a_i'', a_i^{-k,m}), a_{-i})$, in which case $(a_i'', a_i', a_i^{-k,m}) = (a_i^m, a_i^k, a_i^{-k,m}) \in BR_i(a_{-i})$ as well.

Since $a_i \in BR_i(a_{-i})$ was arbitrary, we see that for any $a_i^m \in BR_i^m(a_{-i})$, there exists some $a_i^k \in BR_i^k(a_{-i})$ such that $a_i^k \geq a_i^m$. Likewise, for each $a_i^k \in BR_i^k(a_{-i})$, there exists some $a_i^m \in BR_i^m(a_{-i})$ such that $a_i^k \geq a_i^m$, establishing the result.

□

5.2. Differential Condition

Below we give a differential condition. Notice that unlike the single-crossing condition, this condition is local in nature, but can more useful when strategy spaces are subsets of a Euclidean space. Just as with the single-crossing condition, we need only consider points along the $k-m$ axis. In fact, under slightly stronger assumptions, the differential condition will imply the single-crossing condition. Note that the inquality requirement in the second condition is stronger than in the first condition, but does not require any assumption about quasi-concavity.

Theorem 3. *Suppose that π_i is $\mathcal{C}^1$, quasi-concave in component m, and for all $a_i^k = a_i^m \equiv \overline{a}$, $a_i^{-k,m} \in A_i^{-k,m}$, $\lambda \geq 0$, $\varepsilon \geq 0$, and $a_{-i} \in \mathcal{O}$*

$$\frac{\partial \pi_i}{\partial a_i^m}\left(\left(\overline{a},\, \overline{a}+\lambda,\, a_i^{-k,m}\right),\, a_{-i}\right) \geq \varepsilon \Rightarrow$$

$$\frac{\partial \pi_i}{\partial a_i^k}\left(\left(\overline{a}+\lambda,\, \overline{a},\, a_i^{-k,m}\right),\, a_{-i}\right) \geq \varepsilon$$

Then $k \succeq m$ on $\mathcal{O}$.

Alternatively, if for each $\lambda \geq 0$, we have that

$$\frac{\partial \pi_i}{\partial a_i^k}\left(\left(\overline{a}+\lambda,\, \overline{a},\, a_i^{-k,m}\right),\, a_{-i}\right) \geq \frac{\partial \pi_i}{\partial a_i^m}\left(\left(\overline{a},\, \overline{a}+\lambda,\, a_i^{-k,m}\right),\, a_{-i}\right)$$

then π_i satisfies the single-crossing condition in components k and m on $\mathcal{O}$.

Proof. Fix $a_{-i} \in \mathcal{O}$ and $a_i^{-k,m}$ arbitrarily and suppose that $a_i \in BR_i\left(a_{-i}^*\right)$. Consider the following two cases:

Case 1: $a_i^m \leq a_i^k$.

Case 2: $a_i^m = a_i'' > a_i^k = a_i'$. Observe that for $\lambda \in (0,\, a_i'' - a_i')$, $\frac{\partial \pi_i}{\partial a_i^m}\left(\left(a_i',\, a_i'+\lambda,\, a_i^{-k,m}\right),\, a_{-i}^*\right) \geq 0$: Suppose this is not the case for some λ', or

$$\underset{v\to 0}{lim}\left(\frac{\Delta_m \pi_i(a_i',\, (a_i'+\lambda'+v,\, a_i'+\lambda'),\, a_i^{-km},\, a_{-i})}{v}\right) < 0$$

Then for some $v' \in \left(0,\, a'' - a' - \lambda'\right)$ sufficiently small,

$$\pi_i\left(\left(a_i',\, a_i'+\lambda'+v',\, a_i^{-k,m}\right),\, a_{-i}\right) < \pi_i\left(\left(a_i',\, a_i'+\lambda',\, a_i^{-k,m}\right),\, a_{-i}\right) \leq \pi_i\left(\left(a_i',\, a_i'',\, a_i^{-k,m}\right),\, a_{-i}\right)$$

But by quasi-concavity in the mth component, we must have that for each $\alpha \in [0,\, 1]$,

$$\pi_i\left(\left(a_i',\, \alpha a_i'' + (1-\alpha)\left(a_i'+\lambda'\right),\, a_i^{-k,m}\right),\, a_{-i}\right) \geq$$

$$min\left(\pi_i\left(\left(a_i',\, a_i'',\, a_i^{-k,m}\right),\, a_{-i}\right),\, \pi_i\left(\left(a_i',\, a_i'+\lambda',\, a_i^{-k,m}\right),\, a_{-i}\right)\right) =$$

$$\pi_i\left(\left(a_i',\, a_i'+\lambda',\, a_i^{-k,m}\right),\, a_{-i}^*\right)$$

contradicting the fact that $a_i'+\lambda'+v' \in \left(a_i'+\lambda',\, a_i''\right)$, and establishing the observation. By defining $\varepsilon(\lambda) = \frac{\partial \pi_i}{\partial a_i^m}\left(\left(a_i',\, a_i'+\lambda,\, a_i^{-k,m}\right),\, a_{-i}\right) \geq 0$, then by hypothesis we have that for each $\lambda \in (0,\, a_i'' - a_i')$,

$$\frac{\partial \pi_i}{\partial a_i^k}\left(\left(a_i'+\lambda,\, a_i',\, a_i^{-k,m}\right),\, a_{-i}\right) \geq \frac{\partial \pi_i}{\partial a_i^m}\left(\left(a_i',\, a_i'+\lambda,\, a_i^{-k,m}\right),\, a_{-i}\right)$$

Therefore, by the fundamental theorem of calculus,

$$\Delta_k \pi_i((a_i'', a_i'), a_i', a_i^{-k,m}, a_{-i}) = \int_0^\lambda \left(\frac{\partial \pi_i}{\partial a_i^k} \left(\left(a_i' + z, a_i', a_i^{-k,m} \right), a_{-i} \right) \right) dz \geq$$

$$\int_0^\lambda \left(\frac{\partial \pi_i}{\partial a_i^m} \left(\left(a_i', a_i' + z, a_i^{-k,m} \right), a_{-i} \right) \right) dz = \Delta_m \pi_i(a_i', (a_i'', a_i'), a_i^{-k,m}, a_{-i})$$

or $\pi_i \left(\left(a_i'', a_i', a_i^{-k,m} \right), a_{-i}^* \right) \geq \pi_i \left(\left(a_i', a_i'', a_i^{-k,m} \right), a_{-i}^* \right)$. Thus $\left(a_i'', a_i', a_i^{-k,m} \right) \in BR_i \left(a_{-i} \right)$. In either case, we have found some $a_i'' \in BR_i \left(a_{-i}^* \right)$ which is at least as high in its kth component as a_i is in its mth, and vice versa. This establishes $k \succeq m$.

Suppose the conditions for the second claim are satisfied. Let $a_{-i} \in \mathcal{O}$ and $a_i^{-k,m}$ be arbitrary, and for $a_i^k = a_i'' \geq a_i^m = a_i'$, set $\lambda = a_i'' - a_i' \geq 0$. Then applying the fundamental theorem of calculus as above we have that

$$\Delta_k \pi_i((a_i'', a_i'), a_i', a_i^{-k\,m}, a_{-i}) \geq \Delta_m \pi_i(a_i', (a_i'', a_i'), a_i^{-k,m}, a_{-i})$$

and the result follows.

□

We now give examples of when these conditions can be useful.

5.3. Applications

5.3.1. Bertrand Monopoly with Differentiated Goods

Consider again the monopolist who sells two differentiated products from Section 2. Recall that the price elasticity of demand for firm i at price combination (p_1, p_2) is given by

$$\varepsilon_i (p_1, p_2) = -\frac{p_i}{D_i (p_1, p_2)} \frac{\partial D_i (p_1, p_2)}{\partial p_i}$$

Let us say that market 1 is larger than market 2 if for each price p and $\lambda \geq 0$, the following hold:

1. $D_1 (p + \lambda, p) \geq D_2 (p, p + \lambda)$, and either
2. $\frac{\partial D_2 (p, p+\lambda)}{\partial p_2} \leq \frac{\partial D_1 (p+\lambda, p)}{\partial p_1} \leq 0$ or
3. $0 \leq \varepsilon_1 (p + \lambda, p) \leq \varepsilon_2 (p, p + \lambda) \leq 1$

That is, at similar prices, the larger market will have a higher total demand, and either "less negative" own-price effects, or a lower elasticity. The former case includes demand systems such as the linear model considered in Section 2, whereas the latter includes cases such as CES demand, where higher demand in the larger market must be met with “more negative” own-price effects in order to maintain constant elasticity. While these conditions capture the general intuition of a larger market, we must also consider how a change of price in one market affects demand in the other.

Proposition 1. *If market 1 is larger than market 2 according to the above definition,* $c_1 = c_2 = c$ *and* $\frac{\partial D_1}{\partial p_2} = \frac{\partial D_2}{\partial p_1}$*, then* $p_1 \geq p_2$ *in extremal equilibria.*

Proof. Suppose $c_1 = c_2 = c$ and $\frac{\partial D_1}{\partial p_2} = \frac{\partial D_2}{\partial p_1}$.

Case 1. Also suppose that market 1 is larger than market 2 according to (1) and (2) in the definition above. That is, for all $\bar{p}$ and for all$\lambda \geq 0$, $D_1(\bar{p}+\lambda,\, \bar{p}) \geq D_2(\bar{p},\, \bar{p}+\lambda)$ and $\mid \frac{\partial D_1}{\partial p_1}(\bar{p}+\lambda,\, \bar{p}) \mid \leq \mid \frac{\partial D_2}{\partial p_2}(\bar{p},\, \bar{p}+\lambda) \mid$.

Now consider

$$\frac{\partial \pi}{\partial p_1} = p_1 \frac{\partial D_1}{\partial p_1} + D_1(p_1,\, p_2) - c\frac{\partial D_1}{\partial p_1} + (p_2 - c)\frac{\partial D_2}{\partial p_1}$$

and

$$\frac{\partial \pi}{\partial p_2} = p_2 \frac{\partial D_2}{\partial p_2} + D_2(p_1,\, p_2) - c\frac{\partial D_2}{\partial p_2} + (p_1 - c)\frac{\partial D_1}{\partial p_2}$$

Since $D_1(\bar{p}+\lambda,\, \bar{p}) \geq D_2(\bar{p},\, \bar{p}+\lambda)$, $\mid \frac{\partial D_1}{\partial p_1}(\bar{p}+\lambda,\, \bar{p}) \mid \leq \mid \frac{\partial D_2}{\partial p_2}(\bar{p},\, \bar{p}+\lambda) \mid$, $c_1 = c_2 = c$ and $\frac{\partial D_1}{\partial p_2} = \frac{\partial D_2}{\partial p_1}$, we see that $\frac{\partial \pi}{\partial p_1}(\bar{p}+\lambda,\, \bar{p}) \geq \frac{\partial \pi}{\partial p_2}(\bar{p},\, \bar{p}+\lambda)$ for all $\bar{p}$, $\lambda \geq 0$.

Hence π satisfies the single crossing condition in p_1 and p_2 by Theorem 3, which implies $p_1 \geq p_2$ by Theorem 2.

Case 2. Now suppose that market 1 is larger than market 2 according to (1) and (3) from above.

Then $\epsilon_1 = \frac{-(\bar{p}+\lambda)\frac{\partial D_1}{\partial p_1}(\bar{p}+\lambda,\bar{p})}{D_1(\bar{p}+\lambda,\bar{p})} \leq \frac{-(\bar{p}+\lambda)\frac{\partial 2}{\partial p_2}(\bar{p},\bar{p}+\lambda)}{D_2(\bar{p},\bar{p}+\lambda)} = \epsilon_2 \leq 1$. Equivalently,

$$\frac{-(\bar{p}+\lambda)\frac{\partial D_1}{\partial p_1}(\bar{p}+\lambda,\bar{p}) - D_1(\bar{p}+\lambda,\bar{p})}{D_1(\bar{p}+\lambda,\bar{p})} \leq \frac{-(\bar{p}+\lambda)\frac{\partial 2}{\partial p_2}(\bar{p},\bar{p}+\lambda) - D_2(\bar{p},\bar{p}+\lambda)}{D_2(\bar{p},\bar{p}+\lambda)} \text{ or}$$

$$\frac{(\bar{p}+\lambda)\frac{\partial D_1}{\partial p_1}(\bar{p}+\lambda,\bar{p}) + D_1(\bar{p}+\lambda,\bar{p})}{D_1(\bar{p}+\lambda,\bar{p})} \geq \frac{(\bar{p}+\lambda)\frac{\partial 2}{\partial p_2}(\bar{p},\bar{p}+\lambda) + D_2(\bar{p},\bar{p}+\lambda)}{D_2(\bar{p},\bar{p}+\lambda)} \geq \frac{(\bar{p}+\lambda)\frac{\partial 2}{\partial p_2}(\bar{p},\bar{p}+\lambda) + D_2(\bar{p},\bar{p}+\lambda)}{D_1(\bar{p}+\lambda,\bar{p})},$$

where the last inequality follows from $D_1(\bar{p}+\lambda, \bar{p}) \geq D_2(\bar{p}, \bar{p}+\lambda)$ when demands are inelastic.

This implies

$$(\bar{p}+\lambda) \frac{\partial D_1}{\partial p_1}(\bar{p}+\lambda, \bar{p}) + D_1(\bar{p}+\lambda, \bar{p}) \geq (\bar{p}+\lambda) \frac{\partial 2}{\partial p_2}(\bar{p}, \bar{p}+\lambda) + D_2(\bar{p}, \bar{p}+\lambda) \qquad (\star)$$

Now consider

$$\frac{\partial \pi}{\partial p_1} = p_1 \frac{\partial D_1}{\partial p_1} + D_1(p_1, p_2) - c \frac{\partial D_1}{\partial p_1} + (p_2 - c) \frac{\partial D_2}{\partial p_1}$$

and

$$\frac{\partial \pi}{\partial p_2} = p_2 \frac{\partial D_2}{\partial p_2} + D_2(p_1, p_2) - c \frac{\partial D_2}{\partial p_2} + (p_1 - c) \frac{\partial D_1}{\partial p_2}$$

Using $(\star)$, $\mid \frac{\partial D_1}{\partial p_1}(\bar{p}+\lambda, \bar{p}) \mid \leq \mid \frac{\partial D_2}{\partial p_2}(\bar{p}, \bar{p}+\lambda) \mid$, $c_1 = c_2 = c$ and $\frac{\partial D_1}{\partial p_2} = \frac{\partial D_2}{\partial p_1}$, we see that $\frac{\partial \pi}{\partial p_1}(\bar{p}+\lambda, \bar{p}) \geq \frac{\partial \pi}{\partial p_2}(\bar{p}, \bar{p}+\lambda)$ for all $\bar{p}, \lambda \geq 0$.

Hence π satisfies the single crossing condition in p_1 and p_2 by Theorem 3, which implies $p_1 \geq p_2$ by Theorem 2.

□

6. Local Comparative Statics

In this section, we consider the problem of when a componentwise ordering is preserved under a positive shock to preferences. That is, according to Definition 2, we will assume that the utility function of player i exhibits increasing differences in own action and a paremeter $t \in T$, where T is assumed to be a partially ordered set. Again, we will first consider the preservation of the single-crossing condition of Definition 4, and then consider the differential case.

We first introduce a slightly technical "comparibility" condition. One interpretation is that it is a restriction on the effect that a positive shock can have on the "lower" component m, in the sense that if a positive devation in the mth component was not beneficial at t, it remains not beneficial at t' as well. This condition is automatically satisfied when positive deviations in the mth component are always beneficial.

Definition 5. Suppose that for player i, $\pi_i : A \times T \to \mathbb{R}$ exhibits increasing differences in (a_i, t), and let $t' \geq t$. Then π_i is comparable in components k and m at (t, t') if for each

$a_{-i} \in A_{-i}$ and $a_i \in A_i$ such that $a_i^k = a_i'' \geq a_i^m = a_i'$,

$$\Delta_m \pi_i(a_i', (a_i'', a_i'), a_i^{-k,m}, a_{-i}, t) < 0 \Rightarrow \Delta_m \pi_i(a_i', (a_i'', a_i'), a_i^{-k\,m}, a_{-i}, t') < 0$$

Theorem 4. *Suppose that for player i, $\pi_i : A \times T \to \mathbb{R}$ exhibits increasing differences in (a_i, t), and let $t' \geq t$ be such that π_i is comparable in components k and m at (t, t'). If π_i satisfies the single-crossing property in components k and m at t, then $k \succeq m$ at t'.*

Proof. Let $a_{-i} \in A_{-i}$ be arbitrary and $a_i \in A_i$ satisfy $a_i^k = a_i'' \geq a_i^m = a_i'$. Then increasing differences in (a_i, t) implies

$$\pi_i\left(\left(a_i'', a_i', a_i^{-k,m}\right), a_{-i}, t'\right) - \pi_i\left(\left(a_i', a_i'', a_i^{-k,m}\right), a_{-i}, t'\right) \geq$$

$$\pi_i\left(\left(a_i'', a_i', a_i^{-k,m}\right), a_{-i}, t\right) - \pi_i\left(\left(a_i', a_i'', a_i^{-k,m}\right), a_{-i}, t\right)$$

which is equivalent to

$$\Delta_k \pi_i((a_i'', a_i'), a_i^{-k,m}, a_{-i}, t') - \Delta_m \pi_i(a_i', (a_i'', a_i'), a_i^{-k,m}, a_{-i}, t') \geq$$

$$\Delta_k \pi_i((a_i'', a_i'), a_i^{-k,m}, a_{-i}, t) - \Delta_m \pi_i(a_i', (a_i'', a_i'), a_i^{-k,m}, a_{-i}, t)$$

Let us label the difference terms (1), (2), (3), and (4), respectively. We must show that for $\varepsilon \geq 0$, $(2) \geq \varepsilon \Rightarrow (1) \geq \varepsilon$.

Suppose that $(2) \geq \varepsilon$. By comparability at (t, t'), this implies that we can set $\varepsilon' = (4) \geq 0$. Since the single-crossing property holds at t, we then have that $(3) \geq \varepsilon'$, or $(3) - (4) \geq 0$. Therefore, $(1) - (2) \geq 0$, or $(1) \geq (2) \geq \varepsilon$, giving the result.

□

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