Constrained-Efficient Partnership Agreement*

Vi Cao

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Preliminary Draft

Under suitable assumptions, I design a constrained-efficient partnership agreement between two agents. A partnership agreement is a direct mechanism, which dictates how future profit will be divided and recommends effort contributions from members. Each member has a privately known cost parameter which affects his cost of working for the partnership. Members report their cost parameters to receive their ownership share assignments and wages, then contribute efforts to the partnership. The ownership share allocation dictates how future net profit (which is profit net of wages) will be distributed. In this setting, an agent’s valuation of his share assignment depends on effort contributions, which are endogenously determined by the share allocation and both agents’ private information. The proposed partnership agreement satisfies constrained efficiency, incentive compatibility, budget balance, individual rationality, and Pareto dominates the equal division rule. The partnership agreement design does not depend on distributions of cost parameters and agents’ beliefs.

Keywords: partnership, profit sharing, interdependent values, adverse selection, moral hazard

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1 Introduction

Partnership is a popular legal structure for businesses. Several individuals own and operate the business in the knowledge that any future net profit or loss will be distributed according to a pre-determined ownership share allocation. A frequently asked question by partnership members is how the ownership should be divided.\(^1\) Available answers typically ignore possible private information held by members. As a consequence, in many cases, some member complains that the pre-determined profit sharing rule is unfair as his partner turns out to be unproductive. In this paper, I assume that members hold private information regarding their costs of working for the partnership. Under suitable assumptions, I design a constrained-efficient partnership agreement between two agents. A partnership agreement is a direct mechanism which dictates how future profit will be divided and recommends effort contributions from members.

Specifically, each member has a privately known cost parameter which affects his cost of working for the partnership. A cost parameter takes two values: either low or high. An agent has a low cost parameter if he has high ability or low opportunity cost, or if he enjoys working for the partnership. The goal is to design a direct mechanism that elicits true cost parameters and recommends a constrained-efficient ownership share allocation that maximizes partners’ aggregate welfare. Partners’ welfares depend partly on partnership profit, which is determined by partners’ efforts. Effort contributions, in turn, are endogenously determined (partly) by partners’ true cost parameters and share assignments. We note that ownership division in a partnership is not conditional on efforts as efforts are typically difficult to measure. Hence, our problem belongs to the class of allocation problems with adverse selection, moral hazard, and interdependent values.

Here I explain the desideratum of constrained efficiency. Holmstrom (1982) studies a complete information game in which agents jointly contribute efforts to a team whose output is distributed according to a pre-determined sharing rule. He shows that there exists no sharing rule that

\(^1\)See Deeb (2014), Spolsky (2011), Schall (2016), and Petersen (2015) for some discussions on the challenge of the ownership division problem and how often startup founders seek consultations regarding this matter.
satisfies budget balance\textsuperscript{2} and induces a Pareto optimal Nash equilibrium in the effort-choosing game.\textsuperscript{3} Specifically, since each rational agent does not internalize the positive externality of his effort on his partner, he fails to supply first best effort. Since it is impossible to require rational agents to internalize externality, I only look for a share allocation that maximizes partners’ aggregate welfare given that partners choose efforts rationally. Such allocation is deemed \textit{constrained-efficient}. A partnership agreement is \textit{constrained-efficient} if it recommends a \textit{constrained-efficient} share allocation for each profile of true cost parameters.

To encourage partners to reveal their true cost parameters, I allow them to receive some wages conditional on their reports. A partnership treats wages as expenses, which are deducted from the partnership profit. Hence, budget is always balanced. Under suitable assumptions, I design a partnership agreement that satisfies \textit{constrained efficiency}, \textit{incentive compatibility}, and \textit{individual rationality}. In other words, when the proposed partnership agreement is in effect, there exists a perfect Bayesian equilibrium in which (a) agents report their cost parameters truthfully, (b) the recommended share allocation maximizes the equilibrium aggregate payoff, and (c) individual equilibrium payoffs are non-negative. In addition, the proposed partnership agreement Pareto dominates the equal division rule: (d) everyone always weakly prefers the proposed mechanism, and (e) there exists a profile of cost parameters at which some agent strictly prefers the proposed mechanism. The mechanism design does not depend on distributions of cost parameters and agents’ beliefs.

We note that share allocations and wages play different roles. While a \textit{constrained-efficient} share allocation induces appropriate effort levels that maximizes the aggregate payoff, wages act as an instrument that addresses adverse selection. Suppose each agent prefers to report low cost parameter, regardless of his private information. Then an agent receives some positive wage if and only if he reports high cost parameter and his partner reports low cost parameter. This positive wage encourages an agent with high cost parameter to reveal his private information. The payoff structure ensures

\textsuperscript{2}Budget balance requires that all output is distributed among team members.
\textsuperscript{3}A Pareto optimal Nash equilibrium gives an aggregate payoff that weakly dominates all possible aggregate payoffs.
that this wage arrangement does not distort the report of an agent with low cost parameter, who already prefers to reveal his private information without wages. When members have the same cost parameter, the proposed mechanism and the equal division rule give the same outcome. When members have different cost parameters, (i) a constrained-efficient share allocation rewards the low-cost agent with a higher share, (ii) the proposed mechanism and the equal division rule give the high-cost member the same payoff, and (iii) the low-cost agent strictly prefers the proposed mechanism.

The remaining of the paper is organized as follows. Section 2 presents the model, assumptions, and desirable properties of a partnership agreement. Section 3 characterizes recommended efforts. Section 4 characterizes the proposed partnership agreement. Section 5 concludes.

2 Model

2.1 Timing, Strategies, and Preferences

Timing

Two agents seek to form a partnership. Each agent \( i \in N \equiv \{1, 2\} \) has a privately known cost parameter \( \theta_i \) (his cost of working for the partnership is increasing in \( \theta_i \)). Cost parameters are independently drawn from a cumulative distribution function \( F \) with support \( \Theta \equiv \{\theta_L, \theta_H\} \subset \mathbb{R}^{++} \) (where \( \theta_L < \theta_H \)).

Stage 1. Agents simultaneously report their cost parameters. Let \( \hat{\theta} \equiv (\hat{\theta}_1, \hat{\theta}_2) \) be the reported cost parameters. A partnership agreement \((s, w, \sigma)\)

(a) chooses an ownership share allocation \( s(\hat{\theta}) \equiv [s_1(\hat{\theta}), s_2(\hat{\theta})] \in \Delta^1 \equiv \{(r_1, r_2) \in \mathbb{R}^2_+ \mid r_1 + r_2 = 1\} \) and wages \( w(\hat{\theta}) \equiv [w_1(\hat{\theta}), w_2(\hat{\theta})] \in \mathbb{R}^2_+ \) (the ownership share allocation and wages are enforceable),

(b) recommends effort contributions \( \sigma(\hat{\theta}) \equiv [\sigma_1(\hat{\theta}), \sigma_2(\hat{\theta})] \in \mathbb{R}^2_+ \) (recommended efforts are unenforceable).
**Stage 2.** Agents simultaneously invest efforts \((e_1, e_2) \in \mathbb{R}_+^2\) in the partnership. The cost of effort for each agent \(i\) is \(c(e_i, \theta_i) \in \mathbb{R}_+\). Then each agent 1 receives his wage \(w_i(\hat{\theta})\) and net profit \(\pi(e_1, e_2) - w_1(\hat{\theta}) - w_2(\hat{\theta})\) is divided among partners according to the assigned share allocation \(s(\hat{\theta})\).

**Strategies**

Let \(S \equiv \{s \mid s : \Theta^2 \to \Delta^1\}\) be the collection of feasible share allocation rules and \(W \equiv \{w \mid w : \Theta^2 \to \mathbb{R}_+^2\}\) be the collection of feasible wage rules.

For each agent \(i\), a stage-one information set is his own true cost parameter \(\theta_i \in \Theta\) and a stage-two information set is a pair \((\theta_i, \hat{\theta}) \in \Theta^3\), where \(\theta_i\) is his true cost parameter and \(\hat{\theta}\) is the cost parameters reported in stage 1.

Then for each agent \(i\), a (pure) strategy is a list \((\alpha_{\hat{\theta}}^i, \alpha_{e}^i)\), where the reporting strategy \(\alpha_{\hat{\theta}}^i : \Theta \times S \times W \to \Theta\) assigns to each stage-one information set a cost parameter to be reported and the effort strategy \(\alpha_{e}^i : \Theta^3 \times S \to \mathbb{R}_+\) assigns to each stage-two information set an effort level.

For each effort strategy profile \((\alpha_{e}^1, \alpha_{e}^2)\) and each \((\theta, \hat{\theta}, s) \in \Theta^2 \times \Theta^2 \times S\), let \(\alpha^e(\theta, \hat{\theta}, s) \equiv [\alpha_{e}^i(\theta_i, \hat{\theta}, s)]^2_{i=1}\).

**Remark 2.1** We note that an agent’s effort strategy \(\alpha_{e}^i : \Theta^3 \times S \to \mathbb{R}_+\) does not depend on the wage rule as effort contributions do not affect wages (wages only depend on reported cost parameters).

**Preferences**

Suppose that true cost parameters are \((\theta_1, \theta_2)\), agents report their cost parameters as \(\hat{\theta} \equiv (\hat{\theta}_1, \hat{\theta}_2)\) and choose efforts \((e_1, e_2)\). Then each agent \(i\)’s payoff is

\[
s_i(\hat{\theta}) \left[ \pi(e_i, e_{-i}) - w_i(\hat{\theta}) - w_{-i}(\hat{\theta}) \right] + w_i(\hat{\theta}) - c(e_i, \theta_i).
\]

**2.2 Assumptions**

**Assumption 2.1** The cost function \(c : \mathbb{R}_+ \times \Theta \to \mathbb{R}_+\) is
(a) three times continuously differentiable,

(b) increasing in each argument,

(c) quadratic and strictly convex with respect to the first argument.

**Assumption 2.2** For each \( \theta_i \in \Theta \),

(a) \( c(0, \theta_i) = 0, \frac{\partial c(0, \theta_i)}{\partial e_i} = 0 \), and \( \lim_{e_i \to \infty} \frac{\partial c(e_i, \theta_i)}{\partial e_i} = \infty \),

(b) \( \frac{\partial^2 c(e_i, \theta_i)}{\partial e_i \partial \theta_i} > 0 \) and \( \frac{\partial^3 c(e_i, \theta_i)}{\partial^2 e_i \partial \theta_i} > 0 \) for each \( e_i \in \mathbb{R}_+ \).

### 2.3 Desiderata

Let \( r \equiv (r_1, r_2) \in \Delta^1 \) be a feasible share allocation and \( m \equiv (m_1, m_2) \in \mathbb{R}_+^2 \) be wages. Suppose agents have true cost parameters \((\theta_1, \theta_2)\) and contribute efforts \( e \equiv (e_1, e_2) \in \mathbb{R}_+^2 \). Define

\[
u_i(r, e, \theta_i) = r_i \pi(e) - c(e_i, \theta_i), \text{ and}
\]

\[
u_i(r, m, e, \theta_i) = r_i \left[ \pi(e) - m_i - m_i \right] + m_i - c(e_i, \theta_i).
\]

**Definition 2.1** A partnership agreement \((s, w, \sigma)\) is **efficient** if for each \( \theta \in \Theta^2 \), the arrangement \([s(\theta), \sigma(\theta)]\) solves

\[
\max_{(r, e) \in \Delta^1 \times \mathbb{R}_+^2} \sum_{i=1}^{2} u_i(r, e, \theta_i).
\]

**Definition 2.2** A partnership agreement \((s, w, \sigma)\) is **incentive compatible** if for each \( i \in N \), each \( \theta \in \Theta^2 \), each \( \theta_i' \in \Theta \), and each \( e_i \in \mathbb{R}_+ \),

\[
v_i[s(\theta), w(\theta), \sigma(\theta), \theta_i] \geq v_i[s(\theta_i', \theta_{-i}), w(\theta_i', \theta_{-i}), e_i, \sigma_{-i}(\theta_i', \theta_{-i}), \theta_i].
\]

For each \( \theta \in \Theta^2 \), let
\( \mathcal{IC}(\theta) \equiv \{(r, e) \in \Delta^1 \times \mathbb{R}^2_+ \mid e_i \in \arg \max_{e'_i} u_i(r, e'_i, e_{-i}, \theta_i) \text{ for each } i \in N \} \)

be the collection of partial arrangements that are consistent with incentive compatibility. For each partial arrangement \((r, e) \in \mathcal{IC}(\theta)\) and each agent \(i \in N\), if the assigned share allocation is \(r\) and the other agent takes the recommended effort \(e_{-i}\), then agent \(i\) is better off taking the recommended effort \(e_i\).

**Definition 2.3** A partial agreement \((s, \sigma)\) is constrained-efficient if for each \(\theta \in \Theta^2\), the arrangement \([s(\theta), \sigma(\theta)]\) solves

\[
\max_{(r, e) \in \mathcal{IC}(\theta)} \sum_{i=1}^{2} u_i(r, e, \theta_i).
\]

A partnership agreement \((s, w, \sigma)\) is constrained-efficient if the partial agreement \((s, \sigma)\) is constrained-efficient.

**Definition 2.4** A partnership agreement \((s, w, \sigma)\) satisfies individual rationality if for each \(i \in N\) and each \(\theta \in \Theta^2\),

\[
v_i[s(\theta), w(\theta), \sigma(\theta), \theta_i] \geq 0.
\]

Let \((s^e, w^e, \sigma^e)\) be the equal division rule. That is, \(s^e(\theta) = \left(\frac{1}{2}, \frac{1}{2}\right)\), \(w^e(\theta) = (0, 0)\), and \([s^e(\theta), \sigma^e(\theta)] \in \mathcal{IC}(\theta)\) for each \(\theta \in \Theta^2\). The equal division rule specifies that (a) profit is always equally divided, (b) there are no wages, and (c) recommended efforts are consistent with incentive compatibility given the assigned share allocation.

**Definition 2.5** A partnership agreement \((s, w, \sigma)\) Pareto dominates \((s^e, w^e, \sigma^e)\) if

(a) \(v_i[s(\theta), w(\theta), \sigma(\theta), \theta_i] \geq v_i[s^e(\theta), w^e(\theta), \sigma^e(\theta), \theta_i]\) for each \(i \in N\) and each \(\theta \in \Theta^2\), and

(b) the inequality is strict for some \((i, \theta) \in N \times \Theta^2\).
3 Recommended Efforts

Most of this section devotes to characterizing recommended efforts for agents who have cost parameters $\theta \in \Theta^2$ under some share allocation rule $s \in S$. For Lemmas 3.1 and 3.2, we assume that agents report truthfully and study the existence/behaviors of a profile of recommended efforts $e$ such that $[s(\theta), e]$ is consistent with incentive compatibility, i.e., $[s(\theta), e] \in IC(\theta)$. For Lemma 3.3, we assume that some agent misreports his private information and study his optimal effort after such misreporting. These characterizations are helpful for proving the properties of our proposed partnership agreement.

The following lemma claims that for each profile of true cost parameters $\theta$ and each feasible share allocation $s(\theta) \in \Delta^1$, there is a unique profile of recommended efforts $e$ such that $[s(\theta), e] \in IC(\theta)$. This lemma simplifies the maximization problem of a constrained-efficient partnership agreement. By definition, a constrained-efficient partnership agreement searches for a partial arrangement $(r^*, e^*) \in IC(\theta)$ that gives the highest aggregate payoff. Instead, we can write recommended efforts as some functions of share allocations and search for a share allocation $r^* \in \Delta^1$ that gives the highest aggregate payoff.

**Lemma 3.1** For each $\theta \in \Theta^2$ and each $s \in S$, there is a unique effort profile $(\phi_i[\theta_i, s_i(\theta)])_{i=1}^2 \in \mathbb{R}_+^2$ such that $[s(\theta), (\phi_i[\theta_i, s_i(\theta)])_{i=1}^2] \in IC(\theta)$.

We note that the recommended effort for an agent $\phi_i[\theta_i, s_i(\theta)]$ depends only on his cost parameter $\theta_i$ and his share assignment $s_i(\theta)$. This property comes from the separability of the profit function $\pi(e_i, e_{-i}) = e_i + e_{-i}$.

Let $(s, \sigma)$ be a constrained-efficient partial agreement. For each $\theta \in \Theta^2$, $\sigma(\theta) = (\phi_i[\theta_i, s_i(\theta)])_{i=1}^2$ and $s(\theta)$ solves

$$\max_{r \in \Delta} \sum_{i=1}^2 u_i \left[ r, (\phi_i[\theta_i, r_i])_{i=1}^2, \theta_i \right].$$

The following lemma characterizes recommended efforts.

**Lemma 3.2** For each $\theta \in \Theta^2$, each $s \in S$, and each $i \in N$, 

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Lemma 3.2 implies the following.

For each agent $i \in N$, a bigger share of net profit increases his marginal benefit of effort. Hence, if he is given a bigger share assignment, he is recommended to make greater effort.

Suppose two agents with different cost parameters are given equal share assignments. Then the agent with high cost parameter is recommended to work less as his marginal cost of effort is higher (by Assumption 2.2(b)).

As discussed earlier, an agent is recommended to make more effort if he is given a bigger share. Since the cost function is strictly convex with respect to effort, the marginal cost of effort is strictly increasing in effort level. For an agent with a higher cost parameter, the marginal cost increases faster (by Assumption 2.2(b)); hence, the rise in recommended effort subject to a rise in share assignment is also smaller.

Fix a share allocation rule $s \in S$. Fix an agent $i \in N$ and suppose the other agent reports some $\theta_{-i} = \hat{\theta}_{-i} \in \Theta$. If agent $i$ reports his true cost parameter $\theta_i$, then his optimal effort is $\phi_i[\theta_i, s(\theta)]$. Suppose agent $i$ misreports his cost parameter as $\hat{\theta}_i$. Let $\alpha^*_i(\theta_i, \hat{\theta}, s)$ be agent $i$’s optimal effort after he reports $\hat{\theta}_i$:

$$\alpha^*_i(\theta_i, \hat{\theta}, s) \in \arg\max_{\phi} u_i[s(\hat{\theta}), e_i, \phi_{-i}[\hat{\theta}_{-i}, s_{-i}(\hat{\theta})], \theta_i].$$

**Lemma 3.3** For each $i \in N$, each $\theta_i \in \Theta$, each $\hat{\theta} \in \Theta^2$, and each $s \in S$,

(a) $u_i[s(\hat{\theta}), e_i, \phi_{-i}[\hat{\theta}_{-i}, s_{-i}(\hat{\theta})], \theta_i]$ has a unique maximizer $\phi_i[\theta_i, s_i(\hat{\theta})],

(b) $\frac{\partial \phi_i[\theta_i, s_i(\hat{\theta})]}{\partial s_i(\hat{\theta})} \leq 0$ and $\frac{\partial \phi_i[\theta_i, s(\hat{\theta})]}{\partial \theta_i} \leq 0$. 

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Lemma 3.3 implies that after agent $i$ misreports his cost parameter as $\hat{\theta}_i$, there is a unique optimal effort $\phi_i[\theta_i, s_i(\hat{\theta})]$ for him. This optimal effort depends only on his true cost parameter and his share assignment (due to the separability of the profit function). This effort $\phi_i[\theta_i, s_i(\hat{\theta})]$ responds to a change in share assignment $s_i(\hat{\theta})$ or true cost parameter $\theta_i$ in the same way as the recommended effort $\phi_i[\theta_i, s_i(\theta)]$ does.

4 A Constrained-Efficient Partnership Agreement

In this section, I present the proposed partnership agreement.

Theorem 4.1 There exists a partnership agreement $(s^*, w^*, \sigma^*)$ that satisfies constrained efficiency, incentive compatibility, individual rationality and Pareto dominates $(s^e, w^e, \sigma^e)$.

Theorem 4.1 follows from Propositions 4.1 and 4.2, discussed in the following subsections. Proposition 4.1 claims that a constrained-efficient partial agreement $(s^*, \sigma^*)$ exists. Proposition 4.2 claims that there exists a wage rule $w^*$ such that the partnership agreement $(s^*, w^*, \sigma^*)$ satisfies incentive compatibility, individual rationality, and Pareto dominates the equal division rule. Lemma 4.2 describes some properties of the payoff structure that are helpful for proving our main result.

4.1 A Constrained-Efficient Partial Agreement

In this section, I show that a constrained-efficient partial agreement $(s^*, \sigma^*)$ exists and characterize the constrained-efficient share allocation rule $s^*$. For each profile of reported cost parameters $\hat{\theta} \in \Theta^2$ and each feasible share allocation $s \in \Delta^1$, let $u(\hat{\theta}, s) = \sum_{i=1}^{2} u_i \left[ r_i \left( \phi_i(\hat{\theta}_i, r_i) \right)^2 \right]$.

Let $(s^*, \sigma^*)$ be a constrained-efficient partial agreement. As discussed in Section 3, for each profile of reported cost parameters $\hat{\theta} \in \Theta^2$, recommended efforts $\sigma^*(\hat{\theta}) = \left( \phi_i[\hat{\theta}_i, s^*_i(\hat{\theta})] \right)_{i=1}^{2}$ and recommended share allocation $s^*(\hat{\theta})$ solves $\max_{r \in \Delta^1} u(\hat{\theta}, r)$ (♀).
A share allocation rule $s \in S$ is constrained-efficient if $s(\hat{\theta})$ solves $(\ast)$ for each $\hat{\theta} \in \Theta^2$.

**Proposition 4.1** A constrained-efficient partial agreement $(s^*, \sigma^*)$ exists.

Proposition 4.1 follows from the Weierstrass Theorem. The following lemma characterizes a constrained-efficient share allocation rule.

**Lemma 4.1** If a share allocation rule $s$ is a constrained-efficient, then $s_i(\hat{\theta})$ is decreasing in $\hat{\theta}_i$ for each $i \in N$.

By Lemma 4.1, an agent receives a smaller share if he reports a higher cost parameter. As shown in Appendix B, a constrained-efficient share allocation rule always selects interior share assignments. Hence, agent $-i$ receives a larger share if and only if the gain from assigning a larger share to agent $-i$ is greater than the gain from assigning a larger share to agent $i$, which holds if and only if

\[
\frac{\partial^2 u[\hat{\theta}, s(\hat{\theta})]}{\partial s_i(\hat{\theta}) \partial \hat{\theta}_i} < \frac{\partial^2 u[\hat{\theta}, s(\hat{\theta})]}{\partial s_{-i}(\hat{\theta}) \partial \hat{\theta}_i}.
\]

Since the profit function is separable, the right hand side is zero. By Lemma 3.2(b), the left hand side is negative. We note that a constrained-efficient share allocation rule treats reported cost parameters as true cost parameters. While a rise in agent $i$’s true cost parameter does not affect the other agent, it depresses agent $i$’s response to a higher share. As a result, giving agent $-i$ a higher share generates a higher aggregate payoff.

**4.2 The Payoff Structure**

Fix a constrained-efficient partial agreement $(s^*, \sigma^*)$. Here I describe some properties of the payoff structure that are helpful for proving that the proposed partnership agreement $(s^*, w^*, \sigma^*)$ (presented in Section 4.3) satisfies *incentive compatibility, individual rationality*, and Pareto dominates the equal division rule.
For each $i \in N$, each $\theta_i \in \Theta$, and each $\hat{\theta} \in \Theta^2$, let
\[
\pi_i[s^*(\hat{\theta}), \sigma^*_{-i}(\hat{\theta}), \theta_i] \equiv \max_{e_i \in \mathbb{R}^+} u_i[s^*(\hat{\theta}), e_i, \sigma^*_{-i}(\hat{\theta}), \theta_i].
\]
Suppose there are no wages. Then $\pi_i[s^*(\hat{\theta}), \sigma^*_{-i}(\hat{\theta}), \theta_i]$ is the highest payoff that agent $i$ can achieve if his cost parameter is $\theta_i$ and reported cost parameters are $\hat{\theta}$ (assuming that his partner is truthful and obedient). For notational convenience, let $\beta_i(\hat{\theta}) \equiv [s^*(\hat{\theta}), \sigma^*_{-i}(\hat{\theta})]$ for each $i \in N$ and each $\hat{\theta} \in \Theta^2$.

**Lemma 4.2** If $(s^*, \sigma^*)$ is a constrained-efficient partial agreement, then for each $i \in N$,

(a) $\pi_i[\beta_i(\theta_L, \theta_{-i}), \theta_L] - \pi_i[\beta_i(\theta_H, \theta_{-i}), \theta_L] \\
\quad \geq \pi_i[\beta_i(\theta_L, \theta_{-i}), \theta_H] - \pi_i[\beta_i(\theta_H, \theta_{-i}), \theta_H]$

for each $\theta_{-i} \in \Theta$,

(b) $\pi_i[\beta_i(\theta_L, \theta_L), \theta_H] - \pi_i[\beta_i(\theta_H, \theta_L), \theta_H] \\
\quad > \pi_i[\beta_i(\theta_L, \theta_H), \theta_H] - \pi_i[\beta_i(\theta_H, \theta_H), \theta_H]$,

(c) $\pi_i[\beta_i(\theta_L, \theta_H), \theta_L] - \pi_i[\beta_i(\theta_H, \theta_H), \theta_L] \\
\quad > \pi_i[\beta_i(\theta_L, \theta_L), \theta_H] - \pi_i[\beta_i(\theta_H, \theta_L), \theta_H]$.

To explain the implications of Lemma 4.2, for simplicity, I assume that each agent prefers to report low cost parameter regardless of his private information. Lemma 4.2(a) establishes that a low-cost agent benefits more from reporting a lower cost parameter than a high-cost agent. By Lemma 4.2(b), a high-cost agent benefits more from lying if his partner has a low cost parameter. Lemma 4.2(c) implies that a high-cost agent’s gain from lying is more sensitive to his true cost parameter than to his partner’s true cost parameter.
4.3 The Wage Rule

In this section, I complete the description of the proposed partnership agreement.

**Proposition 4.2** Fix a constrained-efficient partial agreement \((s^*, \sigma^*)\). There exists a wage rule \(w^*\) such that the partnership agreement \((s^*, w^*, \sigma^*)\) satisfies incentive compatibility, individual rationality, and Pareto dominates the equal division rule.

The wage rule \(w^* : \Theta^2 \rightarrow \mathbb{R}^2_+\) is constructed as follows.

**Case 1.** Suppose \(\pi_i[\beta_i(\theta_L, \theta_L), \theta_H] > \pi_i[\beta_i(\theta_H, \theta_L), \theta_H]\) for each \(i \in N\).

For each \(i \in N\), let

\[
w^*_i(\theta_H, \theta_L) = \frac{\pi_i[\beta_i(\theta_H, \theta_L), \theta_H] - \pi_i[\beta_i(\theta_L, \theta_L), \theta_H]}{s^*_i(\theta_H, \theta_L)} > 0, \quad \text{and} \quad w^*_i(\theta_L, \theta_H) = w^*_i(\theta_H, \theta_H) = 0.
\]

**Case 2.** Suppose \(\pi_i[\beta_i(\theta_L, \theta_L), \theta_H] \leq \pi_i[\beta_i(\theta_H, \theta_L), \theta_H]\) for each \(i \in N\).

For each \(i \in N\), let

\[
w^*_i(\theta_L, \theta_H) = \frac{\pi_i[\beta_i(\theta_L, \theta_H), \theta_H] - \pi_i[\beta_i(\theta_L, \theta_L), \theta_H]}{s^*_i(\theta_L, \theta_H)} \geq 0, \quad \text{and} \quad w^*_i(\theta_L, \theta_L) = w^*_i(\theta_H, \theta_H) = 0.
\]

We discuss the wage arrangement for Case 1, where each agent strictly prefers to report low cost parameter regardless of his private information. An agent receives some positive wage if and only if he reports high cost parameter while the other agent reports low cost parameter. This positive wage is for encouraging an agent with high cost parameter to reveal his private information. The properties established in Lemma 4.2 ensure that this positive wage does not distort the report of a low-cost agent who already prefers to reveal his private information without wages. As a result, everyone
is willing to report truthfully as long as his partner is truthful and obedient. Hence, incentive compatibility holds.

If both agents report low cost parameters or if both agents report high cost parameters, the proposed partnership agreement and the equal division rule give the same outcome. If agents have different cost parameters, then the proposed mechanism and the equal division rule gives the high-cost agent the same payoff, while the low-cost agent strictly prefers the proposed mechanism. Specifically, once true cost parameters are revealed, the constrained-efficient share allocation rule $s^*$ rewards the low-cost agent with a bigger share assignment, which reduces free-riding and generates a higher aggregate payoff than the equal division rule. Since budget is always balanced and the high-cost agent earns the same payoff under both mechanisms, the low-cost agent receives all extra surplus generated by the constrained-efficient mechanism; hence, he strictly prefers the proposed mechanism. Thus, the proposed mechanism Pareto dominates the equal division rule. It follows that individual rationality holds.

5 Conclusion

Under suitable assumptions, I design a constrained-efficient partnership agreement between two agents in an environment with adverse selection, moral hazard, and interdependent values. Our setting involves two agents who operate a partnership whose net profit is distributed according to a predetermined ownership share allocation. Each agent has a privately known cost parameter which affects his cost of working for the partnership. In this setting, an agent’s valuation of his share assignment depends on effort contributions, which are endogenously determined (partly) by the assigned share allocation and both agents’ true types. The proposed partnership agreement satisfies constrained efficiency, ex-post incentive compatibility, budget balance, individual rationality, and Pareto dominates the equal division rule. Our proposed mechanism, which is independent of distributions of cost parameters and agents’ beliefs, contributes to address practical concerns regarding ownership division in partnership formation.
References


A Recommended Efforts

Proof for Lemma 3.1.

Fix $\theta \in \Theta$ and $s \in S$. An effort profile $(e^*_1, e^*_2) \in \mathbb{R}^2_+$ satisfies $[s(\theta), (e^*_1, e^*_2)] \in IC(\theta)$ if and only if $s_i(\theta) \frac{\partial \pi_i(e^*_i)}{\partial e_i} = \frac{\partial c[e^*_i, \theta]}{\partial e_i}$ for each $i \in N$.

By Assumptions 2.1 and 2.2(a), for each $i \in N$, there exists a unique effort level $\phi_i[\theta_i, s_i(\theta)] \in \mathbb{R}_+$ such that $s_i(\theta) \frac{\partial \pi_i(\phi_i[\theta_i, s_i(\theta)])}{\partial e_i} = \frac{\partial c[\phi_i[\theta_i, s_i(\theta)], \theta_i]}{\partial e_i}$ (\ast).

It follows that $[\phi_i[\theta_i, s_i(\theta)]^{2}]_{i=1}^{2}$ is the unique effort profile such that $[s(\theta), (\phi_i[\theta_i, s_i(\theta)])^{2}]_{i=1}^{2} \in IC(\theta)$.

Proof for Lemma 3.2.

(a) Applying the Implicit Function Theorem on (\ast) gives:

(i) $1 = \frac{\partial^2 c[\phi_i[\hat{\theta}_i, s_i(\hat{\theta})], \hat{\theta}_i]}{\partial e_i} \frac{\partial \phi_i[\hat{\theta}_i, s_i(\hat{\theta})]}{\partial s_i(\hat{\theta})}$ (\ast\ast),

which implies $\frac{\partial \phi_i[\theta_i, s_i(\theta)]}{\partial s_i(\theta)} > 0$ (by Assumption 2.1(c)),

(ii) $0 = \frac{\partial^2 c[\phi_i[\theta_i, s_i(\theta)], \theta_i]}{\partial e_i} \frac{\partial \phi_i[\theta_i, s_i(\theta)]}{\partial \theta_i} + \frac{\partial^2 c[\phi_i[\theta_i, s_i(\theta)], \theta_i]}{\partial e_i \partial \theta_i}$,

which implies $\frac{\partial \phi_i[\theta_i, s_i(\theta)]}{\partial \theta_i} \leq 0$ (by Assumptions 2.1(c) and 2.2(b)).

(b) Applying the Implicit Function Theorem on (\ast\ast) gives

$$\frac{\partial^2 \phi_i[\hat{\theta}_i, s_i(\hat{\theta})]}{\partial s_i(\hat{\theta}) \partial \theta_i} =$$

$$\frac{\partial^3 c[\phi_i[\hat{\theta}_i, s_i(\hat{\theta})], \hat{\theta}_i]}{\partial e_i \partial \theta_i} \frac{\partial \phi_i[\hat{\theta}_i, s_i(\hat{\theta})]}{\partial s_i(\hat{\theta})} \left( \frac{\partial^2 c[\phi_i[\hat{\theta}_i, s_i(\hat{\theta})], \hat{\theta}_i]}{\partial e_i^2} \right)^{-1} \leq 0$$

(by Assumption 2.1(c), Assumption 2.2(b), and Lemma 3.1).
Proof for Lemma 3.3.

Fix \( i \in N, \theta_i \in \Theta, \hat{\theta} \in \Theta^2 \), and \( s \in S \). An effort level \( e^*_i \) is a maximizer of \( u_i[s(\hat{\theta}), e_i, \phi_{-i}[\hat{\theta}_{-i}, s_{-i}(\hat{\theta})], \theta_i] \) if and only if \( s_i(\hat{\theta}) = \frac{\partial c[e^*_i, \theta_i]}{\partial e_i} \). By Assumptions 2.1 and 2.2(a), there exists a unique effort level \( \phi_i[\theta_i, s_i(\hat{\theta})] \in \mathbb{R}_+ \) such that \( s_i(\hat{\theta}) = \frac{\partial c[\phi_i[\theta_i, s_i(\hat{\theta})], \theta_i]}{\partial e_i} \) (1). It follows that \( \phi_i[\theta_i, s_i(\hat{\theta})] \) is the unique maximizer of \( u_i[s(\hat{\theta}), e_i, \phi_{-i}[\hat{\theta}_{-i}, s_{-i}(\hat{\theta})], \theta_i] \).

Applying the Implicit Function Theorem on (1) gives:

\[
0 = \frac{\partial^2 c[\phi_i[\theta_i, s_i(\hat{\theta})], \theta_i]}{\partial^2 e_i} \frac{\partial \phi_i[\theta_i, s_i(\hat{\theta})]}{\partial \theta_i} + \frac{\partial^2 c[\phi_i[\theta_i, s_i(\hat{\theta})], \theta_i]}{\partial e_i \partial \theta_i},
\]

which implies \( \frac{\partial \phi_i[\theta_i, s_i(\hat{\theta})]}{\partial \theta_i} \leq 0 \) (by Assumptions 2.1(c) and 2.2(b)).

Lemma A.1 For each \( \theta \in \Theta^2 \), each \( s \in S \), and each \( i \in N \),

\[
\frac{\partial^2 \phi_i[\theta_i, s_i(\theta)]}{\partial^2 s_i(\theta)} = 0.
\]

Proof. From the proof for Lemma 3.1,

\[
1 = \frac{\partial^2 c[\phi_i[\theta_i, s_i(\theta)], \theta_i]}{\partial^2 e_i} \frac{\partial \phi_i[\theta_i, s_i(\theta)]}{\partial \theta_i},
\]

Applying the Implicit Function Theorem on (1) gives

\[
0 = \frac{\partial^2 c[\phi_i[\theta_i, s_i(\theta)], \theta_i]}{\partial^2 e_i} \frac{\partial^2 \phi_i[\theta_i, s_i(\theta)]}{\partial^2 s_i(\theta)},
\]

which implies \( \frac{\partial^2 \phi_i[\theta_i, s_i(\theta)]}{\partial^2 s_i(\theta)} = 0 \).

B A Constrained-Efficient Partial Agreement

Proof for Lemma 4.1.

Let \( (s, \sigma) \) be a constrained-efficient partial agreement. For each \( \hat{\theta} \in \Theta^2 \), \( \sigma(\hat{\theta}) = (\phi_i[\hat{\theta}, s_i(\hat{\theta})])^2_{i=1} \) and \( s(\hat{\theta}) \) solves
\[
\max_r \sum_{i=1}^{2} u_i \left[ r_i (\phi_i[\hat{\theta}_i, r_i])^2 \right]
\]

subject to \( r_1 + r_2 = 1 \) and \( 0 \leq r_i \leq 1 \) for each \( i = 1, 2 \).

The Lagrangean function

\[
L[s(\hat{\theta})] = \sum_{i=1}^{2} u_i \left[ s_i(\hat{\theta}), (\phi_i[\hat{\theta}_i, s_i(\hat{\theta})])^2 \right]
\]

\[+ \sum_{i=1}^{2} \lambda_i s_i(\hat{\theta}) - 2 \sum_{i=1}^{2} \gamma_i [s_i(\hat{\theta})] - 1] - \mu [s_1(\hat{\theta}) + s_2(\hat{\theta}) - 1].\]

Let \( u[\hat{\theta}, s(\hat{\theta})] \equiv \sum_{i=1}^{2} u_i \left[ s_i(\hat{\theta}), (\phi_i[\hat{\theta}_i, s_i(\hat{\theta})])^2 \right].\)

The Kuhn-Tucker conditions

\[
\frac{\partial u[\hat{\theta}, s(\hat{\theta})]}{\partial s_i(\hat{\theta})} + \lambda_i - \gamma_i - \mu = 0 \text{ for each } i,
\]

\[
\lambda_i \geq 0, \ s_i(\hat{\theta}) \geq 0, \ \lambda_i s_i(\hat{\theta}) = 0 \text{ for each } i,
\]

\[
\gamma_i \geq 0, \ s_i(\hat{\theta}) \leq 1, \ \gamma_i [s_i(\hat{\theta})] = 0 \text{ for each } i,
\]

\[
s_1(\hat{\theta}) + s_2(\hat{\theta}) = 1.
\]

**Step 1.** We show that \( 0 < s_i(\hat{\theta}) < 1 \) for each \( i \). It suffices to show that \( s_i(\hat{\theta}) \neq 0 \). Suppose \( s_i(\hat{\theta}) = 0 \). Then \( \gamma_i = \lambda_{-i} = 0 \) and the Kuhn-Tucker conditions imply

\[
\frac{\partial u[\hat{\theta}, s(\hat{\theta})]}{\partial s_i(\hat{\theta})} - \frac{\partial u[\hat{\theta}, s(\hat{\theta})]}{\partial s_{-i}(\hat{\theta})} \leq 0 \quad (*).
\]

We have

\[
u[\hat{\theta}, s(\hat{\theta})] = \sum_{j=1}^{2} \phi_j[\hat{\theta}_j, s_j(\hat{\theta})] - \sum_{j=1}^{2} c \left( \phi_j[\hat{\theta}_j, s_j(\hat{\theta})], \hat{\theta}_j \right),
\]

and for each \( j \in N \),

\[
\frac{\partial u[\hat{\theta}, s(\hat{\theta})]}{\partial s_j(\hat{\theta})} = \frac{\partial \phi_j[\hat{\theta}_j, s_j(\hat{\theta})]}{\partial s_j(\hat{\theta})} - \frac{\partial c(\phi_j[\hat{\theta}_j, s_j(\hat{\theta})], \hat{\theta}_j)}{\partial e_j} \frac{\partial \phi_j[\hat{\theta}_j, s_j(\hat{\theta})]}{\partial s_j(\hat{\theta})}
\]

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\[ = [1 - s_j(\hat{\theta})] \frac{\partial \phi_j[\hat{\theta}_j, s_j(\hat{\theta})]}{\partial s_j(\hat{\theta})} \] (since \( s_j(\hat{\theta}) = \frac{\partial c(\phi_j[\hat{\theta}_j, s_j(\hat{\theta})], \hat{\theta}_j)}{\partial e_j} \),

as shown in the proof for Lemma 3.1).

Since \( s_i(\hat{\theta}) = 0 \) and \( s_{-i}(\hat{\theta}) = 1 \), we have
\[
\frac{\partial u[\hat{\theta}, s(\hat{\theta})]}{\partial s_i(\hat{\theta})} = \frac{\partial \phi_i[\hat{\theta}_i, s_i(\hat{\theta})]}{\partial s_i(\hat{\theta})} > 0
\]
(by Lemma 3.1) and \( \frac{\partial u[\hat{\theta}, s(\hat{\theta})]}{\partial s_{-i}(\hat{\theta})} = 0 \), which contradicts (*).

Hence, \( s_i(\hat{\theta}) \neq 0 \).

**Step 2.** Since \( 0 < s_i(\hat{\theta}) < 1 \) for each \( i \), we have \( \lambda_1 = \lambda_2 = \gamma_1 = \gamma_2 = 0 \) and
\[
\frac{\partial u[\hat{\theta}, s(\hat{\theta})]}{\partial s_i(\hat{\theta})} = \frac{\partial u[\hat{\theta}, s(\hat{\theta})]}{\partial s_{-i}(\hat{\theta})}
\]
for each \( i \in N \), which implies
\[
\frac{\partial \phi_i[\hat{\theta}_i, s_i(\hat{\theta})]}{\partial s_i(\hat{\theta})} - \frac{\partial c(\phi_i[\hat{\theta}_i, s_i(\hat{\theta})], \hat{\theta}_i)}{\partial e_i} \frac{\partial \phi_i[\hat{\theta}_i, s_i(\hat{\theta})]}{\partial s_i(\hat{\theta})} = \frac{\partial \phi_{-i}[\hat{\theta}_{-i}, s_{-i}(\hat{\theta})]}{\partial s_{-i}(\hat{\theta})} - \frac{\partial c(\phi_{-i}[\hat{\theta}_{-i}, s_{-i}(\hat{\theta})], \hat{\theta}_{-i})}{\partial e_{-i}} \frac{\partial \phi_{-i}[\hat{\theta}_{-i}, s_{-i}(\hat{\theta})]}{\partial s_{-i}(\hat{\theta})} \tag{*}
\]

Applying the Implicit Function Theorem on (*) gives
\[
\frac{\partial^2 \phi_i[\hat{\theta}_i, s_i(\hat{\theta})]}{\partial s_i(\hat{\theta})^2} = \frac{\partial^2 c(\phi_i[\hat{\theta}_i, s_i(\hat{\theta})], \hat{\theta}_i)}{\partial^2 e_i} \frac{\partial \phi_i[\hat{\theta}_i, s_i(\hat{\theta})]}{\partial s_i(\hat{\theta})} \frac{\partial^2 \phi_i[\hat{\theta}_i, s_i(\hat{\theta})]}{\partial s_i(\hat{\theta})} - \frac{\partial^2 c(\phi_i[\hat{\theta}_i, s_i(\hat{\theta})], \hat{\theta}_i)}{\partial^2 e_i} \frac{\partial^2 \phi_i[\hat{\theta}_i, s_i(\hat{\theta})]}{\partial s_i(\hat{\theta})^2}
\]
\[
+ \frac{\partial^2 c(\phi_{-i}[\hat{\theta}_{-i}, s_{-i}(\hat{\theta})], \hat{\theta}_{-i})}{\partial^2 e_{-i}} \left( \frac{\partial \phi_{-i}[\hat{\theta}_{-i}, s_{-i}(\hat{\theta})]}{\partial s_{-i}(\hat{\theta})} \right)^2 \frac{\partial s_i(\hat{\theta})}{\partial \theta_i} - \frac{\partial^2 c(\phi_i[\hat{\theta}_i, s_i(\hat{\theta})], \hat{\theta}_i)}{\partial^2 e_i} \frac{\partial^2 \phi_i[\hat{\theta}_i, s_i(\hat{\theta})]}{\partial s_i(\hat{\theta})^2}
\]
\[
+ \frac{\partial^2 \phi_{-i}[\hat{\theta}_{-i}, s_{-i}(\hat{\theta})]}{\partial s_{-i}(\hat{\theta})^2} \frac{\partial^2 c(\phi_{-i}[\hat{\theta}_{-i}, s_{-i}(\hat{\theta})], \hat{\theta}_{-i})}{\partial^2 e_{-i}} \left( \frac{\partial \phi_{-i}[\hat{\theta}_{-i}, s_{-i}(\hat{\theta})]}{\partial s_{-i}(\hat{\theta})} \right)^2 \frac{\partial s_{-i}(\hat{\theta})}{\partial \theta_i}
\]
\[
- \frac{\partial \phi_{-i}[\hat{\theta}_{-i}, s_{-i}(\hat{\theta})]}{\partial s_{-i}(\hat{\theta})} \frac{\partial^2 c(\phi_{-i}[\hat{\theta}_{-i}, s_{-i}(\hat{\theta})], \hat{\theta}_{-i})}{\partial^2 e_{-i}} \left( \frac{\partial \phi_{-i}[\hat{\theta}_{-i}, s_{-i}(\hat{\theta})]}{\partial s_{-i}(\hat{\theta})} \right) \frac{\partial s_{-i}(\hat{\theta})}{\partial \theta_i}
\]
\[
- \frac{\partial c(\phi_{-i}[\hat{\theta}_{-i}, s_{-i}(\hat{\theta})], \hat{\theta}_{-i})}{\partial e_{-i}} \frac{\partial^2 \phi_i[\hat{\theta}_i, s_i(\hat{\theta})]}{\partial s_i(\hat{\theta})^2}
\]
Note that
\[
\frac{\partial^2 c(\phi_i[\hat{\theta}_i, s_i(\hat{\theta})], \hat{\theta}_i)}{\partial^2 e_i} \frac{\partial \phi_i[\hat{\theta}_i, s_i(\hat{\theta})]}{\partial \theta_i} + \frac{\partial^2 c(\phi_i[\hat{\theta}_i, s_i(\hat{\theta})], \hat{\theta}_i)}{\partial e_i \partial \theta_i} = 0,
\]
\[
\frac{\partial^2 c(\phi_i[\hat{\theta}_i, s_i(\hat{\theta})], \hat{\theta}_i)}{\partial^2 e_i} \frac{\partial \phi_i[\hat{\theta}_i, s_i(\hat{\theta})]}{\partial s_i(\hat{\theta})} = 1
\]
(as showed in the proof for Lemma 3.1),
\[
\frac{\partial s_{-i}(\hat{\theta})}{\partial \theta_i} = -\frac{\partial s_i(\hat{\theta})}{\partial \theta_i}, \text{ and } s_i(\hat{\theta}) = \frac{\partial c(\phi_i[\hat{\theta}_i, s_i(\hat{\theta})], \hat{\theta}_i)}{\partial e_i}.
\]
Hence, \(\left(\sum_{j \in N} \frac{\partial \phi_j(\hat{\theta}, s_j(\hat{\theta}))}{\partial s_j(\hat{\theta})}\right) \frac{\partial s_i(\hat{\theta})}{\partial \theta_i} = s_{-i}(\hat{\theta}) \frac{\partial^2 \phi_i(\hat{\theta}, s_i(\hat{\theta}))}{\partial s_i(\hat{\theta}) \partial \theta_i}.
\]
By Lemma 3.2(b), \(\frac{\partial^2 \phi_i[\hat{\theta}_i, s_i(\hat{\theta})]}{\partial s_i(\hat{\theta}) \partial \theta_i} \leq 0\). It follows that \(\frac{\partial s_i(\hat{\theta})}{\partial \theta_i} \leq 0\). \(\blacksquare\)

C Proof for Lemma 4.2.

Let \((s^*, \sigma^*)\) be a constrained-efficient partial agreement.

**Lemma C.1** For each \(i \in N\),
\[
\overline{\pi}_i [\beta_i(\theta_L, \theta_{-i}), \theta_L] - \overline{\pi}_i [\beta_i(\theta_H, \theta_{-i}), \theta_L] \\
\geq \overline{\pi}_i [\beta_i(\theta_L, \theta_{-i}), \theta_H] - \overline{\pi}_i [\beta_i(\theta_H, \theta_{-i}), \theta_H]
\]
for each \(\theta_{-i} \in \Theta\),

**Proof.**

Fix \(i \in N\) and \(\{\theta, \hat{\theta}\} \subset \Theta^2\) such that \(\theta_{-i} = \hat{\theta}_{-i}\). Recall that
\[
\pi_i [\beta_i(\hat{\theta})] = s_i(\hat{\theta}) \left( \phi_i[\theta_i, \sigma^*_i(\hat{\theta})] + \phi_{-i}[-\hat{\theta}_{-i}, s^*_i(\hat{\theta})] \right) - c[\phi_i[\theta_i, s^*_i(\hat{\theta})], \theta_i].
\]
We have
Lemma C.2 Let \((s, \sigma)\) be a constrained-efficient partial agreement. For each \(i \in N\) and each \(\{\theta_i, \hat{\theta}_i\} \subset \Theta^2\) such that \(\theta_i = \theta_{-i} = \hat{\theta}_i = \hat{\theta}_{-i},\)

\[
\frac{\partial^2 \pi_i [s(\phi_i, \sigma_{-i} \hat{\theta})]}{\partial \theta_i \partial \theta_{-i}} \geq \frac{\partial^2 \pi_i [s(\phi_i, \sigma_{-i} \hat{\theta}), \theta_i]}{\partial \theta_i \partial \theta_{-i}}.
\]

Proof.

Step 1. We show \(\frac{\partial^2 s_i(\hat{\theta})}{\partial \theta_i \partial \theta_{-i}} = 0\).

From the proof for Proposition 4.1,
\[
\left( \frac{\partial \phi_i[\hat{\theta}_i, s_i(\hat{\theta})]}{\partial s_i(\hat{\theta})} + \frac{\partial \phi_{-i}[\hat{\theta}_{-i}, s_{-i}(\hat{\theta})]}{\partial s_{-i}(\hat{\theta})} \right) \frac{\partial s_i(\hat{\theta})}{\partial \theta_i} = s_{-i}(\hat{\theta}) \frac{\partial^2 \phi_i[\hat{\theta}_i, s_i(\hat{\theta})]}{\partial s_i(\hat{\theta}) \partial \theta_i} \tag{*}
\]

Applying the Implicit Function Theorem on (\ref{eq:implicit}) gives
\[
\left( \frac{\partial \phi_i[\hat{\theta}_i, s_i(\hat{\theta})]}{\partial s_i(\hat{\theta})} + \frac{\partial \phi_{-i}[\hat{\theta}_{-i}, s_{-i}(\hat{\theta})]}{\partial s_{-i}(\hat{\theta})} \right) \frac{\partial^2 s_i(\hat{\theta})}{\partial \theta_i \partial \theta_{-i}} = \frac{\partial s_{-i}(\hat{\theta})}{\partial \theta_{-i}} \frac{\partial^2 \phi_i[\hat{\theta}_i, s_i(\hat{\theta})]}{\partial s_i(\hat{\theta}) \partial \theta_i} - \frac{\partial^2 \phi_{-i}[\hat{\theta}_{-i}, s_{-i}(\hat{\theta})]}{\partial s_{-i}(\hat{\theta}) \partial \theta_{-i}} \frac{\partial s_i(\hat{\theta})}{\partial \theta_i}.
\]

It follows that \(\frac{\partial^2 s_i(\hat{\theta})}{\partial \theta_i \partial \theta_{-i}} = 0\) at \(\theta_i = \theta_{-i} = \hat{\theta}_i = \hat{\theta}_{-i}\).

**Step 2.** Let \(\phi_i[\theta_i, s_i(\hat{\theta})] \equiv \alpha_i^\epsilon(\theta_i, \hat{\theta}, s).\) We have
\[
\frac{\partial^2 \pi_i[s(\hat{\theta}), \sigma_{-i}(\hat{\theta}), \theta_i]}{\partial \theta_i \partial \theta_{-i}} = 
\frac{\partial s_i(\hat{\theta})}{\partial \theta_i} \left( \frac{\partial \phi_i[\theta_i, s_i(\hat{\theta})]}{\partial s_i(\hat{\theta})} \frac{\partial s_i(\hat{\theta})}{\partial \theta_i} - \frac{\partial \phi_{-i}[\hat{\theta}_{-i}, s_{-i}(\hat{\theta})]}{\partial s_{-i}(\hat{\theta})} \frac{\partial s_{-i}(\hat{\theta})}{\partial \theta_{-i}} \right)
+ \frac{\partial s_{-i}(\hat{\theta})}{\partial \theta_{-i}} \frac{\partial \phi_{-i}[\hat{\theta}_{-i}, s_{-i}(\hat{\theta})]}{\partial s_{-i}(\hat{\theta})} \frac{\partial s_{-i}(\hat{\theta})}{\partial \theta_i} - s_i(\hat{\theta}) \frac{\partial^2 \phi_{-i}[\hat{\theta}_{-i}, s_{-i}(\hat{\theta})]}{\partial s_{-i}(\hat{\theta}) \partial \theta_{-i}} \frac{\partial s_i(\hat{\theta})}{\partial \theta_i}.
\]

Since \(\theta_i = \theta_{-i} = \hat{\theta}_i = \hat{\theta}_{-i}\), we have \(\frac{\partial \phi_i[\theta_i, s_i(\hat{\theta})]}{\partial s_i(\hat{\theta})} = \frac{\partial \phi_{-i}[\theta_{-i}, s_{-i}(\hat{\theta})]}{\partial s_{-i}(\hat{\theta})}\).

Hence,
\[
\frac{\partial^2 \pi_i[s(\hat{\theta}), \sigma_{-i}(\hat{\theta}), \theta_i]}{\partial \theta_i \partial \theta_{-i}} = \frac{\partial s_i(\hat{\theta})}{\partial \theta_i} \frac{\partial \phi_{-i}[\hat{\theta}_{-i}, s_{-i}(\hat{\theta})]}{\partial \theta_{-i}}
+ \left[ \frac{\partial s_{-i}(\hat{\theta})}{\partial \theta_{-i}} \frac{\partial \phi_{-i}[\hat{\theta}_{-i}, s_{-i}(\hat{\theta})]}{\partial s_{-i}(\hat{\theta})} - s_i(\hat{\theta}) \frac{\partial^2 \phi_{-i}[\hat{\theta}_{-i}, s_{-i}(\hat{\theta})]}{\partial s_{-i}(\hat{\theta}) \partial \theta_{-i}} \right] \frac{\partial s_i(\hat{\theta})}{\partial \theta_i}.
\]

Since \(\frac{\partial s_{-i}(\hat{\theta})}{\partial \theta_{-i}} = \frac{\partial s_i(\hat{\theta})}{\partial \theta_i} = \frac{s_i(\hat{\theta})}{2} \frac{\partial^2 \phi_i[\hat{\theta}_i, s_i(\hat{\theta})]}{\partial s_i(\hat{\theta}) \partial \theta_i} \left( \frac{\partial \phi_i[\hat{\theta}_i, s_i(\hat{\theta})]}{\partial s_i(\hat{\theta})} \right)^{-1}\), we have
\[
\frac{\partial s_{-i}(\hat{\theta})}{\partial \theta_{-i}} \frac{\partial \phi_{-i}[\hat{\theta}_{-i}, s_{-i}(\hat{\theta})]}{\partial s_{-i}(\hat{\theta})} - s_i(\hat{\theta}) \frac{\partial^2 \phi_{-i}[\hat{\theta}_{-i}, s_{-i}(\hat{\theta})]}{\partial s_{-i}(\hat{\theta}) \partial \theta_{-i}} = - \frac{s_i(\hat{\theta})}{2} \frac{\partial^2 \phi_i[\hat{\theta}_i, s_i(\hat{\theta})]}{\partial s_i(\hat{\theta}) \partial \theta_i}.
\]
It follows that \[ \frac{\partial^2 \pi_i[s(\hat{\theta}), \sigma_{-i}(\hat{\theta}), \theta_i]}{\partial \theta_i \partial \theta_{-i}} \leq \frac{\partial s_i(\hat{\theta})}{\partial \theta_i} \frac{\partial \phi_{-i}[\hat{\theta}_{-i}, s_{-i}(\hat{\theta})]}{\partial \theta_{-i}}. \]

As shown in the proof for Proposition 4.2,
\[ \frac{\partial^2 \pi_i[s(\hat{\theta}), \sigma_{-i}(\hat{\theta}), \theta_i]}{\partial \theta_i \partial \theta_{-i}} = \frac{\partial s_i(\hat{\theta})}{\partial \theta_i} \frac{\partial \phi_i[\theta_i, s_i(\hat{\theta})]}{\partial \theta_i} = \frac{\partial s_i(\hat{\theta})}{\partial \theta_i} \frac{\partial \phi_{-i}[\hat{\theta}_{-i}, s_{-i}(\hat{\theta})]}{\partial \theta_{-i}}. \]

Hence, \[ \frac{\partial^2 \pi_i[s(\hat{\theta}), \sigma_{-i}(\hat{\theta}), \theta_i]}{\partial \theta_i \partial \theta_{-i}} \geq \frac{\partial^2 \pi_i[s(\hat{\theta}), \sigma_{-i}(\hat{\theta}), \theta_i]}{\partial \theta_i \partial \theta_{-i}}. \]

**Lemma C.3** For each \(i \in N,\)
\[ \pi_i[s(\theta_L, \theta_H), \sigma_{-i}(\theta_L, \theta_H), \theta_L] - \pi_i[s(\theta_H, \theta_H), \sigma_{-i}(\theta_H, \theta_H), \theta_L] \]
\[ > \pi_i[s(\theta_L, \theta_L), \sigma_{-i}(\theta_L, \theta_L), \theta_H] - \pi_i[s(\theta_H, \theta_L), \sigma_{-i}(\theta_H, \theta_L), \theta_H]. \]

**Proof.** We have
\begin{align*}
A \equiv & \pi_i[s(\theta_L, \theta_H), \sigma_{-i}(\theta_L, \theta_H), \theta_L] - \pi_i[s(\theta_H, \theta_H), \sigma_{-i}(\theta_H, \theta_H), \theta_L] \\
& = s_i(\theta_L, \theta_H) \left( \phi_i[s_i(\theta_L, \theta_H), \theta_L] + \phi_{-i}[s_{-i}(\theta_L, \theta_H), \theta_H] \right) \\
& \quad - s_i(\theta_H, \theta_H) \left( \phi_i[s_i(\theta_H, \theta_H), \theta_L] - \phi_i[s_i(\theta_H, \theta_H), \theta_H] \right) \\
& \quad - c \left( \phi_i[\sigma_i(\theta_L, \theta_H), \theta_L] \right) + c \left( \phi_i[\sigma_i(\theta_H, \theta_H), \theta_L] \right). \\
B \equiv & \pi_i[s(\theta_L, \theta_L), \sigma_{-i}(\theta_L, \theta_L), \theta_H] - \pi_i[s(\theta_H, \theta_L), \sigma_{-i}(\theta_H, \theta_L), \theta_H] \\
& = s_i(\theta_L, \theta_L) \left( \phi_i[s_i(\theta_L, \theta_L), \theta_H] + \phi_{-i}[s_{-i}(\theta_L, \theta_L), \theta_L] \right) \\
& \quad - s_i(\theta_H, \theta_L) \left( \phi_i[s_i(\theta_H, \theta_L), \theta_H] + \phi_{-i}[s_{-i}(\theta_H, \theta_L), \theta_L] \right) \\
& \quad - c \left( \phi_i[\sigma_i(\theta_L, \theta_L), \theta_H] \right) + c \left( \phi_i[\sigma_i(\theta_H, \theta_L), \theta_H] \right). \\
\end{align*}

Let \( \theta \equiv (\theta_L, \theta_H) \) and \( e^* \equiv \left( \phi_j[s(\theta_H, \theta_H), \theta_j] \right)_{j=1}^2 \). We have
\[ A - B = \sum_{j=1}^2 u_j[s(\theta), \sigma(\theta), \theta_j] - \sum_{j=1}^2 u_j[s(\theta_H, \theta_H), e^*, \theta_j] > 0. \]
Lemma C.4 For each $i \in N$,

$$\pi_i[s(\theta_L, \theta_L), \sigma_{-i}(\theta_L, \theta_L), \theta_H] - \pi_i[s(\theta_H, \theta_L), \sigma_{-i}(\theta_H, \theta_L), \theta_H]$$

$$\geq \pi_i[s(\theta_L, \theta_H), \sigma_{-i}(\theta_L, \theta_H), \theta_H] - \pi_i[s(\theta_H, \theta_H), \sigma_{-i}(\theta_H, \theta_H), \theta_H].$$

Proof.

Step 1. We have

$$A \equiv \pi_i[s(\theta_L, \theta_L), \sigma_{-i}(\theta_L, \theta_L), \theta_H] - \pi_i[s(\theta_H, \theta_L), \sigma_{-i}(\theta_H, \theta_L), \theta_H]$$

$$= s_i(\theta_L, \theta_L) \left( \phi_i[s_i(\theta_L, \theta_L), \theta_H] + \phi_{-i}[s_{-i}(\theta_L, \theta_L), \theta_L] \right)$$

$$- s_i(\theta_H, \theta_L) \left( \phi_i[s_i(\theta_H, \theta_L), \theta_H] + \phi_{-i}[s_{-i}(\theta_H, \theta_L), \theta_L] \right)$$

$$- c \left( \phi_i[s_i(\theta_L, \theta_L), \theta_H], \theta_H \right) + c \left( \phi_i[s_i(\theta_H, \theta_L), \theta_H], \theta_H \right)$$

$$= s_i(\theta_L, \theta_L) \left( \phi_i[s_i(\theta_L, \theta_L), \theta_H] + \phi_{-i}[s_{-i}(\theta_L, \theta_L), \theta_H] \right)$$

$$- s_i(\theta_H, \theta_L) \left( \phi_i[s_i(\theta_H, \theta_L), \theta_H] + \phi_{-i}[s_{-i}(\theta_H, \theta_L), \theta_L] \right)$$

$$- c \left( \phi_i[s_i(\theta_L, \theta_L), \theta_H], \theta_H \right) + c \left( \phi_i[s_i(\theta_H, \theta_L), \theta_H], \theta_H \right) + C,$$

where $C \equiv s_i(\theta_L, \theta_L) \left( \phi_{-i}[s_{-i}(\theta_L, \theta_L), \theta_L] - \phi_{-i}[s_{-i}(\theta_L, \theta_L), \theta_H] \right)$

$$- s_i(\theta_H, \theta_L) \left( \phi_{-i}[s_{-i}(\theta_H, \theta_L), \theta_L] - \phi_{-i}[s_{-i}(\theta_H, \theta_L), \theta_H] \right),$$

$$B \equiv \pi_i[s(\theta_L, \theta_H), \sigma_{-i}(\theta_L, \theta_H), \theta_H] - \pi_i[s(\theta_H, \theta_H), \sigma_{-i}(\theta_H, \theta_H), \theta_H]$$

$$= s_i(\theta_L, \theta_H) \left( \phi_i[s_i(\theta_L, \theta_H), \theta_H] + \phi_{-i}[s_{-i}(\theta_L, \theta_H), \theta_H] \right)$$

$$- s_i(\theta_H, \theta_H) \left( \phi_i[s_i(\theta_H, \theta_H), \theta_H] + \phi_{-i}[s_{-i}(\theta_H, \theta_H), \theta_H] \right)$$

$$- c \left( \phi_i[s_i(\theta_L, \theta_H), \theta_H], \theta_H \right) + c \left( \phi_i[s_i(\theta_H, \theta_H), \theta_H], \theta_H \right).$$

Step 2. We show $C \geq 0$. By Lemma A.1, $\phi_i[\theta_i, s_i(\hat{\theta})]$ is linear in $s_i(\hat{\theta})$ for each $\theta_i \in \Theta$ and each $\hat{\theta} \in \Theta^2$. Hence,

$$C \equiv s_i(\theta_L, \theta_L) \left( \phi_{-i}[s_{-i}(\theta_L, \theta_L), \theta_L] - \phi_{-i}[s_{-i}(\theta_L, \theta_L), \theta_H] \right).$$
\[-s_i(\theta_H, \theta_L)\left(\phi_{-i}[s_{-i}(\theta_H, \theta_L), \theta_L] - \phi_{-i}[s_{-i}(\theta_H, \theta_L), \theta_H]\right)\]

\[= \left[\frac{1}{4} - s_i(\theta_H, \theta_L)s_{-i}(\theta_H, \theta_L)\right] \left(\phi_{-i}[1, \theta_L] - \phi_{-i}[1, \theta_H]\right) \geq 0.\]

**Step 3.** Let \(\theta \equiv (\theta_H, \theta_H)\) and \(e^* \equiv \left(\phi_j[s(\theta_H, \theta_L), \theta_H]\right)^2_{j=1}\). By Step 2,

\[A - B \geq \sum_{j=1}^{2} u_j\left[s(\theta), \sigma(\theta), \theta_j\right] - \sum_{j=1}^{2} u_j\left[s(\theta_H, \theta_L), e^*, \theta_j\right] > 0.\]

### D Proof for Proposition 4.2

Let \((s^*, \sigma^*)\) be a constrained-efficient partial agreement and \(w^*\) be the wage rule constructed as in Section 4.3.

**Lemma D.1** The partnership agreement \((s^*, w^*, \sigma^*)\) is incentive compatible.

**Proof.** Fix \(i \in N\). For each \(\theta_i \in \Theta\) and each \(\hat{\theta} \in \Theta^2\), let

\[v_i[s^*(\hat{\theta}), w^*(\hat{\theta}), \sigma^*_{-i}(\hat{\theta}), \theta_i] = \max_{v_i \in \mathbb{R}_+} v_i[s^*(\hat{\theta}), w^*(\hat{\theta}), e_i, \sigma^*_{-i}(\hat{\theta}), \theta_i].\]

By construction

\[v_i[s^*(\hat{\theta}), w^*(\hat{\theta}), \sigma^*_{-i}(\hat{\theta}), \theta_i] = v_i[\beta_i(\hat{\theta}, \theta_i)] + s^*_{-i}(\hat{\theta})w^*_i(\hat{\theta}) - s^*_i(\hat{\theta})w^*_{-i}(\hat{\theta}).\]

**Step 1.** We have

\[v_i[s^*(\theta_H, \theta_L), w^*(\theta_H, \theta_L), \sigma^*(\theta_H, \theta_L), \theta_H] = v_i[\beta_i(\theta_H, \theta_L)] + s^*_{-i}(\theta_H, \theta_L)w^*_i(\theta_H, \theta_L) - s^*_i(\theta_H, \theta_L)w^*_{-i}(\theta_H, \theta_L).\]

Suppose \(v_i[\beta_i(\theta_H, \theta_L), \theta_H] < v_i[\beta_i(\theta_L, \theta_L), \theta_H]\). By construction,

\[w^*_i(\theta_H, \theta_L) = \frac{v_i[\beta_i(\theta_L, \theta_L), \theta_H] - v_i[\beta_i(\theta_H, \theta_L), \theta_H]}{s^*_{-i}(\theta_H, \theta_L)},\]

and
\[ w^*_{-1}(\theta_H, \theta_L) = w^*_1(\theta_L, \theta_L) = w^*_{-1}(\theta_L, \theta_L) = 0. \]

Hence,

\[
v_i[s^*(\theta_H, \theta_L), w^*(\theta_H, \theta_L), \sigma^*(\theta_H, \theta_L), \theta_H] = \overline{v}_i[\beta_i(\theta_L, \theta_L), \theta_H]
= \overline{v}_i[s^*(\theta_L, \theta_L), w^*(\theta_L, \theta_L), \sigma^*_{-i}(\theta_L, \theta_L), \theta_H] \quad (1).\]

Suppose \( \overline{v}_i[\beta_i(\theta_H, \theta_L), \theta_H] \geq \overline{v}_i[\beta_i(\theta_L, \theta_L), \theta_H] \). By construction,

\[
w^*_{-1}(\theta_H, \theta_L) = \frac{\overline{v}_i[\beta_i(\theta_H, \theta_L), \theta_H] - \overline{v}_i[\beta_i(\theta_L, \theta_L), \theta_H]}{s^*_{-1}(\theta_L, \theta_H)}, \text{ and} \]

\[
w^*_1(\theta_H, \theta_L) = w^*_1(\theta_L, \theta_L) = w^*_{-1}(\theta_L, \theta_L) = 0. \]

Hence,

\[
v_i[s^*(\theta_H, \theta_L), w^*(\theta_H, \theta_L), \sigma^*(\theta_H, \theta_L), \theta_H] = \overline{v}_i[\beta_i(\theta_L, \theta_L), \theta_H]
= \overline{v}_i[s^*(\theta_L, \theta_L), w^*(\theta_L, \theta_L), \sigma^*_{-i}(\theta_L, \theta_L), \theta_H] \quad (2).\]

**Step 2.** We have

\[
v_i[s^*(\theta_H, \theta_H), w^*(\theta_H, \theta_H), \sigma^*(\theta_H, \theta_H), \theta_H]
- \overline{v}_i[s^*(\theta_L, \theta_H), w^*(\theta_L, \theta_H), \sigma^*_{-i}(\theta_L, \theta_H), \theta_H]
= \overline{v}_i[\beta_i(\theta_H, \theta_H), \theta_H] - \overline{v}_i[\beta_i(\theta_L, \theta_H), \theta_H]
- s^*_{-i}(\theta_L, \theta_H)w^*_1(\theta_L, \theta_H) + s^*_i(\theta_L, \theta_H)w^*_{-1}(\theta_L, \theta_H)
= \overline{v}_i[\beta_i(\theta_H, \theta_H), \theta_H] - \overline{v}_i[\beta_i(\theta_L, \theta_H), \theta_H]
- \left( \overline{v}_i[\beta_i(\theta_H, \theta_L), \theta_H] - \overline{v}_i[\beta_i(\theta_L, \theta_L), \theta_H] \right) > 0 \text{ (by Lemma C.4).} \]

Hence, \( v_i[s^*(\theta_H, \theta_H), w^*(\theta_H, \theta_H), \sigma^*(\theta_H, \theta_H), \theta_H] \geq \overline{v}_i[s^*(\theta_L, \theta_H), w^*(\theta_L, \theta_H), \sigma^*_{-i}(\theta_L, \theta_H), \theta_H] \quad (3). \)
Step 3. We have

\[ v_i[s^*(\theta_L, \theta_L), w^*(\theta_L, \theta_L), \sigma^*(\theta_L, \theta_L), \theta_L] \]
\[ \quad - \bar{v}_i[s^*(\theta_H, \theta_L), w^*(\theta_H, \theta_L), \sigma^*_{-i}(\theta_H, \theta_L), \theta_L] \]
\[ = \pi_i[\beta_i(\theta_L, \theta_L), \theta_L] - \bar{\pi}_i[\beta_i(\theta_H, \theta_L), \theta_L] \]
\[ - s_{-i}^*(\theta_L, \theta_L)w_{\theta}^*(\theta_L, \theta_L) + s_{-i}^*(\theta_H, \theta_L)w_{-i}^*(\theta_H, \theta_L) \]
\[ = \pi_i[\beta_i(\theta_L, \theta_L), \theta_L] - \bar{\pi}_i[\beta_i(\theta_H, \theta_L), \theta_L] \]
\[ - \left( \pi_i[\beta_i(\theta_L, \theta_L), \theta_H] - \bar{\pi}_i[\beta_i(\theta_H, \theta_L), \theta_H] \right) \geq 0 \text{ (by Lemma C.1).} \]

Hence, \[ v_i[s^*(\theta_L, \theta_L), w^*(\theta_L, \theta_L), \sigma^*(\theta_L, \theta_L), \theta_L] \]
\[ \geq \bar{v}_i[s^*(\theta_H, \theta_L), w^*(\theta_H, \theta_L), \sigma^*_{-i}(\theta_H, \theta_L), \theta_L] \tag{4} \]

Step 4. We have

\[ v_i[s^*(\theta_L, \theta_H), w^*(\theta_L, \theta_H), \sigma^*(\theta_L, \theta_H), \theta_L] \]
\[ \quad - \bar{v}_i[s^*(\theta_H, \theta_H), w^*(\theta_H, \theta_H), \sigma^*_{-i}(\theta_H, \theta_H), \theta_L] \]
\[ = \pi_i[\beta_i(\theta_L, \theta_H), \theta_L] - \bar{\pi}_i[\beta_i(\theta_H, \theta_H), \theta_L] \]
\[ + s_{-i}^*(\theta_L, \theta_H)w_{\theta}^*(\theta_L, \theta_H) - s_{-i}^*(\theta_H, \theta_H)w_{-i}^*(\theta_L, \theta_H) \]
\[ = \pi_i[\beta_i(\theta_L, \theta_H), \theta_L] - \bar{\pi}_i[\beta_i(\theta_H, \theta_H), \theta_L] \]
\[ - \left( \pi_i[\beta_i(\theta_L, \theta_L), \theta_H] - \bar{\pi}_i[\beta_i(\theta_H, \theta_L), \theta_H] \right) > 0 \text{ (by Lemma C.3).} \]

Hence, \[ v_i[s^*(\theta_L, \theta_H), w^*(\theta_L, \theta_H), \sigma^*(\theta_L, \theta_H), \theta_L] \]
\[ > \bar{v}_i[s^*(\theta_H, \theta_H), w^*(\theta_H, \theta_H), \sigma^*_{-i}(\theta_H, \theta_H), \theta_L] \tag{5} \]

It follows from (1) – (5) that \((s^*, w^*, \sigma^*)\) is 
\textit{incentive compatible}. \blacksquare
Lemma D.2  The partnership agreement \((s^*, w^*, \sigma^*)\) Pareto dominates the equal division rule \((s^e, w^e, \sigma^e)\) and satisfies individual rationality.

Proof.

For \(\theta = (\theta_L, \theta_L)\) or \(\theta = (\theta_H, \theta_H)\), it is clear that
\[
v_i[s^*(\theta), w^*(\theta), \sigma^*(\theta), \theta_i] = v_i[s^e(\theta), w^e(\theta), \sigma^e(\theta), \theta_i]
\]
for each \(i \in N\).

Suppose \(\theta = (\theta_L, \theta_H)\). The proof for Lemma D.1 shows that
\[
v_1[s^*(\theta), w^*(\theta), \sigma^*(\theta), \theta_i] > v_1[s^e(\theta), w^e(\theta), \sigma^e(\theta), \theta_i],
\]
\[
v_2[s^*(\theta), w^*(\theta), \sigma^*(\theta), \theta_i] = v_2[s^e(\theta), w^e(\theta), \sigma^e(\theta), \theta_i].
\]

Hence, \((s^*, w^*, \sigma^*)\) Pareto dominates \((s^e, w^e, \sigma^e)\). It follows that \((s^*, w^*, \sigma^*)\) satisfies individual rationality. ■