Abstract. This paper considers an infinitely repeated three-player Bayesian game with lack of information on two sides, in which an informed player plays two zero-sum games simultaneously at each stage against two uninformed players. Under a correlated prior, the informed player faces the problem of how to optimally disclose information among two uninformed players in order to maximize his long term average payoffs. The objective is to understand the effects of “information spillover” from one game to the other in the Nash-equilibrium payoff set of the informed player. We show that there is a “robust” class of games for which information spillover does not preclude the informed player to obtain the upper bound of his equilibrium payoffs. Moreover, we provide a characterization, through a small adaptation of Hart (1985), of equilibrium payoffs of the game and obtain certain necessary conditions on equilibria that attain that upper bound for the informed player.

1. Introduction and Literature Review

This paper considers a class of games closely related to two-person repeated zero-sum games with lack of information on one side developed in the seminal work of Aumann, Maschler, Stearns (1995). There is not one, but two uninformed players who play at each stage of the repeated game a zero-sum game against one informed player. The informed player collects the payoff of each one of the stage games, and the uninformed players only care (payoffwise) about their own stage games. Players are allowed to observe all the actions (but not the payoffs) realized at each stage of the repeated game. This introduces new strategic problems for the informed player, since the use of information against one uninformed player in one game is observed by the other uninformed player. This is what we call “information

I am grateful to Paulo Barelli and Hari Govindan for their guidance and encouragement. I would like to thank Heng Liu, Bingchao Huangfu, Yu Awaya, Asen Kochov, Nicolas Riquelme, Zhizhen Ma, Rida Laraki, Olivier Gossner, Jérôme Renault and all the participants of the Game Theory Seminar in Paris and the Theory Seminar in Rochester for their suggestions.
spillover”: the ability of an uninformed player to learn payoff-relevant hidden information from payoff-irrelevant information.

Theoretically, we could argue that the modeling of the effects of the information spillover within a zero-sum game framework enables us to draw on preexisting techniques from the literature on zero-sum games in order to settle benchmark results: zero-sum games are usually easier to analyze than their non-zero-sum counterparts, and this allows us to obtain clean results before developing a non-zero-sum theory of this phenomenon.

Empirically, this framework allows us to model a number of relevant political phenomena. As an illustration we could consider a situation where a superpower such as the USA negotiates an agreement with two militarily smaller countries, Iran and Russia. Iran has an expansionist agenda in the region and the USA wants to block it. The optimal action for the USA is to reveal information about its military arsenal in order to threaten Iran. Russia, however, has access to the outcomes of the negotiation between the USA and Iran, and is simultaneously negotiating a reduction of some of its weapon’s arsenal with the USA. Here the optimal action for the USA would be not to reveal anything about its arsenal. How should the military power play in this situation? What is the best possible payoff it could obtain? These are some of the questions that motivate our analysis. Another example for which this model could be used, in the realm of Political Economy, is the competition between a national political party and two smaller parties in elections in two different districts. The idea is that the national party holds private but seminal information regarding the current state of affairs of the country that is important for the smaller parties in the local election and has to choose how much information to disclose simultaneously to the smaller parties.

1.1. Quick Summary of the Results and Organization. The main results of this paper are located in Sections 6, 7 and 8.

They can be summarized as follows:

(1) We provide a sufficient condition on the payoffs of each zero-sum game, called not completely revealing, under which the informed player can always attain the upper bound of equilibrium payoffs (the sum of the (uniform) values in each zero-sum game), no matter what the correlation between the games is. This condition reveals that the problem of attaining the upper bound by the informed player is not only a problem

\footnote{See section 4 for a precise definition.}
of correlation between the games, but also depends in a very specific way on the
genometry of the nonrevealing value function of each of the zero-sum games. This
genometric condition arises from the specific payoff structure of the zero-sum games
and is therefore “orthogonal” to correlation. This is somewhat surprising since the
existence of correlation “impairs” the informed player in his ability to play the two
zero-sum games separately. This impairment is however irrelevant and the informed
player can circumvent it, provided the zero-sum games are not completely revealing.
Moreover, the class of games satisfying this condition encompasses cases in which the
optimal strategies of the informed player in each zero-sum game are not “aligned” in
the sense that in one game the optimal strategy is to reveal and in the other to conceal
information. We show therefore that this apparent tension between concealing and
revealing information does not translate to lower payoffs for the informed player
and there are equilibria of the game in which the informed player can attain the
upper-bound of equilibrium payoffs.

(2) When there is correlation between the zero-sum games, revelation of information in
one game induces revelation in the other game, as was exemplified above. A seminal
problem for the informed player is therefore how to “optimally” disclose information
between games. But what is the precise meaning of optimal information disclosure
in our context? Theorems 7.13 and 8.4 attempt to answer this question.

All these results generalize to a model with one informed player and multiple uninformed
ones. Section 4 provides motivations for the results above. In section 5, besides proving
existence of equilibrium in this model, we establish that the lower bound on equilibrium
payoffs for the informed player is actually always attainable. The Appendices B and C gather
technical results that are important for the analysis but not central. Appendix A concentrates
all the robustness results with respect to payoff perturbations of the class of games that we
introduce in Section 7. Section 2 contains a formal definition of the model; Section 3 recalls
fundamental results from the theory of zero-sum games that are important for the analysis;
section 4 introduces some theoretical motivation to consider the problem and in section 5,
besides proving existence of equilibrium in this model, we establish that the lower bound on
equilibrium payoffs for the informed player is actually always attainable.
1.2. Brief Literature Review. To the best of our knowledge, the model we present here is new. It can be seen as a multi-player generalization of the zero-sum model proposed by Aumann, Maschler, Stearns (1995). The main motivation for our paper comes from Huangfu and Liu (2015). In this paper, the authors are concerned with the “information spillover” phenomenon but not exactly in a repeated setting: they consider a dynamic game in which a seller has two goods of two different qualities – high or low – that he wishes to sell to two short run buyers at each stage, in two different markets (each buyer interacts in only one of the markets and for only one stage). Once goods are sold, the game ends. The “information spillover” effect comes from the fact that the qualities of the goods are correlated and that the buyers are able to observe the past outcomes across markets and use it as a signal to learn about the type of the good being traded in the corresponding market. In what concerns the tools of the analysis, one of our main references is the work Hart (1985) as it provides a simple way to model players’ payoff and posterior processes. In addition to it, the works of Sorin (1983) and Simon, Spiez and Torunczyk (1995) provide theorems that immediately imply existence of equilibrium in our setting; they also suggest that the problem of showing when the “best possible payoff” is attainable lies in the geometry of the nonrevealing value of each of the zero-sum games. A related paper is Renault (2001), which analyzes an almost symmetric situation: 3 players, but two informed ones, play against 1 uninformed player. Renault, however, is concerned with non zero-sum types of games and with the problem of existence of equilibrium.

2. The Model

2.1. Notation. The game we define below has three players, denoted 1, 2 and 3. Action set of player 1 will be $I_A \times I_B$; action set of player 2 will be denoted $J_A$; action set of player 3 will be denoted $J_B$. All action sets are assumed to be finite. There will be two collections of real matrices: \( \{A^{k_A}\}_{k_A \in K_A} \) where $K_A$ is a finite set and $A^{k_A} \in \mathbb{R}^{|I_A| \times |J_A|}$; \( \{B^{k_B}\}_{k_B \in K_B} \) where $K_B$ is a finite set and $B^{k_B} \in \mathbb{R}^{|I_B| \times |J_B|}$. The matrices represent stage payoffs. Moreover, we have the following:

- $\Delta(K)$ will be the set of distributions over $K$; given $k \in K$, $\delta_k$ is the distribution assigning probability 1 to $\{k\}$.
• If $X$ be a topological space, the interior of $X$ will be denoted by $\text{int}X$ and its boundary $\partial X$.

• If $p \in \triangle(K_A \times K_B)$ then we denote the marginal distribution of $p$ on $K_A$ as $\text{marg}_{K_A}p$ and on $K_B$ as $\text{marg}_{K_B}p$.

2.2. Game Form.

• At stage 0, $(k_A, k_B)$ is chosen according to probability $p$ on $K_A \times K_B$ and communicated to player 1 only.

• At stage 1, player 1 chooses a move $(i^1_A, i^1_B) \in I_A \times I_B$, player 2 chooses a move $j^1_A \in J_A$, and player 3 chooses a move $j^1_B \in J_B$. Everyone is informed of the tuple $(i^1_A, i^1_B, j^1_A, j^1_B)$.

• Inductively, at stage $m$, knowing the past history $h_m = (i^1_A, i^1_B, j^1_A, j^1_B, \ldots, i^{m-1}_A, i^{m-1}_B, j^{m-1}_A, j^{m-1}_B)$, player 1 chooses $(i^m_A, i^m_B)$, player 2 chooses $j^m_A$ and player 3 chooses $j^m_B$ in their respective domains. Everyone is informed of $h_{m+1} = (h_m, i^m_A, i^m_B, j^m_A, j^m_B)$.

We reproduce here some of the content of section 4.1 in Hart (1985) adapted to our setting:

For $t \in \mathbb{N}$, we define $H_t := (I_A \times J_A \times I_B \times J_B)^{t-1}$, the set of histories before stage $t$. We also define the set of infinite histories $H_\infty = \Pi_{t=1}^\infty (I_A \times J_A \times I_B \times J_B)$, an element of $H_\infty$ being a sequence $h_\infty = (i^t_A, i^t_B, j^t_A, j^t_B)_{t \in \mathbb{N}}$ of moves by all players at all stages.

On $H_\infty$ we define for each $t \in \mathbb{N}$ the finite field generated by $H_t$ and call it $\mathcal{H}_t$. So two infinite histories belong to the same atom $H_t$ if and only if they coincide up to (but not including) stage $t$. Let $\mathcal{H}_\infty$ be the $\sigma$-field generated by $(\mathcal{H}_t)_{t \in \mathbb{N}}$.

The basic probability space will also include the choice of $\kappa \in K_A \times K_B$ by chance. Thus, let $\Omega = H_\infty \times K$ be endowed with the $\sigma$-field $\mathcal{H}_\infty \otimes 2^{K_A \times K_B}$. Each profile of strategies $(\sigma, \tau_A, \tau_B)$ and each probability vector $p \in \triangle(K_A \times K_B)$ determine a probability distribution on this space. We denote by $\mathbb{P}_{\sigma, \tau_A, \tau_B, p}$ this probability measure, $\mathbb{E}_{\sigma, \tau_A, \tau_B, p}$ the expectation with respect to this measure, and $\mathbb{E}_{\sigma, \tau_A, \tau_B, p}^{k_A, k_B}$ the conditional expectation with respect to $\kappa_A \times \kappa_B = (k_A, k_B)$. Whenever the probability space defining the expectation operator is implicitly understood, we will write $\mathbb{E}$ instead of $\mathbb{E}_{\sigma, \tau_A, \tau_B, p}$.

2.3. Strategies. Pure strategies for player 1 are maps $s^1 : \bigcup_{t \in \mathbb{N} \cup \{0\}} H_t \to I_A \times I_B$; for player 2, pure strategies are maps $s^2 : \bigcup_{t \in \mathbb{N} \cup \{0\}} H_t \to J_A$; for player 3, pure strategies are maps...
s^3 : \bigcup_{t \in \mathbb{N} \cup \{0\}} H_t \to J_B. \text{ Mixed strategies are distributions over pure strategies but taking into account Kuhn’s Theorem}^2 \text{ there is no loss of generality in restricting to behavioral strategies, which are defined for each player below:}

(1) Player 1: \(\sigma : K_A \times K_B \times \bigcup_{n \in \mathbb{N} \cup \{0\}} H_n \to \Delta(I_A \times I_B)\).

(2) Player 2: \(\tau_A : \bigcup_{n \in \mathbb{N} \cup \{0\}} H_n \to \Delta(J_A)\).

(3) Player 3: \(\tau_B : \bigcup_{n \in \mathbb{N} \cup \{0\}} H_n \to \Delta(J_B)\).

2.4. Payoffs.

2.4.1. Stage Payoffs. At each stage \(t \in \mathbb{N}\), when \((k_A, k_B) \in K_A \times K_B\) is chosen and \((i_t^A, i_t^B, j_t^A, j_t^B)\) is played, the stage payoffs are defined by:

- **Player 1:** \(A^{k_A}_{i_t^A, j_t^A} + B^{k_B}_{i_t^B, j_t^B}\).
- **Player 2:** \(-A^{k_A}_{i_t^A, j_t^A}\).
- **Player 3:** \(-B^{k_B}_{i_t^B, j_t^B}\).

We do not have to define payoffs for each history given the definition of Nash equilibrium we present below. It is sufficient to define sequence of payoffs accumulated up to some finite time \(T\). It is possible to provide an equivalent definition of Nash equilibrium through the use of Banach-Limits – this is done in Section 7 – in which there is a payoff associated to every history.

2.4.2. Long-Term Payoffs. Given a profile of strategies \((\sigma, \tau_A, \tau_B)\), the payoffs for each one of the players are:

- **Player 1:** \(\mathbb{E}_{\sigma, \tau_A, \tau_B, p}[\frac{1}{T} \sum_{t=1}^{T} (A^{k_A}_{i_t^A, j_t^A} + B^{k_B}_{i_t^B, j_t^B})]\), for each \((k_A, k_B) \in K_A \times K_B\).
- **Player 2:** \(\mathbb{E}_{\sigma, \tau_A, \tau_B, p}[\frac{1}{T} \sum_{t=1}^{T} (-A^{k_A}_{i_t^A, j_t^A})]\),
- **Player 3:** \(\mathbb{E}_{\sigma, \tau_A, \tau_B, p}[\frac{1}{T} \sum_{t=1}^{T} (-B^{k_B}_{i_t^B, j_t^B})]\),

for all \(T = 1, 2, \ldots\) and all \((k_A, k_B) \in K_A \times K_B\) where \(\kappa_A := \pi_A \circ \kappa, \kappa_B := \pi_B \circ \kappa, \pi_A\) is the projection onto \(K_A\) and \(\pi_B\) is the projection onto \(K_B\).

**Remark 2.1.** We will refer to the game defined by the above payoffs and strategies as \(\mathcal{G}(p)\).

^2A good reference for Kuhn’s Theorem in Repeated Games is in Appendix D of Sorin (2000).
2.5. **Equilibrium Concept.** A triple of strategies \((\sigma, \tau_A, \tau_B)\) is a Nash-equilibrium if:

1. \(\lim\inf_{T \to \infty} \mathbb{E}_{\sigma,\tau_A,\tau_B}^{k_A,k_B} \left[ \frac{1}{T} \sum_{t=1}^{T} (A_{i_t,j_t}^{k_A} + B_{i_t,j_t}^{k_B}) \right] \geq \limsup_{T \to \infty} \mathbb{E}_{\sigma,\tau_A,\tau_B}^{k_A,k_B} \left[ \frac{1}{T} \sum_{t=1}^{T} (A_{i_t,j_t}^{k_A} + B_{i_t,j_t}^{k_B}) \right],\)
   for all \((k_A, k_B) \in K_A \times K_B\).

2. \(\lim\inf_{T \to \infty} \mathbb{E}_{\sigma,\tau_A,\tau_B}^{k_A,k_B} \left[ \frac{1}{T} \sum_{t=1}^{T} (-A_{i_t,j_t}^{k_A}) \right] \geq \limsup_{T \to \infty} \mathbb{E}_{\sigma,\tau_A,\tau_B}^{k_A,k_B} \left[ \frac{1}{T} \sum_{t=1}^{T} (-A_{i_t,j_t}^{k_A}) \right],\)
   for all \(\tau_A\).

3. \(\lim\inf_{T \to \infty} \mathbb{E}_{\sigma,\tau_A,\tau_B}^{k_A,k_B} \left[ \frac{1}{T} \sum_{t=1}^{T} (-B_{i_t,j_t}^{k_B}) \right] \geq \limsup_{T \to \infty} \mathbb{E}_{\sigma,\tau_A,\tau_B}^{k_A,k_B} \left[ \frac{1}{T} \sum_{t=1}^{T} (-B_{i_t,j_t}^{k_B}) \right],\)
   for all \(\tau_B\).

**Definition 2.2.** A vector \((a, b_A, b_B) \in \mathbb{R}^{k_A,k_B} \times \mathbb{R} \times \mathbb{R}\) is a Nash equilibrium-payoff if there exists a Nash equilibrium \((\sigma, \tau_A, \tau_B)\) such that:

1. \(\lim\inf_{T \to \infty} \mathbb{E}_{\sigma,\tau_A,\tau_B}^{k_A,k_B} \left[ \frac{1}{T} \sum_{t=1}^{T} (A_{i_t,j_t}^{k_A} + B_{i_t,j_t}^{k_B}) \right] = a^{k_A,k_B},\) for all \((k_A, k_B) \in K_A \times K_B\).

2. \(\lim\inf_{T \to \infty} \mathbb{E}_{\sigma,\tau_A,\tau_B}^{k_A,k_B} \left[ \frac{1}{T} \sum_{t=1}^{T} (-A_{i_t,j_t}^{k_A}) \right] = b_A\).

3. \(\lim\inf_{T \to \infty} \mathbb{E}_{\sigma,\tau_A,\tau_B}^{k_A,k_B} \left[ \frac{1}{T} \sum_{t=1}^{T} (-B_{i_t,j_t}^{k_B}) \right] = b_B\).

**Remark 2.3.** In our model, if the prior probability is \(p \in \Delta(K_A \times K_B)\) and \((a, b_A, b_B)\) is a Nash-equilibrium-payoff, then we will refer to \(a \cdot p\) as _ex-ante Nash-equilibrium-payoff_ of the informed player.

### 3. Preliminary Results

In this section, we state some fundamental results for our analysis.

#### 3.1. **Zero-sum Games with lack of information on one-side.** We recall some fundamental results on infinitely repeated zero-sum games with lack of information on one side.

An excellent reference is Chapter 3 in Sorin (2000) and we draw the results from there.

**Model of game** \(G(p)\). A finite family \((G^k)_{k \in K}\) of two person zero-sum games is given. Each zero-sum game \(G^k\) is finite and identified with the \(I \times J\) matrix \(G^k\), where \(I = \{1, ..., I\}\) is the pure action set of player 1 (the maximizer) and \(J = \{1, ..., J\}\) is the pure action set of player 2 (the minimizer). For each \(p\) the game form \(\mathcal{G}(p)\) is defined as follows:

1. At stage 0, \(k\) is chosen according to the probability \(p\) on \(K\) and communicated to player 1 only.

2. At stage 1, player 1 chooses a move \(i^1 \in I\), player 2 chooses a move \(j^1 \in J\) and the couple \((i^1, j^1)\) is told to both.

3. Inductively, at stage \(m\), knowing the past history, \(h_m = (i^1, j^1, ..., i^{m-1}, j^{m-1})\), player 1 (player2) chooses a move \(i^m \in I(j^m \in J)\) and the new history \(h_{m+1} = (h_m, i^m, j^m)\) is told to both. The description of the game form is common knowledge.
Payoffs will be taken as undiscounted and defined analogously as above, that is, by taking sample averages.

**Definition 3.1.** Let $p \in \Delta(K)$. An uniform value for game $G(p)$ is the number $f(p) \in \mathbb{R}$, such that player 1 and player 2 can guarantee $f(p)$.

**Signalling and Nonrevealing Strategies.** Because the informed can observe the payoff matrix, he can condition his strategy on the set of types. Therefore we can represent a strategy $\sigma$ of the informed player as a vector $(\sigma_{k_A,k_B})_{(k_A,k_B) \in K_A \times K_B}$. Consider a vector $s \in \Delta(I)$. This corresponds to a one-stage strategy of informed player 1. Toghether with $p \in \Delta(K)$, $s$ induces a transition probability from $K$ to $I$: let $k$ be chosen according to $p \in \Delta(K)$, then $i \in I$ is selected following $s^k$. This generates a posterior probability on $K$ taking the value $p(i) \in \Delta(K)$ with probability $s(i) = \sum_k p_k \cdot s^k$. Note that $p = p(i)$ for all $i$ if and only if, $s^k = \bar{s}$ for all $k$ with $p^k > 0$. Hence the vector $s$ is nonrevealing at $p$ if it belongs to:

$$NR(p) = \{s \in S^K | \forall k, \bar{k}, \quad p^k \cdot s^\bar{k} > 0 \Rightarrow s^k = \bar{s}^k\}$$

If we restrict the strategies of Player 1 to $NR(p)$, Player 1 is unable to use his information. The repeated game with restricted strategies as such has an uniform value called **nonrevealing value** given by $\text{Val}_G(p) := \min_\tau \max_\sigma G(p,\tau)$, where $\sum_{k \in K} p^k C^k$. The optimal strategy for each player in this game is the independent repetition of the optimal strategy of the (one shot) zero-sum game $\sum_{k \in K} p^k C^k$ at each stage.

**Definition 3.2.** Given a finite set $K$ and a continuous map $f : \Delta(K) \to \mathbb{R}$, the concavification of $f$ is the function $\text{Cav}(f) : \Delta(K) \to \mathbb{R}$ with $\text{Cav}(f)(x) = \inf\{g(x) | g \geq f, g : \Delta(K) \to \mathbb{R} \text{ concave}\}$

**Theorem 3.3.** The game $G(p)$ has an uniform value, namely, $\text{Cav}(\text{Val}_G)(p)$.

**Proof.** Proposition 3.31 in Sorin(2000) \hfill $\square$

---

3 Player $i \in \{1, 2\}$ can guarantee $f(p) \in \mathbb{R}$ in $G(p)$ if this player has a strategy such that no matter the strategy of player $j \neq i$ the payoff to player $i$ is at least $f(p)$.

4 The concavification can be equivalently defined as $\text{Cav}(f)(p) = \sup\{\sum_i \lambda_i f(q^i) | \sum_i \lambda_i q^i = p \text{ and } \sum_i \lambda_i = 1\}$. 


Optimal Strategies in $G(p)$. The optimal strategy for the informed player in game $G(p)$ is straightforward: note first that the problem $\sup \{ \sum_i \lambda_i f(q') | \sum_i \lambda_i q^i = p \text{ and } \sum_i \lambda_i = 1 \}$ has a solution $(p_s)_{s \in S}$, $|S|$ finite. As explained above, player 1 can use his strategies to induce a martingale of posteriors. At each of these posteriors we have $\text{Val}_{G}(p_s) = \text{Cav}(\text{Val}_{G})(p_s)$. The strategy of player 1 is defined as follows: first choose a strategy such as to induce posteriors $p_s$ with probabilities $\alpha_s$, where $\sum_{s \in S} \alpha_s = 1$. After posterior $p_s$ is induced, play the nonrevealing optimal strategy in $\text{NR}(p_s)$ of game $G(p_s)$. This guarantees the informed player $\text{Val}_{G}(p_s)$. In expectation, the informed player guarantees $\text{Cav}(\text{Val}_{G})(p)$. The optimal strategy for the uninformed player is obtained from the theory of zero-sum vector payoff games developed by Blackwell (1956) and it is usually referred to as “approachability strategies”. Its construction is more complicated than the above for the informed player so we refer the reader to Appendix B of Sorin (2000).

Example. The graph in blue below is the nonrevealing value $\text{Val}_{B}(\cdot)$ obtained from payoff matrices in a zero-sum game with $|K| = 2$, defined in Example 6.2. The red line corresponds to the part of the graph of $\text{Cav} (\text{Val}_{B})(\cdot)$ that is strictly larger than $\text{Val}_{B}(\cdot)$. Let $p = 1/2$ be the prior probability associated to type 1. The informed player can use his strategies to induce posteriors $1/4$ and $3/4$ with probabilities $1/2$ for each posterior and guarantee in expectation $\frac{1}{2} \text{Val}_{B}(1/4) + \frac{1}{2} \text{Val}_{B}(3/4) = \text{Cav} (\text{Val}_{B})(1/2) = 1$.

\footnote{For more on that, refer to the notion of \textit{splitting procedure} in Sorin (2000), Chapter 2, Proposition 2.3}
Theorem 3.4. Let \( p \in \triangle(K_A \times K_B) \). Let \( G(p) \) be the zero-sum infinitely repeated game with lack of information on one-side with \( G^{k_A,k_B} \) an \( |I_A \times I_B| \times |I_B \times J_B| \)-matrix with \( G^{k_A,k_B}_{i_A,j_A,k_B,j_B} := A^{k_A}_{i_A,j_A} + B^{k_B}_{i_B,j_B} \). Let \( \overline{h}(p) := ValA(marg_{K_A}p) + ValB(marg_{K_B}p) \). The uniform value of \( G(p) \) is \( Cav(\overline{h})(p) \).

Proof. It is straightforward to check that the nonrevealing value of \( G(p) \) is

\[
\overline{h}(p) := ValA(marg_{K_A}p) + ValB(marg_{K_B}p).
\]

The result now follows from Theorem 3.3. \( \square \)

Remark 3.5. The infinitely repeated zero-sum game specified in the statement of the theorem above will be denoted from now on by \( G_{A+B}(p) \). The infinitely repeated zero-sum game \( G(p) \) defined by \( K := K_A, G^{k_A} := A^{k_A} \) for each \( k_A \in K_A \) will be denoted by \( G_A(p) \). Analogously, the infinitely repeated zero-sum game \( G(p) \) defined by \( K := K_B, G^{k_B} := B^{k_B} \) for each \( k_B \in K_B \) will be denoted by \( G_B(p) \).

4. The Benchmark Case: No Correlation

To gain intuition let us start with the benchmark case of independent prior. Intuitively, the product structure of the prior allows the informed player to condition his strategies independently on each set of types, avoiding information spillover from his play from one game to the other.

Let \( p = p_A \otimes p_B \in \triangle(K_A) \times \triangle(K_B) \) be a product prior of game \( G_{A+B}(p) \). The results from zero-sum games with lack of information on one side imply that the informed player can guarantee an expected payoff of \( CavValA(p_A) \) in the game against uninformed player 2 and can guarantee \( CavValB(p_B) \) in the game against uninformed player 3. This is an immediate consequence from the fact that player 1 can use optimal strategies on each of the zero-sum games independently. On the other hand, each of the uninformed players have at their disposal the optimal strategies corresponding to their zero-sum games, which means that player 2 can hold the expected payoff of player 1 at a level not larger than \( CavValA(p_A) \), and player 3 can hold player 1 at a level not larger than \( CavValB(p_B) \).

These observations imply the following result:

Theorem 4.1. If \( p = p_A \otimes p_B \), then every ex-ante Nash-equilibrium-payoff of the game \( G(p) \) of informed player 1 equals \( CavValA(p_A) + CavValB(p_B) \).
No matter what case we analyze – independent prior or not – the optimal strategies of the zero-sum games corresponding to players 2 and 3 imply that an expected equilibrium payoff of the informed player can never be larger than \( \text{CavValA}(p_A) + \text{CavValB}(p_B) \). This is the reason why we shall refer to \( \text{CavValA}(p_A) + \text{CavValB}(p_B) \) as the “upper bound on equilibrium payoffs” to player 1.

Suppose now types are not independently chosen. **Could the informed player obtain the best possible payoff even when he is not able to condition his strategies independently in each of the type spaces of the two games he is playing?** We provide a general answer to this question in Section 7. There are however certain trivial cases in which this question is not interesting: if the optimal strategies in the zero-sum games that form \( G(p) \) are for example to fully reveal information in both games then the correlation between the zero-sum games does not matter and the informed player can always obtain \( \text{CavValA}(p_A) + \text{CavValB}(p_B) \) by simply fully revealing the information received. We are interested therefore in what happens when the optimal strategies in both games are not “perfectly alligned” as such.

Example 6.2 provides a good idea of a game where optimal strategies are not perfectly aligned. At the “perfectly correlated” prior \( p_0 \), the optimal strategy of the informed player in \( G_A(\text{marg}_{K_A} p_0) \) is to not reveal any information. At \( G_B(\text{marg}_{K_B} p_0) \), the optimal strategy is to “partially reveal information”, inducing posteriors 1/4 and 3/4. Will this tension between concealing and revealing information preclude the informed player to attain in equilibrium \( \text{Cav}(\text{ValA})(\text{marg}_{K_A} p) \) and \( \text{Cav}(\text{ValB})(\text{marg}_{K_N} p) \)? To makes things worse for the informed player, the correlated prior prevents him from playing the two zero-sum games independently, which makes us conjecture that he will not be able to attain \( \text{Cav}(\text{ValA})(\text{marg}_{K_A} p) + \text{Cav}(\text{ValB})(\text{marg}_{K_B} p) \) in equilibrium. This is however not true, and we will show why in Section 6.

5. **Equilibrium Existence**

We prove existence of a special class of uniform equilibrium, namely, independent and safe joint-plan Sorin (1983). This type of equilibrium has a very natural interpretation: the informed player uses the outcomes of his individual play, for a finite number of stages, to

---

6We will provide a precise definition of this notion later.
signal information about the underlying type of the game that is being played. Depending on the particular outcome realized, the players then play deterministically in every stage, according to an implicit contract. Deviations are punished by minmax strategies in the corresponding zero-sum games, as usual.

**Definition 5.1.** Let \( h^1_{m} := (i^1_A, i^1_B)_{1 \leq t \leq m-1} \). We define \( H^1_{n} := \bigcup \{ h^1_{n} \} \) and call it the set of individual histories of player 1.

**Definition 5.2.** An independent 2-3-safe-joint-plan is a triple \((S, x, \gamma)\) where:

- (Signals) \( S \) is a set of signals, i.e. a subset of \( H^1_{n} \), for some \( n \in \mathbb{N} \).
- (Signaling strategy) \( x \) is a \( |K_A \times K_B| \)-tuple where for each \((k_A, k_B)\) in \( K_A \times K_B \), \( x^{k_A,k_B} \) is a probability on \( S \).
- (Contracts) \( \gamma = (\gamma^s_A, \gamma^s_B) \), where, \( \gamma_i = (\gamma^s_i)_{s \in S} \), and \( \gamma^s_i = \sigma_i \bigotimes \tau_i, \sigma_i \in \triangle(I_i) \) and \( \tau_i \in \triangle(J_i) \), where \( \tau_i \) is optimal for the corresponding uninformed player at game \( i(\text{marg}_{K,B}) \), for \( i \in \{A, B\} \). We denote by \( \gamma^s_A(i_A, j_A) \) the probability of moves \((i_A, j_A)\) and \( \gamma^s_B(i_B, j_B) \) the probability of moves \((i_B, j_B)\).

Here is the intuition of a joint-plan: player 1 uses his action during finitely many stages to signal about the underlying type of games A and B, which are his private information. After that, conditionally on the observed signal \( s \), players play according to \( \gamma_s \). Following Lemma 2 in Sorin (1983), the independent distribution \( \gamma^s_A = \sigma_A \bigotimes \tau_A \) over \( I_A \times J_A \) can be induced through the play of a deterministic sequence of moves at each stage by players 1 and player 2 with the appropriate frequency. This deterministic path of play is what the contract \( \gamma^s_A \) represents. Similarly, \( \gamma^s_B \) can be induced in the same manner. It is important to note that, in a joint-plan equilibrium, playing a deterministic sequence of moves after signalling stages assures that any deviation after the signaling stages is detectable and can be punished.

**Notation.** Given a joint plan, for each \( s \in S \), \( p(s) \) will stand for the conditional probability on types given \( s \), defined by

\[
p^{k_A,k_B}(s) = \frac{p^{k_A,k_B,x^{k_A,k_B}}(s)}{x(s)},
\]

\(^7\)Compare with section above on Signalling and Nonrevealing Strategies.
where
\[ x(s) = \sum_{k_A, k_B} p^{k_A, k_B} \cdot x^{k_A, k_B}(s). \]

Also, for each \( s \in S \) and \((k_A, k_B) \in K_A \times K_B\) we define:

1. \( \alpha^{k_A, k_B}(s) = \sum_{i_A, j_A, i_B, j_B} A_{i_A, j_A}^{i_A, j_A} \gamma_{i_A, j_A}^s + B_{i_B, j_B}^{i_B, j_B} \gamma_{i_B, j_B}^s \) (the average payoff collected by the informed player after signal \( s \)).
2. \( \alpha^{k_A, k_B} = \max_{t \in S} \alpha^{k_A, k_B}(t) \).
3. \( \alpha = (\alpha^{k_A, k_B})_{k_A, k_B \in K_A \times K_B} \).
4. \( \alpha(s) = \sum_{i_A, j_A} A_{i_A, j_A}^{i_A, j_A} \gamma_{i_A, j_A}^s \) (marginal payoff collected from game \( A^{k_A} \) given \( s \in S \)).
5. \( \alpha_B(s) = \sum_{i_B, j_B} B_{i_B, j_B}^{i_B, j_B} \gamma_{i_B, j_B}^s \) (marginal payoff collected from game \( B^{k_B} \) given \( s \in S \)).
6. \( \beta_A(s) = \sum_{k_A} \text{marg}_{A^{k_A}} p(s)^{k_A} \alpha^{k_A}(s) \).
7. \( \beta_B(s) = \sum_{k_B} \text{marg}_{B^{k_B}} p(s)^{k_B} \alpha^{k_B}(s) \).

A profile of strategies \((\sigma, \tau_A, \tau_B)\) induces a joint-plan \((S, x, \gamma)\) if there exists \( n \in \mathbb{N} \) such that:

1. The probability \( \mathbb{P}_{H_1}^{k_A, k_B} \) induced by \((\sigma, \tau_A, \tau_B)\) over histories \( H_n^1 \) conditioning on \((k_A, k_B)\) being the chosen types satisfies \( x^{k_A, k_B}(s) = \mathbb{P}_{H_n^1}^{k_A, k_B}(s) \) for all \( s \in H_n^1 \).
2. For each \( t \geq 0 \) and \( \mathbb{P}_{\sigma, \tau_A, \tau_B, h_t > 0}^{(i_A, j_A)}(h_t) > 0 \), \( \sigma(k_A, k_B, h_t) = \delta_{i_A^t, j_A^t}, \tau_A(h_t) = \delta_{i_A^t, j_A^t} \), and \( \tau_B(h_t) = \delta_{j_B^t} \) such that \( \frac{1}{T}\mathbb{P} \{ t | 1 \leq t \leq T \text{ s.t. } (i_A^t, j_A^t) = (i_A, j_A) \} \to \gamma_A^s(i_A, j_A), \text{ as } T \to +\infty \) and \( \frac{1}{T}\mathbb{P} \{ t | 1 \leq t \leq T \text{ s.t. } (i_B^t, j_B^t) = (i_B, j_B) \} \to \gamma_B^s(i_B, j_B), \text{ as } T \to +\infty \).

**Theorem 5.3.** Let \( p \in \Delta(K_A \times K_B) \) and \((\sigma, \tau_A, \tau_B)\) inducing a joint-plan \((S, x, \gamma)\) with payoffs \((\alpha, \beta_A, \beta_B)\). Then \((\sigma, \tau_A, \tau_B)\) is a Nash-equilibrium in \( G_{A+B}(p) \) if:

1. \( \beta_A(s) \leq \text{CavValA}(\text{marg}_{A^{k_A}} p(s)) \) and \( \beta_B(s) \leq \text{CavValB}(\text{marg}_{B^{k_B}} p(s)) \), for all \( s \in S \).
2. For all \((k_A, k_B) \in K_A \times K_B\), for all \( s \in S \), \( p^{k_A, k_B} x^{k_A, k_B}(s) > 0 \).
3. \( \alpha \cdot q \geq \overline{h}(q) \), for all \( q \in \triangle(K_A \times K_B) \). \(^8\)

**Proof.** See Sorin (1983) \(\Box\)

---

\(^8\)Condition 3 of Theorem 5.3 is the so called “approachability” condition. This condition is important because in case the informed player deviates from the equilibrium path of play, it allows the uninformed players to punish him, holding his payoff at a level lower than \( \alpha^{k_A, k_B}, \forall (k_A, k_B) \in K_A \times K_B \). For more, refer to the Appendix C.
Condition (1) above is an individual rationality condition for players 2 and 3 and (3) is an individual rationality condition for player 1. Condition (2) is usually called “no cheating” condition and prevents the informed player from profitably deviating at the signalling stages. Deviations at signalling stages are unobservable, so this condition is crucial.

Existence of Equilibrium. The theorem below provides technical conditions we will use in constructing a safe joint-plan that meets the conditions of Theorem 5.3. It essentially means that for a continuous family of affine functions \( h \) on parameter \( p \) that dominates another function \( a \) in the sense of (2), it is possible to find one affine function that also dominates function \( a \) (defined by a vector \( \phi \)) which is almost equal (in the sense of condition (5) below) to some finite subcollection of the continuous parameter family.

**Theorem 5.4.** [Simon, Spiez, Torunczyk, 1995] Let \( K \) be a finite set and \( p_0 \in \triangle(K) \). Let \( a : \triangle(K) \to \mathbb{R} \) and \( h : \triangle(I) \times \triangle(K) \to \mathbb{R}^{|K|} \) be continuous functions such that:

1. The function \( h \) is affine with respect to the variable \( \sigma \in \triangle(I) \), for all \( p \in \triangle(K) \).
2. For all \( p, q \in \triangle(K) \), there is \( \sigma \in \triangle(I) \) such that \( h(\sigma, p) \cdot q \geq a(q) \).

Therefore, there exists a set \( P_0 \subset \triangle(K) \) of cardinality \( \leq |K| \) and vectors \( \sigma_p \in \triangle(I) \) (with \( p \in P_0 \)) and \( \phi \in \mathbb{R}^{|K|} \) such that:

1. \( \phi \cdot q \geq a(q) \) for all \( q \in \triangle(K) \).
2. \( p_0 \in \text{conv}(P_0) \).
3. For all \( p \in P, \) for all \( k \in K \) we have \( \phi^k \geq h^k(\sigma_p, p) \), with equality occurring in place of inequality whenever \( p^k > 0 \).

We will also make use of the following simple version of a topological lemma in Simon, Spiez and Torunczyk (1995):

**Lemma 5.5.** [Simon, Spiez and Torunczyk, 1995] For every \( \epsilon > 0 \) there exists a continuous map \( g_A : \triangle(K_A) \to \triangle(J_A) \) such that \( \sigma_A A(p) g_A(p) \leq ValA(p) + \epsilon \), for all \( (\sigma, p) \in \triangle(I_A) \times \triangle(K_A) \).

This lemma provides us with a function that associates to each \( p \in \triangle(K_A) \) an \( \epsilon \)-optimal strategy for the uninformed player in the zero-sum game \( G_A(p) \). Moreover, it shows that this association can be made continuously.

**Remark 5.6.** Of course, the above result works in the same way for game \( G_B(p) \).
Theorem 5.7. (Existence of Joint-Plan Equilibrium Payoff) There exists a Nash equilibrium inducing a joint-plan in $G(p)$ that is independent and 2-3-safe and has an associated ex-ante payoff for the informed player of $Cav(h)(p_0)$.\footnote{Recall that $h(p) := ValA(marg_{K_A}(p)) + ValB(marg_{K_B}(p))$}

Proof. Assume without loss of generality $p_0 \in \text{int}\Delta(K_A \times K_B)$. If some pair of types has probability zero we can consider a simplex of lower dimension. First we shall write $p_A := marg_{K_A}p$ and $p_B := marg_{K_B}p$. Given $\epsilon > 0$, applying Lemma 5.5 we have that $\sigma_A A(p_A)g_A(p_A) \leq ValA(p_A) + \epsilon$ and $\sigma_B B(p_B)g_B(p_B) \leq ValB(p_B) + \epsilon$ for all $(\sigma_A,p_A) \in \Delta(I_A) \times \Delta(K_A)$ and for all $(\sigma_B,p_B) \in \Delta(I_B) \times \Delta(K_B)$. Define $h(\sigma,p) = ((m_A\sigma)A^{k_A}g_A(p_A)+(m_B\sigma)B^{k_B}g_B(p_B))|_{(k_A,k_B)\in K_A \times K_B}$, where $m_A\sigma = marg_{I_A}\sigma$ and $m_B\sigma = marg_{I_B}\sigma$. Since the marginal operator is affine, the function $h$ is affine on $\sigma$. It is also trivially continuous.

Now, given $p,q \in \Delta(K_A \times K_B)$, let $\tilde{\sigma}_A^p$ be the optimal strategy of the informed player in the zero-sum game $A(p_A) = \sum_{k_A \in K_A} p^{k_A}\lambda^{k_A}$ and let $\tilde{\sigma}_B^p$ be the optimal strategy of the informed player in the zero-sum game $B(p_B) = \sum_{k_B \in K_B} p^{k_B}\lambda^{k_B}$. Define $\sigma := \tilde{\sigma}_A^p \otimes \tilde{\sigma}_B^p$.

Then we have that $h(\sigma,p) \cdot q \geq a(q) =: ValA(q_A) + ValB(q_B)$, where $q_A$ and $q_B$ are analogously defined as $p_A$ and $p_B$. Applying Theorem 5.4, we have that there is $P_0 \subset \Delta(K_A \times K_B)$ with cardinality $\leq |K_A \times K_B|$, $(\sigma_p)_{p \in P_0}$ and $\phi \in \mathbb{R}^{|K_A \times K_B|}$, satisfying (3), (4), (5). From (3) and (4) we have that there exists a nonnegative collection $(\lambda_p)_{p \in P_0}$ such that $\sum_{p \in P_0} \lambda_p p = p_0$ and $\sum_{p \in P_0} \lambda_p = 1$, $\phi \cdot q \geq a(q)$ and $(m_A\sigma_p)A(p_A)g_A(p_A) \leq ValA(p_A) + \epsilon$ and $(m_B\sigma_p)B(p_B)g_B(p_B) \leq ValB(p_B) + \epsilon$ for $\sigma_p \in \Delta(I_A \times I_B)$ and $p \in P_0$.

Notice that the solutions given by the application of theorem 5.4 are all indexed by $\epsilon > 0$. For each $n \in \mathbb{N}$, we can therefore consider a collection $(\sigma_{p_n})_{p_n \in P_0^n}$ with associated solutions: $(m_B\sigma_{p_n})g_B(p_B)$, $(m_A\sigma_{p_n})g_A(p_A)$ and $\phi_n$ satisfying $(m_B\sigma_{p_n})B(p_B)g_B(p_B) \leq ValB(p_B) + 1/n$ and $(m_A\sigma_{p_n})A(p_A)g_A(p_A) \leq ValA(p_A) + 1/n$ such that $\sum_{p_n \in P_0^n} \lambda_{p_n} p_n = p_0$; also, we can obtain $\phi_n$ such that $\phi_n \cdot q \geq a(q), \forall q$ with (5) being satisfied.

Without loss of generality we can assume $|P_0^n| = |K_A \times K_B|$. For each $n$ we can enumerate each element of $P_0^n$ and consider therefore a sequence of vector of probabilities $(p^1_n,\ldots,p^n_{|K_A \times K_B|})$ where $p^1_n \in P_0^n$. Passing to a convergent subsequence if necessary, we can assume the sequences of probability vectors as well as associated solutions converge, since
they all lie in compact sets. Let \((p^i_1, ..., p^i_{|K_A \times K_B|})\), \((\sigma^i_p)\), \((g_A(p^i_A))\), \((g_B(p^i_B))\), and \(\phi\) be the limits. It is easy to check that the limit of the sequences of solutions satisfy (3), (4) and (5). The joint-plan is now defined as follows: consider as contracts \(\gamma^i_A = m_A \sigma^i_p \otimes g_A(p^i_A)\), \(\gamma^i_B = m_B \sigma^i_p \otimes g_B(p^i_B)\).

Let \(\tau^A_p = g_A(p_A)\) and \(\tau^B_p = g_B(p_B)\). By construction, \(\max \sigma A(p_A)\tau^A_p = \text{Val} A(p_A)\) and \(\max \sigma B(p_B)\tau^B_p = \text{Val} B(p_B)\). Therefore it implies that \(\sigma^i_p A(p_A)\tau^A_p \leq \text{Val} A(p^i_A)\) and \(\sigma^i_p B(p_B)\tau^B_p \leq \text{Val} B(p^i_B)\), for each \(i\). This implies in particular (1) of Theorem 5.3 is satisfied. Condition (3) of theorem follows directly from the fact that condition (5) above is satisfied by \(\phi\). Let \((\lambda_{p^i_n})\) be the nonnegative coefficients associated to \((p^i_1, ..., p^i_{|K_A \times K_B|})\) (namely those that satisfy \(\sum_i \lambda_{p^i_n} p^i = p_0\) and \(\sum_i \lambda_{p^i_n} = 1\)). Passing to a subsequence if necessary, we obtain a limit \((\lambda_{p^i})\).

We can now define the following signalling strategy: put first \(S = \{1, ..., |K_A \times K_B|\}\) and define \(x^{K_A \times K_B}(i) = \lambda_{p^i_n} \frac{p^i(k_A, k_B)}{p_0(k_A, k_B)}\), for \((k_A, k_B)\) in the support of \(p_0\). This signalling strategy satisfies condition (2) of Theorem 5.3 and induces the correct posteriors. Now, consider \(n = |K_A \times K_B|\) and consider the individual histories in \(H^1_n\) to use them as signals. Now, in equilibrium, applying condition (5), we have that \(\sum_{p \in P_0} \lambda_p (h(\sigma_p, p) \cdot p) = \phi \cdot p_0 = \text{Cav}\bar{h}(p_0)\). Rewriting the expression for \(h\),

\[
\sum_{p \in P_0} \lambda_p (\sigma^A_p A(p)\tau^A_p + \sigma^B_p B(p)\tau^B_p) = \text{Cav}\bar{h}(p_0).
\]

**Remark 5.8.** The type of joint-plan constructed above is what Sorin (1983) called a safe joint-plan. It is called “safe” because the uninformed players play optimal strategies of the zero-sum games with matrices \(A(\text{margin}_{K_A} p^i)\) and \(B(\text{margin}_{K_B} p^i)\), after \(p^i\) is realized, or more formally, the strategies \(g_A(\text{margin}_{K_A} p^i)\) and \(g_B(\text{margin}_{K_B} p^i)\) that form contracts \(\gamma^i_A\) and \(\gamma^i_B\) respectively, are optimal strategies at \(A(\text{margin}_{K_A} p^i)\) and \(B(\text{margin}_{K_B} p^i)\).

**Remark 5.9.** In Simon, Spiez and Torunckzyk (1995), the objective is to obtain existence of equilibrium for a nonzero sum game with lack of information on one side between two players. What we do here is therefore to extend the application of their theorem to the problem of existence of equilibrium of a nonzero-sum game with lack of information on one-side between three players, where one informed player plays zero-sum games against two uninformed players. Note that the generalization of this result to a model of one informed player and \(n\) uninformed players is immediate from the application of the same theorem. Moreover, this

\[\text{Notice that property (5) of Theorem 5.4 guarantees that the sequence of vectors } \phi \text{ is bounded, so it will also have an accumulation point.}\]
particular zero-sum structure of our nonzero-sum model allows us to calculate the payoff given by the equilibrium constructed and obtain a natural lower bound for equilibrium payoffs of the informed player (see Theorem 6.1). In the equilibrium that we will construct in the next section, we will relax the requirement of “safety”.

6. When can the informed player obtain the best possible payoff?

Having proved the existence of equilibrium for the finite type case in our model, we established as a corollary the payoff associated with the equilibrium. In this section we approach the question of whether $Cav(ValA)(\text{marg}_{K_A}p) + Cav(ValB)(\text{marg}_{K_B}p)$ – referred to in Section 4 as best possible payoff – can be obtained as an equilibrium of a game where there is correlation among types. Somewhat surprisingly, we show that correlation is not enough to preclude the informed player in obtaining the best possible payoff in the game. There is a certain geometry of the game, which is captured by the shape of the nonrevealing value function of each zero-sum game, that in certain cases allow for a play of the informed player that attains the best possible payoff.

Let us first start with a useful result:

**Theorem 6.1.** Let $p_0 \in \Delta(K_A \times K_B)$. The ex-ante Nash equilibrium-payoff set is convex and the ex-ante payoffs of the informed player lie in the interval

$$I(p_0) = [Cav(h)(p_0), Cav(ValA)(\text{marg}_{K_A}p_0) + Cav(ValB)(\text{marg}_{K_B}p_0)]$$

**Proof.** Consider two ex-ante Nash-equilibrium payoffs $(\gamma_1, \alpha_1, \beta_1)$ and $(\gamma_2, \alpha_2, \beta_2)$. Since each of the uninformed players can play his optimal strategies of their respective games $G_A(\text{marg}_{K_A}p)$ and $G_B(\text{marg}_{K_B}p)$, it implies that

$$Cav(ValA)(\text{marg}_{K_A}p_0) + Cav(ValB)(\text{marg}_{K_B}p_0) \geq \gamma_1$$

as well as

$$Cav(ValA)(\text{marg}_{K_A}p_0) + Cav(ValB)(\text{marg}_{K_B}p_0) \geq \gamma_2.$$

Now, the availability of the optimal strategy for the informed player $G_{A+B}(p_0)$ implies that $\gamma_1 \geq Cav(h)(p_0)$ as well as $\gamma_2 \geq Cav(h)(p_0)$. Now, let $\alpha \in (0, 1)$. Consider a jointly controlled lottery (Aumann,Maschler, Stearns (1995), p. 274) that implements the equilibrium profile associated with $\gamma_1$ with probability $\alpha$ and the equilibrium associated with $\gamma_2$ with
probability $1 - \alpha$. By the properties of the jointly controlled lottery, there cannot be profitable undetectable deviations at the stages where the jointly controlled lottery is played. For detectable deviations of one of the uninformed players at the lottery stages, the informed player plays the optimal strategy of the zero-sum game to punish. For detectable deviations of the informed player at the lottery stages, uniformed players play the Blackwell strategy of Appendix C to punish. The strategy profile where a jointly controlled lottery is played at initial stages and, after that, the corresponding strategy profile paying $\gamma_1$ or $\gamma_2$ drawn for the lottery, is trivially an equilibrium of the game, because it satisfies conditions of Theorem 5.3. The payoff of this equilibrium is $\alpha \gamma_1 + (1 - \alpha) \gamma_2$. So convexity follows. □

When is the interval $I(p)$ nondegenerate? Establishing an answer to this question is of fundamental importance to identify the cases where information spillover might “lower” the equilibrium payoffs of the informed player. Indeed, if the interval $I(p)$ is degenerate, then the optimal strategy in the zero-sum game $G_{A+B}(p)$ actually induces optimal strategies in the “marginal” games $G_A(\text{marg}_{K_A}p)$ and $G_B(\text{marg}_{K_B}p)$. Since $\text{Cav}(\overline{h})(p)$ is always attainable in equilibrium by the informed player, it turns out that correlation between the games $G_A(\text{marg}_{K_A}p)$ and $G_B(\text{marg}_{K_B}p)$ has no role. Information spillover has no effect here.

Information spillover is only important when the optimal strategies are not “perfectly aligned”;

**Definition 6.2.** Optimal strategies of $G_A(\text{marg}_{K_A}p)$ and $G_B(\text{marg}_{K_B}p)$ are perfectly aligned when there is an optimal strategy of $G_{A+B}(p)$ that induces the optimal strategies of $G_A(\text{marg}_{K_A}p)$ and $G_B(\text{marg}_{K_B}p)$.

Let $p \in \Delta(K_A \times K_B)$. Suppose the zero-sum games $G_A(\text{marg}_{K_A}p)$ and $G_B(\text{marg}_{K_B}p)$ have no optimal strategies that are perfectly aligned. This means that for every optimal strategy of game $G_{A+B}(p)$, there is one zero-sum game, say $G_A(\text{marg}_{K_A}p)$, and one induced posterior $p_s \in \Delta(K_A \times K_B)$ for which $\text{Val}_A(\text{marg}_{K_A}p_s) < \text{Cav}(\text{Val}_A)(\text{marg}_{K_A}p_s)$. Since for each posterior $\text{Val}_A(p_s') \leq \text{Cav}(\text{Val}_A)(p_s')$, using Jensen’s inequality we have that, $\text{Cav}(\overline{h})(p) < \text{Cav}(\text{Val}_A)(\text{marg}_{K_A}p) + \text{Cav}(\text{Val}_B)(\text{marg}_{K_B}p)$. This observation implies:

**Theorem 6.3.** $I(p)$ is nondegenerate if and only if the optimal strategies of $G_A(\text{marg}_{K_A}p)$ and $G_B(\text{marg}_{K_B}p)$ are not perfectly aligned.

There are cases where the interval $I(p)$ above is degenerate
Below is an example where the interval $I(p)$ could indeed be non-degenerate for a certain prior $p \in \Delta(K_A \times K_B)$. In Theorem 9.6 of Appendix A we show that games where $I(p)$ is nondegenerate are robust to payoff perturbations.

**Example 6.4.** Consider the following example of $\mathcal{G}(p)$:

$$p_0 = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$$

$$A^1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$B^1 = \begin{bmatrix} 4 & 0 & 2 \\ 4 & 0 & -2 \end{bmatrix}, B^2 = \begin{bmatrix} 0 & 4 & -2 \\ 0 & 4 & 2 \end{bmatrix}$$

In $p_0$, the sum of the entries of the rows corresponds to probabilities of types of game $A$; columns correspond to probabilities of types of game $B$. Since the prior assigns perfect correlation between the types of each one of the games, there are only two types to consider effectively: types 1 and 2. Without loss of generality, $p$ will be the probability of type 1 in both games and $(1 - p)$ the probability of type 2 in both games. As we can see from Definition 3.2, in order to calculate the concavification of a function explicitly at a certain point it is not sufficient to know the value of the function at the point. This is why we provide below an explicit arithmetic expression of the functions $\text{ValA}$ and $\text{ValB}$. Knowing this expression, footnote 6 provides us with an equivalent definition of the concavification operation which is suitable to make explicit calculations.

For this example we have that:

$$\text{ValA}(p) = p(1 - p), \text{ for all } p \in [0,1]$$

$$\text{ValB}(p) = \begin{cases} 4p & \text{if } p \in [0,1/4) \\ -4p + 2 & \text{if } p \in [1/4,1/2) \\ 4p - 2 & \text{if } p \in [1/2,3/4) \\ -4p + 4 & \text{if } p \in [3/4,1] \end{cases}$$
These imply that:

\[
CavValA(p) = ValA(p) = p(1 - p), \text{ for all } p \in [0, 1]
\]

\[
CavValB(p) = \begin{cases} 
4p & \text{if } p \in [0, 1/4) \\
1 & \text{if } p \in [1/4, 3/4) \\
-4p + 4 & \text{if } p \in [3/4, 1]
\end{cases}
\]

\[
Cav(ValB + ValA)(p) = \begin{cases} 
4p + p(1 - p) & \text{if } p \in [0, 1/4) \\
1 + 3/16 & \text{if } p \in [1/4, 3/4) \\
-4p + 4 + p(1 - p) & \text{if } p \in [3/4, 1]
\end{cases}
\]

**Figure 1.** Graphs of CavValA and ValA for a "nonrevealing" game A

**Figure 2.** Graphs of CavValB and ValB for a "partially revealing" game B

In Figures 1 and 2 below we depict the graphs of the nonrevealing value in the continuous line. Since the nonrevealing value in game A is concave, it equals its least concave majorant.
In Figure 2, we depict in the dotted line the region where the graphs of the concavification and the nonrevealing value differ in game B. Figures 3 and 4 follow the same notation.

Notice that $CavValA(p_0) + CavValB(p_0) = 1/4 + 1 > 1 + 3/16 = Cav(ValA + ValB)(p_0)$. A natural question is whether we can obtain the whole interval $I(p_0)$ as ex-ante equilibrium payoffs for the informed player. It turns out, as we will see, that games as the one above satisfy a general geometric property that allows the informed player to obtain the whole interval in ex-ante equilibrium payoffs.

We are going to concentrate the analysis in the two-types case for now ($|K_A| = |K_B| = 2$), because it is easier to draw interpretations and state the conditions of the theorems. The general case is presented in the next section. There is a fundamental difference between the general case and the two-types case and we will point out this difference in the next section.

**Definition 6.5.** A two-player zero-sum game with lack of information on one-side is called **completely revealing** if the only optimal strategy of the informed player is to reveal all the information at every prior.

**Remark 6.6.** It is easy to see that, whenever $|K_A| = 2$, the zero-sum repeated game with lack of information on one-side is completely revealing $G_A(p_0)$ if and only if $CavValA(\cdot)$ is affine in $\Delta(K_A)$ and there is no $q \in \text{int}\Delta(K_A)$ for which $CavValA(q) = ValA(q)$. This implies that whenever the zero-sum repeated game is not completely revealing there exists $p \in \text{int}\Delta(K_A)$ such that $ValA(p) = CavValA(p)$.

**Example 6.7.** Here is an example of a completely revealing game: $K = \{1, 2\}$, with payoff matrices $A^1$ and $A^2$. In the continuous line, we have the nonrevealing value function of the zero-sum repeated game and in the dotted line represents the graph of the concavification.

$$A^1 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}; A^2 = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$$

**Example 6.8.** An example of a game that is not completely revealing is in Example 6.2: let $K = \{1, 2\}$, $B^1$ and $B^2$ payoff matrices defined below. According to Example 6.2, $ValA(\cdot)$ (in blue) and $CavValA(\cdot)$ (in red) are depicted as:
\[ B^1 = \begin{bmatrix} 4 & 0 & 2 \\ 4 & 0 & -2 \end{bmatrix}; \quad B^2 = \begin{bmatrix} 0 & 4 & -2 \\ 0 & 4 & 2 \end{bmatrix} \]

It is also important to point out that differently from what happened in the last section, we are not going to construct equilibria that are safe joint-plans in the next theorem. If we stick to this type of equilibrium, it is not possible to attain the upper-bound of \( I(p_0) \) as a Nash equilibrium payoff for the informed player. We will stick to joint-plans that satisfy Definition 5.2 but for which the strategies of the uninformed players defining the contracts do not have to be optimal. Joint-plans satisfying this new definition will be called loose joint-plans. A nonrevealing loose joint-plan is a loose joint-plan for which the set of signals \( S \) of Definition 5.2 is empty – meaning that there is no revelation of information.

\footnote{Indeed, any “safe” joint-plan equilibrium pays at most the \( \text{Cav}(\vec{h})(p) \) for the informed player. See Corollary of Proposition 7.5}
Theorem 6.9. Let \( p_0 \in \triangle(K_A \times K_B) \), where \( |K_A| = |K_B| = 2 \). Assume \( G_A(marg_{K_A}p_0) \) and \( G_B(marg_{K_B}p_0) \) are not completely revealing. Then there exists a Nash equilibrium with an associated ex-ante payoff of \( Cav(ValA)(marg_{K_A}p)\) + \( Cav(ValB)(marg_{K_B}p) \) for the informed player.

The optimal strategy of the informed player in each zero-sum game gives fundamental information about the ways the informed player has to play the zero-sum games simultaneously. Indeed, the proof of the theorem tells us that whenever the optimal strategy in each game is such that the informed player does not have to reveal all the information, this implies, in game \( G(p) \) that he does not have to reveal any information at all. Note also that the conditions are for each of the zero-sum games separately, which makes them easier to check. It should also be mentioned that these conditions are completely orthogonal to correlation i.e, they are based solely on the payoff structure of each zero-sum game and not on the existence of correlation across types of each zero-sum sum repeated game. This reveals the dual nature of the problem: it is not only a problem of correlation but also a problem on the specific payoff structure of each zero-sum game.

Before presenting the actual proof below let us provide an idea with help of the figure above. Let the graph in blue above be the graph of \( ValA(\cdot) \); the horizontal axis is the probability of type 1. Say the \( p_0 \) has marginal \( marg_{K_A}p_0 = (1/2, 1/2) \). Now the point 1/2 in the figure is located in the interval \((1/4, 3, 4)\) where the CavValA(\( \cdot \)) is strictly larger than \( ValA(\cdot) \). Because of the assumption of the game being not completely revealing, there exists

\[12\] The first entry of the vector denotes the probability of type 1 and the second entry the probability of type 2.
a point \( \bar{p} \) such that \( \text{ValA}(\bar{p}) = \text{CavValA}(\bar{p}) \) (see Remark 6.5). This property will imply that vector \( v_A \) above can be generated by payoffs, that is, \( v_A = \sum_{i \in I} \alpha_i (\sigma_i^{A} A \tau_i)_{k_{A}} \) with \( \sum_{i \in I} \alpha_i = 1, \alpha_i \geq 0, I \) finite. Now, the informed player can coordinate the play with the uninformed player in game \( A \) and play a jointly controlled lottery with outcomes \( I \). If outcome \( i \in I \) occurs, they can play, then, a nonrevealing joint-plan that asymptotically approximates \( \sigma_i \otimes \tau_i \). Notice that no information is revealed in this process and the individual rationality conditions pertaining to game \( A \) of the uninformed player is trivially satisfied. Notice also that the jointly controlled lottery procedure guarantees that no deviation on the jointly controlled lottery stage is profitable. If zero-sum game \( G_B(\text{marg}_{K_B}p_0) \) is also not completely revealing zero-sum game, then the same procedure explained can be repeated in that game. Now the payoff associated with this path of play will be \( v_B \cdot p = \text{CavValA}(p) \); summing the payoffs of both games, we have that the upper-bound of \( I(p) \) is attained.

We prove some technical results required for the proof of the theorem.

**Lemma 6.10.** Let \( p \in \Delta(K_A) \) with \( |K_A| = 2 \) and \( \text{Cav}(\text{ValA})(\text{marg}_{K_A}p) > \text{ValA}(\text{marg}_{K_A}p) \). There exists \( \phi \in \mathbb{R}^{K_A} \) such that:

1. \( \phi \cdot q \geq \text{ValA}(q), \forall q \in \Delta(K_A) \).
2. \( \phi \cdot p = \text{Cav}(\text{ValA})(p) \).

**Proof.** Since \( G_A(\text{marg}_{K_A}p) \) is not completely revealing, the optimal strategy of the informed player at \( \text{marg}_{K_A}p \) in \( G_A(\text{marg}_{K_A}p) \) induces at least one posterior \( \bar{p} \in \text{int}\Delta(K_A) \) at which \( \text{ValA}(\bar{p}) = \text{Cav}(\text{ValA})(\bar{p}) \). Notice now that the set \( \{ q \in \Delta(K_A) | \text{ValA}(p) < \text{Cav}(\text{ValA})(p) \} \) is a semi-algebraic set so it is a finite union of open intervals. Let \( Z_A \) be the interval containing \( \text{marg}_{K_A}p \). Then there exists a vector \( \phi \in \mathbb{R}^{K_A} \) \( \text{Cav}(\text{ValA})(q) = \phi \cdot q \forall q \in Z_A \) and \( \phi \cdot q \geq \text{ValA}(q) \), for all \( q \in \Delta(K_A) \).

**Lemma 6.11.** Let \( p \in \Delta(K_A), |K_A| \geq 2 \) and \( \text{Cav}(\text{ValA})(\text{marg}_{K_A}p) > \text{ValA}(\text{marg}_{K_A}p) \). Let \( \phi \in \mathbb{R}^{K_A} \) be such that:

1. \( \phi \cdot q \geq \text{ValA}(q), \forall q \in \Delta(K_A) \).
2. \( \phi \cdot p = \text{Cav}(\text{ValA})(p) \).

---

13See Hart (1985) p.140 for an explanation on this or Aumman, Maschler, Stearns (1995)

14See Bochnak, Coste, Roy (1998), chapter 2, or Neyman (2003).
There exist \( \{ \lambda_k | \lambda_k \geq 0 \text{ with } k = 1, \ldots, |K_A| \text{ and } \sum_k \lambda_k = 1 \} \), \( \{ \sigma^k | k = 1, \ldots, |K_A| \} \subset \Delta(I_A) \) and \( \{ \tau^k | k = 1, \ldots, |K_A| \} \subset \Delta(J_A) \) such that:

\[
\phi = \sum_k \lambda_k (\sigma^k A^k A^k \tau^k)_{kA \in K_A}.
\]

**Proof.** Let \( e_i^{K_A} = (0, \ldots, 1, \ldots, 0) \in \mathbb{R}^{K_A-1} \) with one in the \( i \)-th position. Analogously, \( e_i^{K_A} = (0, \ldots, 1, \ldots, 0) \in \mathbb{R}^{K_A} \). Define \( S : \mathbb{R}^{K_A-1} \to \mathbb{R}^{K_A} \) as the linear transformation that maps \( e_i^{K_A} \mapsto e_i^{K_A} \), for \( i \in \{1, 2, \ldots, K_A - 1\} \). We also use \( S \) to denote the canonical matrix representation of transformation \( S \). Now define the affine transformation \( T := Sx + e_1^{K_A} \).

Notice that \( T \) is injective. Let now \( P = \text{co}\{e_1^{K_A-1}, \ldots, e_{K_A-1}^{K_A-1}\} \cup \{0\} \) and define the restriction \( T^* := T | P \) and \( f := \text{Val}(\cdot) : P \to \mathbb{R} \). Now since \( \text{Val}(\cdot) \) is Lipschitz and \( T^* \) is affine, \( f \) is also Lipschitz. Now extend \( f \) to \( \mathbb{R}^{K_A-1} \) by putting \( f(x) = -\infty \), if \( x \notin P \).

Let \( V \subset \text{int}P \) be an open neighborhood of \( x_0 \in \text{int}P \) such that \( T^*x_0 = \bar{p} \). It follows by Radamacher Theorem\(^{15}\) that \( f \) is almost everywhere differentiable in \( V \). Let \( x_0+h, x_0+h+v \in V \) with \( x_0 + h \) a point of differentiability of \( f \). Then we can write:

\[
f(x_0 + h + v) = f(x_0 + h) + \nabla f(x_0 + h) \cdot v + o(||v||).
\]

Let \( f^A(q) := \max_{\sigma} \sigma A(q) \tau, q \in \Delta(K_A) \). Then, \( g_{\tau} := f^A \circ T^* \). By definition, denoting by \( \tau_x \) an optimal strategy of the minimizer at the zero-sum game with matrix \( A(T^*(x)) \),

\[
g_{\tau x_0+h}^A(x_0 + h + v) \geq \text{Val}(T^*(x_0 + h + v)) = \text{Val}(T^*(x_0 + h)) + \nabla f(x_0 + h) \cdot v + o(||v||).
\]

But \( \text{Val}(T^*(x_0 + h)) = g_{\tau x_0+h}(x_0 + h) \). This gives \( g_{\tau x_0+h}(x_0 + h + v) \geq g_{\tau x_0+h}(x_0 + h) + \nabla f(x_0 + h) \cdot v + o(||v||) \). Since \( g_{\tau x_0+h} \) is a convex function, we have that \( g_{\tau x_0+h}(x_0 + h + v) \geq g_{\tau x_0+h}(x_0 + h) + \nabla f(x_0 + h) \cdot v \). This implies that \( \nabla f(x_0 + h) \) is a subgradient of the convex function \( g_{\tau x_0+h} \) at point \( x_0 + h \).

Consider now \( \phi \) as a row vector and \( S \) the matrix representation in the canonical basis of the linear transformation \( S \) above. Note now that \( g_{\tau x_0+h} \) is piecewise linear and its epigraph \( \mathcal{B} \) is therefore a polyhedral set. This implies every maximal proper face of \( \mathcal{B} \) is contained in a hyperplane \( H_i \) of \( \mathbb{R}^{K_A-1} \times \mathbb{R} \) given by

\[
H_i = \{(x, t) \in \mathbb{R}^{K_A-1} \times \mathbb{R} | t = (s_i A^k A^k \tau_{x_0+h})_{kA \in K_A} Sx_0\} \tag{17}
\]

\(^{15}\)See Sorin (2003), Proposition 2.1.

\(^{16}\)If \( U \) is an open subset of \( \mathbb{R}^n \) and \( f : U \to \mathbb{R}^m \) is Lipschitz continuous, then \( f \) is differentiable (Lebesgue) almost everywhere in \( U \).

\(^{17}\)(\(s_i A^k A^k \tau_{x_0+h})_{kA \in K_A} \) is considered a row vector.
where \( s_i \in I_A \). Define \( \overline{y} = (x_0 + h, f(x_0 + h)) \). Notice that \( (x_0 + h, f(x_0 + h)) = (x_0 + h, g_{r_{x_0+h}}(x_0 + h)) \). Let \( I_0 \) be the set of all pure strategies corresponding to maximal proper faces of \( B \) that contain \( \overline{y} \). The graph \( G \) of the affine function \( r(v) = f(x_0 + h) + \nabla f(x_0 + h) \cdot v \) is a hyperplane of \( \mathbb{R}^{K_A-1} \times \mathbb{R} \) that supports \( B \) at \( \overline{y} = (x_0 + h, f(x_0 + h)) \). Now, an application of Farkas’ lemma\(^{18}\) will give us that there exists \( \sigma_{x_0+h} \in \triangle(I_0) \) such that \( \nabla f(x_0 + h) = \sum_{s \in I_0} \sigma_{x_0+h}(s) sA_{K_A} \cdot r_{x_0+h})k_{A \in K_A}S \).

Notice now that \( \phi.S \) is a supergradient of \( f \) at \( x_0 \), therefore it is in the generalized (Clarke) superdifferential \( \partial f(x_0) := \text{co}\{\lim \nabla f(x + h_i), \text{ as } h_i \to 0 \text{ with } i \to \infty\} \).\(^{19}\) Since \( \partial f(x_0) \) is convex, Carathéodory Theorem allows us to write \( \phi.S \) as a convex combination of \( |K_A| \) points \( N(x_0) := \{d_1, \ldots, d_{|K_A|}\} \) \( \subset \{\lim \nabla f(x + h_i), \text{ as } h_i \to 0 \text{ as } i \to \infty\} \). Therefore, for each \( d_k \in N(x_0) \), there exists \( \{h_i\}_{k \in \mathbb{N}} \) such that \( \lim h_i := \nabla f(x_0 + h_i) = (\sigma_{x_0+h}k_{A \in K_A}S \) such that \( h_i \to d_k \) as \( i \to \infty \). Now, passing to a subsequence if necessary, we can assume that \( \tau_{x_0+h_i} \to \tau_{x_0} \in \triangle(J_A) \) and \( \sigma_{x_0+h_i} \to \sigma_{x_0} \in \triangle(I_A) \), as \( i \to +\infty \). So \( (\sigma_{x_0}k_{A \in K_A}S = d_k \).

This implies that there exists \( \{\lambda_k\} \) such that \( \phi.S = \sum \lambda_k(\sigma_{x_0}k_{A \in K_A}S \) for \( \lambda_k \geq 0 \) and \( \sum \lambda_k = 1 \). For any \( q \in \triangle(K_A) \), we have that there exists \( x \in P \) such that \( T^*x = q \), and

\[
\sum \lambda_k(\sigma_{x_0}k_{A \in K_A}S(x - x_0) = \phi.S \cdot (x - x_0) \geq \text{ValA}(T^*x) - \text{ValA}(T^*x_0),
\]

by the fact that \( \phi.S \) is a supergradient of \( f \) at \( x_0 \). This implies \( \sum \lambda_k(\sigma_{x_0}k_{A \in K_A}S \cdot q \geq \text{ValA}(q), \forall q \in \triangle(K_A) \). Now,

\[
\sum \lambda_k(\sigma_{x_0}k_{A \in K_A}S \cdot \text{marg}_{K_A}p = \phi \cdot \text{marg}_{K_A}p = \text{Cav(ValA)}(\text{marg}_{K_A}p).
\]

This implies \( \phi = \sum \lambda_k(\sigma_{x_0}k_{A \in K_A}S \).

\( \square \)

Proof. of Theorem 6.8

Case 1: Assume

\[
\text{ValA}(\text{marg}_{K_A}p) < \text{Cav(ValA)}(p) \text{ and } \text{ValB}(\text{marg}_{K_A}p) = \text{Cav(ValB)}(\text{marg}_{K_A}p).
\]

If \( \text{ValB}(\text{marg}_{K_A}p) = \text{Cav(ValB)}(p) \), the optimal strategy of the informed player in the zero-sum game \( G_B(p) \) is trivially nonrevealing. Indeed, the optimal strategy \( \sigma^B_p \) of the

---

\(^{18}\)See Rockafeller(1970)

\(^{19}\)See Clarke(1975).
informed player in the one-shot zero-sum game with matrix $B(marg_{K_B} p)$ played independently at each stage is the optimal strategy in $G_B(p)$. Uninformed player 3 plays his optimal “approachability” strategy. The \textit{ex-ante} payoff obtained is $\text{Cav(ValB)}(marg_{K_B} p)$.

We now proceed to the construction of the joint-plan of the joint-plan between players 1 and 2. The joint-plan is described as follows: players 1 and 2 will run a jointly controlled lottery with probability $\lambda_k$ for each of the $k \in \mathcal{O}$, where $\mathcal{O}$ is the set of possible outcomes of the lottery and $|\mathcal{O}| = |K_A|$. If the outcome drawn is $k$, then players will play the deterministic path of play that asymptotically approximates the distribution $\sigma_{x_0}^k \otimes \tau_{x_0}^k$ (recall discussion after Definition 5.2). If the outcome of the lottery is $k$, the payoff corresponding to the implementation of the deterministic sequence of moves approximating asymptotically $\sigma_{x_0}^k \otimes \tau_{x_0}^k$ is $(\sigma_{x_0}^k A^k_{\tau_{x_0}^k})_{k_A \in K_A}$. The expected payoffs of this strategy for each of the types are $\sum_k \lambda_k (\sigma_{x_0}^k A^k_{\tau_{x_0}^k})_{k_A \in K_A} \cdot marg_{K_A} p = \text{Cav(ValB)}(marg_{K_B} p)$, which is equal to

$$\text{Cav(ValA)}(marg_{K_A} p) + \text{Cav(ValB)}(marg_{K_B} p).$$

Notice that since $\sum_k \lambda_k (\sigma_{x_0}^k A^k_{\tau_{x_0}^k})_{k_A \in K_A} \cdot q \geq ValA(q), \forall q \in \Delta(K_A)$, player 2 can punish player 1 in case he deviates from the deterministics path of play, playing the optimal strategy of $G_A(marg_{K_A} p)$. Also $\sum_k \lambda_k (\sigma_{x_0}^k A^k_{\tau_{x_0}^k})_{k_A \in K_A} \cdot marg_{K_A} p = \text{Cav(ValA)}(marg_{K_B} p)$, which makes the payoff individually rational for player 2.

\textit{Case 2:} Assume $\text{ValA}(marg_{K_A} p) < \text{Cav(ValA)}(p)$ and $\text{ValB}(marg_{K_B} p) < \text{Cav(ValB)}(marg_{K_B} p)$.

By the same observation in the first paragraph, there are two vectors $\phi_A$ and $\phi_B$ and intervals $Z_A$ and $Z_B$ containing respectively $marg_{K_A} p$ and $marg_{K_B} p$ with $\text{Cav(ValA)}(q) = \phi_A \cdot q$, for all $q \in Z_A \subset \Delta(K_A)$ and $\text{Cav(ValB)}(q) = \phi_B \cdot q$, for all $q \in Z_B \subset \Delta(K_B)$. Applying Lemma 6.9, there exist $\{\lambda^A_k\}$ and $\{\lambda^B_r\}$ such that $\lambda^A_k \geq 0$, $\lambda^B_r \geq 0$ with $k = 1, ..., |K_A|$ and $r = 1, ..., |K_B|$ and strategies $\sigma^k_A \in \Delta(I_A)$, $\sigma^r_B \in \Delta(I_B)$, $\tau^k_A \in \Delta(J_A)$ and $\tau^r_B \in \Delta(J_B)$ such that

$$\phi_A = \sum_k \lambda^A_k (\sigma^k_A A^k_{\tau^k_A})_{k_A \in K_A}$$
\[ \phi_B = \sum_r \lambda_r^B (\sigma_r^B B^k r B^k \tau_r^B)_{k_B \in K_B}. \]

We now define a loose joint-plan: players 1 and 2 will run a jointly controlled lottery with probability \( \lambda_k^A \) for each of the \( k_A \in \mathcal{O}_A \), where \( \mathcal{O}_A \) is the set of possible outcomes of the lottery and \( |\mathcal{O}_A| = |K_A| \). If the outcome drawn is \( k \), then players will play the deterministic path of play that asymptotically approximates the distribution \( \sigma_k^A \otimes \tau_k^A \) (recall discussion after Definition 5.2). If the outcome of the lottery is \( k \), the payoff corresponding to the implementation of the deterministic sequence of moves approximating asymptotically \( \sigma_k^A \otimes \tau_k^A \) is \( (\sigma_k^A A^k \tau_k^A)_{k_A \in K_A} \). Players 1 and 3 run a jointly controlled lottery with probability \( \lambda_r^B \) for each of the \( r \in \mathcal{O}_B \), where \( \mathcal{O}_B \) is the set of possible outcomes of the lottery and \( |\mathcal{O}_B| = |K_B| \). If the outcome drawn is \( r \), then players will play the deterministic path of play that asymptotically approximates the distribution \( \sigma_r^B \otimes \tau_r^B \) (recall discussion after Definition 5.2). If the outcome of the lottery is \( r \), the payoff corresponding to the implementation of the deterministic sequence of moves approximating asymptotically \( \sigma_r^B \otimes \tau_r^B \) is \( (\sigma_r^B B^k r B^k \tau_r^B)_{k_B \in K_B} \).

We check now that the conditions of Theorem 5.3 are satisfied: for the uninformed player 2 we have that

\[
\sum_k \lambda_k^A (\sigma_k^A A^k \tau_k^A)_{k_A \in K_A} \cdot \text{marg}_{K_A} p = \phi_A \cdot \text{marg}_{K_A} p = \text{Cav}(\text{Val}_A)(\text{marg}_{K_A} p).
\]

For the uninformed player 3,

\[
\sum_r \lambda_r^B (\sigma_r^B B^k r B^k \tau_r^B)_{k_B \in K_B} \cdot \text{marg}_{K_B} p = \phi_B \cdot \text{marg}_{K_B} p = \text{Cav}(\text{Val}_B)(\text{marg}_{K_B} p).
\]

And for each \( q \in \Delta(K_A \times K_B) \), we have that

\[
\sum_k \lambda_k^A (\sigma_k^A A^k \tau_k^A)_{k_A \in K_A} \cdot \text{marg}_{K_A} q + \sum_r \lambda_r^B (\sigma_r^B B^k r B^k \tau_r^B)_{k_B \in K_B} \cdot \text{marg}_{K_B} q \geq \text{Val}_A(\text{marg}_{K_A} q) + \text{Val}_B(\text{marg}_{K_B} q).
\]

**Case 3:** Assume:

\( \text{Val}_A(\text{marg}_{K_A} p) = \text{Cav}(\text{Val}_A)(\text{marg}_{K_A} p) \) and \( \text{Val}_B(\text{marg}_{K_B} p) < \text{Cav}(\text{Val}_B)(\text{marg}_{K_B} p) \).

This case is symmetric to case 1.

**Case 4:** Assume

\( \text{Val}_A(\text{marg}_{K_A} p) = \text{Cav}(\text{Val}_A)(\text{marg}_{K_A} p) \) and \( \text{Val}_B(\text{marg}_{K_B} p) = \text{Cav}(\text{Val}_B)(\text{marg}_{K_B} p) \).
This is case is trivial since the optimal strategies of all the players are nonrevealing. Even more than that, this is the case where \( I(p) \) is degenerate: since the optimal strategy is nonrevealing, players just play the optimal strategy of the zero-sum games. Because strategies are optimal in each zero-sum game, the informed player can guarantee \( \text{Cav}(\text{Val}_B)(\text{marg}_{K_B}p) + \text{Cav}(\text{Val}_A)(\text{marg}_{K_A}p) \). But since \( \text{Cav}(\bar{h}) \) is \textit{ex-ante} Nash-equilibrium payoff, it must be that

\[
\text{Cav}(\text{Val}_A)(\text{marg}_{K_A}p) = \text{Cav}(\text{Val}_B)(\text{marg}_{K_B}p) + \text{Cav}(\text{Val}_A)(\text{marg}_{K_A}p).
\]

\[\square\]

The example below is constructed from two zero-sum games \( G_A(\text{marg}_{K_A}p) \) and \( G_B(\text{marg}_{K_B}p) \) where the former is completely revealing and the latter is not. In this example we have \( I(p) \) nodegenerate but the upper-bound of \( I(p) \) attainable as an equilibrium-payoff by the informed player. This shows that the class of 3-player games defined by our model where \( G_A(\text{marg}_{K_A}p) \) and \( G_B(\text{marg}_{K_B}p) \) are not completely revealing does not identify the cases where the upper-bound of \( I(p) \) can be attained by the informed player.

**Example 6.12.** Let

\[
p_0 = \begin{bmatrix}
0.5 & 0 \\
0 & 0.5
\end{bmatrix}
\]

Consider the payoff matrices:

\[
A^1 = \begin{bmatrix}
0 & 0 \\
0 & -1
\end{bmatrix}; \quad A^2 = \begin{bmatrix}
-1 & 0 \\
0 & 0
\end{bmatrix}
\]

\[
B^1 = \begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix}; \quad B^2 = \begin{bmatrix}
0 & 0 \\
0 & 1
\end{bmatrix}
\]

Since the nonrevealing value of game \( G_B(\text{marg}_{K_B}p_0) \) is concave, to every prior \( p \) on \( \Delta(K_B) \) consider \( \sigma_B \) and \( \tau_B \) the optimal (nonrevealing) strategies of the players 1 and 3 in \( G_B(\text{marg}_{K_B}p) \). Consider, now strategies \( \sigma_A = (1,0) \in \Delta(I_A) \) and \( \tau_A = (0,1)^T \in \Delta(J_A) \). Then, \( \sigma_A A(q) \tau_A = \text{Cav}(\text{Val}_A)(q) = 0, \forall q \in \Delta(K_A) \). Let \( \sigma := \sigma_A \otimes \sigma_B \). The nonrevealing joint-plan induced by strategies \( (\sigma_A, \tau_A) \) – with contracts \( \sigma_A \otimes \tau_A \) – and the nonrevealing equilibrium \( (\sigma_B, \tau_B) \) defines a profile os strategies \( (\sigma, \tau_A, \tau_B) \) that is an equilibrium of the
Figure 3. Graphs of $\text{CavValA}()$ and $\text{ValA}()$ for a "fully revealing" game A

Figure 4. Graphs of $\text{CavValB}()$ and $\text{ValB}()$ for the "nonrevealing" game B

game $\mathcal{G}(p_0)$. Note that we only have to care about punishments to deviations of players in game A. But the joint-plan in this game trivially satisfies the conditions of Theorem 5.3. The ex-ante equilibrium payoff of $\text{Cav(ValA)}(\text{marg}_{K_A}p_0) + \text{Cav(ValB)}(\text{marg}_{K_A}p_0)$.

7. Characterization of Nash-Equilibrium Payoffs

This section is almost entirely expository of the method developed by Hart (1985), but it is important because it allows us to reinterpret the results of the previous section. We reintroduce the notion of $G$-process presented in Hart (1985) and adapt it to our setting. The adaptations are minor and a characterization of equilibrium payoffs can be given with the same ideas present in Hart (1985). One of the interesting corollaries of this characterization is that the set of Uniform Equilibrium payoffs and Nash equilibrium payoffs is the same in $\mathcal{G}(p)^{20}$. This is an immediate corollary of the proof.

The essentially new parts in this section are Theorem 7.11 - generalized in the next section in Theorem 8.2 - and Example 7.9, which is an example of a game where $I(p)$ is nondegenerate but the upper-bound of $I(p)$ can never be attained in equilibrium – actually, only the lower bound can be attained.

---

Definition 7.1. Let \( l_\infty \) be the Banach space of real bounded sequences \( x = (x_n)_{n \in \mathbb{N}} \). A Banach-Limit\(^{21}\) is a real functional \( \mathcal{L} : l_\infty \to \mathbb{R} \) with the following properties:

\begin{enumerate}
    \item \( \mathcal{L}((\lambda x_n + \mu y_n)_{n \in \mathbb{N}}) = \lambda \mathcal{L}((x_n)_{n \in \mathbb{N}}) + \mu \mathcal{L}((y_n)_{n \in \mathbb{N}}) \).
    \item \( \mathcal{L}((x_{n+1})_{n \in \mathbb{N}}) = \mathcal{L}((x_n)_{n \in \mathbb{N}}) \).
    \item \( \liminf_{n \to +\infty} x_n \leq \mathcal{L}((x_n)_{n \in \mathbb{N}}) \leq \limsup_{n \to +\infty} y_n \).
\end{enumerate}

A triple of strategies \((\sigma, \tau_A, \tau_B)\) is an \( \mathcal{L} \)-equilibrium if:

\begin{enumerate}
    \item \( \mathcal{L}(\|\mathbb{E}_{\sigma, \tau_A, \tau_B}(\frac{1}{T} \sum_{t=1}^{T} (A_{i_A,j_B}^{k_A} + B_{i_A,j_B}^{k_B})\|_1)) \geq \mathcal{L}(\|\mathbb{E}_{\sigma, \tau_A, \tau_B}(\frac{1}{T} \sum_{t=1}^{T} (A_{i_A,j_B}^{k_A} + B_{i_A,j_B}^{k_B})\|_1)) \), for all \((k_A, k_B) \in K_A \times K_B\), for all strategies \( \sigma \) of the informed player 1.
    \item \( \mathcal{L}(\|\mathbb{E}_{\sigma, \tau_A, \tau_B}(\frac{1}{T} \sum_{t=1}^{T} (-A_{i_A,j_B}^{k_A})\|_1)) \geq \mathcal{L}(\|\mathbb{E}_{\sigma, \tau_A, \tau_B}(\frac{1}{T} \sum_{t=1}^{T} (-A_{i_A,j_B}^{k_A})\|_1)) \), for all strategies \( \tau_A \) of the uninformed player 2.
    \item \( \mathcal{L}(\|\mathbb{E}_{\sigma, \tau_A, \tau_B}(\frac{1}{T} \sum_{t=1}^{T} (-B_{i_B,j_B}^{k_B})\|_1)) \geq \mathcal{L}(\|\mathbb{E}_{\sigma, \tau_A, \tau_B}(\frac{1}{T} \sum_{t=1}^{T} (-B_{i_B,j_B}^{k_B})\|_1)) \), for all strategies \( \tau_B \) of the uninformed player 3.
\end{enumerate}

The set of equilibrium profiles is the same under any Banach-Limit \( \mathcal{L} \), so there is no loss of generality in fixing a Banach-Limit throughout (see Section 4.2 in Hart (1985)). Indeed, because of (3) of Definition 7.1, a profile of strategies is an \( \mathcal{L} \)-equilibrium if and only if it is a Nash equilibrium.

From now on we fix a Banach-Limit \( \mathcal{L} \).

Notation. Let \( M := \sup\{|A_{i_A,j_B}^{k_A,k_B}|, |B_{i_B,j_B}^{k_A,k_B}|\}_{i_A \in I_A, i_B \in I_B, j_A \in J_A, j_B \in J_B, (k_A, K_B) \in K_A \times K_B} \), then \( \mathbb{R}_M^{K_A,K_B} \) is the set of all vectors in \( \mathbb{R}^{K_A,K_B} \) whose coordinates are bounded by \( M \). We also denote \( \mathbb{R}_M \) will denote the interval \([-M, M]\). Let \( A(i_A, j_A) := (A_{i_A,j_A}^{k_A})_{k_A \in K_A} \).

Then \( F_A = \text{co}\{A(i_A, j_A)|i_A \in I_A, j_A \in J_A\} \) is the set of feasible payoffs in the game between players 1 and 2. The set \( F_B = \text{co}\{B(i_B, j_B)|i_B \in I_B, j_B \in J_B\} \) is analogously defined and corresponds to the set of feasible payoffs in the game between players 1 and 3.

7.1. Nonrevealing Payoff Set. Following Hart (1985), the nonrevealing equilibrium payoff set \( \tilde{G} \) is defined as the set of points \((a, \alpha, \beta, p) \in \mathbb{R}_M^{K_A,K_B} \times \mathbb{R}_M \times \mathbb{R}_M \times \Delta(K_A \times K_B)\) such that:

\begin{enumerate}
    \item \( a \cdot q \geq \text{Val}_A(\text{marg}_{K_A}q) + \text{Val}_B(\text{marg}_{K_B}q), \forall q \in \Delta(K_A \times K_B) \).
    \item \( \alpha \leq \text{CavVal}_A(\text{marg}_{K_A}p) \).
\end{enumerate}

\(^{21}\)The concept of Banach-Limit is introduced in Hart (1985), section 4. A good reference is also Dunford and Schwartz 1958, p. 73.
(3) $\beta \leq \text{CavValB}(\text{marg}_K p)$.
(4) There exists $c_A \in F_A$ and $c_B \in F_B$ and $c^{K_A, K_B} := c_A + c_B$ with $a \geq c$.
(5) $a \cdot p = c \cdot p$.
(6) $c_A \cdot \text{marg}_{K_A} p = \alpha$.
(7) $c_B \cdot \text{marg}_{K_B} p = \beta$.

7.2. $\tilde{G}$-process. A $\tilde{G}$-process starting at $g \in \mathbb{R}_{M}^{K_A, K_B} \times \mathbb{R} \times \mathbb{R} \times \Delta(K_A \times K_B)$ is a sequence of $\mathbb{R}_{M}^{K_A, K_B} \times \mathbb{R} \times \mathbb{R} \times \Delta(K_A \times K_B)$-valued random variables (on some probability space) $g_n = (a_n, \alpha_n, \beta_n, p_n), n \in \mathbb{N}$ satisfying the following properties:

(1) $g_1 = g$ a.s. .
(2) There exists a sequence $(\mathcal{F}_n)_{n \in \mathbb{N}}$ of finite fields with respect to which $(g_n)_{n}$ is a martingale.
(3) Let $g_{\infty}$ be the a.s. limit of $g_{n}$; then $g_{\infty} \in \tilde{G}$ a.s. .
(4) For each $n = 1, 2, \ldots$, either $a_{n+1} = a_n$ a.s., or $p_{n+1} = p_n$ a.s. .

Theorem 7.2. $(a, \alpha_A, \alpha_B)$ is an equilibrium payoff of $G(p)$ then there exists a $\tilde{G}$-process starting at $(a, \alpha_A, \alpha_B, p)$.

The proof of this theorem involves an extensive construction, but exactly analogous to the one presented in Hart (1985). The converse of Theorem 7.2 is also true and the proof is also a small modification of the argument in Hart (1985) – we indicate the modification needed in Appendix B. Since for this section we are only interested in the necessary part of the characterization we reproduce the steps of the construction needed. The proofs of the properties of the stochastic process constructed are essentially no different from those found in Hart (1985) so we abstain from rewriting them and just indicate where they can be found in that paper.

In Theorem 7.12, we use the construction to provide a necessary condition on the asymptotic properties of the stochastic process associated to best-possible payoffs. We present it for the two-types case and leave the general formulation for the next section.

7.3. The Basic Probability Space. For each $t \in \mathbb{N}$, let $H_{t+1/2} = (I_A \times I_B \times J_A \times J_B)^{t-1} \times (I_A \times I_B) = H_t \times (I_A \times I_B)$ and denote by $\mathcal{H}_{t+1/2}$ the finite field it generates. This defines the collection $(\mathcal{H}_s)_{s \in \mathbb{N}_2}$, with $N_2 = \{1, 1 + 1/2, 2, 2 + 1/2, \ldots\}$. Since the probability space is actually $\Omega = H_{\infty} \times K_A \times K_B$, we will denote the field generated by $H_s$ on $\Omega$ also by $\mathcal{H}_s$. 


Note that \((\mathcal{H}_s)_{s \in \mathbb{N}_2}\) is an increasing sequence of finite subfields of \(\mathcal{H}_\infty\) generating \(\mathcal{H}_\infty\). From now on fix \((\sigma, \tau_A, \tau_B)\) and prior \(p\) an equilibrium of the game.

7.4. Martingale of Posteriors. Let \(p^{k_A, k_B}_s := \mathbb{P}_{\sigma, \tau_A, \tau_B, p}(\kappa_A \times \kappa_B = (k_A, k_B)|\mathcal{H}_s)\) and \(p_s := (p^{k_A, k_B}_s)_{k_A, k_B}\).

**Proposition 7.3.** The sequence \((p_s)_{s \in \mathbb{N}_2}\) is a \(\Delta(K_A \times K_B)\)-valued martingale with respect to \((\mathcal{H}_s)_{s \in \mathbb{N}_2}\), satisfying:

1. \(p_1 = p\),
2. \(p_{t+1/2} = p_{t+1}, \forall t \in \mathbb{N}\),
3. There exists \(p_\infty\) such that \(p_s \to p_\infty\) a.s. as \(s \to +\infty\).

**Proof.** See proof of Proposition 4.12 in Hart (1985). \(\square\)

7.5. The expected payoffs martingales. Let \(\alpha_s := L(\mathbb{E}[\alpha_T|\mathcal{H}_s])\), where \(\alpha_T = \frac{1}{T} \sum_{t=1}^{T} A^*_t \sigma A, J_A^t\) and \(\beta_s := L(\mathbb{E}[\beta_T|\mathcal{H}_s])\), where \(\beta_T = \frac{1}{T} \sum_{t=1}^{T} B^*_t \sigma B, J_B^t\).

These martingales correspond to the process of expected payoffs accumulated by the informed player in each game. The martingale property is a consequence of Lemma 4.6 in Hart (1985).

**Proposition 7.4.** \((\alpha_s)_{s \in \mathbb{N}_2}\) and \((\beta_s)_{s \in \mathbb{N}_2}\) are \(\mathbb{R}_M\)-valued martingales with respect to \((\mathcal{H}_s)_{s \in \mathbb{N}_2}\) satisfying \(\alpha_1 + \beta_1 = a \cdot p\)

**Proof.** See proof of Proposition 4.17 in Hart (1985). \(\square\)

**Proposition 7.5.** If \((\sigma, \tau_A, \tau_B)\) is a Nash-equilibrium profile then it must satisfy:

1. \(\alpha_s \leq \text{Cav}(\text{Val}_A)(\text{marg}_{K_A} p_s)\) for all \(s \in \mathbb{N}_2\)
2. \(\beta_s \leq \text{Cav}(\text{Val}_B)(\text{marg}_{K_B} p_s)\) for all \(s \in \mathbb{N}_2\)

**Proof.** This is a consequence of Propositions 4.40 and 3.16 in Hart (1985). \(\square\)

7.6. Expected Vector payoffs Martingales. Let \(c^{k_A}_s := L(\mathbb{E}[a^{k_A}_T|\mathcal{H}_s])\), \(c^{k_B}_s := L(\mathbb{E}[b^{k_A}_T|\mathcal{H}_s])\) (note that here we take conditional expectations over all histories, even those that have probability 0 under \((k_A, k_B)\) to \(k_A\) or \(k_B\)). Let \(c^A_s := (c^{k_A}_s)_{s \in \mathbb{N}_2}\) and \(c^B_s := (c^{k_B}_s)_{s \in \mathbb{N}_2}\). Then, the martingales \(c^{k_A}_s\) and \(c^{k_B}_s\) can be interpreted as the payoffs associated to each matrix \((k_A, k_B)\) of stage payoffs when the informed player plays the equilibrium strategy \(\sigma\) up
to $s - 1/2$ and for $t \geq s$ he switches at every stage to his average nonrevealing strategy

$$\sum_{(k_A, k_B) \in K_A \times K_B} p_s^{k_A, k_B} \sigma(k_A, k_B, h_t)$$
instead of playing $\sigma(k_A, k_B, h_t)$.

**Proposition 7.6.**

1. $c_s^A \to c_\infty^A \in F_A$ a.s.
2. $c_s^B \to c_\infty^B \in F_B$ a.s.

**Proof.** See Proposition 4.20 in Hart (1985).

It is worth to stop here and provide an interpretation of some of the properties of the stochastic process of payoffs induced by equilibria. When defining the martingales associated to an equilibrium payoff, Hart (1985) points out that if $\kappa_A \times \kappa_B = (k_A, k_B)$ and $h_t$ is a history that occurred up to stage $t$ with positive probability, the expected payoff $a^{k_A, k_B}(h_t, i_A^t, i_B^t)$ for the informed player has to be equal for each pair $(i_A^t, i_B^t)$ played with positive probability ($a^{k_A, k_B}(h_t, i_A^t, i_B^t) > 0$). Otherwise, the informed player could give probability one to the pair $(i_A^t, i_B^t)$ leading to the highest payoff; this deviation would be undetected by the uninformed players and would be profitable for the informed player. The expected payoff for the informed player in our environment is the sum of payoffs from both zero-sum games he plays at each stage. This implies that the expected payoff on each game can change, but their sum cannot. This captures the potential signalling tradeoff between each zero-sum game: the signalling procedure played by the informed player might not satisfy the indifference condition mentioned above for each of the zero-sum games considered separately, but when the sum of expected payoffs is considered, it does.

**Proposition 7.7.** $\alpha_s - p_s^A \cdot c_s^A \to 0$ and $\beta_s - p_s^B \cdot c_s^B \to 0$.

**Proof.** See proposition 4.23 in Hart (1985).

**Corollary 7.8.** $\alpha_\infty = p_\infty^A \cdot c_\infty^A$ and $\beta_\infty = p_\infty^B \cdot c_\infty^B$


### 7.7. Equilibrium Martingales.

For each $(k_A, k_B)$ and $s \in N_2$, define $e_s^{k_A, k_B} := \sup_{\sigma^t} L[\mathbb{E}^{k_A, k_B}[a_T^{k_A} + b_T^{k_B} | H_s]]$.

**Proposition 7.9.**

1. $e_1^{k_A, k_B} = a^{k_A, k_B}$

---

LACK OF INFORMATION ON TWO SIDES

\[ e^A_s \geq e_A^K + e^B_s \]

\[ e^A_{t+1/2} = \mathbb{E}[e^A_{t+1} | \mathcal{H}_{t+1/2}] \]

\[ e^A_t(h_t) = \max_{i^A_t, i^B_t} e^A_{t+1/2}(h_t, i^A_t, i^B_t), \text{ for all } h_t \in H_t. \]

**Proof.** See Proposition 4.26 in Hart (1985). □

7.8. The \( \tilde{G} \)-process. Notice that by property (4) above we have that \( (c_s)_{s \in N_2} \) is a super-martingale. Now, following the exact same steps in page 133 in Hart (1985) we are able to obtain a martingale \( (f_s)_{s \in N_2} \) in \( \mathbb{R}^{K_A, K_B} \) with respect to \( \{H_s\}_{s \in N_2} \), satisfying:

\[ f_1 = a \]
\[ f_t = f_{t+1/2}, \text{ for all } t \in \mathbb{N} \]
\[ \text{there exists a random variable } f_\infty \in \mathbb{R}^{K_A, K_B} \text{ such that } f_s \to f_\infty \text{ a.s. as } s \to \infty \]
\[ f_s \geq c_s, \text{ for all } s \in N_2 \]
\[ f_\infty \geq c_\infty \text{ and } p_\infty \cdot f_\infty = p_\infty \cdot c_\infty \text{ a.s.} \]

The \( \tilde{G} \)-process is defined as \( (f_s, \alpha_s, \beta_s, p_s)_{s \in N_2} \). All the required properties are indeed satisfied, which is a consequence of Proposition 4.43 in Hart (1985). As we can see from the definition of a \( \tilde{G} \)-process, the set \( \tilde{G} \) plays a very important role on what payoffs can be attained in equilibrium. For a fixed \( p \), the set \( \tilde{G} \) is actually convex, so the expectation of the martingales starting at \( (a, \alpha, \beta, p) \) with \( p_s = p \), which is of course \( (a, \alpha, \beta, p) \), belongs to \( \tilde{G} \). This implies that the classes of games for which Theorems 6.8 and 8.2 apply essentially imply restrictions on the p-section of \( \tilde{G} \).

With the help of Theorem 7.2 we prove the result claimed in footnote 9.

**Proposition 7.10.** If \( (\sigma, \tau_A, \tau_B) \) is an independent 2-3-safe joint-plan equilibrium of \( G(p) \) then the ex-ante equilibrium payoff for the informed player is \( \text{Cav}(\tilde{h}) \).

**Proof.** By Theorem 7.2 and the construction above, let \( (f_s, \alpha_s, \beta_s, p_s)_{s \in N_2} \) be the \( \tilde{G} \)-process associated to the equilibrium profile \( (\sigma, \tau_A, \tau_B) \) such that \( ((f_1, \alpha_1, \beta_1, p_1)) = (a, \alpha, \beta, p) \). First note that by definition the martingale of posteriors has a.s finite range. So there exists \( s_0 \in N_2 \) such that \( p_s \) is a.s. constant \( \forall s \leq s_0 \). Let \( s \leq s_0 \) and let \( h_s \in H_s \). Then, because the joint-plan is “safe” for players 2 and 3, it implies that \( \alpha_s(h_s) \leq \text{ValA}(\text{marg}_{k_A} p_s(h_s)) \) and \( \beta_s(h_s) \leq \text{ValB}(\text{marg}_{k_B} p_s(h_s)) \). Therefore \( \alpha_s(h_s) + \beta_s(h_s) \leq \text{Cav}(\tilde{h})(p_s(h_s)) \text{ a.s., by definition.} \)

Now taking expectation on both sides, using Proposition 7.4 and Jensen’s Inequality we have that \( \alpha_1 + \beta_1 \leq \text{Cav}(\tilde{h})(p) \). By Theorem 6.1 the result follows. □
Our objective is now to study the asymptotic behavior of the $\tilde{G}$-process and obtain necessary conditions on equilibria that pay the upper bound of $I(p)$. Below, we obtain a necessary condition on the location of the asymptotic posterior of the associated $\tilde{G}$-process. In the next section we generalize this condition to more than two types.

**Definition 7.11.** Let $|K_A| = |K_B| = 2$. Let $p \in \Delta(K_A \times K_B)$. $L(p_A)$ will denote the largest interval of $\Delta(K_A)$ that contains $p_A$ and such that $\text{CavVal}_A(\cdot)$ is linear in $L(p_A)$ and, similarly, we will use $L(p_B)$ to denote such an interval in $\Delta(K_B)$.

**Example 7.12.** In the figure, the interval $(1/4, 3/4)$ in green corresponds to $L((1/2, 1/2))$, where $p_0 = (1/2, 1/2)$. The graph below is taken from Example 6.7.

![Graph](image)

**Theorem 7.13.** Let $|K_A| = |K_B| = 2$. Let $(\sigma, \tau_A, \tau_B)$ be an equilibrium with an ex-ante equilibrium payoff of $\text{CavVal}_A(\text{marg}_{K_A} p) + \text{CavVal}_B(\text{marg}_{K_B} p)$. Then the marginal asymptotic posteriors $\text{marg}_{K_A} p_\infty$ and $\text{marg}_{K_B} p_\infty$ induced by the equilibrium profile satisfy:

1. $\text{marg}_{K_A} p_\infty \in L(\text{marg}_{K_A} p)$ a.s.
2. $\text{marg}_{K_B} p_\infty \in L(\text{marg}_{K_B} p)$ a.s.

**Proof.** First, if both games $G_A(p)$ and $G_B(p)$ are such that the concavification of the non-revealing value is affine, then the result is trivial. Assume therefore without loss of generality that $\text{CavVal}_A(\cdot)$ is not affine and suppose by contradiction that the asymptotic posterior $\text{marg}_{K_A} p_\infty$ takes values in $\Delta(K_A) \setminus L(\text{marg}_{K_A} p)$ with positive probability. Taking limits with $s \to +\infty$ in Theorem 7.5 and applying the Martingale Convergence Theorem to $(\alpha_s)_{s \in \mathbb{N}_2}$, it follows that $\alpha_\infty \leq \text{CavVal}_A(\text{marg}_{K_A} p_\infty)$ a.s. Now taking expectation on both sides $E_{\sigma,\tau_A,\tau_B,p}[\alpha_\infty] \leq E_{\sigma,\tau_A,\tau_B,p}[\text{CavVal}_A(\text{marg}_{K_A} p_\infty)] < \text{CavVal}_A(\text{marg}_{K_A} p)$, because
the distribution of \( \text{marg}_{K_A} p_\infty \) puts positive probability in \( \Delta(K_A) \setminus L(\text{marg}_{K_A} p) \). Because of Proposition 7.5 it follows also that \( \mathbb{E}_{\sigma,\tau_A,\tau_B, p}[\beta_\infty] \leq \mathbb{E}_{\sigma,\tau_A,\tau_B, p}[\text{Cav}(\text{Val}_B)(\text{marg}_{K_B} p_\infty)] \leq \text{Cav}(\text{Val}_B)(\text{marg}_{K_B} p) \). This implies that ex-ante payoff of equilibrium profile \((\sigma, \tau_A, \tau_B)\) is strictly less than \(\text{Cav}(\text{Val}_B)(\text{marg}_{K_B} p) + \text{Cav}(\text{Val}_A)(\text{marg}_{K_A} p)\).

As we commented in section 1.1, this theorem provides a meaning to the notion of “optimal information disclosure” in our model. More precisely, it “quantifies” the notion of optimal information disclosure by establishing a specific region where the asymptotic posterior should be located for an equilibrium paying \(\text{Cav}(\text{Val}_A)(\text{marg}_{K_A} p) + \text{Cav}(\text{Val}_B)(\text{marg}_{K_B} p)\). This region is determined by the specific geometry of \(\text{Cav}(\text{Val}_A)\) and \(\text{Cav}(\text{Val}_B)\), so this necessary condition also captures the interplay between correlation and “payoff structure” of the zero-sum games in our model. Indeed, as we noted before, correlation reduces signalling opportunities for the informed player in the sense that it impairs his abilities of playing the two zero-sum games separately. This however does not determine the attainability of the upper bound of \(I(p)\) by the informed player in equilibrium because the geometry of the nonrevealing value function - which is purely “payoff-dependent” - also plays an important role, determining the \(L(\cdot)\) regions, as we showed above. Therefore, this indicates that for certain payoffs and correlated priors, the attainability of \(\text{Cav}(\text{Val}_A)(\text{marg}_{K_A} p) + \text{Cav}(\text{Val}_B)(\text{marg}_{K_B} p)\) will not be possible. Indeed, as we will show in the example below, it might be the case that \(\text{Cav}(\text{Val}_A)(\text{marg}_{K_A} p) + \text{Cav}(\text{Val}_B)(\text{marg}_{K_B} p)\) is not attainable because any equilibrium reveals “too much information”, that is, the marginal asymptotic posterior induced in equilibrium are never confined to the \(L(\cdot)\) regions.

Indeed, we are going to use Theorem 7.10 above to obtain an example of a game \(G(p)\) where \(I(p)\) is nondegenerate and the \(\text{Cav}(\text{Val}_A)(\text{marg}_{K_A} p) + \text{Cav}(\text{Val}_B)(\text{marg}_{K_B} p)\) is not attainable by the informed player. Indeed, only the lower bound of \(I(p)\) is attainable by the informed player.

7.9. Example.

\[
p_0 = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}
\]

Consider games:
\[ A^1 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, A^2 = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \]

\[ B^1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B^2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \]

**Figure 5.** Graphs of Cav(ValA)(dotted) and ValA(continuous) for a completely revealing game A

**Figure 6.** Graphs of CavValB and ValB for the "nonrevealing" game B
Assume by way of contradiction that \((\sigma, \tau_A, \tau_B)\) is an equilibrium that pays ex-ante \(\text{Cav}(\text{Val}_A)(p^A_0) + \text{Cav}(\text{Val}_B)(p^B_0)\), where \(\text{marg}_{K_A} p_0 = p^A_0\) and \(\text{marg}_{K_B} p_0 = p^B_0\), for the informed player, in the example. Let \(V_A : \triangle(K_A) \to \mathbb{R}, V_A(p) := \max_{\sigma, \tau} \{\sigma A(p) | \sigma A(p) \tau \leq \text{CavVal}_A(p)\}\). For the example, \(V_A(p) = \text{Val}_A(p), \forall p \in \triangle(K_A)\). For each \(s \in \mathbb{N}_2, \alpha_s \leq V_A(p_s)\) a.s.. Therefore, \(\alpha_s \leq \text{Val}_A(p_s)\) a.s.. Letting \(s \to \infty\), we have, by the martingale convergence theorem, \(\alpha_s \to \alpha_\infty, p^A_s \to p^A_\infty\) a.s. and, by the Dominated Convergence Theorem, \(E[\alpha_\infty] \leq E[\text{Val}_A(p^A_\infty)]\). Since, by assumption, \(E[\alpha_\infty] = E[\text{Val}_A(p^A_0)] = \text{CavVal}_A(\text{marg}_{K_A} p_0)\). We have that the distribution of \(p^A_\infty\) has to be concentrated at the boundary of \(\triangle(K_A)\). Also, by theorem Theorem 7.10, \(\text{marg}_{K_B} p_0 = \text{marg}_{K_B} p_\infty\) a.s.

Hence, we have that for any history \(h_\infty \in H_\infty\) outside a set of \(\mathbb{P}_{\sigma, \tau_A, \tau_B}\)-measure zero, the bistochastic matrix representation of \(p_\infty(h_\infty)\) has either the first or the second row filled with zeros. Since \(\text{marg}_{K_B} p_\infty(h_\infty) = \text{marg}_{K_B} p_0\) a.s., then we have that the bi-stochastic representation of \(p_\infty(h_\infty)\) is either:

\[
\begin{bmatrix}
1/2 & 1/2 \\
0 & 0 \\
\end{bmatrix}
or
\begin{bmatrix}
0 & 0 \\
1/2 & 1/2 \\
\end{bmatrix}
\]

Now the process of posteriors is a martingale which implies that the expectation of \(p_\infty\) has to be \(p_0\). In bi-stochastic representation, we then have the following equality for \(\lambda \in [0, 1]\):

\[
\begin{bmatrix}
1/2 & 0 \\
0 & 1/2 \\
\end{bmatrix} = \lambda \begin{bmatrix}
0 & 0 \\
1/2 & 1/2 \\
\end{bmatrix} + (1 - \lambda) \begin{bmatrix}
1/2 & 1/2 \\
0 & 0 \\
\end{bmatrix}.
\]

This equation has no solution in \(\lambda \in [0, 1]\), which finally implies a contradiction. There is no equilibrium paying ex-ante to the informed player \(\text{CavVal}_A(p^A_0) + \text{CavVal}_B(p^B_0)\). The arguments above indeed give us more: remember we had \(\alpha_s \leq V_A(p^A_s) = \text{Val}_A(p^A_s)\) a.s. and \(\beta_s \leq \text{CavVal}_B(p^B_s) = \text{Val}_B(p^B_s)\) a.s.. This implies that \(\alpha_s + \beta_s \leq \text{Val}_A(p^A_s) + \text{Val}_B(p^B_s)\) a.s. and therefore that \(E[\alpha_\infty + \beta_\infty] \leq E[\text{Val}_A(p^A_\infty) + \text{Val}_B(p^B_\infty)] \leq \text{Cav}(\bar{h})(p_0)\). \(\text{Cav}(\bar{h})(p_0)\) is the lowest possible ex-ante payoff to the informed player, the reasoning implies that every equilibrium of the example pays \(\text{Cav}(\bar{h})(p_0)\) to the informed player.
We now allow $|K_A| \geq 2$ and $|K_B| \geq 2$.

**Definition 8.1.** Let $G_A(p)$ denote the zero-sum repeated game with lack of information on one side and prior $p$. $G_A(p)$ game is a *locally not revealing game* whenever there is an optimal strategy of the informed player in this game that induces at least one posterior in the interior of the simplex.

The interpretation is clear: for at least one signal induced with positive probability by the optimal strategy of the informed player, all types that were initially likely are still likely from the point of view of the uninformed player.

The situation in the general case is different from the two-types case: in the two-types case the *locally not revealing* property is equivalent to *not completely revealing*: indeed, because the simplex of probabilities is one dimensional in the two-types case, if there is one point in the interior of the simplex at which the concavification of the nonrevealing value and the nonrevealing value are equal, then for any point in the interior of the simplex, there is an optimal strategy of the informed player at which the induced posteriors are in the interior with positive probability.

**Theorem 8.2.** Let $p \in \Delta(K_A \times K_B)$ with $|K_A| > 2$ and $|K_B| > 2$. Then if $G_A(\text{marg}_{K_A} p)$ and $G_B(\text{marg}_{K_B} p)$ are locally not revealing games at $\text{marg}_{K_A} p$ and $\text{marg}_{K_B} p$ respectively, then the upper-bound of $I(p)$ is an equilibrium payoff for the informed player.

**Proof.** We start with a lemma:

**Lemma 8.3.** Let $p \in \Delta(K_A)$ and $\text{Cav}(\text{Val}_A)(\text{marg}_{K_A} p) > \text{Val}_A(\text{marg}_{K_A} p)$. There exists $\phi \in \mathbb{R}^{K_A}$ such that:

1. $\phi \cdot q \geq \text{Val}_A(q), \forall q \in \Delta(K_A)$.
2. $\phi \cdot p = \text{Cav}(\text{Val}_A)(p)$.

**Proof.** By the theory of zero-sum games with lack of information on one-side, there is a polytope $P_A \subset \Delta(K_A)$ containing $\text{marg}_{K_A} p$ such that the vertices of $P$ are the posteriors induced by the optimal strategy of the informed player in game $G_A(p)$. One of this vertices, say $\overline{p_A}$, is such that $\overline{p_A} \in \text{int}\Delta(K_A)$ by assumption and

$$\text{Val}_A(\overline{p_A}) = \text{Cav}(\text{Val}_A)(\overline{p_A}).$$
Therefore, there exists $\phi_B \in \mathbb{R}^{K_A}$ such that $\text{Cav}(\text{Val}_A)(q) = \phi_A \cdot q, \forall q \in P_A$

$$\text{Cav}(\text{Val}_A)(q) \geq \phi_A \cdot q, \forall q \in \Delta(K_A).$$

\[\square\]

Assume

$$\text{Cav}(\text{Val}_A)(\text{marg}_{K_A}p) > \text{Val}_A(\text{marg}_{K_A}p)$$

and

$$\text{Cav}(\text{Val}_B)(\text{marg}_{K_B}p) > \text{Val}_A(\text{marg}_{K_B}p).$$

Applying the lemma above and Lemma 6.10, there exists $\phi_A \in \mathbb{R}^{K_A}$, $\{\lambda^A_k | \lambda^A_k \geq 0 \text{ with } k = 1, \ldots, |K_A| \}$ and $\sum_k \lambda^A_k = 1$, $\{\sigma^A_k | k = 1, \ldots, |K_A| \} \subset \Delta(I_A)$ and $\{\tau^A_k | k = 1, \ldots, |K_A| \} \subset \Delta(J_A)$ such that:

$$\phi_A = \sum_k \lambda^A_k (\sigma^A_k - \tau^A_k)_{k \in K_A},$$

$$\phi_A \geq \text{Val}_A(q), \forall q \in \Delta(K_A),$$

$$\phi_A \cdot \text{marg}_{K_A}p = \text{Cav}(\text{Val}_A)(\text{marg}_{K_A}p).$$

and $\phi_B \in \mathbb{R}^{K_B}$, $\{\lambda^B_k | \lambda^B_k \geq 0 \text{ with } k = 1, \ldots, |K_B| \}$ and $\sum_k \lambda^B_k = 1$, $\{\sigma^B_k | k = 1, \ldots, |K_B| \} \subset \Delta(I_A)$ and $\{\tau^B_k | k = 1, \ldots, |K_B| \} \subset \Delta(J_B)$ such that:

$$\phi_B = \sum_k \lambda^B_k (\sigma^B_k - \tau^B_k)_{k \in K_B},$$

$$\phi_B \geq \text{Val}_B(q), \forall q \in \Delta(K_B),$$

$$\phi_B \cdot \text{marg}_{K_B}p = \text{Cav}(\text{Val}_B)(\text{marg}_{K_B}p).$$

The construction of equilibria now follows the exact same four cases of Theorem 6.8. \[\square\]

Let $p \in \Delta(K_A)$. Consider $\mathcal{P}(p)$ as the set of convex subsets of $\Delta(K_A)$ that contain $p$ in its relative interior and such that: if $C \in \mathcal{P}(p)$ implies $\text{Cav}(\text{Val}_A)$ is affine in $C$. We can define a partial order on $\mathcal{P}(p)$ as follows: $C \preceq \bar{C}$ if and only if $C \subset \bar{C}$. $\mathcal{P}(p)$ is nonempty since it contains $\{p\}$. Now, for a totally ordered subset $\mathcal{Y}$ of $\mathcal{P}(p)$ it follows that $\mathcal{Y}$ has an upper bound: consider $U = \text{co}(\bigcup_{C \in \mathcal{Y}} C)$ and notice that $U$ is convex and contains $p$ in its relative interior. Moreover, suppose by contradiction that $\text{Cav}(\text{Val}_A)$ is not affine in $U$. Then there exists $p \neq \bar{p}$ and $\alpha \in (0, 1)$ such that $\text{Cav}(\text{Val}_A)(\alpha.p + (1 - \alpha)\bar{p}) > \alpha.\text{Cav}(\text{Val}_A)(p) + (1 - \alpha).\text{Cav}(\text{Val}_A)(\bar{p})$. Without loss of generality we can consider $p \in C$ and $\bar{p} \in \bar{C}$. Since the order relation is total in $\mathcal{Y}$, we can consider $p \in C \subset \bar{C}$. But by definition
\( \alpha.p + (1-\alpha).\tilde{p} \in \tilde{C} \) and \( \text{Cav}(\text{ValA})(\alpha.p + (1-\alpha).\tilde{p}) = \alpha.\text{Cav}(\text{ValA})(p) + (1-\alpha).\text{Cav}(\text{ValA})(\tilde{p}) \).

Contradiction. This implies that \( \mathcal{P}(p) \) is inductive. By Zorn’s Lemma, we have therefore that \( \mathcal{P}(p) \) has a maximal element. Let \( Y(p) \) be the set of maximal elements of \( \mathcal{P}(p) \). We claim \( Y(p) \) is unique. Indeed, note that given \( C \) and \( C' \) in \( Y(p) \), the relative interior \( ri\{C \cap C'\} \) is nonempty, by assumption and \( \text{Cav}(\text{ValA}) \) is affine in this set. Then \( \text{Cav}(\text{ValA}) \) extends (affinely) uniquely to the affine set \( co\{C \cup C'\} \). Since \( \text{Cav}(\text{ValA}) \) is concave, it must indeed be equal to this affine function in \( co\{C \cup C'\} \). Abusing notation, we refer to \( Y(p) \) the maximal element.

The next theorem is a generalization of Theorem 7.11.

**Theorem 8.4.** Let \((\sigma, \tau_A, \tau_B)\) be an equilibrium of \( \mathcal{G}(p) \) paying ex-ante to the informed player \( \text{Cav}(\text{ValA})(\text{marg}_{K_A}p) + \text{CavValB}(\text{marg}_{K_B}p) \). Then the marginal asymptotic posteriors \( \text{marg}_{K_A}p_\infty \) and \( \text{marg}_{K_B}p_\infty \) induced by the equilibrium profile satisfy:

1. \( \text{marg}_{K_A}p_\infty \in Y(\text{marg}_{K_A}p) \) a.s.
2. \( \text{marg}_{K_B}p_\infty \in Y(\text{marg}_{K_B}p) \) a.s.

*Proof.* The proof is exactly analogous to proof of Theorem 7.11.

\[ \Box \]

9. **Appendix A: Robustness Results**

Fix \( K, I \) and \( J \) as finite sets, with \( K, I \) and \( J \) with cardinality larger or equal than 2. A collection of real matrices \( \{(A^k_A)\}_{k_A \in K_A} \), with \( A^k_A \in \mathbb{R}^{I_A \times J_A} \) defines a unique zero-sum game with lack of information on one side \( G(p) \)\(^{23}\) for some \( p \in \Delta(K_A \times K_B) \). We will therefore treat elements of \( \mathcal{G} := \bigcup_{p \in \Delta(K)} G(p) \) as vectors \( \{(A^k_A)\}_{k_A \in K_A} \).

As we mentioned before, if \( |K| = 2 \) the property of a game being *not completely revealing* does not depend on the prior \( p \). Since we want to investigate the robustness of this property with respect to payoff perturbations we will make the following identification: for any \( p, \hat{p} \in \Delta(K_A \times K_B) \) two games of \( G(p) \) and \( G(\hat{p}) \) are equal if and only if for each pair \( k_A \in K_A \) the corresponding matrix is equal. This defines a projection map

\[
\Pi : \mathcal{G} \rightarrow \prod_{k_A \in K_A} (\mathbb{R}^{I_A \times J_A} \times \mathbb{R}^{I_A \times J_A}).
\]

\(^{23}\)Recall section 3.1
Since $\prod_{k_A \times k_B \in K_A \times K_B} (\mathbb{R}^{I_A \times J_A} \times \mathbb{R}^{I_B \times J_B})$ has a natural topology, we can induce a topology in $G$ using inverse images of open sets by $\Pi$. Let $G_2 \subset G$ be the subset of not completely revealing games. We will show that this subset of games has non-empty interior.

First, let $C(\Delta(K))$ denote the class of continuous real maps in $\Delta(K)$. Endow $C(\Delta(K))$ with the $|| \cdot ||_\infty$-norm. Let $f : \mathbb{R}^{I \times J} \times \mathbb{R}^{I \times J} \to C(\Delta(K))$ be defined by $f(A^1, A^2) = ValA(\cdot)$.

**Lemma 9.1.** $f$ is continuous.

**Proof.** Suppose it is not continuous by contradiction. Then there exists $\epsilon > 0$ and a sequence $(A^1_n, A^2_n)_{n \in \mathbb{N}}$ converging to $(A^1, A^2)$ such that $||ValA_n - ValA||_\infty \geq \epsilon$. Since $ValA_n - ValA$ is a continuous function and $\Delta(K)$ is compact, there exists a sequence $(p_n)_{n \in \mathbb{N}}$ such that $|ValA_n(p_n) - ValA(p_n)| \geq \epsilon$. Without loss of generality assume $p_n \to p$ as $n \to \infty$. Then $ValA_n(p_n) - ValA(p_n) = ValA_n(p_n) - ValA(p_n) + ValA_n(p_n) - ValA(p_n) - ValA(p_n) + ValA(p) + ValA(p) - ValA(p_n)$. Using triangle inequality it is sufficient to prove that $ValA_n(p) \to ValA(p)$. Indeed, define $F(A^1, A^2, \sigma, \tau) = \sigma.A(p).\tau$. $F$ is continuous. Applying the maximum theorem twice we obtain that the function $G(A^1, A^2) = \min_{\sigma} \max_{\tau} \sigma.A(p).\tau = ValA(p)$ is continuous. Contradiction.

**Theorem 9.2.** $G_2$ has nonempty interior

**Proof.** Let $(A^1, A^2)$ be such that $\Pi^{-1}((A^1, A^2))$ is a set of games with non-affine concavification. Then, there exists a point $p = (\alpha, 1 - \alpha) \in \text{int}\Delta(K)$ for which $ValA(p) = \text{Cav}(ValA)(p)$ and $\alpha.ValA(p_1) + (1 - \alpha).ValA(p_2) < ValA(p)$, which implies that $(A^1, A^2)$ is not completely revealing. Assume by contradiction that there exists a sequence of pairs $(A^1_n, A^2_n)$ inducing completely revealing games for which $(A^1_n, A^2_n) \to (A^1, A^2)$ as $n \to \infty$. By the continuity of $f$, for $n$ sufficiently large we have that $\alpha.ValA_n(p_1) + (1 - \alpha).ValA_n(p_2) < ValA_n(p)$. This implies that the concavification of $ValA_n(\cdot)$ is not affine. Contradiction. Therefore there exists an open set around $(A^1, A^2)$ such that every element of this open set induces a not completely revealing game. Now the inverse image of this open set by $\Pi$ is an open set around $\Pi^{-1}((A^1, A^2))$.\[\square\]

---

24The assumption $I$ and $J$ with cardinality at least 2 is crucial: if $I$ has cardinality 1, than the nonrevealing value is affine because the informed player has no way to signal. Therefore the concavification is also affine. If $J$ has cardinality 1, then one optimal strategy of the informed player is to completely reveal the information at any prior, since the uninformed player has no other strategy to play. This also implies the concavification is affine.
Now we approach the question of robustness for a more general set of types |K| ≥ 2. We would like to show now that the property of a zero-sum game being locally not revealing at a prior p is robust to payoff perturbation. Recall we consider \(\mathcal{G} := \bigcup_{p \in \Delta(K)} G(p)\) as vectors \(\{A^k_A\}_{k_A \in K_A, p}\). Define \(\mathcal{G}_p\) as the set of matrices \((A^1, ..., A^{[K]}_1)\) such that \((A^1, ..., A^{[K]}_1, p) \in \mathcal{G}\) is locally not revealing at \(p\). We will show that \(\mathcal{G}_p\) is open in \(\prod_k \mathbb{R}^{I \times J}\).

**Lemma 9.3.** Let \((A^1, ..., A^{[K]}_1)\) and \(p \in \Delta(K)\) induce a zero-sum game with lack of information on one-side. There exists an optimal strategy for the informed player such that the induced posteriors \((p_s)_{s \in S}\) with \(|S|\) satisfy \(|S| \leq |K| + 1\).

**Proof.** By the theory of zero-sum games with lack of information on one-side, there is a polytope \(P \subset \Delta(K)\) containing \(p\) such that the vertices of \(P\) are the posteriors induced by the optimal strategy of the informed player. One of this vertices, say \(\hat{p}\), is such that \(\hat{p} \in \text{int}\Delta(K)\) by assumption. Let \((p_s)_{s \in S}\) be the set of posteriors. Then \((p_s, \text{Val}(p_s)) \in \Delta(K) \times \mathbb{R}\). Hence the convex closure of \(\{(p_s, \text{Val}(p_s))\}_{s \in S}\) is by definition a polytope \(\hat{P}\) of \(\mathbb{R}^{|K|+1} \times \mathbb{R}\). It follows that \((p, \text{Val}(p)) \in \hat{P}\), which is a convex combination of at most \(|K| + 1\) affinely independent vertices of \(\hat{P}\) in \(\mathbb{R}^{|K|+1} \times \mathbb{R}\).

**Theorem 9.4.** The operator \(\text{Cav} : C(\Delta(K)) \to C(\Delta(K))\) is continuous.

**Proof.** Let \(f : \Delta(K) \to \mathbb{R}\) be a continuous function. Extend \(f\) to \(\mathbb{R}^K\) putting \(f(x) = -\infty\), if \(x \notin \Delta(K)\). From Appendix A.8 in Sorin(2003), we have that \((\Lambda_s \circ \Lambda_i)(f) = \text{Cav}(f)\) where \(\Lambda_s\) is the (Fenchel-Legendre) conjugation operation \(\Lambda_s(f)(x) = \sup_{y \in \mathbb{R}^n} \{f(y) - < x, y >\}\) and \(\Lambda_i(f)(y) = \inf_{x \in \mathbb{R}^n} \{f(x) + < x, y >\}\). Now the operator \(f \mapsto (\Lambda_s \circ \Lambda_i)(f)\) is continuous.

**Theorem 9.5.** \(\mathcal{G}_p\) has nonempty interior.

**Proof.** Assume by contradiction there is a sequence \((A^n_1, ..., A^n_{|K|})_{n \in \mathbb{N}}\) with \(\lim_{n \to \infty} (A^n_1, ..., A^n_{|K|}) = (A^1, ..., A^{[K]}_1)\), such that for each \(n\) all the optimal strategies at \(p\) induce posteriors in the boundary of the simplex \(\Delta(K)\). Let \((p^n_1, ..., p^n_{|K|+1})\), with

\[
\sum_{i=1}^{|K|+1} \alpha_i^n \cdot p_i^n = p, \quad \sum_{i=1}^{|K|+1} \alpha_i^n = 1, \alpha_i^n \geq 0
\]
be the optimal posteriors for the zero-sum game with lack of information on one side with prior $p$ induced by $(A^1_n,...,A^K_n)$ (we can assume without loss of generality that there are $|K|+1$ of them, by the lemma above). By assumption, $p^n_i \in \partial \Delta(K)$, for all $i = 1,...,|K|+1$. Passing to convergent subsequences if necessary, we can assume that $p^n_i \to p_i \in \partial \Delta(K)$, for each $i = 1,...,|K|$. Similarly, assume $\alpha^n_i \to \alpha_i$, for each $i = 1,...,|K|$. Now, 

$$\sum_{i=1}^{|K|+1} \alpha^n_i . ValA(p^n_i) = CavValA(p^n)$$

Since $Cav(ValA_n(p)) \to CavValA(p)$ as $n \to \infty$, and $\sum_{i=1}^{|K|+1} \alpha^n_i . ValA(p^n_i) \to \sum_{i=1}^{|K|+1} \alpha_i . ValA(p_i)$ as $n \to \infty$, it implies that $\sum_{i=1}^{|K|+1} \alpha_i . ValA(p_i) = CavValA(p)$. Contradiction. □

**Theorem 9.6.** Let $\tilde{G}_p$ be the set of vectors of matrices $(A^1,...,A^K)$ that induce a zero-sum game with lack of information on one-side with prior $p \in \Delta(K)$ for which $I(p)$ is nondegenerate. Then $\tilde{G}_p(K)$ is an open set.

**Proof.** This is an immediate consequence of Theorem 9.4. □

10. **Appendix B: $\tilde{G}$-process**

In this appendix, we offer an idea of how to obtain the proof of the sufficient part of the characterization of Equilibrium-payoffs. As we mentioned before this involves a minor adaptation of Hart’s arguments. In order to fully understand the idea below we refer the reader to section 5 of Hart (1985). Since this section is quite long, we refrain from reproducing the necessary arguments that would make this appendix self-contained.

**Theorem 10.1.** Given a $\tilde{G}$-process starting at $g$ on some probability space, there exists an equilibrium with payoffs $g$.

**Proof.** (Idea of the Proof) The overall idea of this part of the proof is to use the tree representation of the $\tilde{G}$-process as a reference on how to play the game. In section 5.3 of the proof of Hart (1985), a minor modification is needed in the communication stages, since, now, three players are involved in the communication. Communication stages are stages where there is no transmission of information (signalling), but the players play jointly-controlled lotteries in order to determine how to continue playing the game. This case corresponds formally to the steps of the process at which $(f_{n+1} \neq f_n)$ and therefore $p_{n+1} = p_n$. At the communication
stages players 1 and 2 will play a jointly-controlled lottery just as the one defined in section 5.3 of Hart(1985), whose outcomes will indicate which is the node of the tree representation of the \( \tilde{G} \)-process they should “proceed” to. Since player 3 observes the outcomes of this lottery, he can also “proceed” to the same node. No unilateral (unobserved) deviation by the players playing the lottery is profitable in the communication stages, and player 3 play at these stages is payoff irrelevant. Since after the communication stages players play a deterministic path of play, any deviations are observable. □

11. Appendix C: Approachability Strategies

This section is adapted to our setting taking Sorin(2000) as reference but the results are all due to Blackwell(1956).

Let \( C \) be an \( |I_A \times I_B| \times |J_A \times J_B| \)-real matrix with coefficient in \( \mathbb{R}^{K_A \times K_B} \), where

\[
C_{i_n^A,j_n^A,j_n^B} = (C_{i_n^A,i_n^B,j_n^A,j_n^B})_{(k_A,k_B) \in K_A \times K_B} = (a_{i_n^A}^{k_A} + b_{j_n^B}^{k_B})(k_A,k_B)_{(k_A,k_B) \in K_A \times K_B},
\]

where \( a_{i_n^A}^{k_A} \in \mathbb{R} \) and \( b_{j_n^B}^{k_B} \in \mathbb{R} \). We define a vector payoff zero-sum game: at stage \( n \), Player 1(respect. Player 2) chooses a move \((i_n^A,i_n^B)\)(respect. \((j_n^A,j_n^B)\)). The corresponding vector payoff \( g_n = C_{i_n^A,j_n^A,j_n^B} \) is announced. Let \( \overline{g}_n = \frac{1}{n} \sum_{m=1}^{n} g_m \) be the average payoffs up to stage \( n \). Let \( ||C|| = \max_{i_n^A,i_n^B,j_n^A,j_n^B} |C_{i_n^A,j_n^A,j_n^B}| \).

**Definition 11.1.** A set \( P \subset \mathbb{R}^{K_A \times K_B} \) is approachable by Player 2 if for any \( \epsilon > 0 \) there exists strategy \( \tau \) and \( N \) such that, for any strategy \( \sigma \) of Player 1 and \( n \geq N \):

\[
\mathbb{E}_{\sigma,\tau}[d_n] \leq \epsilon,
\]

where \( d_n \) is the euclidean distance \( d(\overline{g}_n,P) \).

Let

\[
C_\tau = co\{ \sum_{j_A,j_B} C_{(i_A,i_B),(j_A,j_B)\tau(j_A,j_B)} |(i_A,i_B) \in I_A \times I_B | \},
\]

where \( coA \) denotes the convex closure of \( A \).

**Definition 11.2.** A closed set \( P \subset \mathbb{R}^{K_A \times K_B} \) is a \( B \)-set for Player 2 if: for any \( z \notin P \) there exists a closest point \( y = y(z) \) in \( P \) to \( z \) and a mixed move \( \tau = \tau(z) \in \Delta(J_A \times J_B) \), such that the hyperplane through \( y \) orthogonal to the segment \([y,z]\) separates \( z \) from \( P_\tau \).
Theorem 11.3. Let $P$ be a $B$-set for Player 2. Then $P$ is approachable by that player. More precisely with a strategy satisfying $\tau(h_{n+1}) = \tau(\overline{g}_n)$, whenever $\overline{g}_n \notin P$, one has:

$$\mathbb{E}_{A,\tau}[d_n] \leq \frac{2||C||}{\sqrt{n}}, \forall \sigma$$

and $d_n$ converges $\mathbb{P}_{A,\tau}$ a.s. 0.

Remark 11.4. The strategy $\tau$ above will be called an approachability strategy.

Theorem 11.5. Let $P = \{z \in \mathbb{R}^{K_A \times K_B} | z \cdot q \geq ValA(margK_Aq) + ValB(margK_Bq), \forall q \in \Delta(K_A \times K_B)\}$. Then $P$ is approachable by player 2. Also, for each $n$, the optimal strategy $\tau$ for player 2 in the theorem above satisfies $\tau(h_n) \in \Delta(J_A) \times \Delta(J_B)$.

Proof. Approachability of $P$ follows from Sorin(2000), Theorem 3.33 where it is checked that $P$ is a $B$-set. Fix $\overline{g}_n$ and let $\tau$ be the $B$-set strategy associated with the $B$-set $P$. Let $\tau_A = marg_{J_A} \tau$ and $\tau_B = marg_{J_B} \tau$. Then,

$$x = \sum_{(i_A,i_B)} \chi^{i_A,i_B} \sum_{J_A,J_B} C_{(i_A,i_B),(J_A,J_B)} \tau_{(J_A,J_B)} = \sum_{(i_A,i_B)} \chi^{i_A,i_B} \sum_{J_A,J_B} a^{k_A}_{(i_A,i_A)} + \sum_{i_B,J_B} b^{k_B}_{(i_B,i_B)} \tau^A_{J_A,J_B} + \sum_{i_B,J_B} b^{k_B}_{(i_B,i_B)} \tau^B_{J_A,J_B} = \sum_{(i_A,i_B)} \chi^{i_A,i_B} \sum_{J_A,J_B} C_{(i_A,i_B),(J_A,J_B)} \tau^A_{J_A,J_B} \tau^B_{J_A,J_B},$$

which shows $C\tau = C(\tau^A \boxtimes \tau^B)$.

So, if player 2 uses at each stage $h_n$ such that $\overline{g}_n(h_n) \notin P$ the strategy $\tau_n(h_n) := \tau_A(h_n) \boxtimes \tau_B(h_n)$, $\tau$ is an approachability strategy.

\qed

12. References


*Department of Economics, University of Rochester, Rochester, NY 14627, USA.*

*E-mail address: pahl.lucas@gmail.com*