Optimal Contracts with Random Auditing*

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Abstract

In this paper we study an optimal contract problem under moral hazard in a principal-agent framework where contracts are implemented through random auditing. A random audit is a monitoring instrument which with some nondegenerate probability r reveals the precise action taken by the agent, and otherwise reveals no information. We consider an environment where the cost of agent's action depends on a state of the world that is disclosed only after the contract is signed and that is private information to the agent. We characterize optimal contracts with random auditing under several information structures that allow for moral hazard and adverse selection. We show that a higher intensity of auditing, as measured by r, always increases the value of a contract when the principal can commit to not void a contract if the agent fails an audit, but can decrease the value of a contract when such commitment is not feasible.

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1 Introduction

Previous literature that examined optimal contract problems under moral hazard considered situations where the principal observes some public signal that is imperfectly correlated with the agent's action (usually the effort exerted). This signal is observed with probability 1 and is employed in the contract design to provide incentives to the agent. In this paper we examine an alternative and, to the best of our knowledge, novel scenario, where contracts are implemented through random auditing. A random audit is a monitoring instrument that allows the principal to observe the precise action taken by the agent with some nondegenerate probability r; with probability $1 - r$, the principal does not observe either this action or any signal correlated with it.

Numerous real-world contracting environments can be captured by this modeling specification. For instance, in certain situations, it may be costly or infeasible for the employer of a large workforce to evaluate the contribution of each worker to the quantity or quality of its aggregate output. Instead, the employer can provide incentives with a monitoring scheme that evaluates the output of randomly selected workers. Another typical example is that of an institution providing a service whose quality is determined by its agents' actions. In many such situations, the service provider may not have the capability to obtain feedback from all customers, but only from a sample of them.

The contribution of this paper is twofold. First, we characterize optimal contracts with random auditing under several standard information structures that allow for moral hazard and adverse selection. Second, we examine how the intensity of auditing, as measured by the probability r, impacts the value of an optimal contract. We show that a higher value of r increases the value of a contract if the principal can credibly commit to not void the contract when the agent fails an audit (we say that the agent *fails an audit* if the audit reveals that he exerted an action not allowed by the terms of the contract; otherwise, the agent *passes the audit*). We show then that a higher value of r may decrease the value of a contract if the principal cannot credibly make this promise.

We investigate this optimal contracting problem in an agency framework with several specific modeling choices. First, the cost incurred by the agent from performing his action (the effort level) depends on a state of nature which only the agent observes. Thus, the model combines features of moral hazard, determined by the hidden action of the agent, with adverse selection, determined by the private information that the agent possesses about his cost type. Second, the contract is signed at an ex-ante stage, before the agent learns his type. Contracts with random auditing can also be characterized with a more typical interim contracting assumption; the ex-ante specification we adopt allows examining the impact of the principal's ability to partially insure the agent against unfavorable draws of his type that induce shirking, by commiting to not void the contract when this shirking is detected through an audit. Third, in our baseline specification of the model, we assume that the principal lacks the ability to make such a commitment. Finally, we assume away pre-play communication and examine separately the optimal contract when communication is feasible.

We characterize the optimal contract in this setting and compare it with two benchmarks, the first-best contract under full information, and the contract with moral hazard but no adverse selection. Under full information, auditing plays no role and the contract specifies a constant wage that perfectly insures the agent. If the agent is risk neutral over monetary transfers (more precisely, if his utility function, which is everywhere assumed separable in monetary transfers and cost of effort, is quasilinear in the transfers), then the first-best contract can be implemented under asymmetric information across all model specifications. This implies that with a risk neutral agent, the intensity of auditing has no impact on the value of the optimal contract. On the other hand, if the agent is risk averse and there is moral hazard, the principal needs to solve the usual trade-off between incentives and risk. Contracts under moral hazard specify a constant wage to be paid to the agent when no audit is performed, which we refer to as a *salary*, and type- or action-contingent wages for situations when an audit is performed. When the agent's cost type is also observable ex-post with an audit, the action-and-type-contingent wage promised if the agent passes the audit is higher than the salary for *some* types - these types receive a reward when they pass an audit - while for the remaining types, it equals their salary. When the type is not observable ex-post, the action-contingent wage is higher than the salary for some actions allowed by the terms of the contract, but lower than it for others. Incentive provision in the presence of adverse selection may thus require that the agent be penalized when audited even if he passes the audit.

The second goal of the paper is to examine how the intensity of auditing impacts the value of a contract and the role played in this context by a credible commitment of the principal to make payments even when the agent fails an audit. While we expect that commitment is valuable for the principal, i.e., its availability should result in a weakly higher value of the contract, it is less clear a priori how the presence of commitment affects the relationship between the value of the contract and the intensity of auditing.

For all information structures that we consider, if it is optimal to induce the agent to exert effort under all possible cost types, implying that an audit is never failed, a higher value of r increases the value of the contract. On the other hand, when an optimal contract induces shirking for some types, more frequent auditing can reduce the value of this contract if the principal cannot make that commitment. This occurs when r is high and thus the agent is likely to be detected when shirking, requiring the principal to pay a large risk premium ex-ante. If the principal can commit, he avails of this tool to reduce the dispersion in the set of ex-post possible wages faced by the agent and thus to lower the risk premium that needs to be paid. In this case, the increased power of incentive determined by the higher probability of auditing renders again the value of the contract be everywhere increasing in r. We conclude that when an employee cannot respond to high-cost realizations by either adjusting the effort level exerted or shirking, a high frequency of auditing may be suboptimal for the employer.

As an extension, we also examine optimal contracts with random auditing when communication is feasible. In this case, the principal can require the agent to declare his private information after he learns it, but before it is determined whether an audit is performed or not. This information can then be employed to adjust the wage paid when an audit is not performed. We show that, as in a situation with no adverse selection, and unlike the case where communication is not feasible, the agent is never penalized when an audit is performed provided that he passes it. Instead, when an audit is passed, some agent types receive a reward, while other types, for which the salary is sufficient to provide incentives to exert effort, receive only their salary.

Our paper contributes to two main streams of economic literature. First, it contributes to the literature on the design of optimal contracts under stochastic monitoring. Second, the paper contributes to the literature that examines situations with both moral hazard and adverse selection. [To be completed].

The framework is introduced in section 2. In section 3 we characterize the optimal contracts with random auditing, including the two benchmarks corresponding to the cases of complete information and of pure moral hazard, respectively. This section is also where we examine the impact that the probability of auditing has on the value of a contract and the role of commitment. In section 4, we study optimal contracts with communication. Section 5 concludes.

2 The Framework

There are two players, a principal ($\mathcal{P}$) and an agent ($\mathcal{A}$). $\mathcal{P}$ owns a firm and offers $\mathcal{A}$ a contract to work for this firm in exchange for monetary compensation. $\mathcal{A}$ can accept or reject the contract. If $\mathcal{A}$ accepts it and exerts effort e in service of the firm during the period of the contract, he produces an output whose value is $y(e)$, where $y'(\cdot) > 0$ and $y''(\cdot) \leq 0$. This output is entirely appropriated by $\mathcal{P}$. $\mathcal{P}$ is risk neutral, and thus his payoff when $\mathcal{A}$ exerts effort e and is paid a wage w is $y(e) - w$. $\mathcal{A}$'s preferences are separable in wages and effort, and are represented by a utility $u(w) - c(s, e)$. The function $u : \mathbb{R} \to \mathbb{R}$ captures $\mathcal{A}$'s preferences over net monetary transfers; we assume $u'(\cdot) > 0$ and $u''(\cdot) \leq 0$, and normalize $u(0) = 0$.[1] The cost for $\mathcal{A}$ of exerting effort e is $c(s, e)$, where (i) s is a random variable that takes values in $[\underline{s}, \overline{s}] \subset \mathbb{R}$, with a continuous density function $f(\cdot) > 0$, and (ii) $c(\cdot, \cdot)$ is a function with $c(s, 0) = 0$, $c_e > 0$, $c_{ee} > 0$, $c_s > 0$ and $c_{es} > 0$, for all $s \in [\underline{s}, \overline{s}]$ and $e \geq 0$. In the following, as standard in the literature, we frequently refer to s as $\mathcal{A}$'s type. $\mathcal{A}$ does not know his type at the time when he is presented with the contract, but upon accepting the contract, he observes it before choosing the effort level. The utility of $\mathcal{A}$'s outside option is $\overline{u}$. The functions $y(\cdot)$, $u(\cdot)$ and $c(\cdot, \cdot)$ are assumed to be twice continuously differentiable. Finally, we assume that the set of feasible effort levels is compact or otherwise that $y(\cdot)$ is bounded on $\mathbb{R}_+$.

$\mathcal{P}$ does not directly observe the effort e exerted by $\mathcal{A}$ or the output $y(e)$.[2] Instead, he owns a monitoring instrument which allows observing e with probability $r \in (0, 1)$. With probability $1 - r$, $\mathcal{P}$ does not observe either e or any signal correlated with e. Since in this paper we aim to examine how the intensity of auditing impacts an optimal contract, we consider the value of r to

[1] To avoid dealing with corner solutions, we assume a set of parameters of the model such that in the optimal contracts the net monetary transfers to $\mathcal{A}$ are positive. For instance, this is the case when $u'(0) = +\infty$ and u_0 high enough. This allows forgoing the need to perform the more tedious and not necessarily more interesting analysis of an optimization problem with a non-negativity constraint on these transfers that may bind.

[2] In line with the motivating examples from Introduction, we assume that $\mathcal{P}$ employs a large number of agents, and that while he may observe an aggregate output, this carries virtually no information of an individual's contribution.

be exogenous and public information. Auditing is random and $\mathcal{A}$ does not know at the time when he chooses the effort level whether or not $\mathcal{P}$ will observe it. At the end of the contract period, it is public information whether or not $\mathcal{P}$ performed the audit and the effort level e if an audit was performed. Unless specified otherwise, we assume that if a contract defines a set of allowed effort levels for $\mathcal{A}$, and $\mathcal{P}$ acquires evidence through an audit that $\mathcal{A}$'s effort level is not in this set, $\mathcal{P}$ can void the contract and make no payment; $\mathcal{P}$ cannot credibly promise ex-ante not to do so.

$\mathcal{P}$ can offer contracts with wage schedules that are defined contingent on all observables.[3] More precisely, $\mathcal{P}$ can offer a contract of the form $\left\{E, w^n, \{w(e)\}_{e\in E}\right\}$, where (i) $E \subset \mathbb{R}_+$ is a set of allowed effort levels, (ii) w^n is the wage paid to $\mathcal{A}$ if no audit is performed, and (iii) $w(e)$ is the wage paid if an audit is performed and it reveals that $\mathcal{A}$ exerted effort level $e \in E$. One can think of w^n as a *salary*, or base wage, offered to the worker as long as he is not caught shirking, and of the difference $w(e) - w^n$ as a wage adjustment performed when an audit is performed and $\mathcal{A}$ passes it. As we show later, under certain information structures, this wage adjustment is nonnegative, i.e, it constitutes a *reward*, but under others, it may be negative for low levels of effort.[4]

We complete the presentation of the framework with several observations. First, we note that the contract defined above is designed on contingencies determined strictly by ex-post observable outcomes. This implicitly assumes away pre-play communication. However, in principle, $\mathcal{P}$ could also offer a contract of the type $\{e(s), w(s), w^n(s)\}_{s\in[\underline{s},\bar{s}]}$, which requires an explicit disclosure of s after $\mathcal{A}$ learns it, but before he is informed whether or not an audit is performed. $\mathcal{P}$ would then employ this message to adjust the wage paid when there is no audit and thus no observable action. While situations with pre-play communication do frequently emerge in real world, in many other

[3] Two assumptions are made here. First, when an audit is performed, $\mathcal{P}$ obtains publicly verifiable evidence of $\mathcal{A}$'s effort level. Second, $\mathcal{P}$ can credibly promise different wages depending on whether or not he performs the audit.

[4] While not explicitly modeled here, one can think of this game as a stage play of a repeated game and of the wages defined in the contract as promised continuation values to the agent under various contingencies. We are currently working on a dynamic version of this model where these specifications are explicitly modelled.

employment situations, auditing is selected as a monitoring instrument precisely so as to reduce the administrative burden. In this case, requiring all workers to disclose their private information, or equivalently to select a contract out of a menu, may be administratively demanding and infeasible. We therefore focus most of the analysis on contracts without communication, and we characterize separately in section 4, as an extension, the optimal contract when communication is feasible.[5]

Second, we assumed for simplicity that the lower bound is zero on the set of effort levels that $\mathcal{A}$ may exert and yet not be detected without an audit. This modeling specification can be modified at the cost of adding some slight complications to have a positive lower bound on this set, and thus to allow capturing more realistic situations where workers cannot "shirk in plain view".

Finally, we assumed that $\mathcal{P}$ can perfectly measure $\mathcal{A}$'s effort with an audit. One can relax this and assume that an audit only reveals a signal correlated with the effort, as in standard moral hazard problems. Thus, our model can be thought of a particular case of a generic principal-agent model where $\mathcal{P}$ only observes a signal informative of $\mathcal{A}$'s action with a nondegenerate probability.

3 Analysis

As benchmarks, we derive first the optimal contracts under two alternative scenarios to the richer model introduced above. First, we elicit the efficient outcome in this framework by examining the case of full information, i.e., with no adverse selection or moral hazard, where both $\mathcal{A}$'s type s and action e are contractible upon. Second, we consider the case of no adverse selection, i.e., with pure moral hazard, where $\mathcal{A}$'s type is observable ex-post when an audit is performed and thus also contractible upon. In both models we maintain our assumption of an ex-ante participation

[5] See, for instance, Melumad and Reichelstein (1989) for a discussion on the value of communication in agencies.

constraint for $\mathcal{A}$.[6] Finally, to simplify the exposition, we focus when studying both benchmarks on the case where it is profitable for $\mathcal{P}$ to induce all types of $\mathcal{A}$ to exert effort, and consider the general case where this assumption is dropped when studying the full-fledged model with moral hazard and adverse selection.

3.1 The Full-Information Benchmark

When $\mathcal{P}$ observes ex-post both $\mathcal{A}$'s type and the effort he exerted, auditing plays no role. $\mathcal{P}$ thus offers a contract $\{e_0(s), w_0(s)\}_{s\in[\underline{s},\overline{s}]}$, where (i) $e_0(s)$ is the effort required from type s, and (ii) $w_0(s)$ is the wage promised to type s in exchange.[7] The only constraint that $\mathcal{P}$ faces is $\mathcal{A}$'s participation constraint, so the optimal contract under full information is the solution to the problem

$$\max_{\{e(s),w(s)\}_{s\in[\underline{s},\overline{s}]}} \int_{\underline{s}}^{\overline{s}} \left[y\left(e(s)\right) - w(s)\right] f(s)ds \tag{1}$$

$$\text{s.t.} \quad \int_{\underline{s}}^{\overline{s}} \left[u\left(w(s)\right) - c(s, e(s))\right] f(s)ds \geq \overline{u} \tag{2}$$

The next proposition, whose proof is straightforward and thus omitted, elicits the conditions defining the corresponding optimal contract when $\mathcal{A}$ is risk averse.[8]

Proposition 1 *Assume $u''(w) < 0$ for all w. The optimal contract under full information is determined by $w_0(s) = w_0$ for all $s \in [\underline{s}, \overline{s}]$ and some $w_0 \in \mathbb{R}$, (2) satisfied with equality, and*

$$\frac{1}{u'(w_0)} c_e(s, e_0(s)) = y'\left(e_0(s)\right) \tag{3}$$

[6] Since random auditing plays a role only under moral hazard, we forgo discussing the less interesting benchmark with adverse selection but no moral hazard.

[7] It is implicitly assumed here and in the rest of the paper that such a contract is binding for $\mathcal{A}$ after he learns his type s, and thus we do not need to account for $\mathcal{A}$'s participation constraint at an interim stage.

[8] By the boundedness of $y(\cdot)$ and the continuity of the relevant functions, an optimal contract exists. Moreover, this is unique up to a set of zero measure.

Since providing $\mathcal{A}$ with incentives to exert effort or reveal information is unnecessary, $\mathcal{P}$ insures $\mathcal{A}$ and offers a constant wage across all states. The condition in (3) equates the marginal cost and marginal benefit for $\mathcal{P}$ of implementing effort. An additional amount of effort Δe increases type s's cost by $c_e(s, e(s))\Delta e$; to compensate it, $\mathcal{P}$ has to increase $\mathcal{A}$'s wage by $\frac{1}{u'(w)} c_e(s, e(s))\Delta e$. Since the return for $\mathcal{P}$ from $\mathcal{A}$'s additional effort is $y'(e(s))\,\Delta e$, $\mathcal{P}$ sets $e_0(s)$ so as to satisfy (3).

Note that since $e_0(s)$ varies across types (it decreases in s), the utility delivered ex-post to each type, $u(w_0) - c(s, e_0(s))$, varies as well (the effect of s on this utility is ambiguous), implying that some types of $\mathcal{A}$ enjoy ex-post more than their reservation utility $\overline{u}$, while others less.

Finally, it is straightforward to see that when $\mathcal{A}$ is risk neutral, the optimal effort schedule is determined by $c_e(s, e_0(s)) = y'(e_0(s))$, and that any wage schedule $\{w_0(s)\}_{s\in[\underline{s},\overline{s}]}$ satisfying $\int_{\underline{s}}^{\overline{s}} w_0(s) f(s) ds = \overline{u} + \int_{\underline{s}}^{\overline{s}} c(s, e_0(s)) f(s) ds$ is optimal.

3.2 The Pure Moral Hazard Benchmark

When an audit reveals both the effort level exerted by $\mathcal{A}$ and his type s,[9] $\mathcal{P}$ offers a contract $\left\{\{e_1(s), w_1(s)\}_{s\in[\underline{s},\overline{s}]}, w_1^n\right\}$ where for each type s, (i) $e_1(s)$ is the effort required, (ii) $w_1(s)$ is the wage paid if an audit reveals that $\mathcal{A}$ exerted at least effort $e_1(s)$, and (iii) w_1^n is the salary, paid if no audit is performed. Since $c_e > 0$, type s of $\mathcal{A}$ exerts either effort $e_1(s)$ or no effort. To implement $e_1(s)$, the contract must thus satisfy the incentive compatibility condition $ru(w_1(s)) + (1-r)\,u(w_1^n) - c\,(s, e_1(s)) \geq (1-r)\,u(w_1^n)$, for any $s \in [\underline{s}, \overline{s}]$. The optimal contract then solves the

[9] An alternative specification of a model with no adverse selection is one where the type s is observable ex-post *without* an audit, i.e., with probability 1. In line with the discussion from section 2, we focus in this paper on modeling situations where it is infeasible for the principal to acquire on a regular basis information about all employees, be that their effort level or their cost type. We therefore chose the modeling specification as defined above.

problem

$$\max_{w^n,\{e(s),w(s)\}_{s\in[\underline{s},\overline{s}]}} \int_{\underline{s}}^{\overline{s}} [y\,(e(s)) - rw(s)]\, f(s)ds - (1-r)\,w^n \tag{4}$$

$$\text{s.t. } ru(w(s)) - c(s, e(s)) \geq 0 \text{ for all } s \in [\underline{s}, \overline{s}] \tag{5}$$

$$\int_{\underline{s}}^{\overline{s}} [ru\,(w(s)) - c\,(s, e\,(s))]\, f(s)ds + (1-r)\,u\,(w^n) \geq \overline{u} \tag{6}$$

The following proposition considers the case when $\mathcal{A}$ is risk neutral. The result follows immediately from the fact that there exists a wage schedule $\left\{\{w_1(s)\}_{s\in[\underline{s},\overline{s}]}\,, w_1^n\right\}$ that implements the full-information effort schedule $\{e_0(s)\}_{s\in[\underline{s},\overline{s}]}$, while delivering $\mathcal{A}$ the same ex-ante expected wage as under full information.[10] Therefore, if $\mathcal{A}$ is risk neutral, $\mathcal{P}$ can attain the same payoff as under full information. This payoff is independent of the intensity of auditing.

Proposition 2 *Assume that $u(w) = w$, for all w. Then $e_1(s) = e_0(s)$, for all $s \in [\underline{s}, \overline{s}]$. Moreover, the value of the optimal contract equals that under full information.*

The next proposition, whose proof is in appendix A1, elicits the conditions that determine the optimal contract when $\mathcal{A}$ is risk averse, and the effect of r on the value of this contract.[11]

Proposition 3 *Assume that $u''(w) < 0$, for all w. Also, assume that it is optimal for $\mathcal{P}$ to induce all types of $\mathcal{A}$ to exert effort. The optimal contract under moral hazard is then determined by (6)*

[10] For instance, the wage schedule defined by $w_1(s) = \frac{1}{r}c(s, e_0(s))$ for all s, and $w_1^n = \frac{\overline{u}}{1-r}$ satisfies (5) and (6), and thus implements $\{e_0(s)\}_{s\in[\underline{s},\overline{s}]}$. Moreover, $\int_{\underline{s}}^{\overline{s}} [rw_1(s) + (1-r)\,w_1^n]\, f(s)ds = \overline{u} + \int_{\underline{s}}^{\overline{s}} [c(s, e_0(s))]\, f(s)ds$. Thus $\mathcal{A}$'s expected wage equals that under full information implying that this contract is optimal since its value attains the theoretical upper bound, the value under full information.

[11] Given proposition 3, the optimal contract can be computed in principle as follows. First, (8) and a binding (5) determine the pairs $(w_1(s), e_1(s))$ for any s in the set on which $w_1(s) > w_1^n$; clearly, this set depends on w_1^n. On the other hand, (8) determines $e_1(s)$ for s with $w_1(s) = w_1^n$ also as a function of w_1^n. Substituting these into the binding constraint from (6) determines w_1^n, and then the rest of the contract.

satisfied with equality, (5), and for all $s \in [\underline{s}, \overline{s}]$,

$$w_1(s) - w_1^n \geq 0, \textit{ and } = 0 \textit{ whenever } ru(w_1(s)) - c(s, e_1(s)) > 0 \tag{7}$$

$$\frac{1}{u'(w_1(s))} c_e(s, e_1(s)) = y'(e_1(s)) \tag{8}$$

The value of the optimal contract is increasing in r.

The participation constraint in (6) is satisfied with equality; otherwise $\mathcal{P}$ can reduce the salary w_1^n without affecting (5). (8) equates again, for each type s, the marginal benefit and marginal cost for $\mathcal{P}$ of implementing additional effort. To understand (7), note that since $\mathcal{A}$ is risk averse, $\mathcal{P}$ aims to minimize the wage risk imposed on $\mathcal{A}$, subject to providing the right incentives. If, contrary to (7), $w_1(s) < w_1^n$, then $\frac{1}{u'(w_1(s))} < \frac{1}{u'(w_1^n)}$ and thus the cost for $\mathcal{P}$ of delivering additional utility to $\mathcal{A}$ is lower when done by means of increasing $w_1(s)$ than by that of w_1^n; therefore, $\mathcal{P}$ can increase $w_1(s)$ and decrease w_1^n so that (6) continues to be satisfied but with a lower expected wage paid. On the other hand, when $w_1(s)$ is set higher than w_1^n, the corresponding risk is imposed on $\mathcal{A}$ so as to create incentives to exert effort; this is again done with a minimum variance in wages, and therefore the incentive constraint in (5) binds, as stated by (7).

As proposition 3 suggests, the optimal contract under moral hazard and no adverse selection essentially specifies a salary w_1^n to be paid to all types s as long as $\mathcal{A}$ is not caught shirking, and a reward $w_1(s) - w_1^n > 0$ offered to some types when they pass an audit. As we show in appendix A1, the effort schedule $e_1(s)$ is decreasing, the wage schedule is generically non-monotonic, but the surplus generated by different types, $y(e(s)) - rw(s) - (1-r)w^n$, is decreasing in s. In terms of ex-post experienced utility, types s with $w_1(s) > w_1^n$ enjoy less than their reservation utility, while some of the remaining types enjoy more.[12]

3.3 The Model with Moral Hazard and Adverse Selection

It is straightforward to see that whenever it is optimal for $\mathcal{P}$ to induce some type s to exert effort, it must be optimal to also do so for all lower cost types. We consider therefore contracts where $\mathcal{P}$ induces types $s \in [\underline{s}, \widehat{s}]$ to exert effort, with the threshold $\widehat{s} \in [\underline{s}, \overline{s}]$ optimally chosen by $\mathcal{P}$.

Principal's Problem By the Revelation Principle, one can think of $\mathcal{P}$'s problem as that of selecting an optimal contract $\left\{ \widehat{s} \in [\underline{s}, \overline{s}], \{e_*(s), w_*(s)\}_{s \in [\underline{s}, \widehat{s}]}, w_*^n \right\}$ that extracts $\mathcal{A}$'s private information from types in the set $[\underline{s}, \widehat{s}]$, induces types in $[\underline{s}, \widehat{s}]$ to work, and the types in $(\widehat{s}, \overline{s}]$ to shirk. The optimal contract is thus the solution to the problem

$$\max_{\widehat{s}, \{e(s), w(s)\}_{s \in [\underline{s}, \widehat{s}]}, w^n} \int_{\underline{s}}^{\widehat{s}} [y(e(s)) - rw(s)] f(s) ds - (1 - r) w^n \tag{9}$$

$$\text{s.t. } s \in \arg\max_{\widetilde{s} \in [\underline{s}, \widehat{s}]} [ru(w(\widetilde{s})) - c(s, e(\widetilde{s}))], \text{ for all } s \in [\underline{s}, \widehat{s}] \tag{10}$$

$$ru(w(s)) - c(s, e(s)) \geq 0, \text{ for all } s \in [\underline{s}, \widehat{s}] \tag{11}$$

$$\max_{\widetilde{s} \in [\underline{s}, \widehat{s}]} [ru(w(\widetilde{s})) - c(s, e(\widetilde{s}))] \leq 0, \text{ for all } s \in (\widehat{s}, \overline{s}] \tag{12}$$

$$\int_{\underline{s}}^{\widehat{s}} [ru(w(s)) - c(s, e(s))] f(s) ds + (1 - r) u(w^n) \geq \overline{u}. \tag{13}$$

where (10) is the incentive compatibility condition that induces types in $[\underline{s}, \widehat{s}]$ to truthfully reveal themselves, while (12) induces types in $(\widehat{s}, \overline{s}]$ to shirk rather than exert an effort level specified for one of the types in $[\underline{s}, \widehat{s}]$. It deserves mentioning here that while (10) is a somewhat standard

[12] To see this, note that the expected utility delivered to type s, i.e., $ru(w_1(s)) + (1 - r) u(w_1^n) - c(s, e_1(s))$ equals $(1 - r) u(w_1^n)$ for s with $w_1(s) > w_1^n$ and (5) binding, and is higher than $(1 - r) u(w_1^n)$ for the rest. Since on average the utility experienced by $\mathcal{A}$ is $\overline{u}$, it must be that $(1 - r) u(w_1^n) < \overline{u}$.

incentive compatibility condition under adverse selection, the specific form of (11) and (12) differs from other types of incentive compatibility constraints under moral hazard from the literature and is due to the particular type of monitoring technology (random auditing) that we examine here.

The following lemma implies that we can replace (11) and (12) with the weaker condition from (14) in the above problem.

Lemma 4 *Any contract that satisfies (10), will satisfy (11) and (12) if and only if*

$$ru\left(w(\widehat{s})\right)-c\left(\widehat{s},e\left(\widehat{s}\right)\right)\geq 0,\ \textit{and}\ =0\ \textit{whenever}\ \widehat{s}<\overline{s}. \tag{14}$$

Proof. Consider first the case when $\widehat{s}<\overline{s}$. We assume throughout that (10) is satisfied and start by showing that then (14) implies (11) and (12). Note first that $ru\left(w(s)\right)-c\left(s,e\left(s\right)\right)\geq ru\left(w(\widehat{s})\right)-c\left(s,e\left(\widehat{s}\right)\right)\geq ru\left(w(\widehat{s})\right)-c\left(\widehat{s},e\left(\widehat{s}\right)\right)$ for all $s\in[\underline{s},\widehat{s}]$, where the first inequality is implied by (10) and the second by $s\leq\widehat{s}$. Therefore, (14) implies (11). Next, since whenever $s\geq\widehat{s}$, we have $ru\left(w(\widetilde{s})\right)-c\left(s,e\left(\widetilde{s}\right)\right)\leq ru\left(w(\widetilde{s})\right)-c\left(\widehat{s},e\left(\widetilde{s}\right)\right)$ for any $\widetilde{s}\in[\underline{s},\widehat{s}]$, it follows that $\max\limits_{\widetilde{s}\in[\underline{s},\widehat{s}]}ru\left(w(\widetilde{s})\right)-c\left(s,e\left(\widetilde{s}\right)\right)\leq\max\limits_{\widetilde{s}\in[\underline{s},\widehat{s}]}ru\left(w(\widetilde{s})\right)-c\left(\widehat{s},e\left(\widetilde{s}\right)\right)=ru\left(w(\widehat{s})\right)-c\left(\widehat{s},e\left(\widehat{s}\right)\right)$, where the equality follows from (10). Thus, (14) implies (12). For the converse, note that (11) immediately implies $ru\left(w(\widehat{s})\right)-c\left(\widehat{s},e\left(\widehat{s}\right)\right)\geq 0$. Assuming by contradiction that $ru\left(w(\widehat{s})\right)-c\left(\widehat{s},e\left(\widehat{s}\right)\right)>0$, by the continuity of $c\left(\cdot\right)$ in s, there exists $\varepsilon>0$ such that for all $s\in(\widehat{s},\widehat{s}+\varepsilon)$, we have $ru\left(w(\widehat{s})\right)-c\left(s,e\left(\widehat{s}\right)\right)>0$, which contradicts (12). When $\widehat{s}=\overline{s}$, then (12) is automatically satisfied, while from the above argument, it is clear that (11) is satisfied if and only if $ru\left(w(\widehat{s})\right)-c\left(\widehat{s},e\left(\widehat{s}\right)\right)\geq 0$. □

To solve for the optimal contract, we employ the standard First-Order Approach. Lemma 5 validates this approach in the current framework by showing the equivalence between the incentive compatibility of a contract with respect to truthful type revelation, on the one hand, and the first order condition of $\mathcal{A}$'s problem in (10), plus the monotonicity of the effort schedule $e(s)$, on the other. Its proof, which builds on a standard strategy in the literature, is presented in appendix A2.

Lemma 5 *A contract induces truthful type revelation for all* $s \in [\underline{s}, \widehat{s}]$ *if and only if*

$$e'(s) \leq 0 \text{ } a.e. \text{ } s \in [\underline{s}, \widehat{s}] \tag{15}$$

$$ru'(w(s))\, w'(s) = c_e(s, e(s))e'(s) \text{ } a.e. \text{ } s \in [\underline{s}, \widehat{s}] \tag{16}$$

To keep the exposition in the main text simple we will ignore the monotonicity constraint in (15) and solve the relaxed problem, as defined by (9), (13), (14) and (16). We present the analysis of the optimal contract problem for the case when the monotonicity constraint from (15) binds in appendix A6. Note also here that (15) and (16) imply that in any incentive compatible contract $w'(s) \leq 0$ (and also $w'(s) < 0$ whenever $e'(s) < 0$).

Finally, also note that the participation constrain in (13) must bind at optimum since otherwise w^n can be reduced to strictly increase the value of the contract. We consider therefore in the following that (13) is satisfied with equality.

Before analyzing the generic case where $\mathcal{A}$ is risk averse, we present proposition 6, which states that if $\mathcal{A}$ is risk neutral, the full-information payoff is attainable by $\mathcal{P}$. For this result, we restrict attention again the simpler case where in a full-information setting, it is optimal for $\mathcal{P}$ to implement positive effort for all types. The proof of the proposition is in appendix A3.[13] An immediate consequence of this result is that the intensity of auditing is again inconsequential for $\mathcal{P}$'s payoff if $\mathcal{A}$ is risk neutral.

[13] Implicitly used in the proof of this result is the assumption that the non-negativity constraint on wages never binds. With limited liability of the agent and binding constraints, the result of proposition 6 no longer holds.

Proposition 6 *Assume that* $u(w) = w$*, for all* w*, and that* $e_0(s) > 0$*, for all* $s \in [\underline{s}, \overline{s}]$*. Then* $e_*(s) = e_0(s)$*, for all* $s \in [\underline{s}, \overline{s}]$*. The value of the optimal contract equals that under full information.*

Optimal Control Approach To solve $\mathcal{P}$'s problem for the case when $\mathcal{A}$ is risk averse, we employ methods from optimal control theory. We first recast the problem in terms of induced utilities $u^n \equiv u(w^n)$ and $u(s) \equiv u(w(s))$, for $s \in [\underline{s}, \widehat{s}]$; these utilities will replace the respective contingent wages as $\mathcal{P}$'s choice variables. By denoting the inverse utility function $h \equiv u^{-1}$, defined on the range of the function u, we have that $w^n = h(u^n)$ and $w(s) = h(u(s))$.[14] Under these transformations, (10) becomes $s \in \arg\max_{\widetilde{s} \in [\underline{s}, \widehat{s}]} [ru(\widetilde{s}) - c(s, e(\widetilde{s}))]$, and so, under the First Order Approach, the incentive compatibility condition in (16) is $ru'(s) = c_e(s, e(s))\, e'(s)$. Given this, the control variable in the optimal control problem is $x(s) \equiv e'(s)$, while the state variables are $e(s)$ and $u(s)$. In addition, to account for the participation condition in (13), we introduce a new state variable

$$v(s) \equiv \int_{\underline{s}}^{s} [ru(\sigma) - c(\sigma, e(\sigma))]\, f(\sigma) d\sigma \tag{17}$$

and rewrite the binding constraint in (13) as the transversality condition $v(\widehat{s}) = \overline{u}^n \equiv \overline{u} - (1 - r)\, u^n$. The other transversality condition on v is $v(\underline{s}) = 0$. There are no transversality conditions on the remaining two state variables, e and u.

We solve for the optimal contract in two steps. First, for any fixed value of u^n, we solve an optimal control problem where the decision variables are $\widehat{s}$ and $\{x(s)\}_{s \in [\underline{s}, \widehat{s}]}$. In the second step, we maximize the resulting optimal value function with respect to u^n, as a standard static optimization

[14] This transformation requires the additional assumption that for every effort level there exists a wage such that the participation constraint of the agent is satisfied (see Bolton and Dewatripont (2005) pp. 154.)

problem. The optimal control problem in the first step is

$$\max_{\widehat{s}\in[\underline{s},\overline{s}],\{x(s)\}_{s\in[\underline{s},\widehat{s}]}} \int_{\underline{s}}^{\widehat{s}} \left[y\left(e(s)\right) - rh\left(u(s)\right)\right] f(s)ds \tag{18}$$

$$\text{s.t. } e'(s) = x(s) \tag{19}$$

$$u'(s) = \frac{1}{r} c_e\left(s, e\left(s\right)\right) x\left(s\right) \tag{20}$$

$$v'(s) = \left[ru\left(s\right) - c\left(s, e\left(s\right)\right)\right] f(s) \tag{21}$$

$$v\left(\underline{s}\right) = 0;\ v\left(\widehat{s}\right) = \overline{u}^n \tag{22}$$

$$ru\left(\widehat{s}\right) - c\left(\widehat{s}, e\left(\widehat{s}\right)\right) \geq 0 \text{ and } = 0 \text{ if } \widehat{s} < \overline{s} \tag{23}$$

Current existence theorems for solutions of optimal control problems do not yield the complete set of properties of the solution to the above problem required in the ensuing analysis. We therefore make the following assumption throughout. The superscript $\widehat{s}$ elicits the fact that the respective trajectory is the solution corresponding to a given cutoff $\widehat{s}$. Part (i) of the assumption can alternatively be derived from some additional sufficient conditions on c_{ee} and c_{es}.[15] Part (ii) ensures that the solution for the optimal cutoff $\widehat{s}$ is determined by the standard in the literature (equality) condition from (33). Part (iii) ensures that u^n is determined optimally by a standard first-order condition. Before presenting the assumption, we introduce some new notation.

Definition 7 *We say a function is $\mathbb{C}_p^{(1)}$ if it is continuous and piecewise continuously differentiable.*

[15] The two Filipov-Cesari type existence theorems that could potentially be applied to a situation where the Hamiltonian is linear in the control variable and the control has an unbounded support are presented in section 11.C on page 392 in Cesari (1983). Theorem 11.4.vii does not apply as stated since none of the growth conditions are satisfied. However, these growth conditions are employed in the proof of the theorem to conclude that the value function of the corresponding optimization problem is bounded. In our case, for any fixed value of $\widehat{s}$, the boundedness follows from that fact that the value is lower than that of the relaxed problem where conditions (19), (20) and (23) are dropped and $r = 1$, i.e., by the value of the contract under full information, which is finite. The theorem can then be applied under the additional assumptions that c_{ee} and c_{es} are bounded, which are used to infer the required properties on what the theorem in Cesari (1983) denotes by $A_0(t, x)$ and $B(t, x)$.

Assumption 8 (Existence and Smoothness) *(i) For any fixed $\widehat{s}$, there exists a solution to (18)-(23) with the corresponding state variables $\left\{e^{\widehat{s}}(s), u^{\widehat{s}}(s)\right\}_{s\in[\underline{s},\widehat{s}]}$ being $\mathbb{C}_p^{(1)}$ functions of s. (ii) The functions $\widehat{s} \to e^{\widehat{s}}(s)$ and $\widehat{s} \to u^{\widehat{s}}(s)$ are $\mathbb{C}_p^{(1)}$ for all $s \in [\underline{s}, \overline{s}]$. (iii) The optimal value of problem (18)-(23) is continuously differentiable as a function of $\overline{u}^n$.*

The Hamiltonian associated with the problem (18)-(23) is

$$H_*\left(e, u, v, x, \lambda_1, \lambda_2, \lambda_3, s\right) \equiv \left[y\left(e\right) - rh\left(u\right)\right] f(s) + \lambda_1 x + \lambda_2 \frac{1}{r} c_e\left(s, e\right) x + \lambda_3 \left[ru - c\left(s, e\right)\right] f(s) \quad (24)$$

Since this Hamiltonian is linear in the control variable x, while the domain of x is unbounded,[16] a solution to this problem necessarily involves a so-called *singular control*, i.e., it must satisfy $\frac{\partial H_*}{\partial x} = 0$ for all s.[17] By the Pontryagin's Maximum Principle,[18] there exist $\mathbb{C}_p^{(1)}$ functions $\lambda_1(s)$, $\lambda_2(s)$ and $\lambda_3(s)$, and a scalar μ, such that the following conditions are necessarily satisfied at the optimum

[16] Recall that we are solving the relaxed problem where we drop the monotonicity condition in (15) and thus the domain is $\mathbb{R}$. If instead we incorporate that condition, the solution may involve a so-called bang-singular-bang control, with $\frac{\partial H_*}{\partial x} = 0$ when $x(s) < 0$, and $\frac{\partial H_*}{\partial x} > 0$ when $x(s) = 0$. See appendix A6 for the details.

[17] See, for instance, page 247 in Bryson and Ho (1975) for a discussion of singular controls on unbounded domains. In regards to that discussion, note that since in our problem there are no initial or terminal conditions on the state variables e and u, which are those affected by the control x, the optimal control will *not* require Dirac function impulses at $\underline{s}$ or $\overline{s}$ meant to generate jumps to the singular solution.

[18] See Theorem 4.2 on page 81 in Caputo (2005) for a more standard version of this result, or Theorem 1 on page 178 in Seierstad and Sydsaeter (1987) for a version that accounts for the state constraint at $\widehat{s}$ in (23).

solution of problem (18)-(23).[19]

$$\frac{\partial H_*}{\partial x} = \lambda_1(s) + \lambda_2(s)\frac{1}{r}c_e(s, e(s)) = 0 \tag{25}$$

$$\lambda_1'(s) = -\frac{\partial H_*}{\partial e} = -y'(e)\, f(s) - \lambda_2(s)\frac{1}{r}c_{ee}(s, e(s))\, x(s) + \lambda_3(s)\, c_e(s, e(s))\, f(s) \tag{26}$$

$$\lambda_2'(s) = -\frac{\partial H_*}{\partial u} = rh'(u(s))f(s) - \lambda_3(s)\, rf(s) \tag{27}$$

$$\lambda_3'(s) = -\frac{\partial H_*}{\partial v} = 0 \tag{28}$$

$$\lambda_1(\underline{s}) = 0;\ \lambda_1(\widehat{s}) = \frac{\partial}{\partial e(\widehat{s})}\mu\left[ru(\widehat{s}) - c(\widehat{s}, e(\widehat{s}))\right] = -\mu c_e(\widehat{s}, e(\widehat{s})) \tag{29}$$

$$\lambda_2(\underline{s}) = 0;\ \lambda_2(\widehat{s}) = \frac{\partial}{\partial u(\widehat{s})}\mu\left[ru(\widehat{s}) - c(\widehat{s}, e(\widehat{s}))\right] = \mu r \tag{30}$$

$$\lambda_3(\underline{s}) \in \mathbb{R};\ \lambda_3(\widehat{s}) \geq 0 \tag{31}$$

$$\mu \geq 0, \text{ with } \mu = 0 \text{ and } \widehat{s} = \overline{s} \text{ if } ru(\widehat{s}) - c(\widehat{s}, e(\widehat{s})) > 0 \tag{32}$$

In addition, since $\widehat{s}$ is a choice variable, we have the condition

$$H_*\left(e(\widehat{s}), u(\widehat{s}), v(\widehat{s}), x(\widehat{s}), \lambda_1(\widehat{s}), \lambda_2(\widehat{s}), \lambda_3(\widehat{s}), \widehat{s}\right) \geq 0, \text{ and } = 0 \text{ if } \widehat{s} < \overline{s} \tag{33}$$

which is the standard necessary condition for free end-time optimal control problems.[20] Condition (25) equates the marginal cost, $-\lambda_1(s)$, and marginal benefit, $\lambda_2(s)\frac{1}{r}c_e(s, e(s))$, for $\mathcal{P}$ of *decreasing* the level of effort required from type s.[21]

There also exists a second-order necessary condition, which in the case of a singular control takes the form of the so-called generalized Legendre-Clebsch condition.[22] As we show in appendix

[19] Note that (29) is redundant given (25) and (30), which is why it is not used when deriving the optimal contract.

[20] See, for instance, Theorem 10.2 on page 266 in Caputo (2005). The interpretation of (33) follows from the fact that the value of the Hamiltonian at s captures the total value to $\mathcal{P}$ generated by type s.

[21] See page 89 Caputo (2005) for an interpretation of the costate variables in dynamic optimization problems. Note that in our case, $\lambda_1(s) < 0$ as that costate variable captures the benefit of decreasing the state variable $e(s)$, rather than increasing it, since $e'(s) < 0$. On the other hand, $\lambda_2(s) > 0$, as it captures the benefit of decreasing $u(s)$.

[22] Also refered to as the Kelley condition; see, for instance, page 246 in Bryson and Ho (1975).

A5, in our problem, this condition is satisfied if, for instance, $c_{ees} \geq 0$ along the trajectory of the solution to (25)-(33).[23] However, this additional assumption is only sufficient, not necessary for the generalized Legendre-Clebsch condition to be satisfied.

Lemma 9 states the sufficiency of conditions in (25)-(33) for the problem in (18)-(23), and the uniqueness of the corresponding solution.

Lemma 9 (Sufficiency and Uniqueness) *If* $\left\{\widehat{s}_*, \{e_*(s), u_*(s), v_*(s), x_*(s)\}_{s\in[\underline{s},\widehat{s}]}\right\}$ *satisfy the conditions in (25)-(33) with costate variables* $\{\lambda_{1*}(s), \lambda_{2*}(s), \lambda_{3*}(s)\}_{s\in[\underline{s},\widehat{s}]}$*, then it is the unique solution to (18)-(23).*

Proof of lemma 9. We employ the Arrow Sufficiency Theorem (see, Theorem 3.4 on page 60 in Caputo (2005)) assuming first that $\widehat{s}$ is fixed, i.e., not a choice variable. In our case, the maximized Hamiltonian evaluated at the costate functions $\left\{\{\lambda_{1*}(s), \lambda_{2*}(s), \lambda_{3*}(s)\}_{s\in[\underline{s},\widehat{s}]}\right\}$ equals $[y(e) - rh(u)]\, f(s) + \lambda_{3*}(s)\,[ru - c(s,e)]\, f(s)$ by (25). Since, as we show later, $\lambda_{3*}(s) > 0$, this maximized Hamiltonian is concave in (e,u,v) and strictly concave in (e,u) by the assumptions imposed on $y(\cdot)$ and $c(\cdot,\cdot)$ in section 2. The Arrow theorem implies that the necessary conditions are sufficient and the uniqueness of the state variables in the solution.[24] The theorem does not state the uniqueness of the control, but since $x_*(s) = e'_*(s)$, the uniqueness of the control follows immediately as well. To account for the fact that $\widehat{s}$ is in fact a choice variable, one can then employ a result from Seierstad (1984) to conclude the claim of the lemma. Since the corresponding details are slightly more technical, they are deferred to appendix A7. □

[23] This condition implies that the marginal cost of effort $c_e(\cdot)$ increases faster in effort for higher s. It is immediately satisfied if, for instance, $c(s,e) = c^1(s) \cdot c^2(e)$, with c^1 increasing and c^2 increasing and convex.

[24] The Arrow Sufficiency Theorem, as stated in Caputo (2005), requires strict concavity of the maximized Hamiltonian in (e,u,v) for the uniqueness of the solution. However, by following its proof, it is evident that if the maximized Hamiltonian is strictly concave in (e,u) and constant in v, as in our case, then $\{e_*(s), u_*(s)\}_{s\in[\underline{s},\widehat{s}]}$ must be unique. The uniqueness of the remaining state variable $\{v_*(s)\}_{s\in[\underline{s},\widehat{s}]}$ follows then from its definition in (17).

To complete the derivation of the necessary conditions for the problem in (9)-(13), we denote by $\mathcal{V}(u^n)$ the value function of the optimal control problem in (18)-(23), as a function of u^n. The necessary first order condition for the choice variable u^n is then

$$\frac{d}{du^n}\left[\mathcal{V}(u^n)-(1-r)h(u^n)\right]=0 \tag{34}$$

The Solution for the Optimal Contract Note first that u^n affects the value of the contract only through $\overline{u}^n$, i.e., $\frac{d\mathcal{V}(u^n)}{du^n}=\frac{\partial\mathcal{V}}{\partial\overline{u}^n}\frac{\partial\overline{u}^n}{\partial u^n}$. Since by the Dynamic Envelope Theorem[25] we have $\frac{\partial\mathcal{V}}{\partial\overline{u}^n}=-\lambda_3\left(\widehat{s}\right)$, while (28) implies that $\lambda_3\left(\cdot\right)$ is constant, it follows by straightforward computations from (34) that $\lambda_3\left(s\right)=h'(u^n)=\frac{1}{u'(w^n)}>0$, for all $s\in[\underline{s},\widehat{s}]$. Employing this result and (25) into the definition of the Hamiltonian H_*, we combine (33) with the requirement that $\widehat{s}=\overline{s}$ if $ru\left(\widehat{s}\right)-c\left(\widehat{s},e\left(\widehat{s}\right)\right)>0$ from (32) to conclude the following result.

Lemma 10 *An optimal contract must satisfy*

$$y\left(e_*\left(\widehat{s}\right)\right)-rw_*\left(\widehat{s}\right)\geq-\frac{1}{u'(w_*^n)}\left[ru\left(w_*\left(\widehat{s}\right)\right)-c\left(\widehat{s},e_*\left(\widehat{s}\right)\right)\right],\ \textit{and}=0\ \textit{if}\ \widehat{s}<\overline{s} \tag{35}$$

Intuitively, there are two effects of $\mathcal{P}$ implementing positive effort for a type s. First, it generates an additional *marginal* revenue to $\mathcal{P}$, $y\left(e_*\left(s\right)\right)-rw_*\left(s\right)$. Second, it delivers a marginal surplus to $\mathcal{A}$ from an ex-ante point of view, $ru\left(w_*\left(s\right)\right)-c\left(s,e_*\left(s\right)\right)$, and allows reducing the salary w_*^n. While the first effect can be negative in an optimal contract, condition (35) requires that the sum of these two effects at $\widehat{s}$ always be non-negative. On the other hand, when $\widehat{s}<\overline{s}$, since the marginal surplus $ru\left(w_*\left(\widehat{s}\right)\right)-c\left(\widehat{s},e_*\left(\widehat{s}\right)\right)$ must be zero by (14) for incentive compatibility reasons, the marginal revenue generated by type $\widehat{s}$ must be zero as well (otherwise $\widehat{s}$ would be increased).

[25] See, for instance, Theorem 9.1 on page 232 in Caputo (2005).

The following lemma, whose proof is in appendix A4, identifies a relationship between w_*^n and the wage schedule $\{w_*(s)\}_{s\in[\underline{s},\widehat{s}]}$ in an optimal contract, representing the counterpart of (7) here. The lemma follows from (32) and the fact proved in appendix that $\mu = \int_{\underline{s}}^{\widehat{s}} \frac{1}{u'(w_*(s))} f(s)ds - \frac{1}{u'(w_*^n)}$.

Lemma 11 *An optimal contract must satisfy*

$$\int_{\underline{s}}^{\widehat{s}} \frac{1}{u'(w_*(s))} f(s)ds - \frac{1}{u'(w_*^n)} \geq 0, \text{ and } = 0 \text{ if } ru(w(\widehat{s})) - c(\widehat{s}, e(\widehat{s})) > 0 \quad (36)$$

When $ru(w(\widehat{s})) - c(\widehat{s}, e(\widehat{s})) > 0$ (which by (14) can occur only when $\widehat{s} = \overline{s}$), at optimum, $\mathcal{P}$ can equalize the marginal utility delivered to $\mathcal{A}$ with an increase of w_*^n by a small amount with the expected increase in utility that could be delivered to $\mathcal{A}$ by increasing each value $w_*(s)$, for $s \in [\underline{s}, \widehat{s}]$, by the same amount. Therefore $\int_{\underline{s}}^{\widehat{s}} \frac{1}{u'(w_*(s))} f(s)ds - \frac{1}{u'(w_*^n)} = 0$. When $ru(w(\widehat{s})) - c(\widehat{s}, e(\widehat{s})) = 0$ (which generically occurs when $\widehat{s} < \overline{s}$), $\mathcal{P}$ may not be able perfectly equalize these inverse marginal utilities. More precisely, he may not be able set the wage schedule $\{w_*(s)\}_{s\in[\underline{s},\widehat{s}]}$ low enough without inducing $\mathcal{A}$ to shirk for at least some types in $[\underline{s}, \widehat{s}]$. To preempt shirking, $\mathcal{P}$ keeps the wages $w_*(s)$ high enough and lowers w_*^n below the level that would equalize the inverse marginal utilities; therefore $\int_{\underline{s}}^{\widehat{s}} \frac{1}{u'(w_*(s))} f(s)ds \geq \frac{1}{u'(w_*^n)}$, as elicited by (36), with the inequality being generically strict.

An implication of lemma 11 is that unlike the case from section 3.2, where the audit also revealed $\mathcal{A}$'s type, in the case with adverse selection studied here, the wage $w_*(s)$ may be lower than the salary w_*^n for some types, i.e., $\mathcal{A}$ may be penalized when audited even if he exerted the level of effort required for his type. This is necessary as if $\mathcal{P}$ were to increase $w_*(s)$ whenever $w_*(s) < w_*^n$ (and simultaneously reduce w_*^n to keep $\mathcal{A}$'s participation constraint binding), aiming to reduce the risk to $\mathcal{A}$, in order to preserve the incentives for truthful type revelation, he would also need to increase the remaining contingent wages, including those higher than w_*^n. On net, this may subject

$\mathcal{A}$ to more risk, thus rendering an increase of $w_*(s)$ suboptimal.

Lemma 12, whose proof is in appendix A5, determines the effort level $e_*(s)$ for each type s as a function of the wage schedule $\left\{w_*^n, \{w_*(s)\}_{s\in[\underline{s},\widehat{s}]}\right\}$. Remark 13 is also proved in appendix A5.

Lemma 12 *An optimal contract must satisfy for all* $s \in [\underline{s}, \widehat{s}]$

$$\frac{c_e\left(s, e_*\left(s\right)\right)}{u'(w_*(s))} f(s) + c_{es}\left(s, e_*\left(s\right)\right) \int_{\underline{s}}^{s} \left[\frac{1}{u'(w_*(\sigma))} - \frac{1}{u'(w_*^n)}\right] f(\sigma) d\sigma = y'(e_*\left(s\right)) f(s) \tag{37}$$

Remark 13 *We have* $\int_{\underline{s}}^{s} \left[\frac{1}{u'(w_*(\sigma))} - \frac{1}{u'(w_*^n)}\right] f(\sigma) d\sigma > 0$ *for all* $s \in (\underline{s}, \widehat{s})$.

While the moral hazard in the model induces a departure from the efficient outcome by requiring a wage schedule that subjects $\mathcal{A}$ to risk, the adverse selection induces inefficiency in the choice of the effort level. In particular, for the given wage schedule $\left\{w_*^n, \{w_*(s)\}_{s\in[\underline{s},\widehat{s}]}\right\}$, the effort level $e_*(s)$ elicited by equation (37) maximizes type s's *virtual surplus* for contracting situations with random auditing, $y(e)f(s) - \frac{c(s,e)}{u'(w_*(s))} f(s) - c_s\left(s, e\right) \int_{\underline{s}}^{s} \left[\frac{1}{u'(w_*(\sigma))} - \frac{1}{u'(w_*^n)}\right] f(\sigma) d\sigma$.[26] Unlike the case of pure moral hazard studied in section 3.2, $\mathcal{P}$ cannot implement the effort that maximizes the social surplus $y(e) - \frac{c(s,e)}{u'(w_*(s))}$. If he did, some of the types in $[\underline{s}, s)$, for which the own marginal cost of effort is lower than that of type s, would choose it instead of their prescribed effort levels. Instead, $\mathcal{P}$ implements the *lower*[27] level of effort $e_*(s)$ which solves (37) and essentially makes the type just below s indifferent between his prescribed effort level and $e_*(s)$ while all other types in $[\underline{s}, s)$ strictly prefer their prescribed effort levels. The magnitude of this downward distortion is affected by the positive factor $\int_{\underline{s}}^{s} \left[\frac{1}{u'(w_*(\sigma))} - \frac{1}{u'(w_*^n)}\right] f(\sigma) d\sigma$. This distorting factor, which for any s is a measure of the cumulative utility gains delivered by the wage adjustments awarded to types

[26] Unlike many other agency models, where the agent's utility is quasilinear in transfers and thus these transfers vanish from the expression of the virtual surplus, here the virtual surplus also depends on wages.

[27] This follows from $c_{es} > 0$, $c_{ee} > 0$, $y'' < 0$ an the result of remark 13,

in $[\underline{s}, s)$ when passing an audit, is increasing in s on $[\underline{s}, \zeta)$, where ζ solves $w_*(\zeta) = w_*^n$, i.e., as long as types receive a bonus when they are audited, and is decreasing on $(\zeta, s]$. The maximum distortion is applied by this factor to the type ζ whose wage when passing an audit equals the salary w_*^n.

Note that at $\underline{s}$, (37) becomes $\frac{c_e(\underline{s}, e_*(\underline{s}))}{u'(w_*(\underline{s}))} = y'(e_*(\underline{s}))$, implying the familiar *no distortion at the top* property; when setting the optimal effort and wage for type $\underline{s}$, $\mathcal{P}$ does not need to account for potential deviations from lower-cost types and thus can implement the efficient effort level for $\underline{s}$. Moreover, when $\widehat{s} = \overline{s}$ and $ru\left(w(\overline{s})\right) - c\left(\overline{s}, e\left(\overline{s}\right)\right) > 0$, employing (36) into (37) it follows that $\frac{c_e(\overline{s}, e_*(\overline{s}))}{u'(w_*(\overline{s}))} = y'(e_*\left(\overline{s}\right))$. Thus, in this case, and unlike most other contracting situations studied in the literature, the optimal contract with random auditing also exhibits *no distortion at the bottom.*

Proposition 14 collects our findings and presents the necessary and sufficient conditions that elicit the optimal contract in this model when $\mathcal{A}$ is risk averse.[28]

Proposition 14 *Assume that $u''(w) < 0$, for all w. The solution for the optimal contract under moral hazard and adverse selection is given by (13) satisfied with equality, (14), (16), (35), (36), and (37).*

It deserves mentioning here that the constraint $ru\left(w(\widehat{s})\right) - c\left(\widehat{s}, e\left(\widehat{s}\right)\right) \geq 0$ from (14) does not necessarily bind in an optimal contract when $\widehat{s} = \overline{s}$. This may occur if, for instance, (i) $\overline{u}$ is sufficiently high, requiring a high wage schedule $\left\{\{w_*(s)\}_{s\in[\underline{s},\widehat{s}]}, w_*^n\right\}$, (ii) the marginal output y'

[28] Given proposition 14, the optimal contract can be computed in principle as follows. When $\widehat{s} < \overline{s}$, then $ru\left(w_*\left(\widehat{s}\right)\right) - c\left(\widehat{s}, e\left(\widehat{s}\right)\right) = 0$ by (35), and thus (36) is satisfied with inequality. Then (37) determines implicitly $e_*(s)$ for each $s \in [\underline{s}, \widehat{s}]$, as a function of $\left\{w_*^n, \{w_*(s)\}_{s\in[\underline{s},\widehat{s}]}\right\}$. This also gives $e_*'(s)$ as a function of the same wage schedule. Then, $\{w_*(s)\}_{s\in[\underline{s},\widehat{s}]}$ is the solution of the differential equation defined by (16), with initial conditions given by (13), $ru\left(w_*\left(\widehat{s}\right)\right) - c\left(\widehat{s}, e\left(\widehat{s}\right)\right) = 0$ and $y\left(e_*\left(\widehat{s}\right)\right) - rw_*\left(\widehat{s}\right)$ (we need three conditions because there are also the two unknowns w_*^n and $\widehat{s}$). When $ru\left(w_*\left(\widehat{s}\right)\right) - c\left(\widehat{s}, e\left(\widehat{s}\right)\right) > 0$, then $\widehat{s} = \overline{s}$, while from the fact that (36) is satisfied with equality, w_*^n is determined as a function of $\{w_*(s)\}_{s\in[\underline{s},\widehat{s}]}$. $\{e_*(s)\}_{s\in[\underline{s},\widehat{s}]}$ and $\{w_*(s)\}_{s\in[\underline{s},\widehat{s}]}$ are derived then as above. While this shows that the optimal contract can in principle be computed with the conditions identified in proposition 14, a practical numerical implementation would involve constructing a system of differential equations in $e(s)$ and $\lambda_2(s)$ and their derivatives, with the two equations obtained from the second derivative of the equality in (25) with respect to s (see the computation of $\frac{d^2}{ds^2}\left(\frac{\partial H_*}{\partial x}\right)$ in appendix A5) and (27), and two initial conditions on $\lambda_2(s)$ given by (30).

is low for *high* levels of effort implying that $\mathcal{P}$ optimally chooses to implement low levels of effort, and (*iii*) the marginal cost of effort c_e is low for low levels of effort, implying that a wage schedule chosen so as satisfy $\mathcal{A}$'s participation constraint *and* to minimize his wage risk[29] is sufficient to also incentivize $\mathcal{A}$ to exert effort. Essentially, when $\mathcal{P}$ needs $\mathcal{A}$ for a job that requires low effort, he offers a contract that satisfies $\mathcal{A}$'s participation constraint with a smooth wage profile across all contingencies and which also provides $\mathcal{A}$ with sufficient incentives to exert that effort.

We close this section with a corollary that states the intuitive fact that the surplus generated by different types of agents is decreasing in the value of the type. Its proof is in appendix A7.

Corollary 15 *The surplus generated by type s, $y\left(e_*(s)\right) - rw_*(s) - (1-r)\,w^n$, is decreasing in s.*

The Effect of r on The Value of the Optimal Contract Denote by $V_*(r)$ the value of the optimal contract as a function of the probability of auditing r. By assumption 8, $V_*(r)$ is $\mathbb{C}_p^{(1)}$. The following lemma, whose proof is in appendix A8, elicits the effect of r on this value.

Lemma 16 *For all $r \in (0,1)$, we have*

$$V_*'(r) = \int_{\underline{s}}^{\widehat{s}} \left[\frac{u(w_*(s))}{u'(w_*(s))} - w_*(s)\right] f(s)ds - \left[\frac{u(w_*^n)}{u'(w_*^n)} - w_*^n\right] \qquad (38)$$

To understand this result, note first that since $\widehat{s}$ and the effort schedule $\{e_*(s)\}_{s\in[\underline{s},\widehat{s}]}$ are chosen optimally, when r increases slightly, the value of the contract is impacted only through the change in the expected payment to $\mathcal{A}$. There are two effects of an increase in r on this expected payment. First, a higher r increases the likelihood of auditing; this effect is captured in (38) by the term

[29] This implies that the utility $\overline{u}$ is delivered to $\mathcal{A}$ not only through a high value of w_*^n, but also through high values for the wages $\{w_*(s)\}_{s\in[\underline{s},\widehat{s}]}$ so to reduce the discrepancy between the wages with an audit and the salary w_*^n.

$-\int_{\underline{s}}^{\widehat{s}} w_*(s)f(s)ds + w_*^n$. Second, the increase in r induces an adjustment in wages, as it relaxes $\mathcal{A}$'s incentive constraints allowing $\mathcal{P}$ to reduce the wage risk by lowering $w_*(s)$; to compensate for this and continue satisfying $\mathcal{A}$'s participation constraint, $\mathcal{P}$ must increase w_*^n. These wage adjustments are captured by the remaining terms in (38).[30]

The next proposition, whose proof is in appendix A9, states that when it is optimal for $\mathcal{P}$ to implement positive effort for *all* types, the value of the contract is increasing in r.

Proposition 17 *Assume that $u''(w) < 0$ for all w. For any $r \in (0,1)$, if $\widehat{s} = \overline{s}$, then $V_*'(r) > 0$.*

Intuitively, when r increases, $\mathcal{P}$ is less reliant on the power of incentives and offers a smoother wage schedule that exposes $\mathcal{A}$ to less risk, thus reducing the risk premium $\mathcal{P}$ needs to pay and increasing the value of the contract.

As a side note here, it deserves mentioning that while we proved the result of proposition 17 under the underlying specification of an ex-ante participation constraint of the agent, it can also be shown that it holds when the agent evaluates the contract only after he learns his type, and thus the optimal contract problem has a more standard interim participation constraint for each type.

The next proposition, which is one of the main results of the paper, states that if the optimal contract does not induce effort for all types, then the value of the contract may be decreasing in the probability of auditing. Its proof constructs a numerical example, under specific functional forms of the fundamentals of the model, with the property that $V_*'(r) < 0$ for high enough values of r.

[30] Since $\widehat{s}$ and $\{e_*(s)\}_{s\in[\underline{s},\widehat{s}]}$ do not change, $ru(w(s))$ stays constant as r increases slightly. In particular, if r increases by some small *percentage* ε, $u(w(s))$ must decrease by the same percentage ε, for each $s \in [\underline{s}, \widehat{s}]$. Therefore, $w(s)$ decreases by an amount $\Delta w(s)$ satisfying $\left[\frac{d}{dw(s)} \ln u(w(s))\right] \Delta w(s) = \varepsilon$, i.e., $\Delta w(s) = \frac{u(w(s))}{u'(w(s))}\varepsilon$. To continue satisfying $\mathcal{A}$'s binding participation constraint, it follows by the same logic that w^n must increase by $\frac{u(w^n)}{u'(w^n)}\varepsilon$. The net impact of these wage adjustments constitutes the second effect of the change in r on $V_*(r)$, as elicited by (38).

Proposition 18 *There exist $f(\cdot)$, $y(\cdot)$, $u(\cdot)$, $c(\cdot)$ and $\overline{u}$, such that $\widehat{s} < \overline{s}$ and the value of the corresponding optimal contract is decreasing in r for all high enough r.*

The numerical example employed in proposition 18 builds on the simple case of a discrete type space $\{s^a, s^c\}$. With no restrictions on the continuous type density function $f(\cdot)$ defined in section 2, one can approximate sufficiently well the discrete distribution from this numerical example with a continuous one for which the corresponding optimal contract maintains the key property stated in proposition 18. Since for any set of remaining parameters of the model, there exist values of s^c high enough that it is optimal for $\mathcal{P}$ to induce effort only from the lower cost type s^a, we considered directly that s^c takes such a high value without setting a particular value for it.[31] In an environment with this particular type distribution, $\mathcal{P}$ offers a contract $\{w^a, w^n, e^a\}$ where (i) w^a is the wage paid to $\mathcal{A}$ when an audit reveals that he exerted at least effort e^a, and (ii) w^n is the salary paid to $\mathcal{A}$ when no audit is performed. The contract is designed so as to be accepted by $\mathcal{A}$ ex-ante and to induce $\mathcal{A}$ to exert effort e^a when his type is s^a and no effort otherwise. The remaining details of the formal analysis and numerical implementation are presented in appendix A10.

Figure 1 presents in *solid lines* key variables from the corresponding optimal contract as functions of the probability r, for the values of r that allow for a positive value of the optimal contract.

[31] The functional forms that we employed are $y(e) = e$, $u(w) = w^{\frac{1}{\beta}}$ and $c(s,e) = s\frac{1}{\theta}e^{\theta}$ with $s \in \{s^a, s^c\}$ and $p^a \equiv \Pr\{s = s^a\}$. The corresponding parameters are set at $\beta = 1.6$, $\theta = 1.2$, $s^a = 0.2$, $p^a = 0.85$ and $\overline{u} = 1$. For the functional forms selected, there are multiple sets of parameters with the property that $V'(r) < 0$ for high enough r.

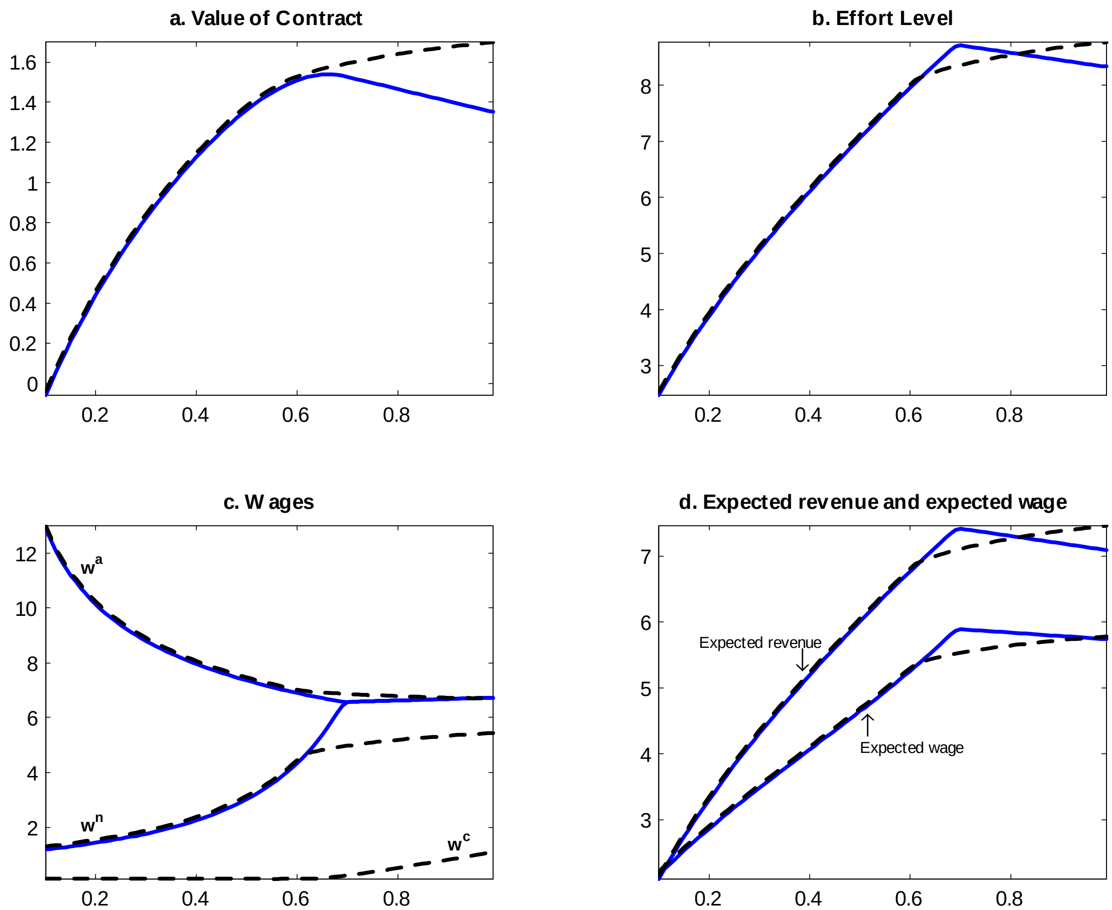

Figure 1: Optimal Contracts as Functions of the Probability of Auditing r

When r is small, the contract specifies a low effort level e^a, and in return promises a high wage w^a if $\mathcal{A}$ passes an audit. Intuitively, the low probability of audit makes it hard to provide incentives since $\mathcal{A}$ knows that even when exerting effort, he is unlikely to be rewarded, as in the absence of an audit, this effort is not observed. Therefore, $\mathcal{P}$ finds it optimal to only induce a low level of effort. Moreover, if r is very small, ($r \leq 0.1$) $\mathcal{P}$ cannot ensure himself a non-negative expected payoff if he were to implement positive effort and has to shut down. As r increases, auditing becomes a more effective monitoring instrument which allows implementing a higher effort with a lower wage w^a; $\mathcal{A}$ is compensated for the higher effort with an increase in the salary w^n. The expected revenue, $p^a y(e^a)$, the expected wage paid, $rp^a w^a + (1-r)w^n$, and the difference between the two, i.e., the value of the contract, all increase over this range.

The increase in r also impacts the risk premium paid to $\mathcal{A}$ through two channels. First, it decreases it by lowering the dispersion in the set of possible wages, $\{w^a, w^n, 0\}$ up to the value of r at which w^a and w^n become equal. Above that value, the wages w^a and w^n are equal and constant, and this first channel shuts down. Second, the increase in r alters the probabilities with which these wages are paid. When r is small, most of the utility is delivered to $\mathcal{A}$ through the salary w^n, while the probabilities of $\mathcal{A}$ being paid either w^a or 0 are small. As r increases, the probability that $\mathcal{A}$ is paid the wage w^a increases, but so does the probability that $\mathcal{A}$ is caught shirking and paid nothing. This second effect tends to increase the wage variance and therefore the risk premium $\mathcal{P}$ needs to pay. For sufficiently high values of r, the consequent increase in risk premium determines a decrease in the value of the contract and can even bring this value to zero.[32]

Next, we argue that when $\mathcal{P}$ can credibly commit not to void the contract, but to make payments even when $\mathcal{A}$ fails an audit, then the optimal value of the contract is again everywhere increasing in r. The following proposition states this result. Its proof, which is presented in appendix A11, considers the standard case adopted in this paper of a continuous type density function.

Proposition 19 *Assume that $u''(w) < 0$ for all w and that $\mathcal{P}$ can credibly commit to make a payment when an audit reveals that $\mathcal{A}$ exerted no effort. The value of the corresponding optimal contract is increasing in r for any $r \in (0, 1)$.*

If $\mathcal{P}$ can commit to make payments even when $\mathcal{A}$ is caught shirking, he can reduce the dispersion in the set of possible wages faced by $\mathcal{A}$ to essentially partially insure him against high-cost

[32] Note the monotonicity of $V(r)$ changes around the value of r where w^n starts increasing at a faster rate. Since the salary w^n is the channel through which this risk premium is paid, this suggests that the risk premium starts increasing significantly, and therefore, that it is indeed the factor driving the decrease of $V(r)$.

realizations. This allows $\mathcal{P}$ to lower the risk premium that he needs to pay. Moreover, and perhaps more surprisingly, the increased power of incentive determined by the higher probability of auditing renders the value of the contract be again everywhere increasing in r.

The optimal contract corresponding to the numerical example employed in proposition 18 is presented in Figure 1 *in broken lines*. The wage w^c is promised to be paid to $\mathcal{A}$ when an audit is performed and it reveals that $\mathcal{A}$ exerted less effort than e^a, i.e., essentially this is the wage paid when $\mathcal{A}$'s type is s^c. As expected, the value of the contract with commitment is at least as high as that without commitment for all values r. When r is low, the two values are equal; in particular, $\mathcal{P}$ implements the same contract as without commitment by setting w^c to zero and thus not availing himself of the possibility of commitment. For the high values, commitment becomes valuable and $\mathcal{P}$ promises a wage $w^c > 0$ to reduce the risk premium paid.

4 Extension: The Optimal Contract with Communication

In this section we analyze the optimal contracts with random auditing when pre-play communication is feasible. As discussed in section 2, in this situation, $\mathcal{P}$ can design a contract that requires $\mathcal{A}$ to declare his type after he learns it, and then define the wage paid to $\mathcal{A}$ when an audit is *not* performed as a function of this type.[33] To focus the exposition, we restrict attention to the case when it is optimal to implement positive effort for all types. $\mathcal{P}$'s problem in this case is to select a

[33] An alternative standard interpretation of the "communication" between $\mathcal{A}$ and $\mathcal{P}$ is that $\mathcal{A}$ chooses a contract $\{w, w^n\}$ out of a menu of contracts offered by $\mathcal{P}$ after he learns his type.

contract $\left\{\left\{e_+(s), w_+(s), w^n_+(s)\right\}_{s\in[\underline{s},\overline{s}]}\right\}$ that solves the problem

$$\max_{\{e(s),w(s),w^n(s)\}_{s\in[\underline{s},\overline{s}]}} \int_{\underline{s}}^{\overline{s}} \left[y\left(e(s)\right) - rw(s) - (1-r)\,w^n\left(s\right)\right] f(s)ds \tag{39}$$

$$\text{s.t. } s \in \arg\max_{\widetilde{s}\in[\underline{s},\overline{s}]} \left[ru\left(w(\widetilde{s})\right) + (1-r)\,u\left(w^n\left(\widetilde{s}\right)\right) - c\left(s, e\left(\widetilde{s}\right)\right)\right] \tag{40}$$

$$ru\left(w(s)\right) - c\left(s, e\left(s\right)\right) \geq 0, \text{ for all } s \in [\underline{s}, \overline{s}] \tag{41}$$

$$\int_{\underline{s}}^{\overline{s}} \left[ru\left(w(s)\right) + (1-r)\,u\left(w^n\left(s\right)\right) - c\left(s, e\left(s\right)\right)\right] f(s)ds \geq \overline{u}. \tag{42}$$

Since the first best outcome can be attained when $\mathcal{A}$ is risk neutral even without communication, it can also be attained if communication is feasible. We consider therefore in the following the case when $\mathcal{A}$ is strictly risk averse.

Following the First Order Approach, we replace (40) with the corresponding first order condition

$$ru'\left(w(s)\right)w'(s) + (1-r)\,u'\left(w^n(s)\right)w^{n\prime}(s) = c_e(s, e(s))e'(s) \text{ a.e. } s \tag{43}$$

and assume that $e'_+(s) < 0$. On the other hand, a counterpart of the lemma 4 from the case without communication does not hold here, and thus we cannot relax (41).

We solve the resulting problem using the same optimal control methods employed in section 3.3. By denoting $u^n(s) \equiv u(w^n(s))$, the first order condition for $\mathcal{A}$'s truthful revelation problem from (43) becomes $ru'\left(s\right) + (1-r)\,u^{n\prime}\left(s\right) = c_e(s, e(s))e'(s)$ a.e. s. In addition to the variables from problem (18)-(23), we introduce a new state variable $u^n(s)$ and a new control $k(s) \equiv u^{n\prime}\left(s\right)$. The optimal control problem is then

$$\max_{\{x(s),k(s)\}_{s\in[\underline{s},\overline{s}]}} \int_{\underline{s}}^{\overline{s}} \left[y\left(e(s)\right) - rh\left(u(s)\right) - (1-r)h(u^n(s))\right] f(s)ds \tag{44}$$

$$\text{s.t. } e'(s) = x(s) \tag{45}$$

$$u^{n\prime}(s) = k(s) \tag{46}$$

$$u'(s) = -\frac{1-r}{r}k(s) + \frac{1}{r}c_e(s, e(s))x(s) \tag{47}$$

$$v'(s) = \left[ru(s) + (1-r)u^n(s) - c(s, e(s))\right] f(s) \tag{48}$$

$$v(\underline{s}) = 0;\ v(\overline{s}) \geq \overline{u} \tag{49}$$

$$ru(s) - c(s, e(s)) \geq 0, \text{ for all } s \in [\underline{s}, \overline{s}] \tag{50}$$

This is an optimal control problem *with pure state constraints* determined by the specific form of (50).[34] The *necessary* conditions delivered by Pontryagin's Maximum Principle for such problems are presented in Theorem 4.1 in Hartl, Sethi and Vickson (1995), with the formal proof of the theorem for our case where we have no constraints with both state and control variables, so condition 2.3 from the text of their problem does not exist, presented in the references cited therein.

Thus, to solve the control problem, we construct the Lagrangian

$$L_+\left(e, u, u^n, v, x, k, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \gamma, s\right) \equiv \left[y\left(e\right) - rh\left(u\right) - (1-r)\, h\left(u^n\right)\right] f(s) + \gamma\left[ru - c\left(s, e\right)\right] f(s)$$
$$+ \lambda_1 x + \lambda_2\left[-\frac{1-r}{r}k + \frac{1}{r}c_e\left(s, e\right) x\right] + \lambda_3\left[ru + (1-r)\, u^n - c\left(s, e\right)\right] f(s) + \lambda_4 k$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ and γ are functions defined on $[\underline{s}, \overline{s}]$. Then, by Theorem 4.1 in Hartl, Sethi and Vickson (1995), there exist almost everywhere differentiable functions $\lambda_1, \lambda_2, \lambda_3, \lambda_4$,[35] and an

[34] See Chapter 6 in Caputo (2005) for a comprehensive discussion of the problems with state and control constraints, and Chapter 5 in Seierstad and Sydsaeter (1987) for a more detailed discussion of problems with pure state constraints.

[35] The theorem does not state that the costate variables $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are continuous, but it states that at all

almost everywhere continuous function γ such that the following conditions are satisfied almost everywhere.[36]

$$\frac{\partial L_+}{\partial x} = \lambda_1(s) + \lambda_2(s)\frac{1}{r}c_e(s,e(s)) = 0 \tag{51}$$

$$\frac{\partial L_+}{\partial k} = -\frac{1-r}{r}\lambda_2(s) + \lambda_4(s) = 0 \tag{52}$$

$$\lambda_1'(s) = -\frac{\partial L_+}{\partial e} = \tag{53}$$

$$= -y'(e)\,f(s) + \gamma(s)\,c_e(s,e(s))\,f(s) - \lambda_2(s)\frac{1}{r}c_{ee}(s,e(s))\,x(s) + \lambda_3(s)\,c_e(s,e(s))\,f(s) \tag{54}$$

$$\lambda_2'(s) = -\frac{\partial L_+}{\partial u} = rh'(u(s))f(s) - r\gamma(s)f(s) - r\lambda_3(s)\,f(s) \tag{55}$$

$$\lambda_3'(s) = -\frac{\partial L_+}{\partial v} = 0 \tag{56}$$

$$\lambda_4'(s) = -\frac{\partial L_+}{\partial u^n} = (1-r)\,h'(u^n(s))\,f(s) - \lambda_3(s)(1-r)\,f(s) \tag{57}$$

$$\lambda_1(\underline{s}) = 0;\ \lambda_1(\overline{s}) = 0;\ \lambda_1(\underline{s}) = 0;\ \lambda_1(\overline{s}) = 0;\ \lambda_3(\underline{s}) \in \mathbb{R};\ \lambda_3(\overline{s}) \geq 0;\ \lambda_4(\underline{s}) = 0;\ \lambda_4(\overline{s}) = 0 \tag{58}$$

$$\gamma(s) \geq 0, \text{ with } \gamma(s) = 0 \text{ if } ru(s) - c(s,e(s)) > 0, \text{ for any } s \in [\underline{s},\overline{s}] \tag{59}$$

In addition, there exists also a variant of the generalized Legendre-Clebsch necessary condition for problems with multiple controls and state constraints, which we show in appendix A12 that it is satisfied if, for instance, we assume again that $c_{ees} \geq 0$ along the trajectory of the solution to (51)-(59).

Lemma 20 states the sufficiency of conditions in (51)-(59) for the problem in (44)-(50) and the uniqueness of the corresponding solution.[37]

points where the constraint (50) binds, the functions λ_1 and λ_2 may have discontinuities given by the following jump conditions $\lambda_1(s^-) = \lambda_1(s^+) + \eta(s)\frac{\partial}{\partial e(s)}[ru(s) - c(s,e(s))]$ and $\lambda_2(s^-) = \lambda_2(s^+) + \eta(s)\frac{\partial}{\partial u(s)}[ru(s) - c(s,e(s))]$ for some positive function $\eta(s)$. Since the same type of result applies to the functions λ_3 and λ_4, but the constraint in (50) is independent of the corresponding state variables $v(s)$ and $u^n(s)$, it follows that λ_3 and λ_4 are continuous everywhere. Given (52), it follows then that λ_2 is also continuous everywhere, and then (51) implies the same for λ_1. We conclude thus that for this problem, as in the case of the necessary conditions from section 3.3, the costate variables are continuous everywhere.

[36] Given (51) and (52), some conditions in (58) are redundant and so are not used in deriving the optimal contract.

[37] The proof of this lemma is simplified by the underlying assumption that it is optimal that all types exert positive

Lemma 20 (Sufficiency and Uniqueness) *If* $\left\{e_+(s), u_+(s), u_+^n(s), v_+(s), x_+(s), k_+(s)\right\}_{s\in[\underline{s},\overline{s}]}$ *satisfy the conditions in (51)-(59) with costate variables* $\{\lambda_{1+}(s), \lambda_{2+}(s), \lambda_{3+}(s), \lambda_{4+}(s)\}_{s\in[\underline{s},\overline{s}]}$*, then it is the unique solution to (44)-(50).*

Proof of lemma 20. The lemma follows from the Arrow Sufficiency Theorem for optimal control problems with *mixed* constraints (see Theorem 6.4 on page 166 in Caputo (2005)).[38] By employing the conditions in (51) and (52), the maximized Hamiltonian evaluated at the corresponding costate variables equals $[y(e) - rh(u) - (1-r)h(u^n)]f(s) + \lambda_{3+}(s)[ru + (1-r)u^n - c(s,e)]f(s)$. From the assumed properties of $y(\cdot)$ and $c(\cdot,\cdot)$ and the fact that for any solution to (51)-(59), we have $\lambda_{3+}(s) \geq 0$ (we prove this in appendix A12), the maximized Hamiltonian is concave in (e, u, u^n, v) and strictly concave in (e, u, u^n). This implies the claims of the lemma 20.[39] □

Proposition 21, proved in appendix A12, is the main result of this section eliciting the conditions that determine the optimal contract with communication when $\mathcal{A}$ is strictly risk averse, and the effect of r on the value of this contract.[40] Remark is proved in the same appendix.

Proposition 21 *Assume that* $u''(w) < 0$*, for all* w*. Also, assume that it is optimal to induce all types of* $\mathcal{A}$ *to exert effort. The solution for the optimal contract with communication under moral*

effort. If that assumption was relaxed, the result to apply for proving sufficiency in an optimal control problem with pure state constraints and a free end time is Theorem 7 on page 377 in Seierstad and Sydsaeter (1987).

[38] Note that while our problem has *pure state* constraints, Theorem 6.4 in Caputo (2005), which deals with optimal problems with *mixed* constraints (where the control also appears in the constraint), applies to it since the rank constraint qualification is not required for that theorem. This rank qualification is not satisfied in problems with pure state constraints and thus we could not apply Theorem 6.1 from Caputo (2005) to conclude that (51)-(59) are necessary conditions for a solution to (44)-(50).

[39] As in the case of lemma 9, the text of Arrow's Sufficiency Theorem from Caputo (2005) requires the maximized Hamiltonian be strictly concave in all state variables, and does not claim the uniqueness of the control variables, but the same argument employed in the proof of lemma 9 completes the proof in this case.

[40] (61) determines $e_+(s)$ as a function of $\{w_+(s), w_+^n(s)\}_{s\in[\underline{s},\overline{s}]}$. Then (43) and (60), subject to the constraint in (41) and with the binding constraint in (42) as initial condition, constitute a system that determine $w_+(s)$ and $w_+^n(s)$.

hazard and adverse selection is given by (42) satisfied with equality, (41), (43), and for any $s \in [\underline{s}, \overline{s}]$

$$w_+(s) - w_+^n(s) \geq 0, \text{ and } = 0 \text{ whenever } ru\left(w_+\left(s\right)\right) - c\left(s, e_+\left(s\right)\right) > 0 \tag{60}$$

$$\frac{c_e\left(s, e_+\left(s\right)\right)}{u'(w_+(s))} f(s) + c_{es}\left(s, e_+\left(s\right)\right) \int_{\underline{s}}^{s} \left[\frac{1}{u'(w_+^n(\sigma))} - \int_{\underline{s}}^{\overline{s}} \frac{f(t)}{u'(w_+^n(t))} dt\right] f(\sigma) d\sigma = y'(e_+(s)) f(s) \tag{61}$$

The value of the optimal contract is increasing in r *for any* $r \in (0, 1)$.

Remark 22 *We have* $\int_{\underline{s}}^{s} \left[\frac{1}{u'(w_+^n(\sigma))} - \int_{\underline{s}}^{\overline{s}} \frac{f(t)}{u'(w_+^n(t))} dt\right] f(\sigma) d\sigma > 0$ *for all* $s \in (\underline{s}, \overline{s})$.

As in the model with pure moral hazard studied in section 3.2, the optimal contract sets $w_+(s) \geq w_+^n(s)$. To see why, note that otherwise, $w_+^n(s)$ could be reduced by an amount ϵ and $w_+(s)$ increased to a value $w'_+(s)$ with the property that $ru\left(w_+(s)\right) + (1-r)\, u\left(w_+^n\left(s\right)\right) = ru\left(w'_+(s)\right) + (1-r)\, u\left(w_+^n\left(s\right) + \epsilon\right)$; this adjustment would not violate any of constraints in problem (39)-(42), but would reduce the amount of risk that $\mathcal{A}$ is subjected to and the risk premium that needs to be paid. Thus, unlike the case with no pre-play communication, where $\mathcal{P}$ needs to use the wage scheme $\{w_*(s)\}_{s \in [\underline{s}, \widehat{s}]}$ both to induce the agent to reveal his type and to exert effort, if communication is feasible, $\mathcal{P}$ can make use of the flexibility offered by his ability to adjust the wage scheme $\left\{w_+^n(s)\right\}_{s \in [\underline{s}, \overline{s}]}$ to tailor the contract so as to not penalize the agent when an audit is performed and the agent passes it. Moreover, as (60) states, the wages with and without audit for a given type s are equal whenever the corresponding level is sufficient to provide incentives for that type to exert effort, i.e., when $ru\left(w_+\left(s\right)\right) - c\left(s, e_+\left(s\right)\right) > 0$. On the other hand, (61) imposes the usual equality between the marginal cost and benefit of requiring an additional effort from type s, with the second term in the left hand side being the (strictly positive, by remark 22) distorting factor that sets type s's induced effort below its efficient level. Finally, under our underlying assumption that all types are induced to exert positive effort, the value of the contract is strictly increasing in r whenever $w_+(s) - w_+^n(s) > 0$ for s in a set of positive measure, and is constant otherwise.

5 Conclusion

In this paper we studied optimal contracts with random auditing defined as a monitoring instrument where the agent's action is observed with some non-degenerate probability, but otherwise the principal has no informative signal of this action. We characterized and compared the optimal contracts under several scenarios that combined moral hazard and adverse selection. We showed that a higher precision of the monitoring instrument, as measured by the probability of auditing, always increases the value of an optimal contract when all agent types are optimally induced to exert effort or when the principal can commit to make payments towards the agent even when the latter fails an audit, but may decrease the value of a contract otherwise. Finally, we characterized the optimal contracts for situations where pre-play communication is possible and thus the principal can adjust the wage paid to the agent when an audit is not performed as a function of the signal transmitted by the agent. As an avenue for future research, we are also currently working on slightly more realistic *repeated* version of this model where a failed audit voids the dynamic contract and leads to a loss for the agent of the promised value of future payments.

Appendix

Appendix A1. Proof of Proposition 3

First, note that in any optimal contract, the participation constraint in (6) must bind since otherwise w_1^n can be decreased without violating any of the constraints. We construct the Lagrangian

$$\begin{aligned} L &= \int_{\underline{s}}^{\overline{s}} \left[y\left(e(s)\right) - rw(s) - \left(1-r\right)w^n\right] f(s)ds + \int_{\underline{s}}^{\overline{s}} \lambda\left(s\right)\left[ru(w(s)) - c(s,e(s))\right] ds \\ &+\mu \left[\int_{\underline{s}}^{\overline{s}} \left[ru\left(w(s)\right) + \left(1-r\right)u\left(w^n\right) - c\left(s,e\left(s\right)\right)\right] f(s)ds - \overline{u}\right] \end{aligned}$$

The necessary first order conditions are then

$$\frac{\partial L}{\partial e(s)} = y'\left(e(s)\right) f(s) - \lambda\left(s\right) c_e(s,e(s)) - \mu c_e\left(s,e\left(s\right)\right) f(s) = 0 \tag{62}$$

$$\frac{\partial L}{\partial w(s)} = -rf(s) + \lambda\left(s\right) ru'(w(s)) + \mu ru'(w(s))f(s) = 0 \tag{63}$$

$$\frac{\partial L}{\partial w^n} = -\left(1-r\right) + \mu\left(1-r\right)u'(w^n) = 0 \tag{64}$$

$$\lambda\left(s\right)\frac{\partial L}{\partial \lambda\left(s\right)} = \lambda\left(s\right)\left[ru(w(s)) - c(s,e(s))\right] = 0 \tag{65}$$

$$\lambda\left(s\right) \geq 0;\ ru(w(s)) - c(s,e(s)) \geq 0;\ \mu \geq 0 \tag{66}$$

From (63), we have $\lambda\left(s\right) + \mu f(s) = \frac{f(s)}{u'(w(s))}$. Substituting this into (62), implies (8) from the text of the proposition. From (64), we have $\mu = \frac{1}{u'(w^n)}$, and thus $\lambda\left(s\right) = \left[\frac{1}{u'(w(s))} - \frac{1}{u'(w^n)}\right] f(s)$. Substituting this into (65), and noting that $\frac{1}{u'(w(s))} - \frac{1}{u'(w^n)} = 0 \Longleftrightarrow w(s) - w^n = 0$, it follows that $w_1(s) - w_1^n = 0$ whenever $ru(w(s)) - c(s,e(s)) > 0$. Finally, $\lambda\left(s\right) \geq 0$ from (66) and the fact that $\frac{1}{u'(w)}$ is increasing in w imply $w_1(s) - w_1^n \geq 0$ as stated by (7). This completes the proof of proposition 3.

Denote now by $V_1\left(r\right)$ the value of the optimal contract; we will show that $V_1'\left(r\right) \geq 0$. Employing the Envelope Theorem, and then substituting μ and $\lambda(s)$ computed above, we have

$$
\begin{aligned}
V_1'(r) &= \frac{\partial L}{\partial r} = \int_{\underline{s}}^{\overline{s}} \left\{ \left[-w(s) + w^n \right] f(s) + \lambda(s) u(w(s)) + \mu \left[u(w(s)) - u(w^n) \right] f(s) \right\} ds \\
&= \int_{\underline{s}}^{\overline{s}} \left\{ \left[-w(s) + w^n \right] + \left[\frac{1}{u'(w(s))} - \frac{1}{u'(w^n)} \right] u(w(s)) + \frac{1}{u'(w^n)} \left[u(w(s)) - u(w^n) \right] \right\} f(s) ds \\
&= \int_{\underline{s}}^{\overline{s}} \left\{ \left[\frac{u(w(s))}{u'(w(s))} - w(s) \right] - \left[\frac{u(w^n)}{u'(w^n)} - w^n \right] \right\} f(s) ds
\end{aligned}
$$

Since $w(s) \geq w^n$ for all s, $V_1'(r) \geq 0$ follows from the fact that $\frac{d}{dw}\left[\frac{u(w)}{u'(w)} - w\right] = \frac{-u(w)u''(w)}{(u'(w))^2} > 0$.

Next, we show that the effort schedule $e_1(s)$ is decreasing, and argue that the wage schedule $w_1(s)$ is not necessarily monotonic. Consider any interval in $[\underline{s}, \overline{s}]$ on which $w_1(s) = w_1^n$,[41] and note that from (8) it follows by the Implicit Function Theorem that $e_1'(s) = -\frac{c_{es}(s,e_1(s))}{c_{ee}(s,e_1(s)) - y''(e_1(s))u'(w_1^n)}$, which is negative since $c_{es} > 0$, $c_{ee} > 0$, $y'' \leq 0$ and $u' > 0$. On the other hand, on an interval in $[\underline{s}, \overline{s}]$ on which $w_1(s) > w_1^n$, we have from (7) that $w_1(s) = u^{-1}\left(\frac{1}{r}c(s, e_1(s))\right)$. Substituting this in (8), we obtain $c_e(s, e_1(s)) - y'(e_1(s))\, u'(u^{-1}\left(\frac{1}{r}c(s, e_1(s))\right)) = 0$. It follows then that $e_1'(s) = -\frac{c_{es}(s,e_1(s)) - y'(e_1(s))\frac{u''(w_1(s))}{u'(w_1(s))}\frac{1}{r}c_s(s,e_1(s))}{c_{ee}(s,e_1(s)) - y''(e_1(s))u'(w_1(s)) - y'(e_1(s))\frac{u''(w_1(s))}{u'(w_1(s))}\frac{1}{r}c_e(s,e_1(s))}$, where we substituted $w_1(s)$ for $u^{-1}\left(\frac{1}{r}c(s, e_1(s))\right)$. Again, the properties of the functions c, y and u imply that $e_1'(s) < 0$. Finally, rewriting (8) as $u'(w_1(s))y'(e_1(s)) - c_e(s, e_1(s)) = 0$ it follows that whenever $w_1(s) > w_1^n$, we have $w_1'(s) = -\frac{u'(w_1(s))y''(e_1(s))e_1'(s) - c_{es}(s,e_1(s)) - c_{ee}(s,e_1(s))e_1'(s)}{u''(w_1(s))y'(e_1(s))}$. Since $c_{es}(s, e_1(s)) > 0$, the numerator is generically unsigned, and thus $w_1(s)$ is not necessarily monotonic. Moreover, the set of values of s for which $w_1(s) = w_1^n$ may be a union of disjoint intervals.

We close by arguing that the surplus generated by the different agent types, $y(e(s)) - rw(s) - (1-r)w^n$, is decreasing in s. Consider first any interval in $[\underline{s}, \overline{s}]$ on which $w_1(s) > w_1^n$, and thus, from (5), $ru(w_1(s)) - c(s, e_1(s)) = 0$, implying $w_1(s) = u^{-1}\left(\frac{1}{r}c(s, e_1(s))\right)$. On this interval, we have

[41] By the continuity of the objective function and constraints in $\mathcal{P}$'s problem, it follows that the sets of s on which $w_1(s) > w_1^n$ and $w_1(s) = w_1^n$, respectively, are unions of intervals.

then $\frac{d}{ds}\left[y\left(e_1(s)\right)-rw_1(s)\right]=\frac{d}{ds}\left[y\left(e_1(s)\right)-ru^{-1}\left(\frac{1}{r}c\left(s,e_1\left(s\right)\right)\right)\right]=\left[y'\left(e_1(s)\right)-\frac{c_e(s,e_1(s))}{u'\left(u^{-1}\left(\frac{1}{r}c(s,e_1(s))\right)\right)}\right]e_1'(s)-\frac{c_s(s,e_1(s))}{u'\left(u^{-1}\left(\frac{1}{r}c(s,e_1(s))\right)\right)}$, which is negative since (8) implies that the first term is zero. On the other hand, on intervals in $[\underline{s},\overline{s}]$ on which $w_1\left(s\right)=w_1^n$, we have $\frac{d}{ds}\left[y\left(e_1(s)\right)-rw_1(s)\right]=y'\left(e_1(s)\right)e_1'(s)$. Since $e_1'(s)<0$, we conclude that $\frac{d}{ds}\left[y\left(e_1(s)\right)-rw_1(s)\right]<0$ for all s. □

Appendix A2. Proof of Lemma 5

We argue first that any incentive compatible contract must satisfy (15) and (16). Consider $\mathcal{A}$'s original problem of choosing an effort level under a contract $\left\{w^n,\{w(e)\}_{e\geq 0}\right\}$ when his type is s, $\max_{e(s)\geq 0}\left[ru\left(w(e)\right)-c\left(s,e\right)\right]$ and note that for any wage schedule $w(e)$, $\mathcal{A}$'s objective function in (10) is submodular in (e,s) because $\frac{\partial^2}{\partial e\partial s}\left[ru\left(w(e)\right)-c\left(s,e\right)\right]=-c_{es}(s,e)<0$. By Topkis' Monotonicity Theorem, it follows then that the maximizer $e\left(s\right)$ is a.e. decreasing in s. Thus, in order for a contract $\left\{\{e(s),w(s)\}_{s\in[\underline{s},\overline{s}]},w^n\right\}$ to be incentive compatible, $e(s)$ must be a.e. decreasing in s (this also implies that $e(s)$ is a.e. differentiable). The necessity of (16) follows from the first order condition in $\mathcal{A}$'s problem in (10).

Next, we show that a contract satisfying (15) and (16) is incentive compatible. Let $\Phi\left(\widetilde{s},s\right)\equiv ru\left(w(\widetilde{s})\right)-c\left(s,e\left(\widetilde{s}\right)\right)$; we will argue that $\Phi\left(s,s\right)\geq\Phi\left(\widetilde{s},s\right)$ for all $\widetilde{s},s\in[\underline{s},\overline{s}]$, which will be enough to complete the proof. We have $\frac{\partial}{\partial\widetilde{s}}\Phi\left(\widetilde{s},s\right)=ru'\left(w(\widetilde{s})\right)w'(\widetilde{s})-c_e\left(s,e\left(\widetilde{s}\right)\right)e'(\widetilde{s})$, and so note that $\frac{\partial}{\partial\widetilde{s}}\Phi\left(\widetilde{s},s\right)\geq\frac{\partial}{\partial\widetilde{s}}\Phi\left(\widetilde{s},\widetilde{s}\right)$ if and only if $-c_e\left(s,e\left(\widetilde{s}\right)\right)e'(\widetilde{s})\geq -c_e\left(\widetilde{s},e\left(\widetilde{s}\right)\right)e'(\widetilde{s})$. Since $c_{es}(\cdot)>0$ and $e'(\cdot)<0$ by (15), it follows that $\frac{\partial}{\partial\widetilde{s}}\Phi\left(\widetilde{s},s\right)\geq\frac{\partial}{\partial\widetilde{s}}\Phi\left(\widetilde{s},\widetilde{s}\right)$ if and only if $\widetilde{s}\leq s$. But by (16), we have $\frac{\partial}{\partial\widetilde{s}}\Phi\left(\widetilde{s},\widetilde{s}\right)=0$. Thus, $\Phi\left(\widetilde{s},s\right)$ is increasing (decreasing) in $\widetilde{s}$ for $\widetilde{s}\leq s$ $(\widetilde{s}\geq s)$, implying immediately that it is indeed maximized when $\widetilde{s}$ equals s. □

Appendix A3. Proof of Proposition 6

We show that there exists a wage schedule that can implement the full-information effort schedule $\{e_0(s)\}_{s\in[\underline{s},\overline{s}]}$, i.e., that can satisfy all the incentive constraints in $\mathcal{P}$'s optimal contract problem under moral hazard and adverse selection, and which delivers to $\mathcal{A}$ the same ex-ante expected wage as in the case with full information. This will imply that the corresponding contract is optimal since its value attains the upper bound, i.e., the value of the optimal contract under full information. Thus, let $w_*(\overline{s})$ be defined by $rw_*(\overline{s}) - c\left(\overline{s}, e_0\left(\overline{s}\right)\right) = 0$, let $w_*(s) = w_*(\overline{s}) - \int_s^{\overline{s}} c_e(\sigma, e_0(\sigma))e_0'(\sigma)d\sigma$ for $s \in [\underline{s}, \overline{s})$, and let $w_*^n = \frac{1}{1-r}\left[\int_{\underline{s}}^{\overline{s}}\left[rw_*(s) - c(s, e_0(s))\right]f(s)ds - \overline{u}\right]$. This wage schedule satisfies (11), (13). Moreover, it also satisfies (16), which combined with $e_0'(s) < 0$ implies that it satisfies (10). Finally, note that $\int_{\underline{s}}^{\overline{s}}\left[rw_*(s) + (1-r)\,w_*^n\right]f(s)ds = \overline{u} + \int_{\underline{s}}^{\overline{s}} c(s, e_0(s))f(s)ds$, which equals the value of $\mathcal{A}$'s expected wage under full information. This completes the proof of proposition 6. □

Appendix A4. Proof of Lemma 11

Since $\lambda_2(\underline{s}) = 0$ by (30), integrating equation (27) we have

$$\lambda_2(s) = r\int_{\underline{s}}^{s}\left[h'(u\left(\sigma\right))f(\sigma) - \lambda_3\left(\sigma\right)f(\sigma)\right]d\sigma, \text{ for all } s \in [\underline{s}, \widehat{s}] \tag{67}$$

Employing $\lambda_2(\widehat{s}) = \mu r$, as required by (30), and $\lambda_3\left(s\right) = h'(u^n)$, we conclude that

$$\mu = \int_{\underline{s}}^{\widehat{s}}\left[h'(u\left(s\right)) - h'(u^n)\right]f(s)ds \tag{68}$$

The claim of lemma 11 follows by substituting for μ from (68) into (32) after observing that $h'(u(s)) = \left(u^{-1}\right)'\left(u(w(s))\right) = \frac{1}{u'(u^{-1}(u(w(s))))} = \frac{1}{u'(w(s))}$, and similarly that $h'(u^n) = \frac{1}{u'(w^n)}$. □

Appendix A5.

Proof of Lemma 12. Differentiating $\lambda_1(s) + \lambda_2(s)\frac{1}{r}c_e(s, e(s)) = 0$ from (25) with respect to s, we get

$$\lambda_1'(s) + \lambda_2'(s)\frac{1}{r}c_e(s, e(s)) + \lambda_2(s)\frac{1}{r}c_{es}(s, e(s)) + \lambda_2(s)\frac{1}{r}c_{ee}(s, e(s))\, e'(s) = 0 \tag{69}$$

Plugging in $\lambda_1'(s)$ and $\lambda_2'(s)$ from (26) and (27), we obtain that

$$\begin{aligned} &\left[-y'(e(s))f(s) - \lambda_2(s)\frac{1}{r}c_{ee}(s, e(s))\, x(s) + \lambda_3(s)\, c_e(s, e(s))\, f(s)\right] + \\ &+ \left[rh'(u(s))f(s) - \lambda_3(s)\, rf(s)\right]\frac{1}{r}c_e(s, e(s)) + \lambda_2(s)\frac{1}{r}c_{es}(s, e(s)) + \lambda_2(s)\frac{1}{r}c_{ee}(s, e(s))\, e'(s) = \\ &= -y'(e(s))f(s) + h'(u(s))c_e(s, e(s))\, f(s) + \lambda_2(s)\frac{1}{r}c_{es}(s, e(s)) = 0 \end{aligned} \tag{70}$$

where we used the fact that $x(s) = e'(s)$. Now, since (67) and $\lambda_3(s) = h'(u^n)$, imply $\frac{\lambda_2(s)}{r} = \int_{\underline{s}}^{s}[h'(u(\sigma)) - h'(u^n)]\, f(\sigma)d\sigma$, it follows that the optimal contract must satisfy

$$h'(u(s))c_e(s, e(s))\, f(s) + \left[\int_{\underline{s}}^{s}\left[h'(u(\sigma)) - h'(u^n)\right] f(\sigma)d\sigma\right] c_{es}(s, e(s)) = y'(e(s))f(s)$$

which can then be immediately rewritten as in (37). This completes the proof of lemma 12. □

Proof of Remark 13. We will show that $\lambda_2(s) \geq 0$ for all $s \in [\underline{s}, \widehat{s}]$. Given the expression computed for $\lambda_2(s)$ above, this will immediately imply the claim of the corollary. Now, to prove that $\lambda_2(s) > 0$ for all $s \in (\underline{s}, \widehat{s})$, since $\lambda_2(\underline{s}) = 0$ and $\lambda_2(\widehat{s}) = \mu r \geq 0$, it would be enough to show that λ_2 is strictly increasing on some interval $[\underline{s}, s']$ and strictly decreasing on $[s', \widehat{s}]$. To this aim, note first that $\frac{d}{ds}[h'(u(s))] = h''(u(s))u'(s) < 0$ because $u(s)$ is decreasing while h is strictly convex as the inverse of a concave and increasing function.[42] Therefore, $h'(u(s))$ is strictly decreasing in s. Since it is also continuous in s it follows that there exists some $s' \in [\underline{s}, \widehat{s}]$ such that $h'(u(s)) - h'(u^n) > 0$ for all $s \in [\underline{s}, s')$ and $h'(u(s)) - h'(u^n) < 0$ for all $s \in (s', \widehat{s}]$. Since $\lambda_2'(s) = [h'(u(s)) - h'(u^n)]\, f(s)$ it follows that $\lambda_2(s)$ is increasing on $[\underline{s}, s']$ and decreasing on $[s', \widehat{s}]$, as desired. □

Verification of the generalized Legendre-Clebsch condition. This condition requires that

$$(-1)^n \frac{\partial}{\partial x}\left[\frac{d^{2n}}{ds^{2n}}\left(\frac{\partial H_*}{\partial x}\right)\right] \leq 0 \tag{71}$$

where $2n$ is the first higher-order derivative of $\frac{\partial H_*}{\partial x}$ with respect to s in which the control x appears (it has been proved that $n \in \mathbb{N}_+$). We already showed above that $\frac{d}{ds}\left(\frac{\partial H_*}{\partial x}\right) = -y'(e(s))f(s) + h'(u(s))c_e(s, e(s))\, f(s) + \lambda_2(s)\, \frac{1}{r} c_{es}(s, e(s))$, so differentiating one more time, we have

$$\begin{aligned}\frac{d^2}{ds^2}\left(\frac{\partial H_*}{\partial x}\right) &= -y''(e(s))x(s)f(s) - y'(e(s))f'(s) + h''(u(s))\frac{1}{r}c_e(s, e(s))\, x(s)\, c_e(s, e(s))\, f(s) + \\ &+ h'(u(s))c_{es}(s, e(s))\, f(s) + h'(u(s))c_{ee}(s, e(s))\, x(s)f(s) + h'(u(s))c_e(s, e(s))\, f'(s) + \\ &+ \left[rh'(u(s))f(s) - \lambda_3(s)\, rf(s)\right]\frac{1}{r}c_{es}(s, e(s)) + \lambda_2(s)\,\frac{1}{r}c_{ess}(s, e(s)) + \lambda_2(s)\,\frac{1}{r}c_{ees}(s, e(s))\, x(s)\end{aligned}$$

where employing (19), (20) and (27), we substituted $x(s)$ for $e'(s)$, $\frac{1}{r}c_e(s, e(s))\, x(s)$ for $u'(s)$ and $rh'(u(s))f(s) - \lambda_3(s)\, rf(s)$ for $\lambda_2'(s)$. Clearly, $n = 1$ and thus the condition in (71) requires that

$$-y''(e(s))f(s) + h''(u(s))\frac{1}{r}\left[c_e(s, e(s))\right]^2 f(s) + h'(u(s))c_{ee}(s, e(s))\, f(s) + \lambda_2(s)\,\frac{1}{r}c_{ees}(s, e(s)) \geq 0 \tag{72}$$

along the solution to (25)-(33). Since $y''(e) < 0$, $h''(u) > 0$, $h'(u) > 0$, $c_{ee}(s, e) > 0$ and $\lambda_2(s) \geq 0$, a sufficient condition for (72) to be satisfied is that $c_{ees}(s, e(s)) \geq 0$ along this solution. Clearly,

[42] Differentiating twice each side of the equality $u(h(v)) = v$, we obtain $h''(v) = -u''(h(v))\, [h'(v)]^2 / u'(h(v)) > 0$.

however, this additional assumption on the cost function $c(s,e)$ is not necessary. □

Appendix A6. The Optimal Contract when the Monotonicity Constraint Binds

Denote by $\left\{ \{e_\times(s), w_\times(s)\}_{s\in[\underline{s},\overline{s}]}, w^n_\times \right\}$ the optimal contract when incorporating the monotonicity constraint in (15) in $\mathcal{P}$'s maximization problem, and assume that there exist intervals of types of $\mathcal{A}$ for which this constraint binds and thus $e_\times(\cdot)$ is constant. To account formally for this additional constraint, we construct a new Hamiltonian as

$$H_\times\left(e,u,v,x,\lambda_1,\lambda_2,\lambda_3,\lambda_4,s\right) \equiv H_*\left(e,u,v,x,\lambda_1,\lambda_2,\lambda_3,s\right) + \lambda_4\left(-x\right) \tag{73}$$

where $H_*\left(\cdot\right)$ is as defined in (24). In addition to the necessary conditions identified in (26)-(91), which must still be satisfied, Pontryagin's Maximum Principle[43] implies that there exists an continuous and piecewise continuously differentiable function $\lambda_4\left(s\right)$ such that (25) is replaced with

$$\frac{\partial H_\times}{\partial x} = \lambda_1\left(s\right) + \lambda_2\left(s\right)\frac{1}{r}c_e\left(s, e\left(s\right)\right) - \lambda_4(s) = 0 \tag{74}$$

while the corresponding complementary slack conditions are

$$x(s) \le 0;\ \lambda_4(s) \ge 0, \text{ with } \lambda_4(s) = 0 \text{ if } x(s) < 0 \tag{75}$$

Moreover, $\lambda_4(s)$ is continuous whenever $x(s)$ is continuous (in this situation, since we do not have a bang-bang solution, the optimal control is continuous everywhere). To understand (74) and (75), note by inspecting the relaxed problem in (18)-(23), that since $H_*\left(\cdot\right)$ is linear, we have $x(s) = 0$ in the optimal solution whenever $\frac{\partial H_*}{\partial x}(s) > 0$. The newly defined function $\lambda_4(s) = \frac{\partial H_*}{\partial x}(s) - \frac{\partial H_\times}{\partial x}\left(s\right)$

[43] See Theorem 6.1 in Caputo (2005) on page 152 for the version with constraints on the control employed here.

captures precisely the nonnegative gradient $\frac{\partial H_*}{\partial x}(s)$. In other words, to deal with the additional monotonicity constraint in (15), rather than defining $H_\times$ and then imposing (74) and (75), we could have instead added a complementary slack condition $\frac{\partial H_*}{\partial x}(s) \cdot x(s) = 0$, together with $\frac{\partial H_*}{\partial x}(s) \geq 0$ and $x(s) \leq 0$ to the set of conditions (25)-(91). The key additional information that Pontryagin's Maximum Principle applied to the non-relaxed problem delivers, and which we do employ below, is that $\frac{\partial H_\times}{\partial x}(s)$ is continuous when $x(s)$ is continuous.

The only impact that these changes have on our previous analysis is on the argument and result of proposition 12. In particular, since $\lambda_4(s)$ is not differentiable, we cannot differentiate $\frac{\partial H_\times}{\partial x} = 0$ with respect to s at all $s \in [\underline{s}, \overline{s}]$, as we did in the proof of that proposition. Consider thus an interval $[s', s''] \subset [\underline{s}, \overline{s}]$ on which $e_\times(s)$ is constant, and thus $e_\times(s) = e_\times(s') = e_\times(s'')$ for all $s \in [s', s'']$, which by (16) also implies that $w_\times(s) = w_\times(s') = w_\times(s'')$ for all $s \in [s', s'']$. Note then that we can replicate the argument from the proof of proposition 12 for all s such that $x_\times(s) < 0$ in the optimal contract. Thus we can determine the effort and wage at every s with $x_\times(s) < 0$ by employing the same conditions identified in the case when the monotonicity constraint does not bind; the fact that $w_\times(s)$ is constant on $[s', s'']$ and continuous everywhere implies that values for $w_\times(s)$ can be imputed for all $s \in [\underline{s}, \overline{s}]$ in (36), (37) and (13) since these values equal $w_\times(s')$ which has been determined (also, values for $e_\times(s)$ can be imputed in (13)). However, the resulting wage and effort schedules will be functions of interval endpoints s' and s''.

To determine these values, note that since $\lambda_4(\cdot)$ is continuous, it must be that $\lambda_4(s') = \lambda_4(s'') = 0$, which from (74) implies then that $\lambda_1(s') + \lambda_2(s')\frac{1}{r}c_e(s', e(s')) = \lambda_1(s'') + \lambda_2(s'')\frac{1}{r}c_e(s'', e(s''))$. Thus, it must be that

$$\int_{s'}^{s''} \frac{d}{ds}\left[\lambda_1(s) + \lambda_2(s)\frac{1}{r}c_e(s, e(s))\right] ds = 0 \tag{76}$$

Noting that λ_4 does not appear in the expressions implied by Pontryagin's Maximum Principle for either $\lambda_1'(s)$ or $\lambda_2'(s)$, by the same argument as in the proof of proposition 12, we have that

$$\frac{d}{ds}\left[\lambda_1(s)+\lambda_2(s)\frac{1}{r}c_e(s,e(s))\right]=$$
$$=\frac{c_e(s,e_\times(s))}{u'(w_\times(s))}f(s)+c_{es}(s,e_\times(s))\int_{\underline{s}}^{s}\left[\frac{1}{u'(w_\times(\sigma))}-\frac{1}{u'(w_\times^n)}\right]f(\sigma)d\sigma-y'(e_\times(s))f(s)$$

It follows then from (76) that we must have

$$\int_{s'}^{s''}\left[\frac{c_e(s,e_\times(s))}{u'(w_\times(s))}f(s)+c_{es}(s,e_\times(s))\int_{\underline{s}}^{s}\left[\frac{1}{u'(w_\times(\sigma))}-\frac{1}{u'(w_\times^n)}\right]f(\sigma)d\sigma-y'(e_\times(s))f(s)\right]ds=0 \tag{77}$$

Equation (77) together with $e_\times(s')=e_\times(s'')$ constitute a system of two equations in two unknowns that allows determining the values for s' and s''. This system may have several solutions, corresponding to a case where there are multiple intervals on which $e_\times(s)$ is constant. It is also worth noticing here that $w_\times(s')=w_\times(s'')$ and $e_\times(s')=e_\times(s'')$ would *not* constitute an alternative system of equations to determine s' and s''. Given (16), these two conditions always hold simultaneously by construction when eliciting the wage and effort schedules as functions of s' and s''. Moreover, the condition in (77) is necessary to be satisfied by s' and s'' and thus needs to be imposed. □

Appendix A7.

Proof of Corollary 15. We have

$$\begin{aligned}\frac{d}{ds}\left[y(e_*(s))-rw_*(s)\right] &= \frac{d}{ds}\left[y(e_*(s))-rh(u_*(s))\right]\\ &= y'(e_*(s))\,e_*'(s)-rh'(u_*(s))u_*'(s)=e_*'(s)\left[y'(e_*(s))-\frac{c_e(s,e_*(s))}{u'(w_*(s))}\right]\end{aligned}$$

where we employed (20) for the last equality. Note that from (37), we have $y'(e_*(s))-\frac{c_e(s,e_*(s))}{u'(w_*(s))}=$

$\frac{c_{es}(s,e_*(s))}{f(s)} \int_{\underline{s}}^{s} \left[\frac{1}{u'(w_*(\sigma))} - \frac{1}{u'(w_*^n)}\right] f(\sigma)d\sigma$. Since the integral was shown to be strictly positive on $(\underline{s}, \widehat{s})$ in remark 13, while $e_*'(s) < 0$, it follows, as required, that $\frac{d}{ds}\left[y\left(e_*(s)\right) - rw_*(s)\right] < 0$. □

Proof of Lemma 9. We present here the remaining details from the proof presented in the main text. To show the sufficiency of the necessary conditions, we appeal to the main theorem in Seierstad (1984). To apply it, we need to show several facts. (*i*) The functions $\widehat{s} \to e_*(\widehat{s})$, $\widehat{s} \to u_*(\widehat{s})$ and $\widehat{s} \to v_*(\widehat{s})$, obtained by solving the necessary conditions in (25)-(32) for each fixed value of $\widehat{s}$, must be $\mathbb{C}_p^{(1)}$. The first two requirements are part of assumption 8, while the last follows from them by the definition of the state variable v. (*ii*) The costate variables should uniquely satisfy (25)-(28) for given trajectories of the state variables, but then the ensuing Note 1 states that this requirement, which is not necessarily obviously satisfied here, can be dropped from the text of the theorem. (*iii*) The function $\beta(\widehat{s}) \equiv H_*\left(e_*\left(\widehat{s}\right), u_*\left(\widehat{s}\right), v_*\left(\widehat{s}\right), x_*\left(\widehat{s}\right), \lambda_{1*}\left(\widehat{s}\right), \lambda_{2*}\left(\widehat{s}\right), \lambda_{3*}\left(\widehat{s}\right), \widehat{s}\right)$ should have the property that there exists some $\widehat{s}' \in [\underline{s}, \overline{s}]$ such that $\beta(\widehat{s}) \geq 0$ for $\widehat{s} < \widehat{s}'$ and $\beta(\widehat{s}) \leq 0$ for $\widehat{s} > \widehat{s}'$. In our case, given the definition of H_* and (25), we have $\beta(\widehat{s}) = \left[y\left(e(\widehat{s})\right) - rh\left(u(\widehat{s})\right)\right] f(\widehat{s}) + \lambda_3\left[ru(\widehat{s}) - c\left(\widehat{s}, e(\widehat{s})\right)\right] f(\widehat{s})$. Note that

$$\begin{aligned}
\frac{d}{d\widehat{s}}\left[\frac{\beta(\widehat{s})}{f(\widehat{s})}\right] &= \frac{d}{d\widehat{s}}\left[y\left(e_*(\widehat{s})\right) - rh(u_*(\widehat{s}))\right] + \lambda_3\left[ru'(\widehat{s}) - c_s\left(\widehat{s}, e(\widehat{s})\right) - c_e\left(\widehat{s}, e(\widehat{s})\right) e'(\widehat{s})\right] \\
&= \frac{d}{d\widehat{s}}\left[y\left(e_*(\widehat{s})\right) - rh(u_*(\widehat{s}))\right] - \lambda_3 c_s\left(\widehat{s}, e(\widehat{s})\right) < 0
\end{aligned}$$

where for the second equality we used (19)-(20) to cancel out the two terms, while for the inequality we used the result obtained in the proof of corollary 15, and the facts $\lambda_3 > 0$ and $c_s > 0$. Thus, by the monotonicity of $\frac{\beta(\widehat{s})}{f(\widehat{s})}$, there exists $\widehat{s}' \in [\underline{s}, \overline{s}]$ such that $\frac{\beta(\widehat{s})}{f(\widehat{s})} \geq 0$ for $\widehat{s} < \widehat{s}'$ and $\frac{\beta(\widehat{s})}{f(\widehat{s})} \leq 0$. Since $f(\widehat{s}) > 0$ for all $\widehat{s}$, the third condition of the theorem in Seierstad (1984) is immediately satisfied. (*iv*) The last requirement of the theorem in Seierstad (1984) is that the control variable has a bounded support, which is clearly not satisfied here. However, this assumption is essentially employed to conclude that the following supremum is finite $\sup_{\{e(s),u(s),v(s)\}_{s\in[\underline{s},\widehat{s}]}\in S} \int_{\underline{s}}^{\widehat{s}} [y(e(s)) - rh(u(s))] f(s)ds$ where $S \equiv \{\{e(s),u(s),v(s)\}_{s\in[\underline{s},\widehat{s}]} : u'(s) = \frac{1}{r}c_e(s,e(s))e'(s),\ v'(s) = [ru(s) - c(s,e(s))] f(s),$ $ru(\widehat{s}) - c(\widehat{s},e(\widehat{s})) = 0,\ v(\underline{s}) = 0,$ and $v(\widehat{s}) = \widehat{v}\}$ for all $\widehat{v}$ in a neighbourhood of $\overline{u}^n$. Since by assumption 8, this supremum is finite when $\widehat{v} = \overline{u}^n$, it is clearly finite also in a neighbourhood of that value by the smoothness of all functions involved. This completes the proof of lemma 9. □

Appendix A8. Proof of Lemma 16

By the Dynamic Envelope Theorem it follows that

$$\begin{aligned}
V_*'(r) &= \int_{\underline{s}}^{\widehat{s}} \frac{\partial}{\partial r} H_*(e(s),u(s),v(s),x(s),s) f(s)ds - \\
&\qquad - \lambda_3(\widehat{s}) \frac{\partial}{\partial r}\overline{u}^n - \frac{\partial}{\partial r}\{(1-r)h(u^n)\} + \frac{\partial}{\partial r}\mu[ru(\overline{s}) - c(\overline{s},e(\overline{s}))] \\
&= \int_{\underline{s}}^{\widehat{s}} \left\{-h(u(s))f(s) - \lambda_2(s)\frac{s}{r^2}c'(e(s))x(s) + \lambda_3(s)u(s)f(s)\right\} ds - \lambda_3(\widehat{s})u^n + h(u^n) + \mu u(\overline{s}) \\
&= \int_{\underline{s}}^{\widehat{s}} \left\{\left[-h(u(s)) + h'(u^n)u(s)\right] f(s) - \lambda_2(s)\frac{1}{r^2}c_e(s,e(s))x(s)\right\} ds + h(u^n) - h'(u^n)u^n + \mu u(\overline{s})
\end{aligned}$$

where for the last equality we used the fact that $\lambda_3(s) = h'(u^n)$.

Now, since as elicited in appendix A5, $\frac{\lambda_2(s)}{r} = -\int_{\underline{s}}^{s} [h'(u^n) - h'(u(\sigma))] f(\sigma)d\sigma$, while by (20), we have $\frac{1}{r}c_e(s,e(s))x(s) = u'(s)$, it follows that

$$\begin{aligned}
V_*'(r) &= \int_{\underline{s}}^{\widehat{s}} \left[-h(u(s)) + h'(u^n)u(s)\right] f(s)ds + h(u^n) - h'(u^n)u^n \\
&\quad + \int_{\underline{s}}^{\widehat{s}} \left[u'(s)\int_{\underline{s}}^{s} \left[h'(u^n) - h'(u(\sigma))\right] f(\sigma)d\sigma\right] ds + \mu u(\overline{s}) \qquad (78)
\end{aligned}$$

Integrating by parts the second integral in (78), we have

$$\begin{aligned}
& \int_{\underline{s}}^{\widehat{s}} \left[u'(s) \int_{\underline{s}}^{s} \left[h'(u^n) - h'(u(\sigma)) \right] f(\sigma) d\sigma \right] ds \\
= & \left[u(s) \int_{\underline{s}}^{s} \left[h'(u^n) - h'(u(\sigma)) \right] f(\sigma) d\sigma \right] \Bigg|_{\underline{s}}^{\widehat{s}} - \int_{\underline{s}}^{\widehat{s}} u(s) \left[h'(u^n) - h'(u(s)) \right] f(s) ds \\
= & - \int_{\underline{s}}^{\widehat{s}} u(s) \left[h'(u^n) - h'(u(s)) \right] f(s) ds - \mu u(\overline{s})
\end{aligned}$$

because $u(\widehat{s}) \int_{\underline{s}}^{\widehat{s}} \left[h'(u^n) - h'(u(\sigma)) \right] f(\sigma) d\sigma = -\mu u(\widehat{s})$ since the integral equals $-\mu$ by (68).

Substituting this result into (78), we obtain that

$$\begin{aligned}
V_*'(r) \;=\; & \int_{\underline{s}}^{\widehat{s}} \left[-h(u(s)) + h'(u^n) u(s) \right] f(s) ds + h(u^n) - h'(u^n) u^n + \mu u(\overline{s}) \\
& - \int_{\underline{s}}^{\widehat{s}} u(s) \left[h'(u^n) - h'(u(s)) \right] f(s) ds - \mu u(\overline{s}) \\
=\; & \int_{\underline{s}}^{\widehat{s}} \left[-h(u(s)) + u(s) h'(u(s)) \right] f(s) ds + h(u^n) - h'(u^n) u^n
\end{aligned}$$

which by using the facts that $u(s) = u(w(s))$, $u^n = u(w^n)$, $h'(u(s)) = \frac{1}{u'(w(s))}$ and $h'(u^n) = \frac{1}{u'(w^n)}$ imply the result of lemma 16. □

Appendix A9. Proof of Proposition 17

Denote by $z^n \equiv \frac{1}{u'(w_*^n)}$ and $z(s) \equiv \frac{1}{u'(w_*(s))}$ and note that (36) with $\widehat{s} = \overline{s}$ implies

$$z^n = \int_{\underline{s}}^{\overline{s}} z\,(s)\, f(s) ds - \mu \tag{79}$$

On the other hand, solving $z=\frac{1}{u'(w)}$ for w to get $w=(u')^{-1}\left(\frac{1}{z}\right)$, it follows that we have $\frac{u(w)}{u'(w)}-w=zu\left[(u')^{-1}\left(\frac{1}{z}\right)\right]-(u')^{-1}\left(\frac{1}{z}\right)$. Thus, (38) for $\widehat{s}=\overline{s}$ becomes

$$\begin{aligned} V_*'(r) &= \int_{\underline{s}}^{\overline{s}}\left[z\left(s\right)u\left[(u')^{-1}\left(\frac{1}{z\left(s\right)}\right)\right]-(u')^{-1}\left(\frac{1}{z\left(s\right)}\right)\right]f(s)ds \\ &\quad -\left\{z^n u\left[(u')^{-1}\left(\frac{1}{z^n}\right)\right]-(u')^{-1}\left(\frac{1}{z^n}\right)\right\} \end{aligned}$$

Denoting by

$$\varphi(z)\equiv zu\left[(u')^{-1}\left(\frac{1}{z}\right)\right]-(u')^{-1}\left(\frac{1}{z}\right)$$

and using (79) for the second equality, we have that

$$V_*'(r)=\int_{\underline{s}}^{\overline{s}}\varphi(z\left(s\right))f(s)ds-\varphi\left(z^n\right)=\int_{\underline{s}}^{\overline{s}}\varphi(z\left(s\right))f(s)ds-\varphi\left(\int_{\underline{s}}^{\overline{s}}z\left(s\right)f(s)ds-\mu\right)$$

We will show next that (i) $\varphi\left(\int_{\underline{s}}^{\overline{s}}z\left(s\right)f(s)ds\right)-\varphi\left(\int_{\underline{s}}^{\overline{s}}z\left(s\right)f(s)ds-\mu\right)\geq 0$, and (ii) $\int_{\underline{s}}^{\overline{s}}\varphi(z\left(s\right))f(s)ds-\varphi\left(\int_{\underline{s}}^{\overline{s}}z\left(s\right)f(s)ds\right)>0$, which would be enough to conclude that $V_*'(r)>0$. To show (i), since $\mu\geq 0$, it is sufficient to prove that φ is increasing. To show (ii), we employ Jensen's Inequality, for which we need to prove that φ is strictly convex.

We have

$$\begin{aligned} \varphi'(z) &= u\left[(u')^{-1}\left(\frac{1}{z}\right)\right]+zu'\left[(u')^{-1}\left(\frac{1}{z}\right)\right]\frac{1}{u''\left[(u')^{-1}\left(\frac{1}{z}\right)\right]}\left(-\frac{1}{z^2}\right)-\frac{1}{u''\left[(u')^{-1}\left(\frac{1}{z}\right)\right]}\left(-\frac{1}{z^2}\right) \\ &= u\left[(u')^{-1}\left(\frac{1}{z}\right)\right] \end{aligned}$$

where we used the fact that $u'\left[(u')^{-1}\left(\frac{1}{z}\right)\right]=\frac{1}{z}$. Since $u\left(w\right)\geq 0$, for $w\geq 0$, it follows that

$\varphi'(z) > 0$, as required for (i). On the other hand,

$$\varphi''(z) = u'\left[(u')^{-1}\left(\frac{1}{z}\right)\right]\frac{1}{u''\left[(u')^{-1}\left(\frac{1}{z}\right)\right]}\left(-\frac{1}{z^2}\right) = -\frac{1}{z^3}\frac{1}{u''\left[(u')^{-1}\left(\frac{1}{z}\right)\right]}$$

Thus, $\varphi''(z) > 0$ when u is strictly concave. Therefore, (ii) is also satisfied. This completes the proof of proposition 17. □

Appendix A10. Proof of Proposition 18

Since $c_e > 0$, if $\mathcal{A}$ accepts the contract $\{w^a, w^n, e^a\}$, then after learning his type, he exerts either effort e^a or 0. By making the standard transformations $u^n \equiv u(w^n)$, $u^a \equiv u(w^a)$ and employing the inverse utility function $h(\cdot)$ defined earlier, $\mathcal{P}$'s problem is to choose e, u^a, u^n so as to maximize $p^a y(e) - rp^a h(u^a) - (1-r)h(u^n)$ subject to $ru^a - c(s^a, e) \geq 0$ and $p^a[ru^a - c(s^a, e)] + (1-r)u^n = \overline{u}$, where we employed the fact that the participation constraint bind at optimum. The Lagrangian for this problem is $L(e, u^a, u^n, \gamma_1, \gamma_2) \equiv \{p^a y(e) - rp^a h(u^a) - (1-r)h(u^n)\} + \gamma_1[ru^a - c(s^a, e)] + \gamma_2\{p^a[ru^a - c(s^a, e)] + (1-r)u^n - \overline{u}\}$. The necessary first order equality conditions are

$$\frac{\partial L}{\partial u^a} = -rp^a h'(u^a) + \gamma_1 r + \gamma_2 rp^a = 0 \tag{80}$$

$$\frac{\partial L}{\partial u^n} = -(1-r)h'(u^n) + \gamma_2(1-r) = 0 \tag{81}$$

$$e\frac{\partial L}{\partial e} = e\left\{p^a y'(e) - \gamma_1 c_e(s^a, e) - \gamma_2 p^a c_e(s, e)\right\} = 0 \tag{82}$$

$$\gamma_1\frac{\partial L}{\partial \gamma_1} = \gamma_1[ru^a - c(s^a, e)] = 0 \tag{83}$$

$$\frac{\partial L}{\partial \gamma_2} = p^a[ru^a - c(s^a, e)] + (1-r)u^n - \overline{u} = 0 \tag{84}$$

while the necessary inequality conditions are $\frac{\partial L}{\partial e} \leq 0$; $ru^a - c(s^a, e) \geq 0$; $\gamma_1 \geq 0$ and $\gamma_2 > 0$. Note that (81) implies $\gamma_2 = h'(u^n)$, and then that (80) implies $\gamma_1 = p^a [h'(u^a) - h'(u^n)]$.[44]

We consider the following functional forms for the fundamentals of the model: $y(e) = e$, $u(w) = w^{\frac{1}{\beta}}$ with $\beta > 1$ (thus, implying $h(u) = u^\beta$), and $c(s, e) = s\frac{1}{\theta}e^\theta$ with $\theta > 1$, where $s \in \{s^a, s^c\}$ and $p^a \equiv \Pr\{s = s^a\}$. With these functional forms, we set the model parameters as follows $\beta = 1.6$, $\theta = 1.2$, $s^a = 0.2$, $p^a = 0.85$ and $\overline{u} = 1$. The values for r are chosen in the interval $[0.11, 0.99]$ where the value of the resulting optimal contract is positive if $\mathcal{P}$ implements positive effort.

To solve for the optimal contract for a particular set of parameters $(\beta, \theta, s^a, p^s, \overline{u}, r)$, we substituted for γ_1 and γ_2, as derived above from (80)-(81), into (82)-(84) and obtained a system of 3 non-linear equations in 3 unknowns (u^a, u^n, e^a) amenable to be solved numerically. We verified that the solution to this system corresponds to positive values for u^a, e^a, γ_1 and γ_2, and that $ru^a - c(s^a, e) \geq 0$ and $p^a y(e) - rp^a h(u^a) - (1-r) h(u^n) > 0$, i.e., that $\mathcal{P}$ has a weakly positive expected payoff from the resulting contract. Since (i) $\mathcal{P}$'s expected payoff is strictly positive at the unique critical point of the Lagrangian that we obtained, (ii) the set of $\mathcal{P}$'s feasible expected payoffs is bounded from above by the finite expected payoff from the first-best case of full information, (iii) $\mathcal{P}$'s expected payoff is continuous as a function of the contract variables, we conclude that the critical point corresponds to a global maximum. $\square$

Appendix A11. Proof of Proposition 19

When $\mathcal{P}$ can credibly promise to make a payment, w_3^c, when an audit is performed and it reveals that $\mathcal{A}$ exerted no effort, $\mathcal{P}$'s problem is to select a contract $\left\{\widehat{s} \in [\underline{s}, \overline{s}], \{e_3(s), w_3(s)\}_{s \in [\underline{s}, \widehat{s}]}, w_3^n, w_3^c\right\}$

[44]It can be shown that $V'(r) = p^a [h'(u^a) u^a - h(u^a)] - [h'(u^n) u^n - h(u^n)]$, which cannot be signed generically.

to solve the problem

$$\max_{\widehat{s},\{e(s),w(s)\}_{s\in[\underline{s},\widehat{s}]},w^n,w^c} \int_{\underline{s}}^{\widehat{s}} [y(e(s)) - rw(s)]\, f(s)ds - (1-r)\, w^n - r\,[1-F(\widehat{s})]\, w^c \tag{85}$$

$$\text{s.t. } s \in \arg\max_{\widetilde{s}\in[\underline{s},\widehat{s}]} [ru(w(\widetilde{s})) - c(s, e(\widetilde{s}))] \tag{86}$$

$$r\,[u(w(s)) - w^c] - c(s, e(s)) \geq 0, \text{ for all } s \in [\underline{s}, \widehat{s}] \tag{87}$$

$$\int_{\underline{s}}^{\widehat{s}} [ru(w(s)) - c(s, e(s))]\, f(s)ds + (1-r)\, u(w^n) + r\,[1-F(\widehat{s})]\, u(w^c) \geq \overline{u}. \tag{88}$$

Making the same transformations as in the solution for the optimal contract without commitment, and denoting by $u^c \equiv u(w^c)$, the optimal control problem for this case is the same as the one defined by (18)-(23), only that $\overline{u}^n$ is replaced in (22) by $\overline{u}^{nc}(\widehat{s}) \equiv \overline{u} - (1-r)u^n - r\,[1-F(\widehat{s})]\, u^c$, and (23) is replaced by $r\,[u(\widehat{s}) - u^c] - c(\widehat{s}, e(\widehat{s})) \geq 0$. The Hamiltonian H_3 associated with this problem takes the same form as H_* defined in (24). The corresponding necessary conditions elicited by the Pontryagin's Maximum Principle are the same as those in (25)-(32), only that (32) is replaced by

$$\mu \geq 0, \text{ with } \mu = 0 \text{ if } r\,[u(\widehat{s}) - u^c] - c(\widehat{s}, e(\widehat{s})) > 0 \tag{89}$$

Next, condition (33) is replaced by

$$H_3(e(\widehat{s}), u(\widehat{s}), v(\widehat{s}), x(\widehat{s}), \lambda_1(\widehat{s}), \lambda_2(\widehat{s}), \lambda_3(\widehat{s}), \widehat{s}) - \lambda_3 r u^c f(\widehat{s}) \geq 0, \text{ and } = 0 \text{ if } \widehat{s} < \overline{s} \tag{90}$$

Finally, denoting by $\mathcal{V}_3(u^n, u^c)$ the optimal value function of this optimal control problem, as a function of the variables u^n and u^c, the following first order conditions $\frac{d}{du^n}[\mathcal{V}_3(u^n, u^c) - (1-r)h(u^n)] = 0$ and $\frac{d}{du^c}[\mathcal{V}_3(u^n, u^c) - r\,[1-F(\widehat{s})]\, h(u^c)] = 0$ are necessary. The first of these two conditions implies by the same argument as in section 3.3 that $\lambda_3(s) = h'(u^n)$, for all $s \in [\underline{s}, \widehat{s}]$. On the other hand, the second condition implies $-\lambda_3(\widehat{s})\frac{\partial \overline{u}^{nc}}{\partial u^c} - \mu r - r[1-F(\widehat{s})]h'(u^c) = 0$, i.e.,

$$[1-F(\widehat{s})]h'(u^c) = [1-F(\widehat{s})]h'(u^n) - \mu \tag{91}$$

By the same argument as in the proof of Lemma 11, it follows that (68) must hold in this situation as well. Substituting in (68) the value of μ from (91), we conclude that the following condition, which is the counterpart of (36), must be satisfied by the optimal contract

$$\int_{\underline{s}}^{\widehat{s}} h'(u(s))f(s)ds + [1-F(\widehat{s})]h'(u^c) - h'(u^n) = 0 \tag{92}$$

Denoting by $V_3(r)$ the value of the optimal contract as a function of r, by the Dynamic Envelope Theorem and employing the fact $\lambda_3(s) = h'(u^n)$, we have that

$$\begin{aligned}
V_3'(r) &= \int_{\underline{s}}^{\widehat{s}} \frac{\partial}{\partial r} H_3(e(s), u(s), v(s), x(s), s) f(s)ds - \lambda_3(\widehat{s})\frac{\partial}{\partial r}\overline{u}^{nc} \\
&\quad -\frac{\partial}{\partial r}\{(1-r)h(u^n) + r[1-F(\widehat{s})]h(u^c)\} + \frac{\partial}{\partial r}\mu\{r[u(\widehat{s}) - u^c] - c(\widehat{s}, e(\widehat{s}))\} \\
&= \int_{\underline{s}}^{\widehat{s}} \left\{-h(u(s))f(s) - \lambda_2(s)\frac{s}{r^2}c'(e(s))x(s) + h'(u^n)u(s)f(s)\right\} ds \\
&\quad -h'(u^n)\{u^n - [1-F(\widehat{s})]u^c\} + h(u^n) - [1-F(\widehat{s})]h(u^c) + \mu[u(\widehat{s}) - u^c] \\
&= \int_{\underline{s}}^{\widehat{s}} \left[-h(u(s)) + h'(u^n)u(s)\right] f(s)ds + \int_{\underline{s}}^{\widehat{s}} u'(s)\left[\int_{\underline{s}}^{s}\left[h'(u^n) - h'(u(\sigma))\right] f(\sigma)d\sigma\right] ds \\
&\quad +h(u^n) - [1-F(\widehat{s})]h(u^c) - h'(u^n)\{u^n - [1-F(\widehat{s})]u^c\} + \mu[u(\widehat{s}) - u^c]
\end{aligned}$$

where for the last equality, we substituted $\frac{\lambda_2(s)}{r} = -\int_{\underline{s}}^{s}[h'(u^n) - h'(u(\sigma))]f(\sigma)d\sigma$ and $\frac{1}{r}c_e(s, e(s))x(s) =$

$u'(s)$. Integrating by parts, as in the proof of lemma 16, we have that

$$\begin{aligned}\int_{\underline{s}}^{\widehat{s}} u'(s)\left[\int_{\underline{s}}^{s}\left[h'(u^n)-h'(u(\sigma))\right]f(\sigma)d\sigma\right]ds &= -\int_{\underline{s}}^{\widehat{s}} u(s)\left[h'(u^n)-h'(u(s))\right]f(s)ds \\ &\quad +u(\widehat{s})\int_{\underline{s}}^{\widehat{s}}\left[h'(u^n)-h'(u(s))\right]f(s)ds\end{aligned}$$

Since by (92), we have $\int_{\underline{s}}^{\widehat{s}}[h'(u^n)-h'(u(\sigma))]f(\sigma)d\sigma = [1-F(\widehat{s})][h'(u^c)-h'(u^n)]$, it follows that

$$\begin{aligned}V_3'(r) &= \int_{\underline{s}}^{\widehat{s}}\left[-h(u(s))+h'(u^n)u(s)\right]f(s)ds - \int_{\underline{s}}^{\widehat{s}} u(s)\left[h'(u^n)-h'(u(s))\right]f(s)ds - [1-F(\widehat{s})]\,h(u^c) \\ &\quad +h(u^n)+u(\widehat{s})[1-F(\widehat{s})]\left[h'(u^c)-h'(u^n)\right]-h'(u^n)\{u^n-[1-F(\widehat{s})]u^c\}+\mu\,[u(\widehat{s})-u^c] \\ &= \int_{\underline{s}}^{\widehat{s}}\left[h'(u(s))u(s)-h(u(s))\right]f(s)\,ds + u(\widehat{s})[1-F(\widehat{s})]\left[h'(u^c)-h'(u^n)\right]-[1-F(\widehat{s})]\,h(u^c) \\ &\quad +h(u^n)-h'(u^n)\{u^n-[1-F(\widehat{s})]u^c\}+[1-F(\widehat{s})]\left[h'(u^n)-h'(u^c)\right][u(\widehat{s})-u^c] \\ &= \int_{\underline{s}}^{\widehat{s}}\left[h'(u(s))u(s)-h(u(s))\right]f(s)\,ds+[1-F(\widehat{s})]\left[h'(u^c)u^c-h(u^c)\right]+\left[h(u^n)-h'(u^n)u^n\right]\end{aligned}\tag{93}$$

where for the second equality, we substituted the value of μ derived from (91). Given then the results in (92) and (93), by the same argument as in the proof of proposition 17, it follows that $V_3'(r)>0$. This completes the proof of proposition 19. $\square$

Appendix A12. Proof of Proposition 21

Differentiating the equality in (52) with respect to s, we have $-\frac{1-r}{r}\lambda_2'(s)+\lambda_4'(s)=0$. Employing (55) and (57), it follows that $\gamma(s)=h'(u(s))-h'(u^n(s))$. Next, (56) and (58) imply that $\lambda_3(s)$ equals a nonnegative constant λ_3 for all s. Employing these results into (55), we conclude that $\lambda_2'(s)=rh'(u^n(s))f(s)-r\lambda_3 f(s)$. Integrating this equality between $\underline{s}$ and any arbitrary $s\in[\underline{s},\overline{s}]$, and accounting for $\lambda_2(\underline{s})=0$ from (58), we have then $\lambda_2(s)=r\int_{\underline{s}}^{s}[h'(u^n(t))-\lambda_3]f(t)dt$. Applying this result at $\overline{s}$, where $\lambda_2(\overline{s})=0$, it follows that $\lambda_3(s)=\lambda_3\equiv\int_{\underline{s}}^{\overline{s}}h'(u^n(t))f(t)dt$ for all $s\in[\underline{s},\overline{s}]$.

Moreover, since $h' > 0$ it follows that $\lambda_3 > 0$ and thus that (42) is satisfied with equality at optimum. Collecting these findings into (59), we obtain condition (60) from the text of proposition 21 once we account for the fact that h is strictly convex and thus h' is strictly increasing.

Differentiating now the equality in (51) with respect to s, as in (69), and then employing (53) and (55) to substitute for $\lambda_1'(s)$ and $\lambda_2'(s)$, we have

$$\left[-y'(e)\,f(s)+\gamma(s)\,c_e(s,e(s))\,f(s)-\lambda_2(s)\,\frac{1}{r}c_{ee}(s,e(s))\,x(s)+\lambda_3(s)\,c_e(s,e(s))\,f(s)\right]+$$
$$+\left[rh'(u(s))f(s)-\gamma(s)rf(s)-\lambda_3(s)\,rf(s)\right]\frac{1}{r}c_e(s,e(s))+\lambda_2(s)\,\frac{1}{r}c_{es}(s,e(s))+\lambda_2(s)\,\frac{1}{r}c_{ee}(s,e(s))\,e'(s)=$$
$$=-y'(e)f(s)+h'(u(s))f(s)c_e(s,e(s))+\lambda_2(s)\,\frac{1}{r}c_{es}(s,e(s))=0 \tag{94}$$

Substituting into (94) the expression for $\lambda_2(s)$ derived above, we obtain

$$h'(u(s))c_e(s,e(s))\,f(s)+\left[\int_{\underline{s}}^{s}\left\{h'(u^n(\sigma))-\int_{\underline{s}}^{\overline{s}}h'(u^n(t))f(t)dt\right\}f(\sigma)d\sigma\right]c_{es}(s,e(s))=y'(e)f(s)$$

which can be rewritten as in (61) in the text of proposition 21. Note also here that since $\lambda_2'(s) = r\left[h'(u^n(s))-\lambda_3\right]f(s)$, $\lambda_2(\underline{s})=0$, $\lambda_2(\overline{s})=0$, and $\frac{d}{ds}\left[h'(u^n(s))\right]<0$, by the same argument as in the proof of remark 13, we can conclude that $\lambda_2(s)>0$ for all $s\in[\underline{s},\overline{s}]$ to prove remark 22.

To evaluate the effect of r on the optimal value of the contract with communication, which we denote here by $V_+(r)$, employing the Dynamic Envelope Theorem, we have

$$\begin{aligned}
V_+'(r) &= \frac{\partial}{\partial r}\int_{\underline{s}}^{\overline{s}} L_+\left(e(s),u(s),u^n(s),v(s),x(s),k(s),\lambda_1(s),\lambda_2(s),\lambda_3(s),\lambda_4(s),\gamma(s),s\right)ds \\
&= \int_{\underline{s}}^{\overline{s}}\left\{\begin{array}{c}\left[-h(u(s))+h(u^n(s))+\gamma(s)u(s)+\lambda_3(s)\,u(s)-\lambda_3(s)\,u^n(s)\right]f(s)+\\ +\lambda_2(s)\,\frac{1}{r^2}\left[k(s)-c_e(s,e(s))x(s)\right]\end{array}\right\}ds \\
&= \int_{\underline{s}}^{\overline{s}}\left\{\left[\begin{array}{c}-h(u(s))+h(u^n(s))+u(s)\left[h'(u(s))-h'(u^n(s))\right]\\ +\lambda_3\left[u(s)-u^n(s)\right]\end{array}\right]f(s)+\frac{1}{r}\lambda_2(s)\left[u^{n\prime}(s)-u'(s)\right]\right\}ds
\end{aligned}$$

where for the third equality we used (47) to substitute for $k(s) - c_e(s, e(s))x(s)$ and then the fact $k(s) = u^{n\prime}(s)$ as well as the result derived above for $\gamma(s)$.

Integrating by parts the second term in the expression for $V_+'(r)$ derived above, we have

$$\int_{\underline{s}}^{\overline{s}} \frac{1}{r}\lambda_2(s)\left[u^{n\prime}(s) - u'(s)\right] ds = \int_{\underline{s}}^{\overline{s}} \left[u^{n\prime}(s) - u'(s)\right] \int_{\underline{s}}^{s} \left[h'(u^n(t)) - \lambda_3\right] f(t)dtds$$

$$= \left\{\left[u^n(s) - u(s)\right] \int_{\underline{s}}^{s} \left[h'(u^n(t)) - \lambda_3\right] f(t)dt\right\}\Big|_{\underline{s}}^{\overline{s}} - \int_{\underline{s}}^{\overline{s}} \left[u^n(s) - u(s)\right] \left[h'(u^n(s)) - \lambda_3\right] f(s)ds$$

$$= -\int_{\underline{s}}^{\overline{s}} \left[u^n(s) - u(s)\right] \left[h'(u^n(s)) - \lambda_3\right] f(s)ds$$

where for the last equality we used the fact that $\lambda_3 = \int_{\underline{s}}^{\overline{s}} h'(u^n(t))f(t)dt$ to conclude that the first term in the previous expression equals zero.

Substituting this result into the expression for $V_+'(r)$ we obtain

$$\begin{aligned} V_+'(r) &= \int_{\underline{s}}^{\overline{s}} \left[\begin{array}{c} -h(u(s)) + h(u^n(s)) + u(s)\left[h'(u(s)) - h'(u^n(s))\right] \\ +\lambda_3\left[u(s) - u^n(s)\right] - \left[u^n(s) - u(s)\right]\left[h'(u^n(s)) - \lambda_3\right] \end{array}\right] f(s)ds \\ &= \int_{\underline{s}}^{\overline{s}} \left[u(s)h'(u(s)) - h(u(s)) - u^n(s)h'(u^n(s)) + h(u^n(s))\right] f(s)ds \end{aligned}$$

Now, note that for the function $\Lambda(u) \equiv uh'(u) - h(u)$ we have $\Lambda'(u) = h''(u)$, which is strictly positive, as argued above. Therefore, using the fact that $u(s) \geq u^n(s)$, we conclude that $V_+'(r) = \int_{\underline{s}}^{\overline{s}} \left[\Lambda(u(s)) - \Lambda(u^n(s))\right] f(s)ds \geq 0$. This completes the proof of proposition 21.

The *generalized Legendre-Clebsch condition* for this problem is a combination of the corresponding condition for problems with multiple controls (see Theorem 6.2 in Krener (1977)) and the condition for problems with state constraints (see conditions (84) or (85) Seywald and Cliff (1993)). Thus, on intervals in $[\underline{s}, \overline{s}]$ on which the constraint in (50) does not bind, the condition stated in

Krener (1977) applied to our problem requires that the following matrix[45]

$$\begin{pmatrix} -\frac{\partial}{\partial x}\left[\frac{d}{ds^2}\left(\frac{\partial L_+}{\partial x}\right)\right] & -\frac{\partial}{\partial x}\left[\frac{d^2}{ds^2}\left(\frac{\partial L_+}{\partial k}\right)\right] \\ -\frac{\partial}{\partial k}\left[\frac{d^2}{ds^2}\left(\frac{\partial L_+}{\partial x}\right)\right] & -\frac{\partial}{\partial k}\left[\frac{d^2}{ds^2}\left(\frac{\partial L_+}{\partial k}\right)\right] \end{pmatrix} \tag{95}$$

be symmetric and negative semidefinite when evaluated at the candidate solution, where we accounted for the fact that x and k appear in the second order derivatives with respect to s of $\frac{\partial H_+}{\partial x}$ and $\frac{\partial H_+}{\partial k}$. On the other hand, on intervals in $[\underline{s}, \overline{s}]$ where the constraint in (50) binds, applying the result from Seywald and Cliff (1993) and other standard results from multivariate constrained optimization (for instance, Theorem 19.7 on page 461 in Simon and Blume (1994)), it follows that it is necessary that the following determinant be nonnegative at the candidate solution[46]

$$\begin{vmatrix} 0 & \frac{\partial}{\partial x}\frac{d}{ds}\left[ru - c(s,e)\right] & \frac{\partial}{\partial k}\frac{d}{ds}\left[ru - c(s,e)\right] \\ \frac{\partial}{\partial x}\frac{d}{ds}\left[ru - c(s,e)\right] & -\frac{\partial}{\partial x}\left[\frac{d}{ds^2}\left(\frac{\partial L_+}{\partial x}\right)\right] & -\frac{\partial}{\partial x}\left[\frac{d^2}{ds^2}\left(\frac{\partial L_+}{\partial k}\right)\right] \\ \frac{\partial}{\partial k}\frac{d}{ds}\left[ru - c(s,e)\right] & -\frac{\partial}{\partial k}\left[\frac{d^2}{ds^2}\left(\frac{\partial L_+}{\partial x}\right)\right] & -\frac{\partial}{\partial k}\left[\frac{d^2}{ds^2}\left(\frac{\partial L_+}{\partial k}\right)\right] \end{vmatrix} \tag{96}$$

We have already showed that $\frac{d}{ds}\left(\frac{\partial L_+}{\partial x}\right) = -y'(e)f(s)+h'(u(s))f(s)c_e(s,e(s))+\lambda_2(s)\frac{1}{r}c_{es}(s,e(s))$ in (94). On the other hand, employing (55) and (57), we have $\frac{d}{ds}\left(\frac{\partial L_+}{\partial k}\right) = -\frac{1-r}{r}\lambda_2'(s) + \lambda_4'(s) = -(1-r)h'(u(s))f(s) + (1-r)\gamma(s)f(s) + (1-r)h'(u^n(s))f(s)$. Then, suppressing arguments of the various functions when there is no risk of confusion, we have

$$\begin{aligned} \frac{d^2}{ds^2}\left(\frac{\partial L_+}{\partial x}\right) &= -y''xf + h''(u)u'fc_e + h'(u)c_{ee}x + \lambda_2'\frac{1}{r}c + \lambda_2\frac{1}{r}c_{ees}x + K_1 \\ &= -y''xf + h''(u)\left[-\frac{1-r}{r}k + \frac{1}{r}c_e x\right]fc_e + h'(u)c_{ee}x + \lambda_2\frac{1}{r}c_{ees}x + K_2 \end{aligned}$$

where for the second equality, we used (47) and (55). K_1 and K_2 are terms that do not depend on

[45] We use here the fact that we have a pure state constraint and thus the first order derivatives with respect to the control variables of the Lagrangian L_+ and Hamiltonian $H_+ \equiv L_+ - \gamma\left[ru - c(s,e)\right]$ are identical.

[46] The function g from the result in Seywald and Cliff (1993) (defined in equation (37)) is here $\frac{d}{ds}\left[ru(s) - c(s,e)\right]$.

either x or k. On the other hand, (K_3 and K_4 again do not depend on the controls) we have

$$\begin{aligned}
\frac{d^2}{ds^2}\left(\frac{\partial L_+}{\partial k}\right) &= -(1-r)\,h''(u)u'f + (1-r)\,h''(u^n)u^{n\prime}f + K_3 \\
&= -(1-r)\,h''(u)\left[-\frac{1-r}{r}k + \frac{1}{r}c_e x\right]f + (1-r)\,h''(u^n)kf + K_4
\end{aligned}$$

where for the second equality, we used (47) and (46). It follows then that

$$\begin{aligned}
\frac{\partial}{\partial x}\frac{d^2}{ds^2}\left(\frac{\partial L_+}{\partial x}\right) &= -y''f + h''(u)\frac{1}{r}\left(c_e\right)^2 f + h'(u)fc_{ee} + \lambda_2\frac{1}{r}c_{ees} \\
\frac{\partial}{\partial k}\frac{d^2}{ds^2}\left(\frac{\partial L_+}{\partial x}\right) &= \frac{\partial}{\partial x}\frac{d^2}{ds^2}\left(\frac{\partial L_+}{\partial k}\right) = -\frac{1-r}{r}h''(u)fc_e \\
\frac{\partial}{\partial k}\frac{d^2}{ds^2}\left(\frac{\partial L_+}{\partial k}\right) &= \frac{(1-r)^2}{r}h''(u)f + (1-r)\,h''(u^n)f
\end{aligned}$$

Finally, we have $\frac{d}{ds}\left[ru - c\right] = ru' - c_s - c_e x = -(1-r)\,k - c_s$ using (47). Therefore,

$$\frac{\partial}{\partial x}\frac{d}{ds}\left[ru - c(s,e)\right] = 0 \text{ and } \frac{\partial}{\partial k}\frac{d}{ds}\left[ru - c(s,e)\right] = -(1-r)$$

Now, assuming again that $c_{ees} \geq 0$, it follows that $-\frac{\partial}{\partial x}\frac{d^2}{ds^2}\left(\frac{\partial L_+}{\partial x}\right) < 0$, whereas the determinant of the matrix in (95) is $\left[-y''f + h''(u)\frac{1}{r}\left(c_e\right)^2 f + h'(u)fc_{ee} + \lambda_2\frac{1}{r}c_{ees}\right]\left[\frac{(1-r)^2}{r}h''(u)f + (1-r)\,h''(u^n)f\right] - \left[-\frac{1-r}{r}h''(u)fc_e\right]^2 > 0$. Therefore, the matrix in (95) is indeed negative semidefinite. On the other hand, the determinant in (96) equals $(1-r)^2\left[-y''f + h''(u)\frac{1}{r}\left(c_e\right)^2 f + h'(u)fc_{ee} + \lambda_2\frac{1}{r}c_{ees}\right] > 0$. We conclude thus that the generalized Legendre-Clebsch condition is indeed satisfied. □

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