EXPERIMENTATION WITH A TECHNOLOGY OF UNKNOWN QUALITY AND STOCHASTIC PROFITABILITY

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ABSTRACT. I study a stopping and resource allocation problem for an experimenter who faces two sources of uncertainty—the quality and profitability of a risky technology. The quality of the technology can be good or bad. If the quality is good, then payoffs arrive at the jump times of a standard Poisson process. If the technology is bad it does not generate payoffs. Payoffs are stochastic and the sizes of the realizations depend on the underlying state of the economy. Some payoffs reveal the state completely, others do not. After the first success, the quality of the technology becomes known for sure, and the optimal stopping policy can be characterized by a cutoff belief about the true state of the economy. Before the quality of the technology becomes known, the stopping region of the experimenter is a subset of the two-dimensional space of beliefs about the quality of the technology and the state of the economy. In the first part of the paper, the Poisson rate of arrival is exogenous; and in the second part, the experimenter chooses the optimal arrival rate of the Poisson process after each observation of the payoffs. The value function and stopping regions resemble those in the exogenous arrival rate problem; and the experimenter’s choice of arrival rate is non-monotonic in the values of key uncertainty parameters. To be more specific, while experimenters always increase their investments as they become more optimistic about the state, experimenters who observe high payoffs more frequently will not, for a given belief about the state, necessarily invest more than experimenters who observe high payoffs less frequently.

Keywords: optimal stopping, jump-diffusion process, endogenous arrival rate, decision under uncertainty

JEL: C61, D81

1. INTRODUCTION

The novelty of this paper is that it introduces an additional source of uncertainty into the traditional setting of experimentation with exponential bandits. Not only does the experimenter not know the quality of his technology initially, but even if the technology is good (that is, it generates positive payoffs at times of arrival of the standard Poisson process), it is risky because the profits it generates evolve stochastically. In addition, in the second part of the paper, the rate of arrival of stochastic payoffs of the good technology is endogenized, which leads to new qualitative comparative statics results. Entrepreneurs and researchers face multiple sources of uncertainty as they experiment with new technologies. In particular, innovators may be uncertain about the quality of their technology and its profitability. Profitability of new technologies can vary with an underlying state of the economy. A technology which competes with a substitute or relies on an input with a fluctuating price is subject to uncertain profitability. Entrepreneurs experimenting

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I am thankful to the participants in the University of Texas at Austin theory writing seminar for comments on drafts of the paper. I am especially grateful to Svetlana Boyarchenko for extensive comments.
with alternative sources of energy, for instance, face uncertainty about fluctuations in the prices of traditional fuel sources. Even an excellent alternative source of energy will face falling demand and profitability when oil prices are low, and the current trajectory of oil prices is often uncertain. Such an experimenter may scale back investment or shut down entirely when the state appears to be unfavorable.

Technologies which rely on inputs with fluctuating prices also face uncertain profitability. The price of stainless steel is volatile due to the volatile price of nickel, a key input. Innovations relying on stainless steel will experience considerable changes in profitability.

In addition to examples which can be strictly defined as “technologies,” the model I develop applies to startups which instead innovate on business models and target markets for which demand is initially unknown. The recent boom in subscription services—which are new to the extent that they are used for products like razor blades, clothing rentals and makeup—presents many such examples. When a new subscription is offered, it faces uncertain long-run demand which can be modeled as unknown quality. The recent emergence of these services means that little data exists for the impact of macro-economic cycles on their profitability, but it is reasonable to believe that consumers’ subscriptions for non-essentials will be the first to go in the face of a pay cut or layoff. Thus, entrepreneurs experimenting with a new subscription service must carefully consider fluctuations in the subscription’s profitability with business cycles when deciding whether to offer the product.

Additionally, sometimes innovation is on the literal quality of a product. A firm may offer an unbranded, lower cost version of a product, or a high end, luxury version. Such experimentation involves uncertainty in the ability to produce and profit from such a product and uncertainty about the profitability during different stages of the business cycle when consumers have varying levels of disposable income.

The literature explores questions about experimenting, unknown quality, and random payoffs along three principle lines: real options theory, bandit problems and sponsored research. The real options literature focuses on exit and entry choices in settings similar to mine. Mordecki (1999) considers the complete information case of optimal exit. Décamps et al. (2005) consider uncertain quality and payoffs, though in their model, uncertainty about quality is never resolved, as the quality takes the form of the drift parameter of a Brownian motion. Thus, while the model does an excellent job of accounting for uncertainty, it lacks the natural “stages” provided by the model presented here.  

Boyarchenko (2016) provides analysis which is closest to mine. She analyzes a model where the jump times of a Poisson process yield costly breakdowns instead of rewarding breakthroughs. She finds similar results for the waiting regions before and after the first jump time. In addition to considering breakthroughs instead of breakdowns, I differ by considering endogenous choice of the parameter governing the arrival process and by providing comparative statics results which illustrate the differential impacts of the two kinds of uncertainty.

There is also an extensive literature on bandit problems which studies unknown quality. Keller et al. (2005) provide the seminal treatment of exponential bandits, which most closely resembles the period before the first payoff in my model. The single agent in their setting

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1By natural “stages” I refer to the natural separation provided by my model, and that of Boyarchenko (2016), between the period of time before the experimenter learns the quality of the technology, and the period of time after quality is known.
uses a cutoff belief in the quality of the arm, and all uncertainty is resolved after a single breakthrough.

The literature on Poisson bandits extends this idea to the case in which the payoffs generated by the good and bad arms are both positive, so that quality uncertainty is not resolved at the instant of the first observation of a payoff. Keller and Rady (2010) provide the seminal treatment of Poisson bandits, and show that a single player also chooses as his optimal strategy a cutoff belief in the probability the arm has the higher arrival rate. Keller and Rady (2015) analyze the Poisson bandit model where the jump times of the Poisson process reveal breakdowns instead of breakthroughs.

The bandit literature has begun to explore the fact that incomplete information about the profitability of a technology can persist beyond the partial or complete resolution of quality uncertainty. To date, when incomplete information about profitability does arise, as in Keller and Rady (2010), a single agent still chooses as his optimal strategy a cutoff belief in the quality of the technology. That is, the uncertainty about the long-run profitability is just unresolved uncertainty about quality.

Bergemann and Hege (1998), Bergemann and Hege (2005) and Hörner and Samuelson (2013) study the problem of exit and resource allocation in the context of sponsored research for projects which are subject to quality uncertainty. In these models, however, once the project has its big breakthrough the experimenting ends and the principal and agent receive their payouts. That is, these models deal only with quality uncertainty. Hörner and Skrzypacz (2016) provide a recent and detailed review of the literature on experimentation and sponsored research. My model does not incorporate the principal-agent dynamics captured by the settings above. The solution to the decision problem is difficult to characterize, and so I leave the principal-agent problem for future work. I focus instead on an self-financed technology.

There have been other papers which jointly consider the impact of market uncertainty and technological uncertainty in models of "staged financing." For example, two recent papers, Lukas et al. (2016) and Zhang et al. (2017) use staged models of financing to think about both technological and market uncertainty. However, these models focus on renegotiation and contracting, not on the optimal decisions of a self-financing agent.

I consider a continuous-time model where a single decision-maker experiments with a technology of unknown quality, which can be either good or bad. A bad technology never generates payoffs, a good technology generates payoffs which arrive at the jump times of the standard Poisson process. The quality of the technology is chosen by nature, and it is not known initially.

If the technology is good, the payoffs it generates are stochastic. The size of these payoffs is governed by a two-state, continuous-time Markov chain which is independent of the Poisson process. The two states are characterized by higher or lower expected payoffs. The realizations of the payoffs can be high, medium or low. The high state only realizes high or medium payoffs, while the low state only realizes medium or low payoffs. Thus, high and low payoffs perfectly reveal the state of the economy while medium payoffs allow the experimenter to update his belief about the current state, but not to determine it exactly.

I first analyze a model where the arrival rate of the Poisson process is exogenous. After the quality of the technology is known—that is, after the arrival of the first payoff—the optimal stopping policy is a cutoff belief in the state of the economy. This optimal stopping policy has three distinct types as a function of beliefs about the state and the values of key parameters:
the experimenter may stop immediately at the realization of a payoff, the experimenter may continue for a finite period of time if no new payoff arrives, or the experimenter may continue indefinitely until the next arrival of a payoff. Comparative statics show that small changes in the parameters governing uncertainty change not just the cutoff belief, but can also change the type of optimal stopping policy.

Before the first arrival of a payoff, the stopping region cannot be characterized by cutoff beliefs. Instead, it is a subset of the two-dimensional space of beliefs about the quality of the technology and the state of the economy, where the subset is not rectangular. Although this result is not entirely new, it is not typical in the previous literature.

I provide numerical comparative statics results which show that the separation of uncertainty about quality and profitability matters. Experimenters make different decisions when the parameters of each type of uncertainty are altered. Adjustments to each parameter governing uncertainty—the persistence of the payoff process, the likelihood of the highest possible payoff and the arrival rate of the Poisson process—yield effects of different magnitudes on the stopping regions, even when these changes have the same effect on long-run expected payoffs. The policy implication of the comparative statics is that a planner who wishes to encourage or discourage experimentation on technologies with externalities should consider the differential effect of each type of uncertainty when deciding how to most efficiently allocate a fixed budget toward altering the parameters.

I also consider the allocation of resources to such a risky technology, that is, I consider an endogenous choice of the parameter of the Poisson process. To keep the problem tractable, I allow the adjustments only at the jump times of the Poisson process.

If the agent devotes no additional resources to the technology, the arrival process is the same as in the model of exogenous rate. The arrival rate can be increased at a convex cost to the agent, where there is a maximum arrival rate that can be achieved given infinite resources. I provide a numerical example to demonstrate the main result: the choice of arrival rate is non-monotonic in the uncertainty parameters. That is, increasing the frequency of the highest payoff raises the choice of arrival rate at some beliefs, and lowers it at others. This result has been established for the period following the first payoff arrival, and results about the period before the first first arrival are forthcoming.

The rest of the paper is organized as follows. In the next section, I analyze the problem of optimal exit by a single agent when the arrival rate of the Poisson process is exogenous and constant. In Section 3 I provide a numerical example for the problem with exogenous arrival rate, which demonstrates the comparative statics results.

In Section 4.4 I consider the problem of endogenous arrival rate. Section 4.4 provides numerical examples for the model with endogenous arrival rate. The final section concludes and outlines future work.

2. Model of Exogenous Arrival Rate

Consider a decision problem, characterized by the following structure. Time is continuous and the time horizon is infinite, $t \in \mathbb{R}_+$. A single, risk-neutral agent experiments with a technology of unknown quality. If the technology is “bad” ($b$), it generates no success, if the technology is “good” ($g$), the agent will receive a stochastic stream of payoffs which arrive at jump times of the standard Poisson process, $N$, with parameter $\lambda > 0$. The quality of the technology is determined by the state of nature $X \in \{g, b\}$. The initial prior assigns probability $\pi_0 \in [0, 1]$ to the event $X = g$. The agent discounts the future at the rate $\rho > 0$. 
In addition to uncertainty about the quality of the technology, there is uncertainty about the state of the economy. If the quality of the technology is good, the payoffs follow a stochastic process $R = \{r_t\}_{t \geq 0}$ which is independent of the Poisson process and depends on the state of the economy. At each date $t$, the payoff can be one of three values: $r_t \in \{r^h, r^m, r^l\}$ where $r^h > r^m > r^l$. If the state of the economy is high ($H$), the expected value of the payoffs is larger than if the state of the economy is low ($L$). Payoffs of size $r^h$ and $r^m$ can occur in the high state, while payoffs of size $r^m$ and $r^l$ can occur in the low state. Denote by $s \in \{H, L\}$ the state of the economy. The payoff distribution is conditioned on the state of the economy at time $t$ and is defined by

\[
\begin{align*}
\text{prob}(r_t = r^h|s = H) &= 1 - \alpha^H \\
\text{prob}(r_t = r^m|s = H) &= \alpha^H \\
\text{prob}(r_t = r^l|s = H) &= 0 \\
\text{prob}(r_t = r^h|s = L) &= 0 \\
\text{prob}(r_t = r^m|s = L) &= \alpha^L \\
\text{prob}(r_t = r^l|s = L) &= 1 - \alpha^L.
\end{align*}
\]

The state of the economy follows a continuous-time Markov chain with transition rate matrix

\[
Q = \begin{bmatrix}
-q^H & q^H \\
q^L & -q^L
\end{bmatrix}
\]

where $q^H > 0$ and $q^L > 0$. Let $p_0 \in [0, 1]$ denote the prior probability that the economy is in state $H$ at time $t = 0$.

Let $\tau_k = \tau_1, \tau_2, \ldots$ denote the jump times of the Poisson process. When a payoff $r_{\tau_k}$ arrives, the agent will learn the true state of the economy if the payoff has size $r^h$ or $r^l$. However, if the payoff has size $r^m$, the agent does not learn the true state of the economy, but may update his belief about it. The relative sizes of $\alpha^L$ and $\alpha^H$ determine the direction of change of the belief after observation of the payoff $r^m$. Beliefs are defined and discussed in detail below.

The agent chooses an irreversible exit time, $\tau \in [0, \infty)$ and incurs the constant flow cost $c$ until he exits. If the agent exits the market at date $\tau$, his costs are given by $\int_0^\tau e^{-\rho t} c \, dt$. At the time of his exit, the agent receives his outside option, which is normalized to 0.

2.1. Beliefs. The Poisson process $N$ and the stochastic process $R$ define a probability measure on the filtered measurable space $(\Omega, F, \{F_t\}_{t \geq 0})$ which is generated by the payoff process. A generic element of $\Omega$ can be defined by a sequence of jump times and payoff realizations $\omega = \{\tau_j, r_{\tau_j}\}_{j \in \mathbb{N}_k}$ where $0 < \tau_1 < \tau_2 < \ldots$ are the jump times of the standard Poisson process $N$, and $r_{\tau_j}$ are the payoff realizations at the jump times.

The evolution of the initial beliefs, $\{p_0, \pi_0\}$ defines the probability measure on the measurable space above. Denote this probability measure $\mathbb{P}_{p_0, \pi_0}$ and the corresponding expectations $\mathbb{E}_{p_0, \pi_0}$.

Beliefs about the quality of the technology evolve according to the function

\[
\pi(\pi_0, t) = \frac{\pi_0 e^{-\lambda t}}{\pi_0 e^{-\lambda t} + 1 - \pi_0}
\]

until the time of the first arrival. The derivation is standard. After the first arrival of a payoff, the posterior belief that the technology is good is $\pi(\pi_0, t) = 1$ if $t \geq \tau_1$. 


The probability that the state of the economy is $j$ at time $t > 0$ if it was $i$ at $t = 0$ is denoted $p_{ij}(t)$ and is given by:

\[
p_{HH}(t) = \frac{q^L}{q^H + q^L} + \frac{q^H}{q^H + q^L}e^{-(q^H+q^L)t}
\]

\[
p_{HL}(t) = \frac{q^H}{q^H + q^L} - \frac{q^H}{q^H + q^L}e^{-(q^H+q^L)t}
\]

\[
p_{LH}(t) = \frac{q^L}{q^H + q^L} - \frac{q^H}{q^H + q^L}e^{-(q^H+q^L)t}
\]

\[
p_{LL}(t) = \frac{q^H}{q^H + q^L} + \frac{q^L}{q^H + q^L}e^{-(q^H+q^L)t}.
\]

Thus, the posterior belief that the state of the economy is high, given that no payoffs have arrived, is given by

\[
\tilde{p}(p_0, t) = p_0p_{HH}(t) + (1 - p_0)p_{ LH}(t)
\]

\[
= \frac{q^L}{q^H + q^L} \left[ 1 - e^{-(q^H+q^L)t} \right] + p_0e^{-(q^H+q^L)t}.
\]

If $p_0 > \frac{q^L}{q^H+q^L}$, then the posterior $\tilde{p}(p_0, t)$ is decreasing in $t$ and approaches the long-run probability that the state of the economy is $H$, $\frac{q^L}{q^H+q^L}$. Otherwise the posterior is increasing in $t$ to $\frac{q^L}{q^H+q^L}$.

The posterior belief jumps at the first arrival time of the Poisson process and is given by

\[
p(p_0, \tau_1|\tau_1) = \begin{cases} 
1 & \text{if } r_{\tau_1} = r^h \\
\frac{\alpha^H \tilde{p}(p_0, \tau_1 - \tau_1)}{\alpha^H \tilde{p}(p_0, \tau_1 - \tau_1) + \alpha^L (1 - \tilde{p}(p_0, \tau_1 - \tau_1))} & \text{if } r_{\tau_1} = r^m \\
0 & \text{if } r_{\tau_1} = r^l
\end{cases}
\]

where $\tilde{p}(p_0, \tau_1 - \tau_1) = \lim_{t \to \tau_1} \tilde{p}(p_0, t)$.

Now consider beliefs between and at arbitrary jumps $\tau_k$, $k \geq 2$. In a slight abuse of notation, let $p_{\tau_k}$ denote the belief at the time of the jump $\tau_k$, although the update of this belief is defined formally in 2.3. When $t \in (\tau_k, \tau_{k+1})$, the belief evolves according to

\[
\tilde{p}(p_{\tau_k}, t) = \frac{q^L}{q^H + q^L} \left[ 1 - e^{-(q^H+q^L)(t-\tau_k)} \right] + p_{\tau_k}e^{-(q^H+q^L)(t-\tau_k)}.
\]

At the moment of an observation, $\tau_k$, the belief is updated according to

\[
p(p_{\tau_{k-1}}, \tau_k|\tau_{k}) = \begin{cases} 
1 & \text{if } r_{\tau_k} = r^h \\
\frac{\alpha^H \tilde{p}(p_{\tau_{k-1}}, \tau_k - \tau_k)}{\alpha^H \tilde{p}(p_{\tau_{k-1}}, \tau_k - \tau_k) + \alpha^L (1 - \tilde{p}(p_{\tau_{k-1}}, \tau_k - \tau_k))} & \text{if } r_{\tau_k} = r^m \\
0 & \text{if } r_{\tau_k} = r^l
\end{cases}
\]

where $\tilde{p}(p_{\tau_{k-1}}, \tau_k - \tau_k) = \lim_{t \to \tau_k} \tilde{p}(p_{\tau_{k-1}}, t)$. Claim 2.1 demonstrates that, upon observation of a payoff of size $r^m$, when $\alpha^H > \alpha^L$, the probability that the high state of the economy generated the payoff is larger, and so the belief jumps up. In contrast, when $\alpha^L > \alpha^H$, the belief jumps down upon observation of $r^m$. 
Claim 2.1. (i) The updated belief at a jump time \( \tau_k \) is non-decreasing in the belief \( p_{\tau_{k-1}} \).
(ii) The updated belief when \( r_{\tau_k} = r^m \) jumps up when \( \alpha^H > \alpha^L \) and jumps down when \( \alpha^H < \alpha^L \).

Proof. (i) By \([2.2]\), \( \bar{p}(p_{\tau_{k-1}}, \tau_k) \) is increasing in \( p_{\tau_{k-1}} \), and by \([2.3]\), \( p(p_{\tau_{k-1}}, \tau_k | r_{\tau_k}) \) is weakly increasing in \( \bar{p}(p_{\tau_{k-1}}, \tau_k) \).

(ii) Subtract the limiting belief just before the jump from the updated jump belief:

\[
p(p_{\tau_{k-1}}, \tau_k | r^m) - \bar{p}(p_{\tau_{k-1}}, \tau_k) = \frac{(\alpha^H - \alpha^L)\bar{p}(p_{\tau_{k-1}}, \tau_k)(1 - \bar{p}(p_{\tau_{k-1}}, \tau_k -))}{\alpha^H \bar{p}(p_{\tau_{k-1}}, \tau_k -) + \alpha^L (1 - \bar{p}(p_{\tau_{k-1}}, \tau_k -))}.
\]

The difference is positive whenever \( \alpha^H > \alpha^L \) and is negative whenever \( \alpha^L > \alpha^H \). \( \square \)

2.2. Experimenting After the First Payoff. For the time being, consider an agent who has observed at least one payoff by time \( t \) so that there is no remaining uncertainty about the state of nature: \( \pi(\pi_0, t) = 1 \) and the quality of the technology is known to be good.

At the moment of each observation, normalize time to \( t = 0 \). Let \( \bar{p}_0 \) denote the belief about the state of the economy which was updated according to \([2.3]\) at the moment of the last arrival. The agent plans to exit at \( \tau \) if he does not observe a new payoff by that date. Let \( \tau' \) denote the random time of the next arrival. Let \( Q \) denote the probability measure associated with \( \tau' \). The value function of the agent with this history is written as the following Bellman Equation:

\[
V(\bar{p}_0) = \sup_{\tau \geq 0} \mathbb{E}_{\bar{p}_0, 1 \times Q} \left[ \int_0^{\tau \wedge \tau'} e^{-\rho t}(-c) \, dt + 1(\tau' \leq \tau) e^{-\rho \tau'} \left( r_{\tau'} + V(p(\bar{p}_0, \tau' | r_{\tau'})) \right) \right].
\]

Let \( \mathcal{V}(\bar{p}_0, t, T) \) denote the value for the agent who observed his most recent payoff at \( t = 0 \), updated his beliefs to \( \bar{p}_0 \) at that moment, is still active at \( t \geq 0 \), and plans to exit at \( T \geq t \) unless a new observation arrives. Then

\[
\mathcal{V}(\bar{p}_0, t, T) = \mathbb{E}_{\bar{p}_0, 1 \times Q} \left[ -\int_t^{T \wedge \tau'} ce^{-\rho(x-t)} \, dx + 1(T \geq \tau') e^{-\rho(\tau'-t)} \left( r_{\tau'} + V(p(\bar{p}_0, \tau' | r_{\tau'})) \right) \right],
\]

or equivalently

\[
(2.4) \quad \mathcal{V}(\bar{p}_0, t, T) = -ce^{\rho t} \int_t^{\infty} xe^{-\lambda y} \left[ 1(T < y) \int_t^T e^{-\rho x} \, dx + 1(T \geq y) \int_t^y e^{-\rho x} \, dx \right] \, dy
\]

\[
+ e^{\rho t} \int_t^{\infty} xe^{-\lambda y} 1(T \geq y) e^{-\rho y} \mathbb{E}_{\bar{p}_0, 1} \left[ r_y + V(p(\bar{p}_0, y | r_y)) \right] \, dy.
\]

This simplifies further to (see Appendix A for details)

\[
(2.5) \quad \mathcal{V}(\bar{p}_0, t, T) = e^{\rho t} \int_t^T e^{-(\lambda+\rho)x} \left( -c + \lambda \mathbb{E}_{\bar{p}_0, 1} \left[ r_x + V(p(\bar{p}_0, x | r_x)) \right] \right) \, dx,
\]
where
\[
\mathbb{E}_{\hat{p}_0,1} \left[ r_x + V \left( p(\hat{p}_0, x|r_x) \right) \right] = (1 - \alpha^H) \hat{p}(\hat{p}_0, x) \left[ r^h + V \left( p(\hat{p}_0, x|r^h) \right) \right] \\
+ \alpha^H \hat{p}(\hat{p}_0, x) + \alpha^L \left( 1 - \hat{p}(\hat{p}_0, x) \right) \left[ r^m + V \left( p(\hat{p}_0, x|r^m) \right) \right] \\
+ \left( 1 - \alpha^L \right) \left( 1 - \hat{p}(\hat{p}_0, x) \right) \left[ r^l + V \left( p(\hat{p}_0, x|r^l) \right) \right].
\]

Thus, the Bellman equation is naturally written
\[
(2.6) \quad V(\hat{p}_0) = \sup_{\tau \geq 0} \mathcal{V}(V, \hat{p}_0, 0, \tau).
\]

Denote by \( \tau^*(\hat{p}_0) \) the optimal policy which is the solution to the problem (2.6). Let \( V_0(\hat{p}_0) = 0 \) and define inductively for \( n = 0, 1, 2... \)
\[
(2.7) \quad V_{n+1}(\hat{p}_0) = \sup_{\tau \geq 0} \left\{ \int_0^\tau e^{-(\rho+\Delta)t} \left( -c + \lambda \mathbb{E}_{\hat{p}_0,1} \left[ r_x + V_n(p(\hat{p}_0, x|r_x)) \right] \right) \, dt \right\}.
\]

The following two pieces of notation will be helpful:
\[
r_H = \mathbb{E}_{p,1} \left[ r_t | p = 1 \right] \\
r_L = \mathbb{E}_{p,1} \left[ r_t | p = 0 \right].
\]

That is, \( r_H \) and \( r_L \) are the expected payoffs, independent of the date, when the state of the economy is known for sure to be \( H \) or \( L \), respectively.

Proposition 2.1 below shows that the value function is well-defined. The sequence defined in (2.7) is uniformly bounded above by \( \frac{r_H}{\rho} \) and below by 0. The lower bound is obvious by the option to exit immediately. The upper bound is the discounted sum of expected payoffs if the state of the economy is always \( H \).

**Proposition 2.1.** i) \( \{V_n\}_{n \in \mathbb{N}_+} \) is non-decreasing (pointwise).

ii) The sequence \( \{V_n\}_{n \in \mathbb{N}_+} \) is uniformly bounded from above by \( \frac{r_H}{\rho} \) and from below by 0.

iii) For any \( \hat{p}_0 \in [0, 1] \), the limit \( V(\hat{p}_0) = \lim_{n \to \infty} V_n(\hat{p}_0) \) exists, \( V(\hat{p}_0) \) admits the bounds \( 0 \leq V(\hat{p}_0) \leq \frac{r_H}{\rho} \) and \( V(\hat{p}_0) \) is a well-defined value function.

**Proof.** i) \( V_0(\hat{p}_0) = 0 \) implies
\[
V_1(\hat{p}_0) = \sup_{\tau \geq 0} \left\{ \int_0^\tau e^{-(\rho+\Delta)t} \left( -c + \lambda \mathbb{E}_{\hat{p}_0,1} \left[ r_x \right] \right) \, dt \right\} \geq 0 \quad \forall \hat{p}_0 \in [0, 1].
\]

Which in turn implies that
\[
V_2(\hat{p}_0) = \sup_{\tau \geq 0} \left\{ \int_0^\tau e^{-(\rho+\Delta)t} \left( -c + \lambda \mathbb{E}_{\hat{p}_0,1} \left[ r_x + V_1(p(\hat{p}_0, x|r_x)) \right] \right) \, dt \right\}
\]
\[
\geq V_1(\hat{p}_0) \quad \forall \hat{p}_0 \in [0, 1].
\]

The result follows by induction on \( n \).
Consider the problem:

\[ V(\hat{p}_0) = \sup_{\tau \geq 0} \mathcal{V}(V, \hat{p}_0, 0, \tau). \]

ii) Suppose \( V(p(\hat{p}_0, 0, t_r)) \leq \frac{\Delta H}{\rho} \) \( \forall \{\hat{p}_0, t, r_t\} \). Then

\[
\mathcal{V}(V, \hat{p}_0, 0, T) \leq \int_0^T e^{-(\rho + \Delta)x} \frac{\lambda x}{\rho} \mathrm{d}x - \int_0^T e^{-(\rho + \Delta)x} c \mathrm{d}x
\]

Observe that \( \mathbb{E}_{\hat{p}_0,1}[r_x] \leq r_H \), where equality holds at \( \hat{p}_0 = 1 \). Thus

\[
\mathcal{V}(V, \hat{p}_0, 0, T) \leq \frac{\lambda H}{\rho} \) \( \forall \{\hat{p}_0, T\}. \)

The sequence \( \{V_n\}_{n \in \mathbb{N}^+} \) is uniformly bounded from above by \( \frac{\lambda H}{\rho} \) because

\[
V_{n+1}(\hat{p}_0) = \sup_{\tau \geq 0} \mathcal{V}(V_n, \hat{p}_0, 0, \tau)
\]

and \( V_0(\hat{p}_0) = 0 \). That the sequence is bounded from below by 0 is obvious from the option to exit immediately.

iii) By i) and ii) above, the limit exists as \( n \) approaches infinity in (2.7). The existence of this limit guarantees that \( V \) is a well-defined value function.

Proposition 2.2 demonstrates that the choice of stopping time, \( \tau \), is time consistent in the sense that the agent does not change his optimal stopping policy between the jump times of the Poisson process. This follows naturally from the fact that, while the belief about the state of the economy evolves between jump times, the agent obtains no new information about the state of either nature or the economy between jumps.

Proposition 2.2. \( \tau^*(\hat{p}_0) \) is time consistent: if \( \tau^*(\hat{p}_0) \) maximizes \( \mathcal{V}(V, \hat{p}_0, 0, \tau) \) with respect to \( \tau \), then \( \tau^*(\hat{p}_0) \) maximizes \( \mathcal{V}(V, \hat{p}_0, t, \tau) \) \( \forall t \leq \tau^*(\hat{p}_0) \) with respect to \( \tau \).

Proof. Suppose \( \tau^*(\hat{p}_0) \) solves

\[
V(\hat{p}_0) = \sup_{\tau \geq 0} \mathcal{V}(V, \hat{p}_0, 0, \tau).
\]

Consider the problem:

\[
V(\bar{p}(\hat{p}_0, t)) = \sup_{\tau \geq t} \mathcal{V}(V, \hat{p}_0, t, \tau) \text{ where } t \leq \tau^*(\hat{p}_0).
\]

Note that

\[
\mathcal{V}(V, \hat{p}_0, 0, \tau) = \mathcal{V}(V, \hat{p}_0, 0, t) + e^{-\rho t} \mathcal{V}(V, \hat{p}_0, t, \tau),
\]

and the result follows because \( \mathcal{V}(V, \hat{p}_0, 0, t) \) and \( e^{-\rho t} \) are independent of \( \tau \). 

Three types of stopping solutions are possible in the model. In the first type, the agent continues indefinitely until the arrival of the next payoff. In the second, the agent stops in finite time if another payoff does not arrive. Finally, for some beliefs and parameters, the agent exits immediately at the observation of a payoff. Proposition 2.3 demonstrates that the waiting region, defined by \( G := \{\hat{p}_0 \text{ s.t. } \tau^*(\hat{p}_0) > 0\} \subset [0, 1] \), is characterized by a cutoff belief \( \bar{p}^* \) and discusses a special case of the first solution type, where the agent does not exit before the next arrival of a payoff for all beliefs.
Proposition 2.3. If \( \exists \bar{p} \in [0,1] \) such that \( V(\bar{p}) = 0 \), then the waiting region \( G = (\bar{p}^*, 1) \) where \( \bar{p}^* \) is the supremum of all such \( \bar{p} \) and the value function \( V \) is increasing over \( G \). Additionally, if \( V(\bar{p}) > 0 \) \( \forall \bar{p} \in [0,1] \), it is never optimal to exit before a new observation, that is, \( (\tau^*(\bar{p}) = \infty \) \( \forall \bar{p} \).

Proof. For the remainder of the proof, suppose \( V(\bar{p}) = 0 \). Consider \( \bar{p} \) and \( \bar{p}' \) such that \( \bar{p} < \bar{p}' \). Suppose \( V(\bar{p}) = 0 \). Let \( \tau^*(\bar{p}) \) be the optimal exit time corresponding to \( \bar{p} \). Now suppose at belief \( \bar{p} \) the agent decides to exit at \( \tau^*(\bar{p}) \), which may be suboptimal. The expected payoff at an arrival time \( \tau' < \tau^*(\bar{p}) \) is given by

\[
p(\bar{p}', \tau')(r_H - r_L) + r_L > p(\bar{p}, \tau^*)(r_H - r_L) + r_L,
\]

because \( p(\bar{p}, \tau') \) increases in \( \bar{p} \). Thus, \( \forall \), \( V(\bar{p}', 0, T^*(\bar{p})) > V(\bar{p}, 0, T^*(\bar{p})) \) and \( V(\bar{p}') > V(\bar{p}) \), because replacing \( \tau^*(\bar{p}) \) with \( \tau^*(\bar{p}') \) can only increase \( V \). Thus, \( V \) is increasing in \( \bar{p} \) on the region \( \bar{p} > \bar{p}^* \).

Similarly, suppose \( \bar{p}'' < \bar{p}' \) and \( V(\bar{p}'') > 0 \). By the argument above, \( V(\bar{p}'') > V(\bar{p}'') > 0 \), contradicting \( V(\bar{p}^*) = 0 \).

Finally, suppose \( V(\bar{p}) > 0 \) \( \forall \bar{p} \in [0,1] \). Set \( \bar{p} = 0 \). Suppose by way of contradiction that \( \tau^*(\bar{p}) < \infty \) for some \( \bar{p} \in [0,1] \). Then at \( \tau^*(\bar{p}), V(\bar{p}^*\tau\bar{p}, \tau^*(\bar{p}))) = 0 \) because the agent exits immediately, which contradicts the assumption that \( V(\bar{p}) > 0 \) \( \forall \bar{p} \).

Proposition 2.4. characterizes the case in which the waiting region is non-empty and also provides a lower bound on \( \bar{p}^* \). Define by

\[
\mathcal{U}(V, \bar{p}, t) = \lambda E_{\bar{p}, t} \left[ r_t + V \left( p(\bar{p}, t) \right) \right] - c,
\]

the difference between the marginal benefit and the marginal cost of waiting at time \( t \). The function \( \mathcal{U} \) is also the integrand (before discounting) of the value function in (2.5).

Proposition 2.4. Assume \( c < \lambda r_H \). Then \( \exists \bar{p} \in (0,1) \) such that \( \tau^*(\bar{p}) > 0 \) and the cutoff belief for the waiting region satisfies

\[
\bar{p}^* < \frac{c - \lambda r_L}{\lambda (r_H - r_L)}
\]

Proof. Suppose, by way of contradiction, that the optimal policy is \( \tau^*(\bar{p}_0) = 0 \) \( \forall \bar{p}_0 \in [0,1] \). Thus \( V(\bar{p}_0) = 0 \) \( \forall \bar{p}_0 \). Then, at \( \bar{p}_0 = 1 \), the following condition must hold:

\[
\sup_{\tau \geq 0} \int_0^\tau e^{-(\rho + \lambda)t} (-c + \lambda E_1[r_\tau]) \, dx = 0.
\]

At \( t = 0 \) and \( \bar{p}_0 = 1 \), the belief is strictly decreasing in time, so \( \tau^*(1) = 0 \) implies that \( \mathcal{U}(V, 1, 0) < 0 \) and \( c \geq \lambda E_1[r_\tau] = \lambda r_H \). This constitutes a contradiction. Thus, whenever, \( c < \lambda r_H \), there is a belief above which the agent continues for a strictly positive amount of time.

We can show that the waiting region satisfies \( \bar{p}^* < \frac{c - \lambda r_L}{\lambda (r_H - r_L)} \) by noting that the instantaneous payoff to an agent who remains in the market at time \( t = 0 \) is given by

\[
-c + \lambda \left( \bar{p}_0 (r_H - r_L) + r_L \right).
\]

Thus the agent is willing to remain in the market at least as long as \( \bar{p}_0 > \frac{c - \lambda r_L}{\lambda (r_H - r_L)} \).

Proposition 2.5 characterizes a set of parameters for which, if an agent does not exit immediately, he also does not exit before the next arrival of a payoff.
Proposition 2.5. Assume

\[ c < \lambda r_H \quad \text{and} \quad \frac{q^L}{q^H + q^L} > \frac{c - \lambda r_L}{\lambda (r_H - r_L)}. \]

Then whenever \( \hat{p}_0 > \bar{p}^* \), \( \tau^*(\hat{p}_0) = \infty \).

Proof. By Proposition 2.4, \( \bar{p}^* < \frac{q^L}{q^H + q^L} \). When \( \hat{p}_0 > \frac{q^L}{q^H + q^L} \) the belief \( \bar{p}(p_{\tau_k}, t) \) is decreasing in \( t \) and bounded from below by \( \frac{q^L}{q^H + q^L} > \bar{p}^* \). When \( \bar{p}^* < \hat{p}_0 < \frac{q^L}{q^H + q^L} \), the belief \( \bar{p}(p_{\tau_k}, t) \) is increasing in \( t \). Thus, \( \tau^*(\hat{p}_0 > \bar{p}^*) = \infty \), as the agent’s belief never exits the waiting region. \( \square \)

If we add the assumption \( \alpha^H > \alpha^L \) to Proposition 2.5, we see that such an agent whose belief jumps above the long-run average probability of the high state does not exit until he observes the lowest payoff, \( r_l \).

Figure 2.1 illustrates Proposition 2.5. The red curve represents beliefs beginning above the long-run average probability of the high state and the blue curve represents beliefs beginning below the long-run average probability. If the waiting region extends below the long-run probability of the high state, then the agent with a belief greater than the cutoff, \( \bar{p}^* \), never exits before the next payoff. If the waiting region begins above the long-run probability of the high state, the agent continues only until his belief drifts below the cutoff, which occurs in finite time if no new payoff arrives.

- **Beliefs when the cutoff is low**
- **Beliefs when the cutoff is high**

**Figure 2.1.** Illustration of Proposition 2.5. Agents who do not exit immediately continue indefinitely until the next observation when the cutoff is below the long-run average, and stop in finite time if the cutoff is above the long-run average.

In this section, I demonstrated that the value function of an agent who has learned the quality of the technology is well behaved. I have characterized the waiting regions and three types of exit solutions: exit immediately, continue for a finite time and continue indefinitely until the next payoff. In the next section, I work backward to solve the problem of the agent who faces a technology of unknown quality.

2.3. **Experimenting Before the First Payoff.** Now consider the problem of optimal exit before the resolution of uncertainty about the quality of the technology. That is, let \( \pi_0 \in [0, 1] \) and \( p_0 \in [0, 1] \). The agent chooses a time \( \tau \) at which he will exit if he does not observe a payoff. Let \( \tau_1 \) denote the random time of the observation of the first payoff, and let \( Q' \) denote
the probability measure associated with \( \tau_1 \). The value function \( W \) can be written using the value function \( V \) characterized in Section 2.2:

\[
W(p_0, \pi_0) = \sup_{\tau \geq 0} \mathbb{E}_{p_0, \pi_0} \left[ \int_0^{\tau \wedge \tau_1} e^{-\rho x} (-c) \, dx + \mathbb{I}(X = g \wedge \tau_1 < \tau)e^{-\rho \tau_1} \left[ r_{\tau_1} + V(p_0, \tau_1|\tau_1) \right] \right].
\]

Let \( \mathcal{W}(p_0, \pi_0, t, T, V) \) denote the value of the agent who has not observed any payoffs, had prior beliefs \( p_0 \) and \( \pi_0 \) at \( t = 0 \), is still active at \( t \geq 0 \), and plans to exit at \( T \geq t \) unless a payoff arrives.

\[
\mathcal{W}(p_0, \pi_0, t, T, V) = \mathbb{E}_{p_0, \pi_0} \left[ \int_t^{T \wedge \tau_1} e^{-\rho(x-t)} (-c) \, dx + \mathbb{I}(X = g \wedge \tau_1 < T)e^{-\rho(\tau_1-t)} \left[ r_{\tau_1} + V(p_0, \tau_1|\tau_1) \right] \right].
\]

This value can be rewritten, using the same methods above:

\[
\mathcal{W}(p_0, \pi_0, t, T, V) = e^{\rho t} \int_t^T e^{-(\rho+\lambda)x} \pi(\pi_0, x) \left( \lambda \mathbb{E}_{p_0} \left[ r_x + V(p_0, x|\tau_x) \right] - c \right) \, dx + e^{\rho t} \int_t^T e^{-\rho x} (1 - \pi(\pi_0, x)) (-c) \, dx.
\]

Thus, the value function for the case of unknown quality is conveniently written:

\[
W(p_0, \pi_0) = \sup_{\tau \geq 0} \mathcal{W}(p_0, \pi_0, 0, \tau, V).
\]

Proposition 2.6 describes the properties of the value function and the optimal stopping policy. The value function is non-decreasing, time consistent and well defined. Let \( \tau^*(p_0, \pi_0) \) denote the solution to (2.9).

**Proposition 2.6.** i) \( 0 \leq W(p_0, \pi_0) \leq \frac{\lambda \mathbb{E}_p}{\rho} \).

ii) \( W(p_0, \pi_0) \) is non-decreasing in \( p_0 \) and \( \pi_0 \).

iii) \( W(p_0, \pi_0) \) is time consistent: if \( \tau^*(p_0, \pi_0) \) solves \( \sup_{\tau \geq 0} \mathcal{W}(p_0, \pi_0, 0, \tau, V) \), then it also solves \( \sup_{\tau \geq t} \mathcal{W}(p_0, \pi_0, \tau, V) \).

**Proof.** i) \( W(p_0, \pi_0) \geq 0 \) is guaranteed by the option of immediate exit. The upper bound is calculated by observing that \( V(p_0) \leq \frac{\lambda \mathbb{E}_p}{\rho} \) by Proposition 2.1 and proceeding as in Proposition 2.1.

ii) Assume, by way of contradiction \( p < p' \), \( \tau^*(p, \pi_0) \) is the solution to \( \sup_{\tau \geq 0} \mathcal{W}(p, \pi_0, 0, \tau, V) \) and \( W(p, \pi_0) > W(p', \pi_0) \). \( \mathcal{W}(p', \pi_0, 0, \tau^*(p, \pi_0), V) \geq \mathcal{W}(p, \pi_0, 0, \tau^*(p, \pi_0), V) \) because the expected value of \( r_t \) is increasing in \( p_0 \) and \( V(p_0, t|\tau_t) \) is non-decreasing in \( p_0 \) by Proposition 2.3. Choosing the optimal \( \tau^*(p', \pi_0) \) can only increase \( W(p', \pi_0) \) so that \( W(p, \pi_0) \leq W(p', \pi_0) \), and thus we have a contradiction.
Suppose by way of contradiction $\pi < \pi'$, $\tau^{**}(p_0, \pi)$ is the solution to $\sup_{\tau \geq 0} W(p_0, \pi, 0, \tau, V)$ and $W(p_0, \pi) > W(p_0, \pi')$. $\pi(\pi_0, t)$ is increasing in $\pi_0$. $\pi(\pi_0, t)$ multiplies the net payoff while $(1 - \pi(\pi_0, t))$ multiplies the costs when the technology is bad.

Thus $W(p_0, \pi', 0, \tau^{**}(p_0, \pi), V) \geq W(p_0, \pi, 0, \tau^{**}(p_0, \pi), V)$. Choosing the $\tau^{**}(p_0, \pi')$ can only increase $W(p_0, \pi')$, thus we have a contradiction.

iii) The proof proceeds in the same manner as Proposition 2.2.

For the agent’s problem before the first payoff arrival, we define the waiting region by $J := \{p_0, \pi_0 | \tau^{**}(p_0, \pi_0) > 0\}$. I will use this definition in the next section, but do not provide any theoretical results for this region here.

3. NUMERICAL EXAMPLES

In Appendix B, I present an algorithm for calculating the value functions for each stage (known and unknown arrival rate). Below I give some results from that analysis to demonstrate visually the theoretical results presented above, and to discuss the effects of the two kinds of uncertainty on the outcomes observed.

Let us begin with what I will refer to as the “baseline” parameters. Let the initial parameter values be given by:

$$
\alpha^H = 0.4 \quad \alpha^L = 0.3 \\
\rho^H = 40 \quad \rho^m = 15 \quad \rho^l = 5 \quad c = 30 \\
q^H = 1.2 \quad q^L = 0.8 \\
\lambda = 1.5 \quad \rho = 0.05 \quad \epsilon = 0.001,
$$

where $\epsilon$ is a computational parameter which determines when the algorithm stops the iteration process described in (2.7). In addition, please note that some of the graphs appear somewhat “irregular” in their current state. This is due entirely to limitations on computing power and time which restrict the fineness of the grids over beliefs used to produce them. I will provide better graphics in the near future.

Figure 3.1 shows the value function by iteration for the baseline parameter values above. The value function is smooth because $0.4 = \frac{q^H}{q^H + q^L} < \bar{p}^* = 0.46$, so that the agent exits immediately if the updated belief at the last payoff was less than 0.46, and continues for a finite period of time if it was greater. You can see that the value function lies strictly below $\frac{\lambda m}{\lambda} = 900$ and in fact lies below 6.
In the baseline model, the optimal exit time in the waiting region is finite and increasing in \( p_0 \). The iteration procedure on the exit times is illustrated in Figure 3.2. Note that due to a change of variables where \( y = e^{-t} \) to make the calculations numerically tractable, the time axis on the graph ranges from 0 to 1 where \( y = 0 \) corresponds to \( T = \infty \) and \( y = 1 \) corresponds to \( T = 0 \). Note that the change of variables also restricts analysis to situations where \( \lambda + \rho > 1 \).

Finally, I examine the outcomes before the first arrival of a payoff. The value function \( W \) depends on \( p_0 \) and \( \pi_0 \), and the waiting region is also a function of each. In Figure 3.3 I provide an illustration of the value of \( W(p_0, \pi_0) \).
In Figure 3.4 we can see that the waiting region depends both on $p_0$ and $\pi_0$ and that the values which constitute the boundary are interdependent, that is, they cannot be characterized as a set of cutoff beliefs.

3.1. Varying the Uncertainty Parameters. There are eight parameters in the model which determine the degree of uncertainty faced by the agent. $\alpha^L$ and $\alpha^H$ determine the probabilities of receiving the payoff $r^m$ in the low and high states, respectively. $q^L$ and $q^H$ determine the persistence of the states of the economy. An increase in either $q^L$ or $q^H$ corresponds to an decrease in the persistence of the corresponding state, and an increase in the rate at which the economy transitions out of that state. $r^h$, $r^m$ and $r^l$ determine the size of the payoffs received and the closer these parameters are to each other, the smaller
the variance of the payoffs. Finally, \( \lambda \) governs the arrival rate when the technology is good. Higher values of \( \lambda \) correspond to higher profitability, as payoffs arrive more frequently. In the following analysis, I will present an in-depth look at the effects of three of these parameters, \( q^H \), \( \alpha^H \) and \( \lambda \). Figures for the remainder of the section are large, and have been included as Appendix C.

Figure C.1 illustrates the effects of changes in \( q^H \) on the value function and stopping times after the first arrival, as well as the effect of these changes on the stopping region for agents before the first arrival of a payoff.

Figure C.1 demonstrates the possible shapes of the value function—either the value function is kinked and the agent’s optimal stopping strategy is \( T^*(\hat{p}_0) \in \{0, \infty\} \) when the agent faces low levels of uncertainty (persistent high state), or the value function is smooth and the agent exits in finite time when the agent faces high levels of uncertainty.

Next, consider changes to the parameter \( \alpha^H \), the probability that a payoff realized in the high state has size \( r^m \). Increasing \( \alpha^H \) reduces the expected payoff of the agent. Figure C.2 shows the value function and stopping time policy after the first payoff, and the stopping region before the first payoff.

Finally, varying \( \lambda \) causes the changes shown in Figure C.3. Note that the changes in the \( \lambda \) and \( \alpha^H \) parameters are more or less shifts in the functions of interest, while changes in \( q^H \) appear to rotate and shift these functions. In order to further investigate these differences, I consider changes in the parameter values such that each change has the same effect on the expected value of the payoff at the long-run belief that the state is high (\( \frac{q^L}{q^H + q^L} \)).

3.2. Comparing Changes in the Uncertainty Parameters. Consider changes to the baseline parameters which result in an increase (decrease) in the long-run expected payoff from 25.2 to 28.5 (23) where the long-run expected payoff is defined by \( \lambda \left( \mathbb{E}_{q^L / (q^H + q^L)} [r_t] \right) \). The increase in the long-run expected payoff can be achieved by alternatively changing each of the 3 parameters studied: \( q^H = 0.8, \alpha^H = 0.18 \) or \( \lambda = 1.696 \). The decrease can be achieved by changing each of the parameters in the opposite direction: \( q^H = 1.6, \alpha^H = 0.547 \) or \( \lambda = 1.369 \).

For each parameter change which increases the expected payoff of the agent, the agent facing such a parameter chooses, after the first arrival, to continue indefinitely until the next arrival, or to exit immediately. That is, his value function becomes “kinked” and he chooses infinite stopping times if he does not exit.

Importantly, the change in \( \alpha^H \) has an effect of similar magnitude on the waiting region as the effect on the change in \( \lambda \), but the change in \( q^H \) has a smaller effect.

The numerical examples reveal the importance of each of the parameters governing uncertainty. Changes in these parameters shift value functions and stopping regions, but also, importantly, alter the structure of the best response functions. These examples provide motivation for expanding the simple model of technological and business cycle uncertainty to include the case in which the agent can pay to change these parameters. In the next section, I focus on the agent’s willingness to pay to change \( \lambda \).

4. Model of Endogenous Arrival Rate

Suppose that the agent, in addition to choosing his stopping time, may adjust the parameter \( \tilde{\lambda} \) of the Poisson process governing the arrivals of payoffs for a good technology. To
do so, he pays an additional cost to increase the rate above the “natural” rate assumed in Section 2. The rate $\tilde{\lambda}$ is chosen at $t = 0$ and at each subsequent arrival of a payoff. It cannot be changed between arrivals, and is thus properly written $\tilde{\lambda}(\hat{p}_{\tau_k})$ where $\hat{p}_{\tau_k}$ is the belief at the time of the last arrival (or $t = 0$ if no payoff has arrived). In a slight abuse of notation, I drop the dependence on $\hat{p}_{\tau_k}$ and write $\tilde{\lambda}$, sometimes denoted $\tilde{\lambda}_t$ to denote the value of $\tilde{\lambda}$ chosen at date $t$.

Assume $\lambda \leq \tilde{\lambda} \leq \lambda < \infty$ where $\lambda$ has flow cost $c$. Reasons for specifying a minimum, instead of allowing a reduction to zero, are twofold. The first reason is that the problem can always be scaled so that this minimum is not chosen except where the agent is indifferent between waiting and exit. The second is purely mathematical, I will need to impose $\tilde{\lambda} > 1$ in order to do the change of variables for numerical examples.

Let the cost function be given by $k(\tilde{\lambda}) = c + f(\tilde{\lambda} - \lambda)$, where $f'(\tilde{\lambda}) > 0, f''(\tilde{\lambda}) > 0$ and $\lim_{\lambda \to \tilde{\lambda}} k(\tilde{\lambda}) = \infty$. That is, costs are convex and achieving the maximal arrival rate requires prohibitively high levels of resources. Other than these changes to the choice of arrival rate and the cost function, all other assumptions and notation from Section 2 remain unchanged.

4.1. Beliefs. Let $N(\tilde{\lambda})$ denote the stochastic process generated by the choice of $\tilde{\lambda}$, where each inter-arrival time is exponentially distributed with the parameter $\tilde{\lambda}$ chosen either at $t = 0$ or the most recent arrival time. At the arrival times $N(\tilde{\lambda})$, the stochastic process $R$ (the payoff process from Section 2) governs the size of the payoffs. $N(\tilde{\lambda})$ and $R$ define a probability measure on the measurable space $(\Omega, F, \{F_t\}_{t \geq 0})$ which is generated by the payoff process and the choice of $\tilde{\lambda}$. A generic element of $\Omega$ is given by $\omega = \{\tau_j, r_{\tau_j}, \tilde{\lambda}_{\tau_j}\}_{j \in \mathbb{Z}^+}$ where $\{\tau_j\}_{j \in \mathbb{Z}^+}$ are the jump times of the Poisson process $N(\tilde{\lambda})$ and $\tau_0 = 0$. Let $\{r_{\tau_j}\}_{j \in \mathbb{Z}^+}$ be the payoff realizations at the jump times where $\tau_0 = 0$. Finally, $\tilde{\lambda}_{\tau_j}$ are the parameter values chosen by the agent at each jump time, including an initial choice, $\tilde{\lambda}_0$ at $t = 0$.

Beliefs about the quality of the technology evolve according to the function

$$\pi(\pi_0, t, \tilde{\lambda}) = \frac{\pi_0 e^{-\tilde{\lambda}t}}{\pi_0 e^{-\tilde{\lambda}t} + 1 - \pi_0}$$

until the time of the first arrival. After the first arrival of a payoff, the posterior belief that the technology is good is $\pi(\pi_0, t, \tilde{\lambda}) = 1 \ \forall t \geq \tau_1$. Beliefs about the state of the economy evolve as in (2.2) and (2.3).

4.2. Experimenting After the First Payoff. For the time being, consider an agent who has observed at least one payoff by time $t$ and thus $\pi(\pi_0, t, \tilde{\lambda}) = 1$. At the moment of each arrival of a payoff, normalize time to $t = 0$. Let $\hat{p}_0$ denote the belief, updated at the time of the most recent arrival of a payoff, that the state of the economy was good at that moment. The agent chooses a date $\tau$ at which he exits if an additional payoff has not arrived.

Let $\tau'$ denote the random arrival time of the next payoff. Let $Q'(\tilde{\lambda})$ denote the probability measure associated with the arrival $\tau'$, which depends on the choice of $\tilde{\lambda}$. The value function
for such an agent at \( t = 0 \) can be written as the following Bellman equation:

\[
Z(\hat{\rho}_0) = \sup_{\tau \geq 0, \tilde{\lambda} \in [\Delta, \bar{\lambda}]} \mathbb{E}_{p(\hat{\rho}_0, 1) \times Q'(\tilde{\lambda})} \left[ \int_0^{\tau \wedge \tau'} e^{-\rho x} (-k(\tilde{\lambda})) \, dx \right.
\]
\[
+ \mathbb{1}(\tau > \tau') e^{-\rho \tau'} \left( r_{\tau'} + Z(p(\hat{\rho}_0, \tau'|r_{\tau'})) \right) \bigg].
\]

Let \( \mathcal{D}(\hat{Z}, \hat{\rho}_0, \tilde{\lambda}, t, T) \) denote the value of the agent who: had last observation at time normalized to \( t = 0 \), updated his beliefs to \( \hat{\rho}_0 \) at \( t = 0 \), chose \( \tilde{\lambda}_0 \) at \( t = 0 \), is still active at \( t \geq 0 \), and plans to exit at \( T \geq t \) unless an additional payoff arrives. Then

\[
\mathcal{D}(\hat{Z}, \hat{\rho}_0, \tilde{\lambda}_0, t, T) = \mathbb{E}_{p(\hat{\rho}_0, 1) \times Q'(\tilde{\lambda}_0)} \left[ \int_t^{T \wedge \tau'} e^{-\rho (x-t)} (-k(\tau'_0)) \, dx \right.
\]
\[
+ \mathbb{1}(T > \tau') e^{-\rho (\tau' - t)} \left( r_{\tau'} + Z(p(\hat{\rho}_0, \tau'|r_{\tau'})) \right) \bigg].
\]

This value simplifies to the following using the same methods as Appendix A

\[
\mathcal{D}(\hat{Z}, \hat{\rho}_0, \tilde{\lambda}_0, t, T) = e^{\rho t} \int_t^T e^{-(\rho + \tilde{\lambda}_0) x} \left[ -k(\tilde{\lambda}_0) + \tilde{\lambda}_0 \mathbb{E}_{\hat{\rho}_0, 1} \left[ r_x + Z \left( p(\hat{\rho}_0, x| r_x) \right) \right] \right] \, dx
\]

where

\[
\mathbb{E}_{\hat{\rho}_0, 1} \left[ r_x + Z \left( p(\hat{\rho}_0, x| r_x) \right) \right] = \left( 1 - \alpha^H \right) \bar{p}(\hat{\rho}_0, x) \left[ r^h + Z \left( p(\hat{\rho}_0, x| r^h) \right) \right]
\]
\[
+ \left[ \alpha^H \bar{p}(\hat{\rho}_0, x) + \alpha^L \left( 1 - \bar{p}(\hat{\rho}_0, x) \right) \right] \left[ r^m + Z \left( p(\hat{\rho}_0, x| r^m) \right) \right]
\]
\[
+ \left( 1 - \alpha^L \right) \left( 1 - \bar{p}(\hat{\rho}_0, x) \right) \left[ r^l + Z \left( p(\hat{\rho}_0, x| r^l) \right) \right].
\]

Thus, the Bellman equation can be written:

\[
(4.1) \quad Z(\hat{\rho}_0) = \sup_{\tau \geq 0, \tilde{\lambda} \in [\Delta, \bar{\lambda}]} \mathcal{D}(\hat{Z}, \hat{\rho}_0, \tilde{\lambda}, 0, \tau).
\]

Denote by \( \left\{ \tau^+(\hat{\rho}_0), \tilde{\lambda}^+(\hat{\rho}_0) \right\} \) the optimal policy which is the solution to \((4.1)\). Here, as in Section 2 I start in Proposition 4.1 by showing that \( Z \) is a well-defined value function using induction. Let \( Z_0(\hat{\rho}_0) = 0 \) and define inductively, for \( n = 0, 1, 2... \)

\[
(4.2) \quad Z_{n+1}(\hat{\rho}_0) = \sup_{\tau \geq 0, \tilde{\lambda} \in [\Delta, \bar{\lambda}]} \int_0^\tau e^{-(\rho + \tilde{\lambda}) x} \left[ -k(\tilde{\lambda}) + \tilde{\lambda} \mathbb{E}_{\hat{\rho}_0, 1} \left[ r_x + Z_n(p(\hat{\rho}_0, x| r_x)) \right] \right] \, dx.
\]

**Proposition 4.1.**

i) \( \left\{ Z_n \right\}_{n \in \mathbb{N}^+} \) is non-decreasing (pointwise).

ii) The sequence \( \left\{ Z_n \right\}_{n \in \mathbb{N}^+} \) is uniformly bounded from above by \( \bar{\lambda} \frac{r_k}{\rho} \) and from below by 0.

iii) For any \( \hat{\rho}_0 \in [0, 1] \), the limit \( Z(\hat{\rho}_0) = \lim_{n \to \infty} Z_n(\hat{\rho}_0) \) exists, \( Z \) admits the bounds \( 0 \leq Z(\hat{\rho}_0) \leq \bar{\lambda} \frac{r_k}{\rho} \) and \( Z \) is a well-defined value function.
Proof. i) \( Z_0(\hat{p}_0) = 0 \) implies
\[
Z_1(\hat{p}_0) = \sup_{\tau \geq 0, \lambda \in [\Delta, \bar{\lambda}]} \int_0^\tau e^{-(\rho+\tilde{\lambda})x} \left[ -k(\tilde{\lambda}) + \lambda \mathbb{E}_{\hat{p}_0,1} [r_x] \right] \, dx \geq 0 \quad \forall \hat{p}_0 \in [0, 1].
\]
This implies that
\[
Z_2(\hat{p}_0) = \sup_{\tau \geq 0, \lambda \in [\Delta, \bar{\lambda}]} \int_0^\tau e^{-(\rho+\tilde{\lambda})x} \left[ -k(\tilde{\lambda}) + \lambda \mathbb{E}_{\hat{p}_0,1} [r_x] \right] \, dx
+ \int_0^\tau e^{-(\rho+\tilde{\lambda})x} \tilde{\lambda} \mathbb{E}_{\hat{p}_0,1} Z_1(p(\hat{p}_0, x|r_x)) \, dx \geq Z_1(\hat{p}_0) \quad \forall \hat{p}_0 \in [0, 1].
\]
The result follows by induction in \( n \).

ii) Suppose \( Z(p(\hat{p}_0, t|r_t)) < \frac{X_{\rho t_H}}{\rho} \forall \{\hat{p}_0, t, r_t\} \). Then
\[
\mathcal{Z}(Z, \hat{p}_0, \bar{\lambda}, 0, T) = \int_0^T e^{-(\rho+\tilde{\lambda})u} \left[ -k(\tilde{\lambda}) + \lambda \mathbb{E}_{\hat{p}_0,1} [r_x + Z(p(\hat{p}_0, x|r_x))] \right] \, du
\leq \int_0^T e^{-(\rho+\tilde{\lambda})u} \left[ \tilde{\lambda} \mathbb{E}_{\hat{p}_0,1} [r_x + Z(p(\hat{p}_0, x|r_x))] \right] \, du
\leq \int_0^T e^{-(\rho+\tilde{\lambda})u} \lambda \left[ r_H + \frac{r_H}{\rho} \right] \, du
= \left[ 1 - e^{-(\rho+\tilde{\lambda})T} \right] \frac{\tilde{\lambda}}{\rho} \left[ \frac{r_H}{\rho} \right]
\leq \frac{X_{\rho t_H}}{\rho} \forall \{\hat{p}_0, T\}.
\]
The sequence \( \{Z_n\}_{n \in \mathbb{N}_+} \) is thus uniformly bounded from above by \( \frac{X_{\rho t_H}}{\rho} \) because
\[
Z_{n+1}(\hat{p}_0) = \sup_{\tau \geq 0, \lambda \in [\Delta, \bar{\lambda}]} \mathcal{Z}(Z, \hat{p}_0, \bar{\lambda}, 0, \tau).
\]

iii) By i) and ii) above, the limit exists as \( n \) approaches infinity in (4.2). The existence of this limit guarantees that \( Z \) is a well-defined value function.

Proposition 4.2 demonstrates that the value function \( Z \) has properties similar to those of \( V \) from Section 2. In particular, the waiting region for \( \tau(\hat{p}_0) \) is characterized by a cutoff belief and the value function \( Z \) is increasing over the waiting region. Define the waiting region by \( G' := \{\hat{p}_0 \text{ s.t. } \tau(\hat{p}_0) > 0\} \subset [0, 1] \).

Proposition 4.2. If \( \exists \bar{p} \in [0, 1] \) such that \( Z(\bar{p}) = 0 \), then the waiting region \( G' \) has the form \( G' = (\bar{p}', 1] \), where \( \bar{p}' \) is the supremum of all such \( \bar{p} \) and the value function \( Z \) is non-decreasing over \( G' \). Additionally, if \( Z(\bar{p}) > 0 \forall \bar{p} \in [0, 1] \), then \( \tau(\bar{p}) = \infty \forall \bar{p} \in [0, 1] \).

Proof. Consider \( \bar{p} \) and \( \bar{p}' \) such that \( \bar{p} < \bar{p}' \). Suppose \( Z(\bar{p}) > 0 \). Let \( \{\tau^\dagger(\bar{p}), \lambda^\dagger(\bar{p})\} \) be the optimal policy corresponding to \( \bar{p} \). Now suppose at belief \( \bar{p}' \) the agent chooses the policy \( \{\tau^\dagger(\bar{p}), \lambda^\dagger(\bar{p})\} \), which may be suboptimal. The expected payoff at an arrival time \( \tau' < \tau^\dagger(\bar{p}) \) is given by
\[
\lambda^\dagger(\bar{p}) \left[ p(\bar{p}', \tau')(r_H - r_L) + r_L \right] > \lambda^\dagger(\bar{p}) \left[ p(\bar{p}, \tau')(r_H - r_L) + r_L \right].
\]
The inequality holds because \( p(\bar{p}, \tau') \) is increasing in \( \bar{p} \). Thus, \( \mathcal{E}(Z, \hat{p}', \hat{\lambda}'(\bar{p}), 0, \tau'(\bar{p})) > \mathcal{E}(Z, \bar{p}, \hat{\lambda}'(\bar{p}), 0, \tau'(\bar{p})) \). Replacing \( \{ \tau'(\bar{p}), \hat{\lambda}'(\bar{p}) \} \) with \( \{ \tau'(\bar{p}'), \hat{\lambda}'(\bar{p}') \} \) can only increase \( Z \). Thus, \( Z \) is increasing in \( \bar{p} \) on the region \( \bar{p} > \bar{p}^\dagger \).

Similarly, suppose \( \bar{p}'' < \bar{p}^\dagger \) and \( Z(\bar{p}'') > 0 \). By the argument above, \( Z(\bar{p}^\dagger) > Z(\bar{p}'') > 0 \), which contradicts the assumption that \( Z \) is positive for all \( \bar{p} \).

Finally, suppose \( Z(\bar{p}) > 0 \) \( \forall \bar{p} \in [0, 1] \). Set \( \bar{p}^\dagger = 0 \). Suppose by way of contradiction that \( \tau'(\bar{p}) < \infty \) for some \( \bar{p} \in [0, 1] \). Then at \( \tau'(\bar{p}) \), \( Z(p_{\tau'(\bar{p})}(\bar{p}, \tau'(\bar{p}))) = 0 \) because the agent exits immediately, which contradicts the assumption that \( Z \) is positive for all \( \bar{p} \).

Thus, \( Z \) shares the properties in Proposition 4.2 with those of the value function in the exogenous arrival rate problem, but the optimal policies in the problem with endogenous arrival rate are not time consistent in the following sense: an agent who has set an optimal \( \hat{\lambda} \) at time \( t \) normalized to 0, may wish to revise his choice if given an opportunity at \( t > 0 \).

**Proposition 4.3.** Suppose \( \tau'(\hat{p}_0) > 0 \) and \( \hat{\lambda}'(\hat{p}_0) > \lambda \). Then, \( \hat{\lambda}'(\hat{p}_0) \not= \hat{\lambda}'(\hat{p}(\hat{p}_0, t)) \), holding \( \tau'(\hat{p}_0) \) fixed. However, \( \tau'(\hat{p}_0, \hat{\lambda}) = \tau'(\hat{p}(\hat{p}_0, t), \hat{\lambda}), \) holding \( \hat{\lambda}'(\hat{p}_0) \) fixed.

**Proof.** Suppose \( \{ \tau'(\hat{p}_0), \hat{\lambda}'(\hat{p}_0) \} \) solves
\[
Z(\hat{p}_0) = \sup_{\tau \geq 0, \lambda \in [\Delta, \lambda]} \mathcal{E}(Z, \hat{p}_0, \lambda, 0, \tau).
\]

Fix \( \tau'(\hat{p}_0) \) and let the agent re-optimize on \( \hat{\lambda} \). The problem at time \( t \) with updated belief \( \hat{p}(\hat{p}_0, t) \) is given by:
\[
Z(\hat{p}(\hat{p}_0, t)) = \sup_{\lambda \in [\Delta, \lambda]} \mathcal{E}(Z, \hat{p}_0, \lambda, t, \tau'(\hat{p}_0)).
\]

Note that
\[
\mathcal{E}(Z, \hat{p}_0, \lambda, 0, \tau'(\hat{p}_0)) = \mathcal{E}(Z, \hat{p}_0, \lambda, 0, t) + e^{-\rho t} \mathcal{E}(Z, \hat{p}_0, \lambda, t, \tau'(\hat{p}_0)).
\]

Thus the time \( t \) problem will have a different solution for \( \hat{\lambda}'(\hat{p}_0) \) because the time 0 problem is not independent of \( \lambda \) until time \( t \), and thus the agent may choose a different \( \hat{\lambda} \) if given the choice to re-optimize.

Now, fix \( \hat{\lambda}'(\hat{p}_0) \) and let the agent re-optimize on \( \tau \). The problem at time \( t \) with updated belief \( \hat{p}(\hat{p}_0, t) \) is given by:
\[
Z(\hat{p}(\hat{p}_0, t)) = \sup_{\tau \geq 0} \mathcal{E}(Z, \hat{p}_0, \hat{\lambda}'(\hat{p}_0), t, \tau).
\]

Note that
\[
\mathcal{E}(Z, \hat{p}_0, \hat{\lambda}'(\hat{p}_0), 0, \tau) = \mathcal{E}(Z, \hat{p}_0, \hat{\lambda}'(\hat{p}_0), 0, t) + e^{-\rho t} \mathcal{E}(Z, \hat{p}_0, \hat{\lambda}'(\hat{p}_0), t, \tau'(\hat{p}_0)).
\]

Thus the time \( t \) problem will have the same solution for \( \tau \), because the problem until time \( t \) is independent of \( \tau \).

Consider the objective function \( \mathcal{E}(Z, \hat{p}_0, \lambda, 0, \tau) \). The derivatives of this objective function are given by:
\[
(4.3) \quad \frac{\partial \mathcal{E}(Z, \hat{p}_0, \lambda, 0, \tau)}{\partial \tau} = e^{-(\rho + \lambda)\tau} \left[ -k(\lambda) + \hat{\lambda}_{\hat{p}_0, 0, 1} \left[ r_\tau + Z \left( p(\hat{p}_0, \tau | r_\tau) \right) \right] \right]
\]
Let $Q$ does not observe a payoff. Let $\tau$ function from date $t \in \mathbb{Z}$ that $\pi$ and endogenous choice of arrival rate when the quality of the technology is not yet known, 4.3. Experimenting Before the First Payoff.

Consider (4.4). The first component is the effect of increasing $\lambda$ on the discount factor. The last two components are the instantaneous marginal cost and marginal benefit of increasing $\lambda$ respectively. The second derivative of the objective function with respect to $\lambda$ is:

\[
\frac{\partial^2 \mathcal{L}(Z, \hat{p}_0, \tilde{\lambda}, 0, \tau)}{\partial \lambda^2} = \int_0^\tau e^{-(\rho+\tilde{\lambda})t} \left[ t^2 \left( -k(\tilde{\lambda}) + \tilde{\lambda} \mathbb{E}_{p_0,1} \left[ r_t + Z(p(\hat{p}_0, t|r_t)) \right] \right) - 2t \left( \mathbb{E}_{p_0,1} \left[ r_t + Z(p(\hat{p}_0, t|r_t)) \right] - k'(\tilde{\lambda}) \right) \right] dt
\]

Note that when $k''(\tilde{\lambda})$ is very large, the objective function is concave in $\tilde{\lambda}$ for a fixed $\tau$. However, the value function is not necessarily concave in $\tau$, by (4.3), and thus I know global concavity does not necessarily exist in $\tau$ and $\tilde{\lambda}$ and thus I have not been able to show uniqueness of the solution. See the numerical example in Section 4.4 below for illustration.

4.3. Experimenting Before the First Payoff. Now consider the problem of optimal exit and endogenous choice of arrival rate when the quality of the technology is not yet known, that is, $\pi_0 \in [0, 1]$ and $p_0 \in [0, 1]$. The agent chooses a time $\tau$ at which he will exit if he does not observe a payoff. Let $\tau_1$ denote the random time of the first observation of a payoff. Let $Q'(\tilde{\lambda})$ denote the probability measure associated with $\tau_1$ and the choice of $\tilde{\lambda}$. Suppose that $Z(p_0)$ is the value function calculated using the iteration in Section 4.2. Then the value function from date $t = 0$ is written:

\[
Y(p_0, \pi_0) = \sup_{\tau \geq 0, \tilde{\lambda} \in [\Delta, \lambda]} \mathbb{E}_{p_0, \pi_0 \times Q'(\tilde{\lambda})} \left[ \int_0^{\tau \wedge \tau_1} e^{-\rho t} \left( -k(\tilde{\lambda}) \right) dt + \mathbb{I}(X = g \wedge \tau_1 \leq \tau) e^{-\rho \tau_1} \left[ r_{\tau_1} + Z \left( p(p_0, \tau_1|r_{\tau_1}) \right) \right] \right].
\]

Let $\mathcal{Y}(p_0, \pi_0, t, T, \tilde{\lambda}_0, Z)$ denote the value of the agent who has not observed any payoffs, had prior beliefs $p_0$ and $\pi_0$, is still active at $t \geq 0$, chose $\tilde{\lambda}_0$ at $t = 0$ and plans to exit at $T \geq 0$ unless an observation arrives. Then

\[
\mathcal{Y}(p_0, \pi_0, t, T, \tilde{\lambda}, Z) = \mathbb{E}_{p_0, \pi_0} \left[ \int_t^{T \wedge \tau_1} e^{-\rho(t-t)} \left( -k(\tilde{\lambda}) \right) dt + \mathbb{I}(X = g \wedge \tau_1 < T) e^{-\rho \tau_1 - t} \left[ r_{\tau_1} + Z \left( p(p_t, \tau_1|r_{\tau_1}) \right) \right] \right].
\]
Which can be rewritten using integration by parts as:

\[
\mathcal{Y}(p_0, \pi_0, t, T, \tilde{\lambda}, Z) = e^{\rho t} \int_{t}^{T} e^{-(\rho+\tilde{\lambda})x} \pi(\pi_0, x, \tilde{\lambda}) \left( -k(\tilde{\lambda}) + \tilde{\lambda} \mathbb{E}_{p_0} \left[ r_x + Z(p_0, x|r_x) \right] \right) \, dx 
\]

\[
- e^{\rho t} \int_{t}^{T} e^{-\rho x} \left( 1 - \pi(\pi_0, x, \tilde{\lambda}) \right) k(\tilde{\lambda}) \, dx
\]

Thus the value function at \( t = 0 \) is given by:

\[(4.5) \quad Y(p_0, \pi_0) = \sup_{\tau \geq 0, \tilde{\lambda} \in [\lambda, \bar{\lambda}]} \mathcal{Y}(p_0, \pi_0, 0, \tau, \tilde{\lambda}, Z).\]

**Claim 4.1.**

i) \( Y(p_0, \pi_0) \) is bounded from below by 0 and above by \( \bar{X} \frac{mu}{\rho} \)

ii) \( Y(p_0, \pi_0) \) is non-decreasing in \( p_0 \) and \( \pi_0 \)

iii) \( Y(p_0, \pi_0) \) is time consistent in \( \tau \).

**Proof.** The proof proceeds in the same manner as Propositions 2.6 and 4.3. \( \square \)

**4.4. Numerical Example.** Let the baseline model be given by the following parameters:

- \( a^H = 0.4 \quad a^L = 0.3 \)
- \( r^h = 40 \quad r^m = 15 \quad r^l = 5 \)
- \( q^H = 1.2 \quad q^L = 0.8 \)
- \( \lambda = 1 \quad \bar{\lambda} = 2 \quad \rho = 0.05 \quad \epsilon = 0.1 \)

I use the cost function \( k(\tilde{\lambda}) = 100(\tilde{\lambda} - \lambda)^2 + 20 \). Quick first calculations show that, when searching for optimal \( \tilde{\lambda} \), it is not necessary to look at values of \( \tilde{\lambda} \) lower than 1.05 or greater than 1.125 because they will not be selected in any of the variations presented.

Figure 4.1 illustrates the value function, where it is shown as a function of both the prior and the possible values of \( \tilde{\lambda} \). The optimal value of \( \tilde{\lambda} \) is unique in this example.

**Figure 4.1.** The Value Function over \( p_0 \) and a subset of \( \tilde{\lambda} \).
In addition, in the baseline model, the stopping times are finite, as illustrated in Figure 4.2. Note that due to the change of variables (see Section 3), the top of the $z$ axis indicates immediate exit, while the bottom indicates an infinite stopping time.

Figure 4.2. Optimal exit times over $p_0$ and a subset of $\tilde{\lambda}$, where $y = 0$ corresponds to $\tau^* = \infty$.

Figure 4.3 shows the objective function in $\tilde{\lambda}$ and $\tau$ for an initial belief of $\hat{p}_0 = 0.5$ after the first arrival of a payoff. It is quite irregular, and not weakly concave, although a unique maximum exists in this example.

Figure 4.3. The objective function in $\tilde{\lambda}$ and $\tau$ when $\hat{p}_0 = 0.5$. Note that the $\tau$ in this graph is true time, not the change of variables, and thus the value at $\tau = \infty$ is excluded.
Consider now changes in the uncertainty parameters $\alpha^H$ and $q^H$ which are similar to those used in Section 3. The remaining figures for this section have been moved to Appendix D due to their sizes.

Figure D.1 shows the effects of changing $\alpha^H$ on the value function, stopping times and optimal $\bar{\lambda}$ after the first payoff. Figure D.2 shows the effects of changing $q^H$. In panel (c) of Figures D.1 and D.2 we can see that the choice of $\bar{\lambda}$ is non-monotonic in the expected long-run payoff. That is, for the same beliefs, with higher long-run payoffs, the agent may choose a lower level of $\bar{\lambda}$. By contrast, in the both models (exogenous and endogenous $\bar{\lambda}$), increasing the expected long-run payoff by decreasing $q^H$ or $\alpha^H$ causes the agent to remain in the market at lower beliefs, increases his optimal stopping time, and increases his value for every belief.

Finally, in Figure D.3, I compare the effects of changing different uncertainty parameters, holding long-run expected payoffs constant, on the value function and policy functions after the first arrival. Consider an decrease of $q^H$ to 0.8 or a decrease of $\alpha^H$ to 0.18 which raise the long-run expected payoff, or an increase of $q^H$ to 1.6 or increase of $\alpha^H$ to 0.547 which lower the long-run expected payoff. I illustrate in the effects of these changes in uncertainty on three outcomes of interest: the value of the agent, the optimal exit time of the agent, and the optimal choice of $\bar{\lambda}$ in Figure D.3. This figure illustrates clearly that the agent cares directly about the sources of uncertainty, and not just about long-run payoffs. The agent chooses different stopping times and $\bar{\lambda}$ for each change.

5. CONCLUSION AND ONGOING WORK

I study a stopping and resource allocation problem for an experimenter who faces two sources of uncertainty—the quality and profitability of a risky technology. The quality of the technology can be good or bad. If the quality is good, then payoffs arrive at the jump times of a standard Poisson process. If the technology is bad it does not generate payoffs. Payoffs are stochastic and the sizes of realizations depend on the underlying state of the economy, which follows a two-state continuous-time Markov chain. Some payoffs reveal the state completely, others do not. High payoffs are only generated in the high state and low payoffs are only generated in the low state, but medium payoffs can be generated in either state. In this sense, even when the first payoff is generated for a good technology, the experimenter keeps working in a risky environment, not only because a medium payoff does not fully reveal the state, but because the state may change in the future.

In a model with a fixed rate of arrival of the Poisson process, after the quality of the technology is known, I find that the optimal policy can be characterized by a cutoff in the beliefs about the true state of the economy. Before the quality of the technology becomes known, the stopping region of the experimenter is a subset of the two-dimensional space of beliefs about the quality of the technology and the state of the economy. In a numerical example I provided additional evidence that uncertain profitability matters because the effects of changing parameters governing each type of uncertainty alter the stopping region by different amounts, even when the long-run profitability is held constant.

These results differ substantially from the current literature which finds cutoff beliefs for single agents in environments of uncertainty, both before and after the first information about quality arrives.

I also solve a version of a resource allocation problem where I allow the experimenter to choose endogenously the arrival rate of the Poisson process after each observation of the
payoffs. I find that the choice of arrival rate is non-monotonic in the uncertainty parameters. That is, for the same beliefs, with higher long-run payoffs due to a change in an uncertainty parameter, the agent may choose a lower level of $\tilde{\lambda}$.

Future work will focus on analyzing sponsored research in the presence of quality and profitability uncertainty. I will study, in an environment such as Bergemann and Hege (2005) with moral hazard, the provision of financing at the jump times of the Poisson process.
References


Appendix A. Simplifying the Value Function

Begin with (2.4)

\[ V(V, \hat{p}_0, t, T) = -ce^\rho t \int_t^\infty \lambda e^{-\lambda x} \left[ \mathbb{1}_{T<x} \int_t^T e^{-\rho y} dy + \mathbb{1}_{T \geq x} \int_t^x e^{-\rho y} dy \right] dx \]

(A.2) 

\[ + e^\rho t \int_t^\infty \lambda e^{-\lambda x} \mathbb{1}_{x \leq T} e^{-\rho x} \mathbb{E}_{\hat{p}_0,1} \left[ r_x + V(p(p_0, x | r_x)) \right] dx. \]

(A.1) above can be simplified to the following sum:

(A.3) 

\[ -ce^\rho t \int_t^\infty \lambda x e^{-\lambda x} \int_t^T e^{-\rho y} dy dx \]

(A.4) 

\[ -ce^\rho t \int_t^T \lambda x e^{-\lambda x} \int_t^x e^{-\rho y} dy dx. \]

Define \( u(x) = \int_t^x e^{-\rho y} dy \) and \( v'(x) = \lambda e^{-\lambda x} \) and use integration by parts to further simplify (A.4):

(A.5) 

\[ ce^\rho t \left[ e^{-\lambda T} \int_t^T e^{-\rho x} dx - \int_t^T e^{-(\lambda + \rho)x} dx \right]. \]

So that the cost part of the value function (A.1) is simplified to:

(A.6) 

\[ -ce^\rho t \int_t^T e^{-(\lambda + \rho)x} dx \]

Additionally, simplifying (A.2) and combining, we get the result ((2.5).

\[ V(V, \hat{p}_0, t, T) = e^\rho t \int_t^T e^{-(\lambda + \rho)x} \left( -c + \mathbb{E}_{\hat{p}_0,1} \left[ r_x + V(p(p_0, x | r_x)) \right] \right) dx \]

Appendix B. Algorithm to Compute the Value Functions

The following is an algorithm for calculating the value functions in the second stage when the quality of the technology is known to be good for sure, after the arrival of the first payoff.

Given the value for \( V \), calculating \( W \) is straightforward. Note that a change of variables was implemented so that in the actual procedure, the variable \( R = e^{-t} \) was used and the time grid went from 0 to 1.

1. Set the values of the parameters, \( \{\alpha^H, \alpha^L\}, \{r^H, r^m, r^L\}, \{q^H, q^L\}, \tilde{\lambda}, c, \rho \).
2. Assign computational parameters: a minimum and maximum for the time grid (\( t_{\text{min}} \) and \( t_{\text{max}} \)), the size of the time grid increment (\( t_{\text{inc}} \)), and corresponding parameters for the belief grid (\( p_{\text{min}}, p_{\text{max}}, p_{\text{inc}} \)). Also assign the precision, \( \epsilon \), for stopping the value function iteration.
3. Create two vectors, \( V^* = 0 \) and \( \tau^* = 0 \) with the same length as the \( p \)-grid.
4. Define a function \( p(t, p_0) \) which calculates the belief at each time moment between arrivals, given the prior, \( p_0 \), and the time, \( t \), since the last arrival.
5. Calculate the first iteration:

\[ \text{Algorithm to Compute the Value Functions} \]

Please note that this exposition of the algorithm is preliminary. Additional details (code written in R) are available from the author upon request.
(a) First, make an outer loop through the p-grid:
   (i) Set a temporary \( p_0 \) to the current value on the p-grid.
   (ii) Define a temporary v-vector equal to zero for each point on t-grid.
   (iii) Make an inner loop through the t-grid:
      (A) Set the temporary variable \( x \) to the current value on t-grid.
      (B) Calculate the expected value of stopping at \( x \) using the belief function \( p(t, p_0) \) and assign it to the temporary v-vector in the position where \( p = p_0 \). I use the built-in integration function in R.
   (iv) Find the maximum on the temporary v-vector. Assign it to the \( p_0 \) position on the value function \( (V^*) \). Assign the maximizing \( t \) to the \( p_0 \) position on the stopping time function \( (\tau^*) \).

(6) Create a function \((2.3)\) of \( t \) and \( p_0 \) for the beliefs after the arrival of a return of size \( r_i \) at time \( t \).

(7) Define the approximated \( V \) function:
   (a) Create functions \( \tilde{p}^{up} \) and \( \tilde{p}^{down} \) that find the points on the p grid just above and below any input belief.
   (b) Use the line formed by connecting \( V^* \) between \( \tilde{p}^{up} \) and \( \tilde{p}^{down} \) to evaluate beliefs not on the p grid (linear interpolation). Call this function \( V_{approx}(p_0) \).

(8) Begin the iteration procedure:
   • While the difference between the new value function \( (V^*_n) \) and the previous value function \( (V^*_{n-1}) \) is less than \( \epsilon \):
     (a) Outer loop through p-grid:
        (i) Set a temporary \( p_0 \) to the current value on the p-grid.
        (ii) Define a temporary v-vector equal to zero for each point on t-grid.
        (iii) Inner loop through t-grid:
           (A) Set a temporary \( x \) to the current value on t-grid.
           (B) Calculate the expected value of stopping at \( x \) using the belief function \( p(t, p_0) \) and \( V_{approx}(p_0) \) and assign it to the temporary v-vector in the position where \( p = p_0 \). I use the built-in integration function in R.
           (iv) Find the maximum on the temporary v-vector. Assign it to the \( p_0 \) position on the value function \( (V^*_n) \). Assign the maximizing \( t \) to the \( p_0 \) position on the stopping time function \( (\tau^*_n) \).
     (b) Calculate the absolute value of the difference between the current \( V^*_n \) and \( V^*_{n-1} \) point-wise on the p-grid. If this difference is greater than \( \epsilon \), repeat the procedure.
     (c) Update \( V^* \) to \( V^*_n \).

I keep track of the \( V^*_n \) and \( \tau^*_n \) at each iteration to check that convergence looks reasonable. Given the function \( \tilde{V}(\tilde{p}) \), I calculate \( W(p_0, \pi_0) \) by computing the integral in \((2.8)\) for each \( p_0, \pi_0 \) and stopping time \( \tau \) and taking the maximum across the stopping times. The \( \pi_0 \) are also calculated over a grid.
Appendix C. Numerical Examples in the Model with Exogenous Arrival Rate

Figure C.1. This figure presents the value function and stopping time for various values of $q^H$, where all other parameters are constant. $q^H = 1.2$ in the baseline model.
Figure C.2. This figure presents the value function and stopping time for various values of $\alpha^H$, where all other parameters are constant. $\alpha^H = 0.4$ in the baseline model.
Figure C.3. This figure presents the value function and stopping time for various values of $\lambda$, where all other parameters are constant. $\lambda = 1.5$ in the baseline model.
Figure C.4. This figure presents the value function and stopping time for parameter changes holding the long-run expected payoffs of the technology constant. $\alpha^H = 0.18$, $q^H = 0.8$ and $\lambda = 1.696$ correspond to an increase in expected payoffs. $\alpha^H = 0.547$, $q^H = 1.6$ and $\lambda = 1.389$ correspond to a decrease in payoffs. In subfigure (b), the plot for $\alpha^H = 0.18$ is identical to that for $\lambda = 1.696$. 
Appendix D. Numerical Examples in the Model with Endogenous Arrival Rate

Figure D.1. This figure presents the value function, stopping time and choice of $\lambda$ for values of $\alpha^H$. $\alpha^H = 0.4$ in the baseline model.
Figure D.2. This figure presents the value function, stopping time and choice of $\lambda$ for values of $q^H$. $q^H = 1.2$ in the baseline model.
Figure D.3. This figure presents the value function, stopping time and choice of $\lambda$ for parameter changes holding the long-run expected payoffs of the technology constant. $\alpha^H = 0.18$ and $q^H = 0.8$ correspond to an increase in expected payoffs. $\alpha^H = 0.547$ and $q^H = 1.6$ correspond to a decrease in payoffs.