Efficiency-based measures of inequality

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Roadmap

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- A society is characterized by a distribution of income (or wealth) levels.
- Which society do you prefer?

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- This gives rise to *utilitarianism*: you care about the sum of utilities
- Suppose your utility function has decreasing marginal utility for money
- Then we could improve your ex-ante well-being by moving money from the rich to the poor.
    - a preference for equity
    - comparison to choice under uncertainty

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- We propose a still different notion of fate, while otherwise remaining faithful to Harsanyi's thought experiment
- Specifically, we suppose that incomes are allocated through a *contest*
- Evaluate a society on the basis of EU, using beliefs derived from the equilibrium properties of a suitable non-cooperative game

A brief review of inequality measures

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- Axiomatic literature on what we want from inequality measure: Fields and Fei (1978), Shorrocks (1980), Shorrocks (1984), Foster (1983), Sen and Foster (1997), ..., Vega, Urrutia and Volij (2013)
- Quite distant from utilitariansim (Sen (1973): Weak equity axiom)

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- Let X be the set of allocations
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- Notes:
  - Reduces inequality to one number
  - We will fix n

Central Axioms (1 & 2 of 3)

See e.g. Fields and Fei (1978), Foster (1983), Sen and Foster (1997), Vega, Urrutia and Volij (2013):

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Let $x, x' \in X$ be the same except for some pair $x_i \leq x_j$ we have $x'_i = x_i - \Delta$ and $x'_j = x_j + \Delta$, for some $\Delta > 0$. Then $M(x) < M(x')$.

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- Essential defining property
- Corresponds to Lorenze domination (Dasgupta et al (1973), Rothschild and Stiglitz (1973))
- Some measures violate it (var. of logarithms, Foster and Ok (1999))

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Imposing homogeneity precludes a welfarist approach

- Atkinson (1970): judgments should come from social welfare function, enabling one to consider a tradeoff between total income and unequal income
- Sen (1997): "objective" versus "normative" measures

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- We argue that utilitarianism and the axiomatic literature can be reconciled
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- One-shot *allocation game* to show that equilibrium (expected) payoff corresponds to a well-known inequality measure
- Repeated allocation game to show that multiple measures can be induced, among them Rawls (1971) maxmin-measure
- More general allocation contests to provide a better understanding of the limits of this approach

Roadmap

Allocation games

Kuzmics, Palfrey and Rogers (2014)

- fix $x \in X$ (recall n is fixed)
- n players, $I = \{1, \ldots, n\}$
- Action set $A_i = A = \{1, ..., n\}$ for each i
- Payoffs to all players are zero unless a permutation of $(1, 2, 3, ..., n)$ is played, in which case the player who plays action i receives payoff x_i.

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Solution

- Many equilibria
- We consider the (unique) symmetric equilibrium with positive payoffs

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- Many variations possible
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- Proof of concept

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- Note: $V(\alpha x) = \alpha V(x)$, for $\alpha > 0$
- We define normalized payoff as $V_*(x) = V(x)/\bar{x} = P(x)$

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- Shorrocks (1980): the mean log deviation measure satisfies all three axioms and others
  - Additive decomposability
  - Decomposition coefficients independent of subgroup incomes (equal to population shares)

Tradeoff between inequality and total wealth

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- Use equilibrium payoff $V(x)$
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- Severe requirement for larger n: $\left(\frac{n-1}{n}\right) x_i < \bar{x}$
  - But not Rawlsian

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- To satisfy homogeneity it is sufficient that multiplying all prizes x_i with the same $\alpha > 0$ changes all payoffs in the game affinely
  - then individual behavior in the game is unchanged
- Less equal societies are judged worse for reasons independent of "risk aversion"

Roadmap

Repeated allocation games

- n players play an allocation game (with $x \in X$) repeatedly
- at discrete points in time
- without any feedback
- until "coordination" is achieved (end of game)

- Players discount with $\delta \in [0, 1)$
- We consider stationary symmetric equilibrium

Equilibrium

Lemma (uniqueness)

The repeated allocation game with allocation $x \in X$ has a unique stationary symmetric equilibrium that generates positive payoffs. Its expected payoff is given by the value V that solves:

$$(1-\delta)V(\sum_j x_j - n\delta V)^{n-1} = (n-1)! \prod_j (x_j - \delta V)$$

(Proof)

Measures of Inequality

- Let $V^\delta(x)$ denote the equilibrium payoff
- It is homogeneous of degree one: $V^\delta(\alpha x) = \alpha V^\delta(x)$
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- Another measure: per-period probability of coordination $Q^\delta(x)$
- $V^\delta(x) = Q^\delta(x)\bar{x} + \delta(1 - Q^\delta(x))Q^\delta(x)\bar{x} + \ldots$

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- $V^\delta_*(x) = \frac{V^\delta(x)}{\bar{x}} = \frac{Q^\delta(x)}{1-\delta(1-Q^\delta(x))}$
- Monotone transformation (so same ranking)

Pigou-Dalton transfer principle

Proposition (Pigou-Dalton)

The measure of efficiency defined by V_*^δ satisfies symmetry, homogeneity and the Pigou-Dalton axiom, for all $\delta \in [0, 1)$.

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- Includes the one shot allocation game
- V_*^δ can be viewed as a (one parameter) class of relative inequality measures

(Proof)

Extreme points of $V_*^\delta(x)$

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- Thus, the Harsanyi (1953) utilitarian DM, using the contest as $\delta \to 1$, has Rawls (1971) maxmin preferences under expected utility
  - No "ambiguity aversion"

Tradeoff between inequality and total wealth

- As before, we use non-normalized payoff $V^\delta(x)$

Proposition (increasing the value of one prize)

The expected payoff in the repeated allocation game, $V^\delta(x)$, *decreases* in the value of prize x_i if and only if $\frac{n-1}{n}\left(x_i - \delta V^\delta(x)\right) > \bar{x} - \delta V^\delta(x)$.

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- One shot: $\frac{n-1}{n} x_i > \bar{x}$
- Repeated: $\frac{n-1}{n} x_i > \bar{x} - \frac{1}{n}\delta V^\delta(x)$
- The latter is generally weaker (but equivalent for $n = 2$ and $n \to \infty$)
- An increase in x_i may decrease expected payoff in the repeated game yet not in the one-shot game

Proof: From lemma, take logs, differentiate w.r.t. x_i; use shape of the function

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Allocation Contests

- In the spirit of Frank and Cook (1995) and contest literature (e.g. Konrad (2009))
- Fix allocation $x \in X$
- Players $I = \{1, ..., n\}$
- Set of effort levels $E = \mathbb{R}_+$

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- The utility function for player i is given by

$$u_i(e_i, e_{-i}) = \sum_j x_j \varphi_i^j(e_i, e_{-i}) - c(e_i) \sum_j x_j$$

Further assumptions

- φ is *anonymous*
- φ and c are smooth functions (twice continuously differentiable)
- c is increasing and convex
- Inada-like conditions $c'(0) = 0$ and $c'(\infty) = \infty$

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- Inada-like conditions $c'(0) = 0$ and $c'(\infty) = \infty$
- $\sum_j x_j \frac{\partial}{\partial e_i} \varphi_i^j(e, \ldots, e)$ is a decreasing function in e
- $\sum_j x_j \frac{\partial}{\partial e_i} \varphi_i^j(0, \ldots, 0) = \infty$
- $\sum_j x_j \frac{\partial}{\partial e_i} \varphi_i^j(\infty, \ldots, \infty) = 0$
- $\sum_j x_j \frac{\partial^2}{\partial e_i \partial e_k} \varphi_1^j(e, \ldots, e)$ is non-positive

Equilibrium

- Moreover, we assume existence of an interior pure strategy symmetric equilibrium
- From before: $u_i(e_i, e_{-i}) = \sum_j x_j \varphi_i^j(e_i, e_{-i}) - c(e_i) \sum_j x_j$

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- Symmetry: $\sum_j x_j \left(\frac{\partial}{\partial e_i} \varphi_i^j(e^*, ..., e^*) - c'(e^*) \right) = 0$

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- Moreover, we assume existence of an interior pure strategy symmetric equilibrium
- From before: $u_i(e_i, e_{-i}) = \sum_j x_j \varphi_i^j(e_i, e_{-i}) - c(e_i) \sum_j x_j$
- FOC: $\sum_j x_j \frac{\partial}{\partial e_i} \varphi_i^j(e_i, e_{-i}) - c'(e_i) \sum_j x_j = 0$
- Symmetry: $\sum_j x_j \left(\frac{\partial}{\partial e_i} \varphi_i^j(e^*, ..., e^*) - c'(e^*) \right) = 0$
- Let $E(x) = e^*$
- By construction, $E(x)$ satisfies anonymity and homogeneity
- Expected payoff $U(x) = \bar{x} - c(e^*) \sum_j x_j$

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Pigou-Dalton

The measure $E(x)$ (and $U(x)$) satisfies Pigou-Dalton if and only if, for all possible symmetric equilibrium effort levels e^* there is an $\bar{\epsilon} > 0$ such that for every $\epsilon \in (0, \bar{\epsilon})$ the distribution $\varphi_1(e^* + \epsilon, e^*, ..., e^*)$ dominates the uniform distribution in terms of the likelihood-ratio.

(Proof)

Tradeoff between inequality and total wealth

...is not very clean, but

Lemma

$U(x)$ increases in x_1 (highest prize) if and only if

$$\frac{1}{n} - c(e^*) - \frac{\partial e^*}{\partial x_1} c'(e^*) \sum_j x_j > 0.$$

A final example

- Three prizes: $x_1 \geq x_2 \geq x_3$
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- Note: ϕ satisfies LR-dominance
- Cost function: $c(f_1) = (x_1 + x_2 + x_3)\frac{1}{2}(f_1)^{\frac{2}{\alpha}}$, for some $\alpha > 0$.
- Utility of, e.g., player 1:

$$U_1(f_1, f_{-1}) = x_1\varphi_1^1(f) + x_2\varphi_1^2(f) + x_3\varphi_1^3(f) - (x_1 + x_2 + x_3)c(f_1)$$

A final example

Equilibrium payoff

$U(x) = \frac{1}{3}\left((1 - \frac{\alpha}{3})x_1 + (1 - \frac{\alpha}{12})x_2 + (1 + \frac{5\alpha}{12})x_3\right)$

- For $\alpha \leq 3$ then for any values x_1, x_2, x_3 this is the (unique) equilibrium payoff.
- For $3 < \alpha \leq \frac{24}{5}$, if second highest prize x_2 is large enough, then this is the (unique) equilibrium payoff.
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- For $\alpha > \frac{24}{5}$, there is no symmetric pure strategy equilibrium.

- Notice: $U(x)$ is increasing in x_1 iff $\alpha < 3$
- Intuition: increasing α induces fiercer competition

Roadmap

Future directions

- Explore the mappings between axioms and classes of games
  - Have some preliminary observations already
  - Mechanism design approach
- Deeper modeling of inequality issues
  - Social mobility, *opportunity*
  - Income/leisure tradeoffs
- Incorporating productive effort
  - What forces govern an observed income (wage) distribution?
  - If effort is both wasteful and productive, incentives are necessary but costly

(Population size)

Conclusion

- Behind the veil of ignorance
- if income levels are distributed through a contest
- the ex-ante expected utility (given a symmetric equilibrium of this contest)
- can be seen as an efficiency-based measure of inequality
- which satisfies the axioms of anonymity, homogeneity (after normalizing), and the Pigou-Dalton transfer principle (under some conditions)
- with the added benefit that one can naturally talk about welfare.

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V &= (n-1)!\alpha_2\alpha_3...\alpha_n x_1 + (1-(n-1)!\alpha_2\alpha_3...\alpha_n)\delta V \\
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- Plugging back in: $(1-\delta)V(\sum_j x_j - n\delta V)^{n-1} = (n-1)!\Pi_j(x_j - \delta V)$

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- which follows from Cauchy-Schwarz inequality: $\sum_j (a_j)^2 \sum_j (b_j)^2 \geq (\sum a_j b_j)^2$ where $a_j = \sqrt{x_j - \delta V}$ and $b_j = \frac{1}{\sqrt{x_j - \delta V}}$.

(Return)

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- Take i and j with $x_i < x_j$

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- As g is increasing this implies the zero of g increases [picture]

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- $MU_1(x'; e^*) = \sum_l x_l' a_l = \sum_l x_l a_l - a_i\Delta + a_j\Delta = -a_i\Delta + a_j\Delta$
- $MU_1(x'; e^*) > 0$ iff $a_j > a_i$

Proof of P-D in the allocation contest (2)

- $MU_1(x'; e^*) > 0$ iff $a_j > a_i$ iff

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- Take $x_j \geq x_i$. PD requires $E(x') > E(x)$, which is equivalent to LR-dominance, and conversely.

(Return)

Population size

- There is no need to fix n
- The traditional axiom is one of "replication invariance": replicating a society leaves the measure unchanged, i.e., $M(x; x) = M(x)$
- We could normalize our measures to satisfy such an axiom

Population size

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- The traditional axiom is one of "replication invariance": replicating a society leaves the measure unchanged, i.e., $M(x; x) = M(x)$
- We could normalize our measures to satisfy such an axiom
- But it is more natural in our framework to satisfy "linear replication variance": duplicating a society doubles the extent of inequality in the world, i.e. $M(x; x) = 2M(x)$
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- Result: $V_*(x)$ agrees with MLD(x) for populations of equal size, but does not generally agree when comparing populations of different sizes.

(Return)