This paper studies two-player games in continuous time with imperfect public monitoring, where information may arrive both continuously, governed by a Brownian motion, and discontinuously, according to Poisson processes. For this general class of two-player games, we characterize the equilibrium payoff set via a differential equation of its boundary. The equilibrium payoff set is obtained from an iterative procedure similar to that known in discrete time. In our setting, however, the resulting payoff set in each step of the iteration is characterized explicitly. Our analysis reveals the drastic influence of abrupt information on the equilibrium payoff set: because of the additional possibility for value burning, the equilibrium payoff set extends much closer to the efficient frontier and its boundary may contain additional straight line segments.

Keywords: Repeated games, continuous time, imperfect observability, equilibrium characterization, abrupt information.

1 Introduction

Continuous-time games with imperfect information exhibit the realistic feature that information may arrive both continuously through a noisy signal and intermittently as occurrences of rare but informative events. Consider a partnership between two firms, where each firm chooses hidden effort levels and observes only the total revenue of the partnership. The total revenue moves continuously due to day-to-day fluctuations in supply and demand conditions, and it is subject to demand shocks when one of the firms receives bad press. In a continuous-time model, these two different sources of information can be cleanly separated: the public signal moves continuously under normal market conditions and suddenly jumps when a demand shock...
Figure 1: The total revenue of the partnership (top panel) consists of normal market fluctuations (bottom left panel) and shocks due to bad press (bottom right panel). Both components carry important information about past play: the expected rise in continuous revenue $\mu$ and the intensity of the scandals $\lambda$ depend on the chosen effort levels.

occurs. This reflects the fact that bad press itself is publicly observable, and hence the impact of the demand shock on the total revenue can be isolated. As illustrated in Figure 1, a decomposition of the information leads to two separate signals that are both relevant: the continuous increase in total revenue without the impact of demand shocks and the frequency at which scandals occur. In a discrete-time setting, such a distinction is not unambiguously possible and these two sources of movement in revenue may at least partially be confounded for each other.

The fundamentally different nature of these two types of information arrival lead to different ways in which players can use the observed information to structure intertemporal incentives. At the boundary of the equilibrium payoff set, the continuous information may be used only to transfer value tangentially between players as in Fudenberg, Levine and Maskin [6], whereas the discontinuous information may be used to transfer value tangentially and to burn value by mutually punishing all players as in Green and Porter [7]. In his introduction of continuous-time repeated games with imperfect information, Sannikov [13] shows that the continuous-time techniques make it possible to relate the tangential transfers to the curvature of the equilibrium payoff set. For two-player games with Brownian information only, he obtains an explicit characterization of the equilibrium payoff set $\mathcal{E}(r)$ through an ordinary differential equation for any discount rate $r > 0$. This result illustrates the potential of continuous-time models — such an exact description of $\mathcal{E}(r)$ also for impatient players is a result without analogue in discrete time. However, since Brownian information can be used to create incentives through value transfers only, his description of $\mathcal{E}(r)$ does not capture the effect that value burning may have on equilibrium payoffs. Such an analysis is possible only if the signal structure is extended to include infrequent but informative events, whose frequencies are affected by players’ actions.
The purpose of this paper is to extend the class of games, for which an exact
description of $E(r)$ is known, to games in which players have both means of incentive
provision available. We prove a characterization of $E(r)$ when information arrives
both continuously and abruptly, modelled through Brownian and Poisson processes
as in Sannikov and Skrzypacz [15]. We show that value burning can be a very
effective way to provide incentives and lead to a great increase of efficiency, even if
the observed events are of purely informational nature and do not themselves affect
players’ payoffs.

The fundamental difference in terms of incentives between the use of continuous
and abrupt information has first been observed in Sannikov and Skrzypacz [15] in their
treatment of discrete-time games with frequent actions. In [15], these informational
restrictions are used to establish a payoff bound for discrete-time games when the
length of the time period approaches zero. By working directly in a continuous-time
setting, we are able to relate these informational restrictions to $\partial E(r)$ and find a
precise description of $E(r)$ for any discount rate $r > 0$. Our main result is thus a
generalization of Theorem 2 in Sannikov [13] to a more general information structure
that includes abrupt information, as well as an improvement from a payoff bound in

In terms of provided incentives, this paper differs from Sannikov and Skrzypac-
za [15] by the fact that only bounded amounts of value can be transferred or
destroyed upon the arrival of a rare event. This is not a difference in the underlying
model, but rather in the result that we prove: Incentives through rare events result in
payoff jumps that reward or punish the appropriate players after the observation of a
rare event. For the construction of the payoff bound in [15], it is sufficient that these
payoff jumps lie in a suitable dominating half-space, whereas for our characterization
of $E(r)$, it is necessary that the jumps remain in the bounded set $E(r)$. Because
the jump sizes are proportional to the provided incentives, only bounded amounts of
value can be transferred or destroyed. This has a major impact on the equilibrium
payoff set for impatient players: because the jump size of the continuation value is
proportional to the discount rate, as the discount rate $r$ increases and players get
more impatient, fewer incentives can be provided through the observation of rare
events — a feature that does not arise in [15].

Our characterization of the boundary involves three main steps. In a first step,
we show a degeneracy result for points on the boundary, where incentives can be
provided with rare events exclusively. These points consist entirely of straight line
segments and isolated points. The second step is the characterization of $\partial E(r)$, where
tangential transfers are necessary to provide incentives. This step works similarly to
Sannikov [13]: Because $E(r)$ is self-generating, the continuation value of an equilib-
rium profile has to remain in $E(r)$. This restricts the possible incentives that can be
provided at the boundary, which gives rise to an expression for the curvature of the
set. In a model with abrupt information, however, the restriction of incentives at the
boundary creates a non-trivial fixed point problem: Self-generation implies that the jumps of the continuation value have to remain within $\mathcal{E}(r)$, which means that the set of admissible incentives depend on $\mathcal{E}(r)$ as well. Thus, at any point on the boundary, the differential equation describing the boundary depends on its entire solution $\partial \mathcal{E}(r)$. We solve this fixed point problem with an iterative procedure over the arrival of rare events. In each step, we relax the condition on the jumps to land in some fixed payoff set $\mathcal{W}$. This gives rise to the largest payoff set $\mathcal{B}(r, \mathcal{W})$ that is self-generating until the arrival of the first event, at which point the continuation value jumps to $\mathcal{W}$. This is a continuous-time analogue to the standard set operator $\mathcal{B}$ in Abreu, Pearce and Stacchetti [1]: $\mathcal{W} \subseteq \mathcal{B}(r, \mathcal{W})$ implies that $\mathcal{W}$ is self-generating, and we prove that a successive application of $\mathcal{B}(r, \cdot)$ to the set of all feasible and individually rational payoffs converges to $\mathcal{E}(r)$. This algorithm is similar to the algorithm known from discrete time. However, contrary to its discrete-time counterpart, the sets in every step of the continuous-time algorithm can be computed efficiently as the numerical solution to the explicit differential equation of $\partial \mathcal{B}(r, \mathcal{W})$.

The third step of the characterization is the proof that each $\mathcal{B}(r, \mathcal{W})$ is closed. This step is necessary to ensure that in the next step of the algorithm, the set of incentives, over which we maximize, is compact. To show closedness for any convex payoff set $\mathcal{W}$, we need to make two assumptions on the information that is carried in the discontinuous information of the public signal. Roughly speaking, these conditions are (1) that the information in the discontinuous component is suitably dominated by the information of the continuous component and (2) a convexity condition for the provision of incentives using rare events only. While these assumptions are sufficient such that $\mathcal{B}(r, \mathcal{W})$ is closed for any convex payoff set $\mathcal{W}$, they are by no means necessary for our main result as we need closedness only for a suitable sequence of payoff sets. We show in the online supplemental material [4] how closedness of $\mathcal{B}(r, \mathcal{W})$ can be verified on a case-to-case basis if these assumptions are not met.

Our result generalizes Theorem 2 in Sannikov not only by extending it to a more general information structure, but also by lowering the conditions on the game primitives. Most notably, we do not require that action profiles are pairwise identifiable, that is, deviations of two players may not be statistically distinguishable. Therefore, our result is also applicable when the signal is continuous but one-dimensional. This extension contains the important applications of a Cournot duopoly in a single homogeneous good and partnership games, where only the total revenue is observed. When action profiles fail to be pairwise identifiable, they may not be enforceable on any regular tangent hyperplane (cf. Fudenberg, Levine ans Maskin [6]). This means that players may not be willing to transfer future payoffs at any rate and instead have a cap on the exchange rate. While this may result in a collapse of $\mathcal{E}(r)$ to the set of static Nash payoff in some games, in other games the players locally keep exchanging payoffs at these limiting rates. In these games, the equilibrium payoff set may be flat, i.e., the boundary may have straight line segments. Equilibria that attain payoffs on
these line segments have a continuation value that enters the interior of $\mathcal{E}(r)$ with probability 1. This is different from Sannikov [13], where continuation payoffs are absorbed on the boundary, and shows that in continuous-time games, a bang-bang result may fail to hold if some action profiles fail to be pairwise identifiable. We present such an example in Section 6.2.

The remainder of the paper is organized as follows. We introduce the continuous-time model with the general information structure in Section 2. In Section 3, we present two examples, for which the characterization of the equilibrium payoff set is new: a climate agreement, where information arrives both continuously and discontinuously, and a Cournot duopoly, where the publicly observable market price is one-dimensional. Section 4 contains the important concepts of enforceability and self-generation in our setting. In Section 5, we present our main result, Theorem 5.2, and an algorithm for the computation of $\mathcal{E}(r)$. A description of how to implement the numerical solution of our main result is presented in Section 6 together with several examples. Appendices A–D contain the proof of our results, whereas auxiliary results and alternatives to our Assumption 2 are contained in the Online Appendix [4].

2 THE SETTING

Consider a game where two players $i = 1, 2$ continuously choose actions from the finite sets $\mathcal{A}^i$ at each point in time $t \in [0, \infty)$. The set of all pure action profiles $a = (a^1, a^2)$ is denoted by $\mathcal{A} = \mathcal{A}^1 \times \mathcal{A}^2$. Rather than directly observing each other’s actions, players see only the impact of the chosen actions on the distribution of a random signal. The public signal contains continuous, but noisy information modelled by a $d$-dimensional Brownian motion $Z$ and informative, but infrequent observations of events of type $y \in Y$. We assume that there are finitely many (possibly zero) different types of events in $Y = \{y_1, \ldots, y_m\}$. Events arrive according to Poisson processes $(J^y)_{y \in Y}$ that are independent from each other and independent of the Brownian motion $Z$. An event of type $y$ leads to a jump in the public signal of size $h(y)$ so that the public signal is given by $X = Z + \sum_{y \in Y} h(y) J^y$.

The public information at time $t$ is a $\sigma$-algebra $\mathcal{F}_t$ that contains the history of the processes $Z, (J^y)_{y \in Y}$ up to time $t$, as well as orthogonal information that players may use as a public randomization device. Events of different types are thus observable but their underlying intensities are not. Because we study perfect public equilibria, a player’s choice of action at time $t$ must be based solely on information in $\mathcal{F}_t$, which is formalized in the following definition.

---

1Public randomization allows us to conclude that the equilibrium payoff set $\mathcal{E}(r)$ is convex early on. Corollary 5.8 shows that public randomization is not needed to achieve payoffs on the boundary of $\mathcal{E}(r)$ and hence the result is also valid without public randomization.
**Definition 2.1.** A *(public) pure strategy* $A^i$ for player $i$ is an $(\mathcal{F}_t)_{t \geq 0}$-predictable stochastic process with values in $\mathcal{A}^i$.

The game primitives $\mu : \mathcal{A} \rightarrow \mathbb{R}^d$ and $\lambda(y \mid \cdot) : \mathcal{A} \rightarrow (0, \infty)$ determine the impact of a chosen action profile on the drift rate of the public signal and the intensity of events of type $y \in Y$, respectively. Let $\lambda(a) := (\lambda(y_1 \mid a), \ldots, \lambda(y_m \mid a))^\top$ denote the vector of all intensities. We assume that events of any type $y$ are possible after any history, that is, it is a game of full support public monitoring.

**Assumption 1** (Full support). $\lambda(y \mid a) > 0$ for all $a \in \mathcal{A}$ and all $y \in Y$.

Because at any time $t$, the chosen strategy profile affects the future distribution of the public signal, play of a strategy profile $A = (A^1, A^2)$ induces a family of probability measures $Q^A = (Q_t^A)_{t \geq 0}$, under which players observe the public signal. On $[0, T]$ for any $T > 0$, the public signal signal takes the form

$$X_t = \int_0^t \mu(A_s) \, ds + Z_t^A + \sum_{y \in Y} h(y) \, J^y_t,$$

under $Q_T^A$, where $Z^A = Z - \int \mu(A_s) \, ds$ is a $Q_T^A$-Brownian motion describing noise in the continuous component and $J^y$ has instantaneous intensity $\lambda(y \mid A)$ under $Q_T^A$.

**Remark 2.1.** Note that it is possible to consider signals of the slightly more general form $X = \sigma Z + \sum_{y \in Y} h(y) \, J^y$ for a $k$-dimensional Brownian motion $Z$ and covariance matrix $\sigma \in \mathbb{R}^{d \times k}$ with rank $d$. Then $\sigma$ has right-inverse $\sigma^\top (\sigma \sigma^\top)^{-1}$ and the game is equivalent to the game with public signal

$$\tilde{X}_t = \int_0^t \sigma^\top (\sigma \sigma^\top)^{-1} \mu(A_s) \, ds + Z_t^A + \sum_{y \in Y} \sigma^\top (\sigma \sigma^\top)^{-1} h(y) \, J^y_t.$$

Indeed, the information carried by $\tilde{X}$ is identical to the information in $X = \sigma \tilde{X}$.

Anderson [2] and Simon and Stinchcombe [16] demonstrate that seemingly simple strategies need not necessarily lead to unique outcomes in continuous time. This is not a problem in our model because actions taken by agents do not immediately generate information. Indeed, this class of games are games of full support public monitoring: Assumption 1 in conjunction with the unbounded support of the normal

$$\exp\left(\int_0^t \mu(A_s) \, dZ_s - \int_0^t \left(\frac{1}{2} \left| \mu(A_s) \right|^2 + \sum_{y \in Y} \lambda(y \mid A_{s-}) - 1 \right) \, ds\right) \prod_{0 < s \leq T} (1 + (\lambda(y \mid A_{s-}) - 1) \Delta J^y_s).$$
distribution implies that any outcome is possible after play of any strategy profile. In public monitoring games one can identify the probability space with the path space of all publicly observable processes, and hence, a realized path \( \omega \in \Omega \) leads to the unique outcome \( A(\omega) \). This is analogous to discrete-time repeated games with full-support public monitoring; see Mailath and Samuelson [11] for a thorough exposition of discrete-time games.

Each player \( i \) receives an expected flow payoff \( g^i : \mathcal{A} \rightarrow \mathbb{R} \). In a game of imperfect information, players’ expected payoffs depend on their opponents’ actions only through their effect on the distribution of the public signal. That is, player \( i \)’s expected flow payoff is of the form

\[
g^i(a) = f^i(a^i, \mu(a), \lambda(a))\]

Definition 2.2. Let \( r > 0 \) be a discount rate common to both players.

(i) Player \( i \)’s discounted expected future payoff (or continuation value) under strategy profile \( A \) at any time \( t \geq 0 \) is given by

\[
W^i_t(A) = \int_t^{\infty} re^{-r(s-t)} \mathbb{E}_{Q^A_s} \left[ g^i(A_s) \mid \mathcal{F}_t \right] \, ds. \tag{1}
\]

(ii) A strategy profile \( A \) is a perfect public equilibrium (PPE) for discount rate \( r \) if for every player \( i \) and all possible deviations \( \tilde{A}^i \),

\[
W^i_t(A) \geq W^i_t(\tilde{A}^i, A^{-i}) \text{ a.s.}
\]

holds at all times \( t \), where \( A^{-i} \) denotes the strategy of player \( i \)’s opponent.

(iii) We denote the set of all payoffs achievable by PPE by

\[
\mathcal{E}(r) := \{ w \in \mathbb{R}^2 \mid \text{there exists a PPE } A \text{ with } W_0(A) = w \text{ a.s.} \}.
\]

Because the weights \( re^{-r(s-t)} \) in (1) integrate up to one, the continuation value of a strategy profile is a convex combination of stage game payoffs. The set of feasible payoffs is thus given by the convex hull of pure action payoffs \( \mathcal{V} := \text{conv} \{ g(a) \mid a \in \mathcal{A} \} \). By deviating to her strategy of myopic best responses \( \arg \max g^i(\cdot, A^{-i}) \), player \( i \) can ensure that her payoff in equilibrium dominates her minmax payoff

\[
v^i = \min_{a^{-i} \in A^{-i}} \max_{a^i \in A^i} g^i(a^i, a^{-i}).
\]

The set of equilibrium payoffs is thus contained in the set of all feasible and individually rational payoffs \( \mathcal{V}^* := \{ w \in \mathcal{V} \mid w^i \geq v^i \text{ for all } i \} \). Let \( \mathcal{A}^N \subseteq \mathcal{A} \) denote the set of static Nash equilibria and denote by \( \mathcal{V}^N := \text{conv} \{ g(a) \mid a \in \mathcal{A}^N \} \) the corresponding payoffs. Clearly, the inclusions \( \mathcal{V}^N \subseteq \mathcal{E}(r) \subseteq \mathcal{V}^* \subseteq \mathcal{V} \) hold. Observe that \( \mathcal{E}(r) \) is convex because players are allowed to use public randomization. Indeed, for any two PPE \( A \) and \( A' \) with expected payoffs \( W_0(A) \) and \( W_0(A') \), respectively, any payoff \( \nu W_0(A) + (1-\nu)W_0(A') \) for \( \nu \in (0,1) \) can be attained by selecting either \( A \) or \( A' \) according to the outcome of a public randomization device at time 0.
Consider a climate agreement between two neighbouring countries that obligates each signatory to reduce its greenhouse gas (GHG) emissions. While the reduction of the atmospheric GHG concentration is of mutual benefit, the cost of reducing the GHG output is borne by the countries individually. This creates an incentive for the countries to violate the agreement and “free-ride” on the efforts of the other signatory. Because countries cannot measure each other’s GHG output directly, such a violation is not immediately detected. Countries may, however, observe an increase in industrial production or an increase in atmospheric GHG concentrations — information that is suggestive, but not conclusive proof, that their counterparty has violated the agreement. In addition to the observation of these continuous processes, countries also observe infrequent but informative political and economic events such as the passing of an environmental bill or the commissioning of a coal power plant. While these events may not significantly affect a country’s total GHG emission, they may be a good indication of a country’s overall stance on climate matters.

To formalize this setting, suppose that a country can either cooperate (C) or defect on the agreement (D). The public signal has three continuous components \( (X_1, X_2, X_3) \). The first two are given by industrial production indices of the respective countries and the third component is the atmospheric GHG concentration. Suppose that the expected annual increase of country \( i \)'s industrial production is 1.6\% or 2\% under policies C and D, respectively, so that \( X_i \) satisfy
\[
\frac{dX_i^t}{X_i^t} = (1.6 + 0.4 \cdot 1_{\{A_i^t=D\}}) \, dt + dZ_i^{A,i}, \quad i = 1, 2,
\]
where \( Z_i^{A,1}, Z_i^{A,2} \) are Brownian motions under \( Q^A \) with correlation coefficient 0.6. Let \( G \) denote the atmospheric GHG-concentration and suppose that its expected annual increase is 1.8\% under compliance with the climate agreement. A violation by one country amplifies the increase by 0.1\%. Suppose further that the atmospheric GHG concentration is less volatile than the industrial production, so that its law of motion under strategy profile \( A = (A^1, A^2) \) is given by
\[
\frac{dG_t}{G_t} = (1.8 + 0.1 \cdot 1_{\{A_i^1=D\}} + 0.1 \cdot 1_{\{A_i^2=D\}}) \, dt + \frac{1}{3} dZ_i^{A,3}
\]

\[\begin{array}{|c|cc|cc|}
\hline
\mu(a) & C & D & g(a) & C & D \\
\hline
C & (1, 1, 5.4) & (0.625, 1.625, 5.7) & C & (0.7, 0.7) & (0.025, 1.025) \\
D & (1.625, 0.625, 5.7) & (1.25, 1.25, 6) & D & (1.025, 0.025) & (0.35, 0.35) \\
\hline
\end{array}\]

**Table 1**: Drift rate of the public signal \( X \) in the left table and its induced flow payoff to the right.

3 Examples

3.1 Climate agreement
Figure 2: Example 1 – Climate agreement. Our main result precisely quantifies the value of the additional observation of the rare events: The equilibrium payoff set $\mathcal{E}(r)$ extends a lot closer to the efficient frontier than the equilibrium payoff set $\mathcal{E}_c(r)$ when information is restricted to the continuous component of the public signal. This becomes most apparent in the right panel where $\mathcal{E}_c(0.2) = \mathcal{V}^N$. Moreover, the region on the boundary $\partial \mathcal{E}(r)$ where the climate agreement is honoured by both countries (solid lines) becomes larger as more information is available.

for a Brownian motion $Z^{A,3}$ under $Q^A$ independent of $Z^{A,1}, Z^{A,2}$. After a normalization of the volatility matrix (see Remark 2.1), this leads to a drift function $\mu$ as in the left panel of Table 1. Suppose that a country’s payoff is linear in its industrial production and that each country suffers an externality due to environmental damages. That is, the discounted future increase in country $i$’s welfare is then given by

$$\int_t^\infty r e^{-r(s-t)} \left( \frac{dX_i^t}{X_i^t} - 3 \left( \frac{dG_t}{G_t} - 1.7 dt \right) \right).$$

In addition to the continuous processes $X^1, X^2, X^3$, the countries observe infrequent but informative events about a country’s policy. Suppose that for $i = 1, 2$, a climate-friendly event $\bar{y}^i$ occurs in country $i$ on average every three years under $C$ and every four years under $D$. An indicator $\bar{y}^i$ of an environmentally unfriendly policy in country $i$ happens on average every six years under $D$ and every ten years under $C$. In summary, for $i = 1, 2$,

$$\lambda(\bar{y}^i|a^i = C) = \frac{1}{3}, \quad \lambda(\bar{y}^i|a^i = D) = \frac{1}{4}, \quad \lambda(\bar{y}^i|a^i = C) = \frac{1}{10}, \quad \lambda(\bar{y}^i|a^i = D) = \frac{1}{6}.$$  

When only the continuous components are observed, these are, the industrial production indices and the atmospheric greenhouse gas concentration, then the boundary is characterized by Theorem 2 in Sannikov [13]. When, in addition, economic and political events are observed, the equilibrium payoff set is characterized by our main
Figure 3: Example 2 – Duopoly. The sum of payoffs in every possible equilibrium dominates the static Nash payoff, even though collusion of the firms is impossible as shown by Sannikov and Skrzypacz [14]. Indeed, the right panel shows the optimal action profiles on the boundary $\partial \mathcal{E}(r)$: since $(4,4)$ is not enforceable on the negative diagonal, collusion of the firms is impossible.

result, Theorem 5.2. Figure 2 shows and compares the computed payoff sets for different discount rates $r = 0.1$ and $r = 0.2$. We see that self-regulation of the two countries is more efficient with the observation of the political and economic events. This is not surprising since the additional information makes it more difficult for countries to cheat on the agreement undetectedly. What may be surprising, however, is the amount of gained efficiency. As shown in the left panel of Figure 2, $\mathcal{E}(0.1)$ extends a lot closer to the efficient frontier than its continuous counterpart even though the necessary value burnt at payoff $A$ is minimal. This effect is even more pronounced for discount rate $0.2$ in the right panel of Figure 2: no strategy profile is enforceable without value burning with the exception of the static Nash profile. Nevertheless, non-trivial equilibria exist and their associated payoffs can be computed with our Theorem 5.2.

The strength of our result lies in its quantitative nature. By comparing the gained efficiency, it is possible to assign a value to the observation of abrupt information given the continuous information, which may have important policy implications. A mechanism designer, for example, may compare the value of observing certain events to the cost of reporting them. If the gained efficiency outweighs the cost of reporting, requiring participants to report these events will increase overall efficiency. In this example of the climate agreement, these considerations should be taken into account when drafting the agreement as the efficient frontier corresponds to payoffs, where the climate agreement is enforced; see also Figure 2.

3.2 Cournot Duopoly

In this section, we present an example which shows that even in the absence of abrupt information, our equilibrium characterization is applicable to a wider class of games than previously known from Sannikov [13]. Consider the example of a Cournot
duopoly in a single homogeneous good. At any point in time, firm $i = 1, 2$ chooses its individual supply rate from the sets $A^i = \{0, 1, 2, 3, 4, 5\}$ and the demand for the product is stochastic. Since the products of the two firms are indistinguishable, their market prices coincide and depend on the firms’ actions only through the total supply. Under strategy profile $(A^1, A^2)$, the price is given by

$$dP_t = (20 - 2(A^1_t + A^2_t)) dt + dZ^A_t,$$

where $Z^A$ is a one-dimensional Brownian motion under $Q^A$. We suppose that the production costs are linear in the supply rate and given by $c_i(A^i_t) = 3A^i_t$. Then, the discounted future payoff of firm $i$ at time $t$ equals

$$\int_t^\infty e^{-r(s-t)} (A^i_s dP_s - 3A^i_s ds).$$

In this game, the public information is one-dimensional, and hence the pairwise-identifiability assumption in Sannikov [13] is not satisfied. Nevertheless, one can compute $\partial E(r)$ as shown in Figure 3, based on our Theorem 5.2.

## 4 Enforceability and Self-generation

In games of imperfect monitoring, players’ incentives are necessarily tied to the public signal and thus, we first need to understand the dependence of the continuation value on the public signal. The following stochastic differential representation is the extension of Proposition 1 in Sannikov [13] to games with abrupt information.

**Lemma 4.1.** For a two-dimensional process $W$ and a pure strategy profile $A$, the following are equivalent:

(i) $W$ is the discounted expected payoff under $A$.

(ii) $W$ is a bounded semimartingale which satisfies for $i = 1, 2$ that

$$dW^i_t = r(W^i_t - g_i(A_t)) dt + r\beta^i_t (dZ_t - \mu(A_t) dt)$$

$$+ r \sum_{y \in Y} \delta^i_t(y) (dJ^y_t - \lambda(y|A_t) dt) + dM^i_t$$

for a martingale $M^i$ (strongly) orthogonal to $Z$ and all $J^y$ with $M^i_0 = 0$, predictable processes $\beta^i$ and $\delta^i(y)$, $y \in Y$, satisfying $E_{Q^A} \left[ \int_0^T |\beta^i_t|^2 \right] < \infty$ and $E_{Q^A} \left[ \int_0^T |\delta^i_t(y)|^2 \lambda(y|A_t) dt \right] < \infty$ for any $T \geq 0$. 

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The process \( r\beta_i \) is the sensitivity of player \( i \)'s continuation value to the continuous component of the public signal, and the processes \( r\delta_i(y) \) are the impacts on player \( i \)'s continuation value when an event of type \( y \in Y \) occurs. In discrete-time games, incentives are provided by a \textit{continuation promise} that maps the public signal to a promised continuation payoff for every player; see, for example, Abreu, Pearce and Stacchetti [1]. The representation in (2) shows that in continuous-time games, the continuation value is linear in the public signal and hence, so is the continuation promise. Similarly to Sannikov [13] and Sannikov and Skrzypacz [15], the incentive compatibility conditions take the following form.

**Definition 4.2.** An action profile \( a \in \mathcal{A} \) is \textit{enforceable} if there exists a \textit{continuation promise} \((\beta, \delta)\) with \( \beta = (\beta_1, \beta_2)\top \in \mathbb{R}^{2 \times d} \) and \( \delta = (\delta_1, \delta_2)\top \in \mathbb{R}^{2 \times m} \) such that for every player \( i \), the sum of expected instantaneous payoff rate \( g^i(a) \) and promised continuation rate \( \beta_i \mu(a) + \delta_i \lambda(a) \) is maximized in \( a^i \). That is, for every \( \tilde{a}^i \notin \mathcal{A}^i \setminus \{a^i\} \),

\[
g^i(a) + \beta_i \mu(a) + \delta_i \lambda(a) \geq g^i(\tilde{a}^i, a^{-i}) + \beta_i \mu(\tilde{a}^i, a^{-i}) + \delta_i \lambda(\tilde{a}^i, a^{-i}).
\]

(3)

We say such a pair \((\beta, \delta)\) \textit{enforces} \( a \). A strategy profile is \textit{enforceable} if and only if it takes values in enforceable action profiles almost everywhere.

If players keep their promises and the continuation promises used to enforce \( A \) are, in fact, the sensitivities of its continuation value, then no player has an incentive to deviate as formalized in the following lemma. It is a generalization of Proposition 2 in Sannikov [13] to our setting.

**Lemma 4.3.** A strategy profile is a \textit{PPE} if and only if \((\beta_1, \beta_2)\) and \((\delta_1(y), \delta_2(y))\) \( y \in Y \) related to \( W(A) \) by (2) enforce \( A \).

Lemmas 4.1 and 4.3 motivate how we construct equilibrium profiles in continuous time — as the solution to (2) subject to the enforceability constraint in (3). However, because we consider repeated games with an infinite time horizon, we cannot apply results from the theory of backward stochastic differential equations to find a solution. Instead, we use time-homogeneity of repeated games to construct forward solutions using self-generating payoff sets similarly as in discrete time. The following is the definition of a self-generating payoff set in a continuous-time setting.

**Definition 4.4.** A payoff set \( \mathcal{W} \subset \mathbb{R}^2 \) is called \textit{self-generating} if for every \( w \in \mathcal{W} \), there exists an enforceable strategy profile \( A \), enforced by \((\beta, \delta)\) related to \( W(A) \) by (2) such that \( W_0(A) = w \) a.s. and \( W_\tau(A) \in \mathcal{W} \) a.s. for every stopping time \( \tau \).

**Lemma 4.5.** The set \( \mathcal{E}(r) \) is the largest bounded self-generating set.

This lemma is the analogue of Theorem 1 in Abreu, Pearce and Stacchetti [1]. The proof of Lemma 4.5 is easily derived from the proof of Lemma 2 in Bernard and Frei [5], as that proof works more generally for signals given by any Lévy process.
Figure 4: Since $\mathcal{E}(r)$ is self-generating, the drift rate $w - g(a) - \delta \lambda(a)$ has to point into the interior of the set, that is, $N_w^T (g(a) + \delta \lambda(a) - w) \geq 0$. Moreover, the diffusion $r \beta_t (dZ_t - \mu(a) \, dt)$ has to be tangential to $\partial \mathcal{E}(r)$, as the continuation value would escape $\mathcal{E}(r)$ immediately otherwise. Finally, an event of type $y \in Y$ incurs a jump in the continuation value of size $r \delta(y)$. Since $W$ cannot jump outside of $\mathcal{E}(r)$, it is necessary that $w + r \delta(y) \in \mathcal{E}(r)$.

5 CHARACTERIZATION OF $\mathcal{E}(r)$

The characterization of $\mathcal{E}(r)$ as the largest bounded self-generating set makes it possible to relate the continuation value of a PPE to its boundary $\partial \mathcal{E}(r)$: Because the continuation value of a PPE can never escape $\mathcal{E}(r)$, we obtain certain restrictions on the continuation promise $(\beta, \delta)$ used to provide incentives to the players at a payoff $w \in \partial \mathcal{E}(r)$. Motivated by the SDE characterization in [2] and illustrated in Figure 4, these restrictions are:

(i) Inward-pointing drift: $N_w^T (g(a) + \delta \lambda(a) - w) \geq 0$,

(ii) Tangential volatility: $N_w^T \beta = 0$,

(iii) Jumps within the set: $w + r \delta(y) \in \mathcal{E}(r)$ for every $y \in Y$.

Sannikov [13] shows that when information arrives continuously only, the boundary of the equilibrium payoff set is explicitly characterized by an ordinary differential equation (ODE) using restrictions (i) and (ii). Such an explicit form of the ODE arises because these restrictions are local restrictions on the use of information and depend on the geometry of $\mathcal{E}(r)$ only through the normal vector $N_w$ at $w$. Thus, given the state $(w, N_w)$, one can solve the ODE locally. In our setting, however, the third restriction is a global restriction that depends on the precise shape of $\mathcal{E}(r)$. In order to solve a differential equation involving restriction (iii) at some point $w \in \partial \mathcal{E}(r)$, one would have to know its solution $\partial \mathcal{E}(r)$ already. This creates a non-trivial fixed point problem. We solve it with an iteration over the arrival of rare events, where restriction (iii) is relaxed to the condition that jumps land in a fixed payoff set $\mathcal{W}$.

In Section 5.2, we will formally introduce the largest sets that are self-generating until the arrival of the first event, such that the continuation value jumps into $\mathcal{W}$ at the arrival of the event. This is similar to the standard set-valued operator in
discrete time introduced in Abreu, Pearce and Stacchetti [1] and, in analogy, we
denote said set by $B(r, \mathcal{W})$. Similarly to Abreu, Pearce and Stacchetti [1], we show
that a successive application of $B$ to $V^*$ converges to $E(r)$. However, unlike its discrete-
time counterpart, the boundary of the resulting set at each step of the iteration is
characterized by an ODE like in Sannikov [13]. Since the condition on the incentives
provided through jumps is fixed, the characterization of $B(r, \mathcal{W})$ is explicit.

The arrival of rare events carries many similarities to the public signal in discrete-
time games. There exists a canonical embedding into discrete time, where the periods
are given by $[\sigma_{n-1}, \sigma_n)$ for $\sigma_n$ indicating the $n$th jump amongst $(J^y)_{y \in Y}$. At time $\sigma_n$, the signal $y$ is observed if and only if an event $y$ happens at that time. Because $Y$
is finite in our setting, the discrete information does not satisfy a bang-bang result
and the jump may go into the interior of $E(r)$. However, since two or more jumps
of independent Poisson processes happen at the same time with probability 0, it
is always possible to use public randomization between events to jump back to the
boundary. It is therefore sufficient to check restriction (iii) above only at the boundary
of the set, which makes a characterization via $\partial E(r)$ possible. This motivates the
following definition of restricted-enforceable incentives that have to be provided on
the boundaries of $E(r)$ and $B(r, \mathcal{W})$, respectively. Denote by $S^1$ the unit circle.

**Definition 5.1.** An action profile $a \in \mathcal{A}$ is restricted-enforceable for any $w, N, r$ and
$\mathcal{W} \subset \mathbb{R}^2$ if it is enforced by $(T\phi, \delta)$ such that $N^T(g(a) + \delta \lambda(a) - w) \geq 0$ as well as
$w + r\delta(y) \in \mathcal{W}$ and $N^T\delta(y) \leq 0$ for every $y \in Y$, where $\phi$ is a row vector in $\mathbb{R}^d$ and
$T \in S^1$ is orthogonal to $N$. Let $\Xi_a(w, N, r, \mathcal{W})$ denote the set of all pairs $(\phi, \delta)$ that
restricted-enforce $a$.

**Remark 5.1.** The condition $N^T\delta(y) \leq 0$ means that upon the arrival of event $y \in Y$,
value can only be transferred tangentially or burned relative to the direction $N$.
For the equilibrium payoff set $E(r)$, these are the only ways how the information of
events can be used because $E(r)$ is convex. Imposing this condition also for the use of incentives on $\partial B(r, \mathcal{W})$ will both simplify the theoretical characterization of $B(r, \mathcal{W})$
and improve the speed of convergence of the algorithm to $E(r)$.

On each of the sets $[\sigma_{n-1}, \sigma_n)$, we construct equilibrium profiles attaining extremal
payoffs. Where the boundary is continuously differentiable and the continuous com-
ponent of the public signal is used to provide incentives, this is done similarly as in Sannikov [13]. The amount of tangential transfers needed to structure incentives is
inversely proportional to the curvature of $\partial E(r)$. In a setting with abrupt informa-
tion, the continuous component of the public signal may not be necessary to provide
incentives and $E(r)$ may have corners where this is the case. Denote by $\mathcal{G}(r) \subseteq E(r)$
the set of all of these payoffs, that is,

$$
\mathcal{G}(r) := \left\{ w \in \partial E(r) \mid \exists (a, N, \delta) \text{ with } (0, \delta) \in \Xi_a(w, N, r, E(r)), \right. \\
\text{where } N^T(w - \bar{w}) \geq 0 \ \forall \bar{w} \in E(r) \right\}.
$$
5.1 Main result

For any action profile \( a \in A \) and any player \( i = 1, 2 \), let \( M^i(a) \) and \( \Lambda^i(a) \) denote the matrices containing column vectors \( \mu(\tilde{a}^i, a^{-i}) - \mu(a) \) and \( \lambda(\tilde{a}^i, a^{-i}) - \lambda(a) \), respectively. \( M^i(a) \) and \( \Lambda^i(a) \) contain the informational changes induced by player \( i \)'s deviations from \( a^i \) to any other action. We make the following assumptions, which we motivate in Section 5.3 after the introduction of the algorithm that approximates \( E(r) \).

Assumption 2.

(i) Suppose that \( \text{span } \Lambda^i(a)^\top \subseteq \text{span } M^i(a)^\top Q^{-i}(a)^\top \), where \( Q^i(a) \) is any matrix whose row vectors form a basis of \( \text{ker } M^i(a) \).

(ii) If there exists a direction \( T \in S^1 \) and a vector \( \psi \in \mathbb{R}^{|Y|} \) such that \( (0, T \psi) \) enforces \( a \) with \( \psi \lambda(a) \leq 0 \), then there exists \( \tilde{\psi} \) with \( \psi(y) \wedge 0 \leq \tilde{\psi}(y) \leq 0 \) such that \( (0, T \tilde{\psi}) \) enforces \( a \).

Assumption 2(i) is the conjunction of an identifiability assumption together with the assumption that the information driving the continuous component of the public signal dominates the information underlying the rare events. Indeed, if the continuous component of the public signal makes all action profiles pairwise identifiable, i.e., \( \text{span } M^1(a) \cap \text{span } M^2(a) = \{0\} \) for every action profile \( a \in A \), then there exist matrices \( Q^1, Q^2 \) isolating incentives for players 1 and 2 by Lemma 1 of Sannikov [13]. Therefore, for such a game, Assumption 2(i) reduces to \( \text{span } \Lambda^i(a) \subseteq \text{span } M^i(a) \) for \( i = 1, 2 \), that is, any change of incentives through the rare events can be compensated by incentives arising from the continuous information.

The interpretation of Assumption 2(ii) is the following: if an action profile \( a \) is enforced by attaching rewards/punishments to rare events relative to a fixed direction \( T \), then \( a \) can be enforced also by attaching only rewards or only punishments to rare events (depending on whether or not the average \( \psi \lambda(a) \) is a reward or a punishment). Note that Assumption 2(ii) is certainly satisfied if there is at most one type of rare event.

Assumption 3. Suppose that

(i) best responses are unique, and

(ii) if \( a \in A \) is enforced by \( (\beta, \delta) \) with \( N^\top \beta = 0 \) and \( N^\top \delta(y) \leq 0 \) for every \( y \in Y \) and some coordinate direction \( N \in \{\pm e_i\} \), then \( (\hat{a}^i, a^{-i}) \) is enforceable, where \( \hat{a}^i \) is player \( i \)'s best response to \( a^{-i} \).

 Remark 5.2. Assumptions 2 and 3 are generalizations of the assumptions in Sannikov [13] to our framework. Note, however, that our assumptions are less stringent when \( Y = \emptyset \), that is, no rare events are observed. Indeed, in that case Assumption 2 is always fulfilled and Assumption 3 reduces to Assumption 3(i), corresponding to Assumption 2(i) in Sannikov [13].
Theorem 5.2. Under Assumptions 1–3, \( \mathcal{E}(r) \) is the largest closed subset of \( \mathcal{V}^* \) such that \( \partial \mathcal{E}(r) \setminus \mathcal{G}(r) \) is continuously differentiable with curvature at almost every payoff \( w \) given by

\[
\kappa(w) = \max_{a \in A} \max_{(\phi, \delta) \in \Xi_a(w, r, N_w, \mathcal{E}(r))} \frac{2N_w^\top (g(a) + \delta \lambda(a) - w)}{r \| \phi \|^2},
\]

where we set \( \kappa(w) = 0 \) if the maxima are taken over empty sets. \( \mathcal{G}(r) \) consists of straight line segments and isolated points only, and all corners of \( \mathcal{E}(r) \) are contained in the set of static Nash payoffs.

Compared to games where information arrives continuously only, the observation of rare events enlarges the equilibrium payoff set \( \mathcal{E}(r) \). Indeed, if incentives provided through the continuous information exclusively are sufficient to enforce a strategy profile, these incentives are still sufficient when additional events are observed. At points on the boundary, where incentives are provided through both continuous and discontinuous information, the optimality equation looks similar to that in Sannikov [13]. However, the possible tradeoffs between incentives provided through the continuous and abrupt information increases efficiency and enlarges \( \mathcal{E}(r) \): The numerator of (4) is a measure for the inefficiency, that is, the amount that \( \partial \mathcal{E}(r) \) is below the stage game payoff \( g(a) \). Even though value burning can only increase this amount of inefficiency, the tradeoff between incentives may significantly reduce the amount of tangential volatility required to provide sufficient incentives, leading to an increase in overall efficiency.

Even though the curvature is characterized only at almost every \( w \in \partial \mathcal{E}(r) \setminus \mathcal{G}(r) \), a solution is unique with the additional requirement that it be continuously differentiable. This implies that \( \partial \mathcal{E}(r) \) is twice continuously differentiable almost everywhere, which is important for the numerical solution of (4), as numerical procedures rely on discretizations. We will elaborate on the numerical implementation in Section 6.

5.2 Algorithm

In this section, we present a continuous-time analogue of the algorithm in Abreu, Pearce and Stacchetti [1] and show how it can be used to solve the aforementioned fixed-point problem. It should be noted that this algorithm is used not just for the computation of \( \mathcal{E}(r) \) in specific games, but also for the proof of our main result, Theorem 5.2. We begin by defining the set-valued operator \( \mathcal{B}(r, \mathcal{W}) \) in our setting.

Definition 5.3. For a payoff set \( \mathcal{W} \subseteq \mathbb{R}^2 \), let \( \mathcal{B}(r, \mathcal{W}) \) denote the largest bounded payoff set that is self-generating up to the arrival of the first rare event, together with the condition that the first jump lands in \( \mathcal{W} \) and is directed inwards at \( \partial \mathcal{B}(r, \mathcal{W}) \).

What stands out in comparison to discrete time is the condition that the jumps are directed inwards at the boundary of \( \mathcal{B}(r, \mathcal{W}) \). This is related to the discussion in
Remark 5.1. Because we do not know a priori, whether $B(r, W)$ is closed or not, a strategic definition of $B(r, W)$ is rather technical and hence deferred to Appendix A.2 together with the proofs of the following two lemmas. The next lemma shows that similarly to discrete time, the operator $B$ is closely related to self-generation.

Lemma 5.4. Let $W \subseteq V$. If $W \subseteq B(r, W)$, then $B(r, W)$ is self-generating. On the other hand, if $W$ is self-generating and convex, then $W \subseteq B(r, W)$.

Any payoff in $B(r, W)$ is attainable by an enforceable strategy profile such that its continuation value after the first rare event is in $W$. An $n$-fold application of $B$ to a set $W$ thus ensures that the continuation value after the first $n$ events is in $W$. Because Poisson processes have only countably many jumps, taking the limit as $n$ goes to infinity covers all events. We thus obtain the following algorithm to compute $E(r)$ iteratively.

Lemma 5.5. Let $W_0 = V^*$ and $W_n = B(W_{n-1})$ for $n \geq 1$. Then $(W_n)_{n \geq 0}$ is decreasing in the set-inclusion sense with $\bigcap_{n \geq 0} W_n = E(r)$.

This algorithm is similar to Abreu, Pearce and Stacchetti [1]. However, unlike its discrete-time counterpart, the boundary of the resulting set at each step of the iteration is characterized by an ODE. Since the condition on the incentives provided through jumps is fixed, the characterization of $B(r, W)$ is explicit. Let

$$G(r, W) := \left\{ w \in \partial B(r, W) \mid \exists (a, \delta) \text{ with } (0, \delta) \in \Xi_a(w, N, r, W) \text{ for all normals } N \text{ to } \partial B(r, W) \text{ at } w \right\}$$

denote the set of all payoffs $w \in \partial B(r, W)$, where incentives can be provided through the observation of rare events only. We say that such a pair $(a, \delta)$ decomposes $w$, and, in turn, that $w$ is decomposable. We obtain the following characterization of $\partial B(r, W)$.

Proposition 5.6. Suppose that Assumptions 1–3 hold and that $W \subseteq V^*$ is compact and convex with non-empty interior. Then $B(r, W)$ is the largest closed subset of $V^*$ such that $\partial B(r, W) \setminus G(r, W)$ is continuously differentiable with curvature at almost every payoff $w$ given by

$$\kappa(w) = \max_{a \in A} \max_{(\phi, \delta) \in \Xi_a(w, r, N, W)} \frac{2N_w^T(g(a) + \delta \lambda(a) - w)}{r \|\phi\|^2},$$

(5)

where we set $\kappa(w) = 0$ if the maxima are taken over empty sets. Moreover, $G(r, W)$ consists of straight line segments and isolated points only, and all corners of $B(r, W)$ are contained in the set of static Nash payoffs. If a straight line segment is decomposed by some action profile $a$, its infinite continuation goes through $g(a)$.

Observe that Theorem 5.2 follows from Lemma 5.5 and Proposition 5.6. Indeed, because $B(r, W)$ preserves compactness by Proposition 5.6, Lemma 5.5 shows that $E(r)$ is compact. Therefore, an application of Proposition 5.6 for $W = E(r)$ provides a description of the boundary of $B(E(r)) = E(r)$. 

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5.3 Discussion of Assumption 2

Assumption 2(i) implies a transferability of incentives from the discontinuous component to the continuous component of the public signal, which is a powerful tool in the proof of Proposition 5.6.

Lemma 5.7. Suppose that Assumption 2(i) is satisfied.

(i) If \( a \in A \) has the best response property for player \( i \), then \( a \) is enforceable by \((\beta, 0)\) with \( \beta^i = 0 \).

(ii) If \( a \in A \) is enforced by \((T\phi, \delta)\) for some non-coordinate direction \( T \), then for any \( \delta \) there exists \( \phi \) such that \( a \) is also enforced by \((T\phi, \delta)\).

In this section, we outline the proof of Proposition 5.6 in broad strokes and illustrate in which steps this transferal of incentives is needed. Based on this discussion, we propose a candidate for \( B(r, W) \) in games where Assumption 2 is not satisfied. The detailed verification argument for that candidate is contained in Online Appendix 4 together with the verification for a specific example introduced in Section 6.2. The characterization of \( \partial B(r, W) \) can be divided into three main parts:

- Characterization through (5) outside of \( G(r, W) \),
- Characterization of \( G(r, W) \),
- Closedness of \( B(r, W) \).

Similarly to Sannikov [13], extremal payoffs in \( \partial B(r, W) \setminus G(r, W) \) are attained by a strategy profile \( A \) with incentive processes \((\beta, \delta)\) that maximize (5). Its continuation value remains on the boundary where the boundary is strictly curved until the next rare event occurs. Despite our general framework, this step does not rely on Assumption 2. A natural candidate for \( B(r, W) \) is thus a set, whose boundary is characterized by (5) where the boundary is curved. Let \( \tilde{B}(r, W) \) denote this candidate, that is, it is the largest closed payoff set in \( V^* \), for which the boundary is given by (5) outside of

\[
\tilde{G}(r, W) := \left\{ w \in \partial \tilde{B}(r, W) \left| \exists (a, \delta) \text{ with } (0, \delta) \in \Xi_a(w, N, r, W) \text{ for all normals } N \text{ to } \partial \tilde{B}(r, W) \text{ at } w \right. \right\}
\]

and such that \( \tilde{G}(r, W) \) consists of straight line segments and isolated points only. Observe that without the transferability of incentives, it is possible that \( \tilde{B}(r, W) \) has corners outside the set of static Nash payoffs. We will elaborate on this fact in the example in Section 6.2. Let \( \tilde{P}(r, W) \) denote the set of payoffs, where \( \partial \tilde{B}(r, W) \) changes from being a solution to (5) to being a straight line segment in a continuously differentiable way. With the following steps one can verify whether \( \tilde{B}(r, W) = B(r, W) \):
Figure 5: The left panel illustrates that if \( w \in \mathcal{G}(r, \mathcal{W}) \) is decomposed by \((a, \delta_0)\) with \( N_w^\top (g(a) - w) > 0 \), then there exists \( \delta \) close to \( \delta_0 \) with \((\beta, \delta) \in \Xi_a(v, N, r, \mathcal{W})\) for \((v,N)\) in a neighbourhood of \((w,N_w)\). Thus, \( \mathcal{B}(r, \mathcal{W}) \) could be enlarged by a solution \( \mathcal{C} \) to \((5)\), a contradiction. The right panel illustrates that the controls \( a^*, \delta^* \) lead to an inward-pointing drift at \( w \in \mathcal{P}(r, \mathcal{W}) \). For \( w \) to be in \( \mathcal{B}(r, \mathcal{W}) \), it is thus necessary that \( a^* \) is restricted-enforceable in a neighbourhood of \( w \).

(i) Show that \( \mathcal{P}(r, \mathcal{W}) \) and all corners of \( \mathcal{B}(r, \mathcal{W}) \) are contained in \( \mathcal{B}(r, \mathcal{W}) \). Straight line segments in \( \partial \mathcal{B}(r, \mathcal{W}) \) are thus contained in \( \mathcal{B}(r, \mathcal{W}) \) as well by public randomization. Since curved segments in \( \partial \mathcal{B}(r, \mathcal{W}) \setminus \mathcal{G}(r, \mathcal{W}) \) are attainable with a continuation value that remains in \( \partial \mathcal{B}(r, \mathcal{W}) \setminus \mathcal{G}(r, \mathcal{W}) \) until the arrival of the first event, it follows that \( \mathcal{B}(r, \mathcal{W}) \subseteq \mathcal{B}(r, \mathcal{W}) \).

(ii) Show that no payoffs outside of \( \mathcal{B}(r, \mathcal{W}) \) are decomposable. This means that \( \partial \mathcal{B}(r, \mathcal{W}) \setminus \mathcal{B}(r, \mathcal{W}) \) is a solution to \((5)\), contradicting the definition of \( \mathcal{B}(r, \mathcal{W}) \).

The proof of this verification procedure is contained in Online Appendix [4]. In the remainder of this section we show how Assumption [2] helps to overcome these steps.

The transfer of incentives between continuous and abrupt information is the key ingredient in the characterization of \( \mathcal{G}(r, \mathcal{W}) \). Suppose that \( \mathcal{G}(r, \mathcal{W}) \) is \( C^1 \) in a neighbourhood of some payoff \( w \in \mathcal{G}(r, \mathcal{W}) \) that is decomposed by \((a, \delta_0)\) with \( N_w^\top (g(a) - w) > 0 \). Due to Lemma [5.7] sufficient incentives can be provided in any direction for any perturbation \( \delta \) of \( \delta_0 \). Thus, for any \((v,N)\) in a neighbourhood of \((w,N_w)\), there exists \((\beta, \delta)\) restricted-enforcing \( a \), where \( \beta \) can be made arbitrarily small by choosing \( v \) and \( \delta \) close to \( w \) and \( \delta_0 \). Therefore, there exists a solution \( \mathcal{C} \) to \((5)\) with arbitrarily large curvature in a neighbourhood of \( w \), hence such a solution intersects \( \partial \mathcal{B}(r, \mathcal{W}) \) on both sides of \( w \). Any payoff on \( \mathcal{C} \) can be attained by a suitable solution to \((2)\) that remains in \( \mathcal{C} \cup \mathcal{B}(r, \mathcal{W}) \) until the arrival of the first event, at which point the continuation value jumps to \( \mathcal{W} \). This contradicts maximality of \( \mathcal{B}(r, \mathcal{W}) \). Therefore, \( N_w^\top (g(a) - w) = 0 \) for all action profile \( a \in \mathcal{A} \) that decompose \( w \), implying that \( \mathcal{G}(r, \mathcal{W}) \) consists of straight line segments and isolated points only. As a consequence, all corners of \( \mathcal{B}(r, \mathcal{W}) \) are contained in \( \mathcal{V}^N \subseteq \mathcal{B}(r, \mathcal{W}) \). See Lemma [C.6] and its corollaries for details.

Lemma [5.7] is also sufficient to show step (ii) above. Let \( w \in \mathcal{P}(r, \mathcal{W}) \), then by definition of \( \mathcal{P}(r, \mathcal{W}) \), the boundary of \( \mathcal{B}(r, \mathcal{W}) \) is a solution to \((5)\) on one side of \( w \).
Let $a^*, \phi^*$ and $\delta^*$ be the maximizers in (5) as $v$ approaches $w$. One can show that $N_w^\top(g(a^*) + \delta^*\lambda(a^*) - w) > 0$, and hence the continuation value of a strategy profile attaining $w$ may fall strictly into the interior of $\mathcal{B}(r, \mathcal{W})$. For $w$ to be in $\mathcal{B}(r, \mathcal{W})$, it is necessary that $a^*$ is restricted-enforceable in a neighbourhood of $w$. This is clearly satisfied under Assumption 2.(i) as $a^*$ is enforceable by $(T_w \phi, 0)$ due to Lemma 5.7.

Finally, Lemma 5.7 implies that any payoff in the interior of $\mathcal{B}(r, \mathcal{W})$ is attainable as discounted expected future payoff of an enforceable strategy profile with continuous incentive process $(\beta, 0)$. Public randomization is thus not needed to attain payoffs in the interior, and hence the characterizations in Theorem 5.2 and Proposition 5.6 are also valid in games without public randomization. In the proof of the following corollary, which is deferred to Appendix A.1, we deal with the fact that $\partial \mathcal{B}(r, \mathcal{W})$ may contain straight line segments.

**Corollary 5.8.** Under Assumptions 1–3, any payoff in $\mathcal{B}(r, \mathcal{W})$ is achievable without public randomization.

### 6 Computation

In this section, we illustrate how to numerically compute $\partial \mathcal{E}(r)$ with two examples where the characterization is new. We focus mainly on the case where $Y$ is non-empty and $\mathcal{E}(r)$ is computed with the algorithm in Lemma 5.5 as the continuous monitoring case is described in detail in Section 8 of Sannikov [13]. In Section 6.1 we illustrate how the algorithm of Lemma 5.5 converges to $\mathcal{E}(r)$ in the climate agreement example of Section 3.1. In Section 6.2 we present a partnership example with a one-dimensional public signal both when monitoring is continuous and when, in addition, infrequent events are observed. The characterization is new even in the continuous-monitoring case and $\partial \mathcal{E}(r)$ may have straight line segments that are precluded under the conditions in Sannikov [13]. When information contains both continuous and discontinuous information, Assumption 2 is not satisfied in this game and we show how to verify the candidate described in Section 5.3.

At payoffs where $\partial \mathcal{B}(r, \mathcal{W})$ is the solution to (5), the implementation works similarly to Sannikov [13], that is, the boundary of $\mathcal{B}(r, \mathcal{W})$ is parametrized in terms of the tangential angle $\theta$. Let $w(\theta)$ denote the set of payoffs in $\mathcal{B}(r, \mathcal{W})$ with normal vector $N(\theta) = (\cos(\theta), \sin(\theta))^\top$. Then $w(\theta)$ is unique where the curvature of $\partial \mathcal{B}(r, \mathcal{W})$ is strictly positive and one can solve

$$\frac{dw(\theta)}{d\theta} = \frac{T(\theta)}{\kappa(\theta)}$$

numerically, where $T(\theta) = (-\sin(\theta), \cos(\theta))^\top$ and $\kappa(\theta) = \kappa(w(\theta))$ is given by the optimality equation (5). Because $\mathcal{B}(r, \mathcal{W})$ is the largest bounded set with the curvature
given by (5), we search for an extremal pair of initial values \((\theta, w(\theta))\), for which (6) has a closed solution. We can thus search either for the extremal initial angle \(\theta\) for a fixed starting point \(u_0 \in \partial B(r, W)\), or for the extremal starting point \(w\) given a fixed initial angle \(\theta_0\). The former is practical if there exists a static Nash payoff \(g(a_e)\) in \(\partial V^*\). Because \(V^N \subseteq B(r, W) \subseteq V^*\), it follows that \(g(a_e) \in \partial B(r, W)\) and hence we know a possible starting point. The latter is useful if the game is symmetric, hence there has to be a point \(w \in \partial B(r, W)\) on the positive diagonal with \(\theta = \pi/4\).

### 6.1 Climate agreement

The climate agreement example of Section 3.1 is symmetric, hence we can search for a starting value \((w_0, \theta_0)\) on the positive diagonal with \(\theta_0 = \pi/4\). Because the game has a static Nash equilibrium at \((0.35, 0.35) \in \partial V^*\), a continuous solution of the optimality equation must reach \((0.35, 0.35)\) since \((0.35, 0.35) \in \partial B(r, W_n)\) for any \(n \geq 0\) as argued above. To find the largest of these solutions in the first iteration, we first perform a grid search on the positive diagonal and check whether a solution crosses the line \(\{w_2 = 0.35\}\) below or above \(w_1 = 0.35\). Because (5) is continuous in initial conditions, the starting value has to lie between 0.65 and 0.7 as illustrated in the left panel of Figure 6. Then we perform a binary search between 0.65 and 0.7 to find the exact starting value of 0.6593. Solving the optimality equation for starting value 0.6593, we obtain \(W_1\), the largest of the sets depicted in the right panel of Figure 6. An iteration of this procedure leads to the decreasing sequence \((W_n)_{n \geq 0}\), which converges to \(E(r)\). We stop the numerical iteration when we consider the difference in area between \(W_n\) and \(W_{n-1}\) as small enough.
Consider a simple partnership game amongst two players, where each player continuously chooses an effort level from $A_i = \{0, 1\}$ at every point in time $t$. Players observe only the total revenue $2X_t$, where $dX_t = \mu(A_t) dt + dZ^A_t$ for a $Q^A$-Brownian motion $Z^A$ and $\mu(a) = 4a^1 + 4a^2 - a^1a^2$. Suppose that players share the revenue equally and are subject to a cost of effort of $5a^i$ so that the expected flow payoff of player $i$ is $g^i(a) = 4(a^1 + a^2) - a^1a^2 - 5a^i$. Because Assumptions 1–3 are satisfied, $\mathcal{E}(r)$ is closed and hence contained in the payoff bound $\mathcal{M}$ from Sannikov and Skrzypacz [15]; see Figure 7. This speeds up the search for the straight line segments significantly.

The static Nash payoff is contained in $\partial \mathcal{M}$ and hence has to be contained in $\partial \mathcal{E}(r)$. However, since action profiles are not pairwise identifiable, $\partial \mathcal{E}(r)$ may contain straight line segments that we have to search for. From Proposition 5.6 we know that straight line segments go through $g(a)$ for some action profile $a$. Thus, $\partial \mathcal{E}(r)$ could have a straight line segment starting right at $\mathcal{V}^N$. To find the largest possible solution, we first perform a grid search for a starting value on edge $K$ of $\mathcal{M}$ as indicated in the right panel of Figure 7, where $\theta$ is such that $N(\theta)$ is normal to $K$. If the grid search is successful, we perform a binary search to find the exact starting value. If the grid search is not successful even after refining the grid size, then $\partial \mathcal{E}(r) \cap \partial \mathcal{M} = \mathcal{V}^N$, hence we perform a search over initial angles for fixed starting value $w_0 = (0, 0)$.

Next, suppose that revenue arrives continuously, but accidents may occur at a rate of $\lambda(a) = 21 - 4(a^1 + a^2) - 12a^1a^2$ that cost 0.1 each. The total revenue is given by $2X_t$ where $dX_t = \mu(A_t) dt + dZ^A_t - 0.05 dJ^t$ and $\mu(a) = 1.05 + 3.8(a^1 + a^2) - 1.6a^1a^2$ so that the expected flow payoff is the same as in the continuous monitoring example.
This information structure violates Assumption 2 and hence we need to verify that the largest closed solution $\tilde{B}(r, \mathcal{W}_n)$ to \eqref{5} coincides with $B(r, \mathcal{W}_n)$ for any $n \geq 0$ as elaborated in Section 5.3. We show this in the online appendix \cite{4} and focus here on the numerical implementation. We have to search jointly over initial angle and starting value to find $\tilde{B}(r, \mathcal{W})$ as illustrated in the left panel of Figure 8. We perform a search over the initial angles, say, $\theta_1, \ldots, \theta_m$, and for each $j = 1, \ldots, m$, we search for the starting value for initial angle $\theta_j$ to find the largest closed solution to \eqref{4}. If the curvature of the optimality equation ever equals 0, as in the point $A$ in the left panel of Figure 8, we perform a search on the tangent $T_A$ through $A$ to find the payoff furthest away from $A$, from which a solution to \eqref{1} connects to $\mathcal{V}^N$. This could potentially be a complicated procedure since at every payoff in $\tilde{G}(r, \mathcal{W})$, the set $\tilde{B}(r, \mathcal{W})$ could have a corner and one has to search jointly for initial angle and starting value again. In this specific example, the search is concluded when we find a continuous solution connecting to $B \in \tilde{G}(r, \mathcal{W})$ due to symmetry.

Figure 8 shows once more the drastic effect that abrupt information can have on the equilibrium payoff set. Because the action profile of mutual effort is not enforceable on the negative diagonal by observing the total revenue only, $E_c(r)$ remains below $L$ for any discount rate. This also follows form the payoff bound of Sannikov and Skrzypacz \cite{15}. What is new, however, is the fact the same bound holds when abrupt information is observed but players are sufficiently impatient: With the observation of the accidents, $(1, 1)$ can be enforced by burning 0.125 units of payoff of each player upon the arrival of an accident. However, this punishment is consistent with equilibrium behaviour only when $r \leq 15$. For higher discount rates, the induced
jump in the continuation value lands outside of \( \mathcal{V}^* \). This shows that \( \mathcal{E}(r) \) is not lower hemi-continuous in the discount rate when players observe abrupt information.

Let us discuss what happens when the continuation value of a PPE reaches a corner of \( \mathcal{E}(r) \). If it ever reaches a corner within the set of static Nash payoffs \( \mathcal{V}^N \), then it is absorbed there forever similarly to Sannikov [13]. At corners outside \( \mathcal{V}^N \), such as \( B \) in this example, the situation looks surprisingly similar. Because \( B = g(a) + \delta_0 \lambda(a) \) for the minimal amount of value burning \( \delta_0 \) necessary to enforce \( a \) in that direction, incentives \((0, \delta_0)\) are unique: no smaller amount of value burning enforces \( a \) and no larger amount of value burning has an inward-pointing (or zero) drift. Thus, the solution \( W \) to (2) is locally unique with drift rate \( w - g(a) - \delta_0 \lambda(a) = 0 \) and hence \( W \) remains in \( B \) until the arrival of the next event. Because we consider games of full support public monitoring, the event will occur eventually, after which play becomes dynamic again. Corners outside \( \mathcal{V}^N \) can thus be viewed as locally static Nash payoffs.

In the above partnership game, absorption in the static Nash equilibrium corresponds to the termination of the partnership. At the payoff \( B \), the partnership is going at its best. The sum of payoffs is maximized in equilibrium and players trust each other to exert effort without monitoring the continuous revenue. The reward level in this situation is so high that no player wishes to deviate, even though both players know an accident will arrive eventually, with potentially disastrous effects. For \( r = 15 \), the continuation value jumps to the static Nash payoff, which terminates the partnership.

7 Conclusion

This paper studies a class of continuous-time two-player games with imperfect public observation, where information may arrive both continuously through the observation of a noisy signal, and discontinuously as the occurrences of infrequent but informative events. For this class of games, we characterize the equilibrium payoff set \( \mathcal{E}(r) \) and show how to compute it efficiently. In the presence of abrupt information, this involves an algorithm that relies on a continuous-time analogue to the standard set-operator in Abreu, Pearce and Stacchetti [1]. The result is new even when the signal is continuous but one-dimensional and thereby vastly generalizes the class of games for which an explicit characterization of \( \mathcal{E}(r) \) is known.

The application of these games are numerous, as in many situations there is a very intuitive decomposition of the available information into continuous and discontinuous information. Moreover, there could be interesting generalizations to situations where instead of the frequency of the jumps, the size of the jumps depends on players actions. Such a situation would arise if firms can choose whether they disclose certain information continuously or only the accumulated information at intermediate time points. This work could thus lead to future research on information disclosure.
A Proofs of auxiliary results in the main text

A.1 Dynamics of the continuation value, incentives and public randomization

Proof of Lemma 4.7. The proof is similar to the proof of Lemma 1 in Bernard and Frei [5], with the following additional arguments for the jumps. Because \((J^y)_y \in Y\) are pairwise orthogonal and orthogonal to \(Z\), the stable subspace generated by \(Z\) and \((J^y)_y \in Y\) is the space of all stochastic integrals with respect to these processes (Theorem IV.36 in Protter [12]). Therefore, we obtain the unique martingale representation property for a square-integrable martingale by Corollary 1 to Theorem IV.37 in Protter [12]. That is, for a bounded \( \mathcal{F}_T \)-measurable \( c_T \), there exists an \( \mathcal{F}_0 \)-measurable \( c_T \), predictable processes \((\beta^i_t, \delta^i_t, \lambda_i(y)|A_t) dt\) for all \( y \in Y \) with \( \mathbb{E}^{Q^i_y}_A \left[ \int_0^T \beta^i_t, \delta^i_t, \lambda_i(y)|A_t \right] dt < \infty \) and \( \mathbb{E}^{Q^i_y}_A \left[ \int_0^T \delta^i_t, \lambda_i(y)|A_t \right] dt < \infty \) and a \( Q^i_T \)-martingale \( M^i \) orthogonal to \( Z \), \((J^y)_y \in Y\), with \( M^i_0 = 0 \) such that

\[
 w_T = c_T + \int_0^T \beta^i_t, \delta^i_t, \lambda_i(y)|A_t \right) dt + \sum_{y \in Y} \int_0^T \delta^i_t, \lambda_i(y)|A_t \right) dt + M^i_{T,T}.
\]

The remainder of the equivalence (i) \( \iff \) (ii) works analogously to the arguments in Bernard and Frei [5], with the following additional arguments for the jumps: Since \( \int r \delta^i(y)(dJ^y_t - dt) \) has bounded jumps by construction for any \( y \in Y \), it follows that \( \int_t^s r \delta^i(y)(dJ^y_t - \lambda(y)|A_t) ds \) is a BMO-martingale under \( Q^i_u \) up to any time \( u \in (t, \infty) \). Assumption 1 implies that the jumps of \( \lambda(y)|A_t - 1 \Delta J^y_t \) in Footnote 2 are bounded from below by \(-1 + \varepsilon\) for any \( y \in Y \). Therefore, Remark 3.3 and Theorem 3.6 in Kazamaki [10] imply that \( \int_t^s r \delta^i(y)(dJ^y_t - \lambda(y)|A_t) ds \) is a BMO-martingale under \( Q^i_u \).

Proof of Lemma 4.3. This works analogously to the second statement of Lemma 1 in Bernard and Frei [5].

Proof of Lemma 5.7. Let \( G^i(a) \) denote the row vector with entries \( g^i(\tilde{a}^i, a^i) - g^i(a) \) for \( \tilde{a}^i \in A^i \setminus \{a^i\} \). It is player \( i \)'s change in expected flow payoff by deviating from \( a^i \) to any other action. Then \((\beta, \delta)\) enforcing \( a \) is equivalent to the condition that for both players \( i = 1, 2 \),

\[
 G^i(a) + \beta^i M^i(a) + \delta^i \Lambda^i(a) \geq 0,
\]

where the inequality is understood componentwise. For the first statement, the best-response property for player \( i \) implies that \( a \) is enforceable by \((\beta, \delta)\) with \( \beta^i \) and \( \delta^i \) equal to 0. Since \( \delta^i \Lambda^i(a) \) is a linear combination of the row vectors in \( \Lambda^-i(a) \), Assumption 2(i) implies that there exists \( \tilde{\beta} \) with \( \tilde{\beta}^i Q^i(a) M^-i(a) = \delta^-i \Lambda^-i(a) \) and \( \tilde{\beta}^i \) = 0. Therefore, \((\beta + \tilde{\beta}^i Q^i(a), 0)\) enforces \( a \). For the second statement,
Assumption 2.(i) implies that for any \( \tilde{\delta} \), there exists \( \hat{\beta} \) such that \( \hat{\beta} Q^i(a) M^i(a) = (\tilde{\delta}^i - \delta^i) \Lambda^i(a) \) for \( i = 1, 2 \). It is straightforward to check that \( (T\tilde{\phi}, \tilde{\delta}) \) enforces \( a \), where

\[
\tilde{\phi} = \phi + \frac{1}{T^1} \hat{\beta} Q^1(a) + \frac{1}{T^2} \hat{\beta} Q^2(a).
\]

**Proof of Corollary 5.8.** We need to show that any payoff in \( B(r, W) \) is attainable by a solution \( W \) to (2) without public randomization. Lemma 5.7 shows that any action profile is enforceable with \( \delta = 0 \). Therefore, any payoff in the interior of \( B(r, W) \) can be attained by an enforceable strategy profile without jumps or public randomization until the continuation value reaches the boundary of \( B(r, W) \). On the boundary, it is clear that while such a solution is in extremal payoffs of \( B(r, W) \), it makes no use of public randomization. It thus remains to show that any payoff in the relative interior of a straight line segment \( L \subseteq \partial B(r, W) \) is attainable without public randomization such that an extremal payoff or the interior of \( B(r, W) \) is reached with certainty. This follows from another application of Lemma 5.7: Action profile \( a \) restricted-enforceable at an end point of \( L \) is restricted-enforceable orthogonal to that line segment with \( \delta = 0 \) by Lemma 5.7. Since for \( \delta = 0 \) the controls are location-independent, \( a \) is restricted-enforceable on all of \( L \). Play of the constant strategy profile \( A \equiv a \) with \( \delta \equiv 0 \) thus either enters the interior of \( B(r, W) \) if the drift is strictly inward pointing, or reaches an end point with certainty when the drift is parallel to \( L \). \[ \square \]

### A.2 Convergence of the Algorithm

In this appendix we prove the convergence of the algorithm in Lemma 5.5 to \( E(r) \). We start with a strategic definition of \( B(r, W) \). Because it is not a priori clear whether \( B(r, W) \) is closed or not, the condition that the jumps are directed towards the interior on \( \partial B(r, W) \) has to be defined over a limit as the distance to the boundary becomes smaller.

**Definition A.1.** For any \( \varepsilon > 0 \), \( r > 0 \) and \( W \subseteq \mathbb{R}^2 \), call a payoff set \( \tilde{W} \) locally \((\varepsilon, r, W)\)-admissible if for every payoff \( w \in \tilde{W} \), there exist a stopping time \( \tau \) and a solution \( (W, A, \beta, \delta, M, Z, (J^y)_{y \in Y}) \) to (2) with \( W_0 = w \) such that for \( 0 \leq t < \sigma \wedge \tau \),

(i) \( W_t \in \tilde{W} \),

(ii) \( (\beta_t, \delta_t) \) enforces \( A_t \),

(iii) \( W_t + r\delta_t(y) \in W \) for every \( y \in Y \), and

(iv) \( N^T \delta_t(y) \leq 0 \) for every \( y \in Y \) and every outward normal \( N \) to \( \partial \tilde{W} \) in \( B_\varepsilon(W_t) \).

We say that a payoff set is \((\varepsilon, r, W)\)-admissible if for every payoff \( w \) within the set, there exists a solution to (2) attaining \( w \) satisfying the above properties for \( \tau = \infty \). Denote by \( B_\varepsilon(r, W) \) the largest \((\varepsilon, r, W)\)-admissible payoff set. The following lemma and its corollary establish that this set is well defined.
Lemma A.2. Let $\mathcal{W}_1, \ldots, \mathcal{W}_n$ be locally $(\varepsilon, r, \mathcal{W})$-admissible and let $\overline{\mathcal{W}}$ denote the convex hull of $\mathcal{W}_1, \ldots, \mathcal{W}_n$. For any $\eta > 0$ and $k = 1, \ldots, n$, define

$$D_{k, \eta} := \mathcal{W}_k \cap B_\eta(\partial \mathcal{W}_k \cap \partial \overline{\mathcal{W}}).$$

If $\tau_k = \inf_{w \in D_{k, \varepsilon/2}} \sup_\mathcal{W} \inf \{ t \geq 0 \mid W_t \notin D_{k, \varepsilon/2} \}$, then $\tau_k > 0$ a.s., where the supremum is taken over all solutions to (2) attaining $w$ satisfying properties 1–4 in the definition of local admissibility. If $\tau_k = 0$ a.s., then $\overline{\mathcal{W}}$ is $(\varepsilon, r, \mathcal{W})$-admissible.

Corollary A.3. If $\mathcal{W}_1, \mathcal{W}_2$ are $(\varepsilon, r, \mathcal{W})$-admissible, so is the convex hull $\text{co}(\mathcal{W}_1, \mathcal{W}_2)$.

Proof of Lemma A.2. We show that any payoff in $w \in \overline{\mathcal{W}}$ can be attained by a suitable solution to (2). Observe that any payoff outside of $D_{\varepsilon/2} := \bigcup_{k=1}^n D_{k, \varepsilon/2}$ is attainable by a public randomization device at time 0 with values in $D_{1, \varepsilon/2}, \ldots, D_{n, \varepsilon/2}$. We may thus assume that $w \in D_{k, \varepsilon/2}$ for some $k$. By local admissibility, there exists a solution $(\mathcal{W}, A, \beta, \delta, M, Z, (\check{y}_y)_{y \in Y})$ to (2) starting in $w$ such that on $[0, \sigma \lor \tau_k]$, $w \in \mathcal{W}_k$, $\beta$ enforces $A$ and $\delta(y) \in \partial\mathcal{W}(\check{W}) \cap H(N)$ for normal vectors $N$ to $\partial \mathcal{W}_k$ within distance $\varepsilon$ of $\check{W}$. This solution also satisfies the inward-jump condition with respect to $\partial \overline{\mathcal{W}}$ outside of the set $\mathcal{H} := \bigcup_{k=1}^n \mathcal{H}_k$, where

$$\mathcal{H}_k := \mathcal{W}_k \cap B_\varepsilon(\partial \overline{\mathcal{W}}) \setminus D_{k, \varepsilon}$$

see Figure 9 for an illustration of $\mathcal{H}$. The main idea of the proof is to concatenate solutions to (2) that exist by local admissibility and to use public randomization whenever the solution reaches $\mathcal{H}$. We do this in two steps.

Step 1: We show that if $\tau_k < \infty$ with positive probability, it is possible to find concatenations of solutions to (2) that satisfy the inward-jump condition until either an event occurs or the continuation value reaches $\mathcal{H}$. Indeed, on the set $\{ W_{t_k} \notin D_{k, \varepsilon/2} \}$ the payoff $W_{t_k}$ is attainable by a public randomization device at time $t_k$ with values in $D_{k, \varepsilon/2}$. We may thus assume that $W_{t_k} \in D_{j, \varepsilon/2}$ for some $j \in \{1, \ldots, n\}$ without loss of generality. Therefore, there exists a solution $(\check{W}, \tilde{A}, \tilde{\beta}, \tilde{\delta}, \check{M}, \check{Z}, (\check{y}_y)_{y \in Y})$ to (2) attaining $W_{t_k}$ such that on $[0, \sigma \lor \tau_j]$, $\check{W} \in \mathcal{W}_j$, $\tilde{\beta}$ enforces $A$, $\check{W} + r\tilde{\delta}(y) \in \mathcal{W}$ for every $y \in Y$ and $N^T\tilde{\delta}(y) \leq 0$ for every $y \in Y$ and every outward normal $N$ to $\partial \mathcal{W}_j$ within $B_\varepsilon(\check{W})$. Define the concatenations $\check{Z}_t := \check{Z}_{t \lor t_k} \lor \check{Z}_{t \lor t_k} \mathbf{1}_{\{ t > t_k \}}$ and similarly for $\check{M}$ and $(\check{y}_y)_{y \in Y}$. Moreover, set $\check{W}_t := W_{t \lor t_k} \lor \check{W}_{t \lor t_k} \mathbf{1}_{\{ t > t_k \}}$ and define $\tilde{A}$, $\tilde{\beta}$ and $\tilde{\delta}$ analogously to $\check{W}$. Then, the tilde-processes are a solution to (2) up to a stopping
time \( \tilde{\tau} \) such that \( \tilde{\tau} - \tau_k \) is identically distributed as \( \tau_j \) and on \([0, \tilde{\tau}]\), \((\tilde{\beta}, \tilde{\delta})\) enforces \( \tilde{A} \), \( \tilde{W} + r(\tilde{y}) \in W \) for every \( y \in Y \) and \( N^T\tilde{y} \leq 0 \) for every \( y \in Y \) and every outward normal \( N \) to \( \partial W \) within \( B(\tilde{W}) \) where \( \tilde{W} \notin H \). Since \( \tau_1, \ldots, \tau_n \) are strictly positive, so is \( \tau_0 := \min(\tau_1, \ldots, \tau_n) \). By repeating this procedure, we extend the solution on a time interval of at least \( \tau_0 \) for every repetition. A countable repetition thus extends to infinity with probability 1 by Lemma 11 in Bernard and Frei [5]. We have thus constructed a concatenation of solutions that satisfies the inward-jump condition with respect to \( \partial W \) outside of \( H \).

Step 2: Let \( \rho := \inf\{t \geq 0 \mid W_t \in \mathcal{H}\} \). Since \( \mathcal{H} \) is bounded away from \( \partial W \), \( W_\rho \) is attainable by a public randomization device with values in some finite set \( \{w_1, \ldots, w_m\} \subseteq D_{\varepsilon/2}^r \). For each \( w_\ell \), denote by \((W^{\ell}, \mathcal{A}^{\ell}, \alpha^{\ell}, \delta^{\ell}, M^{\ell}, Z^{\ell}, (J^{\ell}y)_{y \in Y}) \) the concatenation of solutions to (2) obtained through step 1 above. Observe that none of these solutions reach \( \mathcal{H} \) again before time \( \tau_0 \) by assumption. Thus, concatenating independent copies of solutions attaining \( w_1, \ldots, w_n \) yields a solution on \([0, \infty)\) again by Lemma 11 in Bernard and Frei [5].

Finally, define \( B(r, W) := \bigcup_{\varepsilon > 0} B_\varepsilon(r, W) \). Note that \( B_\varepsilon(r, W) \) is monotone in \( \varepsilon \). Indeed, if the jumps are inward-pointing within distance \( \varepsilon \) of the boundary, they are also inward-pointing within distance \( \varepsilon' < \varepsilon \). As a consequence, \( B(r, W) \) is convex as the limit of a non-decreasing sequence \( B_{1/n}(r, W) \) of convex sets. We first show that \( B \) is monotonic in the set-inclusion sense.

**Lemma A.4.** Let \( W \subseteq W' \). Then \( B(r, W) \subseteq B(r, W') \).

**Proof.** By definition, any payoff \( w \in B(r, W) \) can be attained by an enforceable strategy profile with continuation value \( W \) such that \( W \in B(r, W) \) up to time \( \sigma \) with \( W_\sigma \in W \). Moreover, if \( W_{\sigma-} \in \partial B(r, W) \) then \( W_\sigma \in H(W_{\sigma-}, N_{W_{\sigma-}}) \). Since \( W \subseteq W' \) implies that \( W_\sigma \in W' \) or \( W_\sigma \in W' \cap H(W_{\sigma-}, N_{W_{\sigma-}}) \), respectively, hence \( B(r, W) \subseteq B(r, W') \) by maximality of \( B(r, W') \).

We are now ready to prove Lemmas [5.4] and [5.5].

**Proof of Lemma [5.4].** Fix \( w \in B(r, W) \) arbitrary. By definition, there exists an enforceable strategy profile \( A \) attaining \( w \) such that \( W \) remains in \( B(r, W) \) on \([0, \sigma]\) and \( W_\sigma \in W \) a.s. This implies that \( W_\sigma(A) \in B(r, W) \), hence there exists an enforceable strategy profile \( A \) attaining \( W_\sigma(A) \) such that \( W \) remains in \( B(r, W) \) up to the second jump time \( \sigma_2 \) with \( W_{\sigma_2} \in W \) a.s. Therefore, the concatenation \( \hat{A} := A_1[0, \sigma] + A_{\sigma-1}[\sigma, \infty) \) is enforceable and remains in \( B(r, W) \) up to \( \sigma_2 \). Because Poisson processes have only finitely many jumps on any finite time interval and \( Y \) is finite, a countable iteration of this procedure will lead to an enforceable strategy profile, whose continuation value remains in \( B(r, W) \) forever. This shows that \( B(r, W) \) is self-generating.
For the second statement, observe that self-generation implies that for any \( w \in W \), there exists an enforceable strategy profile with continuations that remain in \( W \). In particular, \( W_{\sigma} \in W \) a.s., and by convexity, at the boundary the jump cannot be directed outwards. Therefore, \( W \subseteq B(r, W) \) by maximality of \( B(r, W) \).

**Proof of Lemma 5.5.** Since \((W_n)_{n \geq 0}\) is decreasing and bounded from below by the empty set, it must converge and its limit satisfies \( W_\infty = B(r, W_\infty) \). Because this implies that \( W_\infty \) is self-generating by Lemma 5.4, it follows from Lemma 4.5 that \( W_\infty \subseteq E(r) \). To show that also \( E(r) \subseteq W_\infty \), observe that \( E(r) \subseteq W_n \) for some \( n \geq 0 \) implies \( E(r) \subseteq B(E(r)) \subseteq B(r, W_n) = W_{n+1} \) by self-generation and Lemmas 5.4 and A.4. Since \( E(r) \subseteq W_0 \) it follows that \( E(r) \) is contained in \( W_n \) for every \( n \geq 0 \) and hence \( W_\infty = E(r) \).

**B Regularity of the optimality equation**

The purpose of this appendix is to prove that the optimality equation is locally Lipschitz continuous at almost every point, so that locally, it admits a unique solution. We show in Lemma C.5 that \( \partial B(r, W) \setminus G(r, W) \) is \( C^1 \), hence \( \partial B(r, W) \setminus G(r, W) \) is the unique \( C^1 \) solution to the optimality equation. For any fixed \( r > 0, a \in A \), and closed and convex \( W \subseteq V \), consider the optimality equation in the following form:

\[
\kappa_a(w, N) = \max_{(\phi, \delta) \in \Xi_a(w, N, r, W)} \frac{2N^\top (g(a) + \delta \lambda(a) - w)}{r \|\phi\|^2}.
\]

(7)

We start by reducing the two-variable optimization problem to a one-variable optimization by expressing the control \( \phi \) in terms of \( \delta \). For player \( i = 1, 2 \), define

\[
I_i^a(N, \delta) := \{ \phi \in \mathbb{R}^d \mid (T_i^\phi, \delta^i) \text{ satisfies } (3) \text{ for player } i \}
\]

for any direction \( N \in S^1 \) and \( \delta^i \in \mathbb{R}^{|Y_i|} \). Because \( I_i^a(N, \delta^i) \) is the intersection of closed half-spaces, it is a (possibly unbounded or empty) closed convex polytope. Therefore, so is \( \Phi_a(N, \delta) := I_1^a(N, \delta_1) \cap I_2^a(N, \delta_2) \), the set of all vectors \( \phi \in \mathbb{R}^d \) such that \( (T\phi, \delta) \) enforces \( a \). Let \( \phi(a, N, \delta) \) denote the vector of smallest length in \( \Phi_a(N, \delta) \).

**Lemma B.1.** Fix \( a \in A \). Then \( (N, \delta) \mapsto \phi(a, N, \delta) \) is locally Lipschitz continuous where \( \Phi_a(N, \delta) \neq \emptyset \) and \( N \) is different from a coordinate direction.

In an intermediate step, we will show that the set-valued map \( (N, \delta) \mapsto \Phi_a(N, \delta) \) is locally Lipschitz continuous for \( N \) different from coordinate directions. We refer to Aubin and Frankowska [3] for a detailed overview of set-valued maps and their properties and state here only the most central property.

**Definition B.2.** A set-valued map \( G : x \mapsto G(x) \) is said to be Lipschitz continuous if \( G(x) \subseteq G(x^\ast) + K \|x - x^\ast\|B_1(0) \) for some constant \( K \).
Proof of Lemma B.1. Let \( \mathcal{I}_i(\delta^i) := \{ \beta \in \mathbb{R}^d \mid (\beta, \delta^i) \text{ satisfies } \textbf{(3)} \text{ for player } i \} \) be the solution set to \( \textbf{(3)} \) for player \( i \) and observe that it is a closed convex polytope. Its hyperfaces have normal vectors \( \Delta \mu_j^i := \mu(a) - \mu(a_j^i, a^{-i}) \), where \( a_1, \ldots, a_m^i \) is an enumeration of \( A^i \setminus \{ a^i \} \). The parameter \( \delta^i \) determines the location of these hyperfaces. Observe that a change from \( \delta^i \) to \( \bar{\delta}^i \) shifts face \( j_i \) by \( (\bar{\delta}^i - \delta^i) \Delta \lambda_j^i \), where \( \Delta \lambda_j^i := \lambda(a) - \lambda(a_j^i, a^{-i}) \). Therefore, the triangle inequality implies that

\[
\mathcal{I}_i(\delta^i) \subseteq \mathcal{I}_i(\bar{\delta}^i) + B_1(0) \sum_{j_i=1,\ldots,m_i} \| \Delta \lambda_j^i \| \| \bar{\delta}^i - \delta^i \|
\]

i.e., \( \mathcal{I}_i(\delta^i) \) is Lipschitz continuous in \( \delta^i \). It is clear that \( \mathcal{I}_i(N, \delta^i) = \frac{1}{T^i} \mathcal{I}_i(\delta^i) \) for \( i = 1, 2 \) is locally Lipschitz continuous in \( (N, \delta^i) \) for \( N \) different from coordinate directions. Since the stretching does not affect the direction of the normal vectors, the normal vectors of \( \mathcal{I}_i(N, \delta^i) \) are constant, hence \( (N, \delta) \mapsto \Phi_a(N, \delta) = \mathcal{I}_i(N, \delta^1) \cap \mathcal{I}_i(N, \delta^2) \) is locally Lipschitz continuous by Lemma B.1. The statement now follows from the following Lemma.

Lemma B.3. Let \( f(x,y) \) be a single-valued Lipschitz continuous map and let \( G(x) \) be a set-valued (locally) Lipschitz continuous map. Then \( h(x) = \max_{y \in G(x)} f(x,y) \) is (locally) Lipschitz continuous.

Proof. For any \( x \), let \( U \) be a neighbourhood of \( x \) such that \( G \) is Lipschitz continuous on \( U \) with Lipschitz constant \( K_G \). Let \( x_1, x_2 \in U \) and suppose without loss of generality that \( h(x_1) \geq h(x_2) \). Let \( K_f \) be the Lipschitz constant of \( f \). Then

\[
h(x_1) - h(x_2) \leq K_f \| x_2 - x_1 \| + \max_{y \in G(x_1)} f(x_2, y) - \max_{y \in G(x_2)} f(x_2, y)
\]

\[
\leq K_f \| x_2 - x_1 \| + \max_{y \in G(x_2) + K_G \| x_2 - x_1 \| B_1(0)} f(x_2, y) - \max_{y \in G(x_2)} f(x_2, y)
\]

\[
\leq K_f \| x_2 - x_1 \| + K_f K_G \| x_2 - x_1 \|. \quad \Box
\]

Lemma B.1 significantly simplifies the constraints in the maximization in \( \textbf{(7)} \) because we are left with a maximization over \( \delta \) only. We subsequently characterize the set of all eligible \( \delta \), over which the maximization takes place. For any \( \varepsilon \geq 0 \) and any set-valued map \( w \mapsto \mathcal{D}(w) \subseteq \mathbb{R}^2 \), define

\[
\Psi_a^\varepsilon(w, N, r, \mathcal{D}) := \left\{ \delta \in \mathbb{R}^{2\times|Y|} \mid \Phi_a(N, \delta) \neq \emptyset, N^\top (g(a) + \delta \lambda(a) - w) \geq 0, \right. \\
\left. \delta(y) \in \mathcal{D}(w) \cap H(\tilde{N}) \forall \tilde{N} \in B_\varepsilon(N), \forall y \in Y \right\}
\]

so that \( \textbf{(7)} \) is equivalent to

\[
\kappa_a(w, N) = \max_{\delta \in \Psi_a^\varepsilon(w, N, r, \mathcal{D})} \frac{2N^\top (g(a) + \delta \lambda(a) - w)}{r \| \phi(a, N, \delta) \|^2}. \quad (8)
\]

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for $\varepsilon = 0$ and $D(w) = (W - w)/r$. In Appendix C we will be needing regularity of the optimality equation for different choices of $\varepsilon$ and $w \mapsto D(w)$, which is why we do not limit ourselves to $(W - w)/r$ in this appendix. We say a map $w \mapsto D(w) \subseteq \mathbb{R}^2$ is of class $B$ if it is affine, convex- and compact-valued with a uniform bound for $w \in V$.

**Remark B.1.** Note here that in the definition of $\Psi^\varepsilon$, the dependency on normal vectors close to $N$ differs from the definition of $B_\varepsilon(r, W)$. For $w \in B_\varepsilon(r, W)$, we require that $N^T\delta(y) \leq 0$ for all $N \in \partial B_\varepsilon(r, W) \cap B_\varepsilon(w)$, whereas in $\Psi^\varepsilon$ we choose a definition that depends neither on the boundary of a certain set, nor on the location $w$. This greatly simplifies establishing regularity properties for $\Psi^\varepsilon$. The two definitions are easily related, however, for $C^1$ parts of the boundary $C \subseteq \partial B_\varepsilon(r, W)$. Then there exists $\eta(C, \varepsilon) > 0$ such that for $w \in C$, $N \in C \cap B_\varepsilon(w)$ implies that $N \in B_{\eta(C, \varepsilon)}(N_w)$.

**Lemma B.4.** Let $w \mapsto D(w)$ be of class $B$. Then for any $a \in A$ and $\varepsilon \geq 0$, the map $(w, N) \mapsto \Psi^\varepsilon(w, N, r, D)$ is compact- and convex-valued. Moreover, it is locally Lipschitz continuous for $N$ different from coordinate directions and for $\varepsilon$ such that extremal vectors in $B_\varepsilon(N) \cap S^1$ are different from coordinate directions.

**Proof.** Identify $\mathbb{R}^{2 \times |V|}$ with $\mathbb{R}^{2|V|}$ by setting $\delta \approx (\delta^1, \delta^2)$. For any subset $W$ of $\mathbb{R}^2$, define $W^{|V|} := \{ (\delta^1, \delta^2) \in \mathbb{R}^{2|V|} \mid (\delta^1(y), \delta^2(y))^T \in W \forall y \in Y \}$. Let $\Psi^\varepsilon_a(w, N)$ and $J_a(N)$ denote the set of all $\delta$, for which $N^T(g(a) + \delta \lambda(a) - w) \geq 0$ and $\Phi^\varepsilon_a(N, \delta) \neq \emptyset$, respectively, are satisfied. We begin by showing that $J_a(N)$ is closed and convex, hence so is $\Psi^\varepsilon_a(w, N, r, D(w)) = J_a(N) \cap \Psi^\varepsilon_a(w, N) \cap D(w)^{|V|} \cap H_\varepsilon(N)^{|V|}$ as intersection of such sets, where $H_\varepsilon(N) = \bigcap_{N \in B_\varepsilon(N)} H(N)$. Indeed, let $\delta_1, \delta_2 \in J_a(N)$. Then there exist $\phi_1, \phi_2$ such that $(\delta_j, T\phi_j)$ for $j = 1, 2$ satisfy (3) for every $\bar{a} \in A \setminus \{a^i\}$ and $i = 1, 2$. By linearity of (3), so does $(\delta_\nu, T\phi_\nu)$ for $\nu \in [0, 1]$, where we set $\delta_\nu := \nu\delta_1 + (1 - \nu)\delta_2$ and $\phi_\nu := \nu\phi_1 + (1 - \nu)\phi_2$. This shows that $\delta_\nu \in J_a(N)$, i.e., $J_a(N)$ is convex. Let $(\delta_n)_{n \geq 0}$ be a sequence in $J_a(N)$. Then there exists $\phi_n$ such that $(\delta_n, T\phi_n)$ satisfies (3). Since the inequalities in (3) are not strict, $\lim_{n \to \infty} \delta_n = \lim_{n \to \infty} \phi_n$ satisfies (3), hence $\lim_{n \to \infty} \delta_n \in J_a(N)$ and $J_a(N)$ is closed. Compactness of $\Psi^\varepsilon_a(w, N, r, D(w))$ now follows from boundedness of $D(w)^{|V|}$.

For $\phi \in \mathbb{R}^d$, introduce the auxiliary sets $J_a(N, \phi)$ of those $\delta \in \mathbb{R}^{2|V|}$, for which $(T\phi, \delta)$ enforces $a$. For $i = 1, 2$, let $a_1^i, \ldots, a_m^i$ be an enumeration of $A \setminus \{a^i\}$ and abbreviate $\Delta \mu^i := \mu(a) - \mu(a_1^i, a^{-i})$ and $\Delta \lambda^i := \lambda(a) - \lambda(a_1^i, a^{-i})$ as in the proof of Lemma B.1. Then $J_a(N, \phi)$ is a closed convex polytope, whose hyperfaces have normal vectors

$$
\begin{pmatrix}
\Delta \lambda_j^1 \\
0
\end{pmatrix}, \quad j_1 = 1, \ldots, m_1, \quad \begin{pmatrix}
0 \\
\Delta \lambda_j^2
\end{pmatrix}, \quad j_2 = 1, \ldots, m_2
$$

(9)

and $N$ only determines the position of these hyperfaces. Thus, similarly as in the proof of Lemma B.1, $N \mapsto J_a(N, \phi)$ is Lipschitz continuous with a Lipschitz constant that
depends only on $\Delta \mu^i_{\phi}$. In particular, the Lipschitz constant of $N \mapsto \mathcal{J}_a(N, \phi)$ is uniformly bounded in $\phi$.

Observe that $w \mapsto \Psi_a(w, N)$ is an affine function, and so are the constant maps $w \mapsto H_\varepsilon(N)^{|Y|}$ and $w \mapsto \mathcal{J}_a(N)$. Lipschitz continuity in $w$ thus follows from Lemma ???. For Lipschitz-continuity in $N$, observe that $H_\varepsilon(N)$ is the intersection of two half-spaces with extremal normal vectors $N_-, N_+$ in $B_\varepsilon(N) \cap S^1$, where in the case $\varepsilon = 0$, $N_- = N_+ = N$. In particular, $H_\varepsilon(N)^{|Y|}$ is a convex polytope with normal vectors $(N^1 e_y, N^2 e_y)^\top$ and $(N^1_+ e_y, N^2_+ e_y)^\top$, where $e_y \in \mathbb{R}^{|Y|}$ is the row unit vector in coordinate $y$. The normal vector $-(N^1 \lambda(a)^\top, N^2 \lambda(a)^\top)^\top$ of $\Psi_a(w, N)$ is a linear combination of all normal vectors of $H_\varepsilon(N)^{|Y|}$ for all $N \in S^1$ since $N = k(\varepsilon)(N_+ + N_-)$ for some constant $k(\varepsilon)$. Observe however, that $-(N^1 \lambda(a)^\top, N^2 \lambda(a)^\top)^\top$ is not a linear combination of a proper subset of normal vectors of $H_\varepsilon(N)^{|Y|}$ because $\lambda(y|a) > 0$. Therefore, the ranks of the matrices formed by any combination of normal vectors of $H_\varepsilon(N)^{|Y|}$ and $\Psi_a(w, N)$ are constant for all $N \in S^1$ as required by Lemma ???. While $H_\varepsilon(N)^{|Y|}$ and $\Psi_a(w, N)$ may fail to be Lipschitz continuous because they are unbounded, their intersection with a constant and bounded polytope is. Let $\bar{\mathcal{W}}$ be such a polytope containing $\mathcal{D}(w)^{|Y|}$ such that none of its normal vectors are arbitrarily close to being linearly dependent to any $2|Y| - 1$ normal vectors of $H_\varepsilon(N)^{|Y|}$, $\Psi_a(w, N)$ or any of the vectors in $[9]$. Then $H_\varepsilon(N)^{|Y|} \cap \bar{\mathcal{W}}$ and $\Psi_a(w, N) \cap \bar{\mathcal{W}}$ satisfy the conditions in Lemma ?? and hence $N \mapsto H_\varepsilon(N)^{|Y|} \cap \Psi_a(w, N) \cap \bar{\mathcal{W}}$ is Lipschitz continuous. For any $\phi \in \mathbb{R}^d$, $\mathcal{J}_a(N, \phi)$ is a polytope with normal vectors in $[9]$. Let $N$ be different from a coordinate direction and suppose that $\varepsilon$ is such that neither $N_-$ or $N_+$ are coordinate directions. Let $X(N)$ be any subset of normal vectors to $H_\varepsilon(N)^{|Y|}$, $\Psi_a(w, N)$ and $\mathcal{J}_a(N, \phi)$. If there exists a linear combination amongst the vectors in $X(N)$, then there exists a linear combination also in $X(N)$ for $N$ arbitrarily close to $N$ by multiplying the coefficients by $\bar{N}_i / N_i$ or $\bar{N}_i / N_i$ and $\bar{N}_i / N_i$, respectively. Therefore, Lemma ?? applies and shows that $N \mapsto H_\varepsilon(N)^{|Y|} \cap \Psi_a(w, N) \cap \mathcal{J}_a(N, \phi) \cap \bar{\mathcal{W}}$ is locally Lipschitz continuous in $N$ except for coordinate directions. Since $\mathcal{D}(w)^{|Y|} \subseteq \bar{\mathcal{W}}$ and the intersection of a Lipschitz continuous map with a convex set is Lipschitz continuous, it follows that for any $\phi \in \mathbb{R}^d$, $N \mapsto H_\varepsilon(N)^{|Y|} \cap \Psi_a(w, N) \cap \mathcal{J}_a(N, \phi) \cap \mathcal{D}(w)^{|Y|}$ is Lipschitz continuous. Local Lipschitz continuity of $N \mapsto \Psi_\varepsilon^a(w, N, r, D)$ now follows from the fact that the arbitrary union of Lipschitz continuous maps with uniformly bounded Lipschitz constants is Lipschitz again.

So far we have shown that $[8]$ is locally Lipschitz continuous for almost every direction $N$, where $\phi(a, N, \delta)$ is well defined and bounded away from 0. Define

$$E_a^r(r, D) := \{ (w, N) \in \mathbb{R}^2 \times S^1 \mid \Psi_a^\varepsilon(w, N, r, D) \neq \emptyset \}$$

$$\Gamma_a^r(r, D) := \{ (w, N) \in \mathbb{R}^2 \times S^1 \mid \exists \delta \in \Psi_a^\varepsilon(w, N, r, D) \text{ with } \phi(a, N, \delta) = 0 \}$$

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and $\Gamma_\varepsilon(r, \mathcal{D}) := \bigcup_{a \in A} \Gamma_\varepsilon^a(r, \mathcal{D})$. Denote by $\mathcal{P} := \mathbb{R}^2 \times \{ \pm e_1, \pm e_2 \}$ the set of points $(w, N) \in \mathbb{R}^2 \times S^1$ with a coordinate normal vector $N$.

**Lemma B.5.** Let $\varepsilon \geq 0$ and let $\mathcal{D}$ be of class $B$. If a sequence $(w_n, N_n)_{n \geq 0}$ converges to $(w, N) \notin \mathcal{P}$ such that $\Psi_\varepsilon^a(w_n, N_n, r, \mathcal{D}) \neq \emptyset$ for all $n \geq 0$, then $\Psi_\varepsilon^a(w, N, r, \mathcal{D}) \neq \emptyset$.

*Proof.* Let $\delta_n \in \Psi_\varepsilon^a(w_n, N_n, r, \mathcal{D})$. Because $\mathcal{D}(w_n)$ is uniformly bounded, so is $\delta_n$, hence $(\delta_n)_{n \geq 0}$ converges along a subsequence $(n_k)_{k \geq 0}$ to some finite limit $\delta$ with $N^T(g(a) + \delta \lambda(a) - w) \geq 0$ and $\tilde{N}^T \delta(y) \leq 0$ for every $\tilde{N} \in B_\varepsilon(N) \cap S^1$ and $y \in Y$. Since $\mathcal{D}$ is closed-valued and Lipschitz continuous, $\delta(y) \in \mathcal{D}(w)$ for every $y \in Y$. It remains to show that $\Phi_a(N, \delta) \neq \emptyset$. Suppose towards a contradiction that the converse is true. Then closedness of $\mathcal{I}_a^i(N, \delta^i)$ for $i = 1, 2$ implies that $\mathcal{I}_a^1(N, \delta^1)$ and $\mathcal{I}_a^2(N, \delta^2)$ are strictly separated. By continuity, $\mathcal{I}_a^1(N_{\ell_k}, \delta_{\ell_k}^1)$ and $\mathcal{I}_a^2(N_{\ell_k}, \delta_{\ell_k}^2)$ are separated as well for $k$ sufficiently large, a contradiction. \hfill $\square$

**Corollary B.6.** For any $a \in A$ and $\varepsilon \geq 0$, $E_\varepsilon^a(r, \mathcal{D}) \cup \mathcal{P}$ and $\Gamma_\varepsilon^a(r, \mathcal{D})$ are closed. Therefore, so is $\Gamma_\varepsilon^a(r, \mathcal{D})$.

*Proof.* Indeed, $\Gamma_\varepsilon^a(r, \mathcal{D})$ is closed since $0 \in \Phi_a(N, \delta)$ for some $N \in S^1$ if and only if $0 \in \Phi_a(N, \delta)$ for all $N \in S^1$. \hfill $\square$

**Proposition B.7.** For $\varepsilon \geq 0$ and $w \mapsto \mathcal{D}(w)$ of class $B$,

$$
\kappa(w, N) = \max_{a \in A} \max_{\delta \in \Psi_\varepsilon^a(w, N, r, \mathcal{D})} \frac{2N^T(g(a) + \delta \lambda(a) - w)}{r \|\phi(a, N, \delta)\|^2}
$$

(10)

is locally Lipschitz continuous outside of $\Gamma_\varepsilon^a(r, \mathcal{D})$, except where $(w, N)$ leaves or enters $E_\varepsilon^a(r, \mathcal{D})$ of the maximizing action profile $a$. Here, we interpret $\kappa(w, N) = 0$ on $\bigcap_{a \in A} E_\varepsilon^a(r, \mathcal{D})^c$, i.e., where the maxima are taken over empty sets.

*Proof.* We first show local Lipschitz continuity of $\kappa_a$ in (8) for fixed $a \in A$. Suppose first that $N$ is not a coordinate direction, that is, $(w, N) \in E_\varepsilon^a(r, \mathcal{D}) \setminus (\Gamma_\varepsilon^a(r, \mathcal{D}) \cup \mathcal{P})$. Since $\Gamma_\varepsilon^a(r, \mathcal{D})$ is closed by Corollary B.6, there exists an open neighbourhood $U$ of $(w, N)$ bounded away from $\Gamma_\varepsilon^a(r, \mathcal{D}) \cup \mathcal{P}$. Therefore, $\inf_{N, \delta} \|\phi(a, N, \delta)\| \geq c$ and hence the function that is maximized in the right hand side of (8) is Lipschitz continuous on $U$ by Lemma B.1. It follows that $\kappa_a$ is Lipschitz continuous by Lemmas B.3 and B.4. Because (10) is the maximum over finitely many functions $\kappa_a$, it is Lipschitz continuous except where $(w, N)$ leaves the domain of the maximal function $\kappa_a$. \hfill $\square$

When we refer to a solution to (10), we will always mention explicitly with respect to what $\varepsilon$ and which map $\mathcal{D}$ (10) is being solved.

**Lemma B.8.** Let $a \notin A^N$ have the unique best response property for player $i$ with $\kappa_a(w, N) > 0$ for $(w, N) \in \mathcal{P}$. Then for every $\varepsilon \in (0, \kappa_a(w, N))$ there exists a neighbourhood $U$ of $(w, N)$ such that $\kappa_a(\bar{w}, \bar{N}) \geq \varepsilon$ for any $(\bar{w}, \bar{N}) \in U$. 33
Proof. Let \( N = \pm e_i \). Let \( \beta \) be the vector with minimal length such that \( (\beta, 0) \) enforces \( a \). Observe that such a vector exists by Lemma 5.7. Moreover, \( \beta \neq 0 \) since \( a \notin A^N \). Since \( 0 \in \text{int} \mathcal{I}_a(0) \), it follows that \( (\beta/\tilde{T}^{-i}, 0) \) restricted-enforces \( a \) at \( (\tilde{w}, \tilde{N}) \) in a neighbourhood of \( (w, N) \). Let \( K \) be the Lipschitz constant of

\[
\tilde{k}_a(\tilde{w}, \tilde{N}) := \frac{2\tilde{N}^\top(g(a) - \tilde{w})}{r\|\beta\|^2}(\tilde{T}^{-i})^2.
\]

Then for every \((\tilde{w}, \tilde{N}) \in B_{\epsilon/K}(w, N)\) it follows that \( \epsilon \leq \tilde{k}_a(\tilde{w}, \tilde{N}) \leq k_a(\tilde{w}, \tilde{N}) \). \( \square \)

### C Characterization of \( \partial \mathcal{B}(r, \mathcal{W}) \)

We start by showing that \( \partial \mathcal{B}(r, \mathcal{W}) \setminus \mathcal{G}(r, \mathcal{W}) \) is given by the optimality equation. Because the continuous part of the signal is what creates the curvature, these steps are similar in ideas to Sannikov [13]. Some technical bounds on the provision of incentives and proximity of solutions to (10) for different choices of \( \mathcal{D} \) are deferred to Appendix ???. The first lemma asserts that for any solution to (10), there exists an enforceable strategy profile such that its continuation value remains on the curve until an end point is reached or a rare event occurs.

**Lemma C.1.** For \( \epsilon \geq 0 \) and \( \mathcal{D} \) of class \( B \), let \( \mathcal{C} \) be a \( C^1 \) solution to

\[
\kappa(w) = \frac{2N_w^\top(g(a^*(w)) + \delta^*(w)\lambda(a^*(w)) - w)}{r\|\phi(a^*(w), N_w, \delta^*(w))\|^2} \tag{11}
\]

oriented by \( w \mapsto N_w \), where \( a^*, \delta^* \) are such that for every \( w \) in the relative interior of \( \mathcal{C} \), the expression in (11) is strictly positive and \( \delta^*(w) \in \Psi_{a^*(w)}^\epsilon(w, N_w, r, \mathcal{D}) \). Then the solution \( W \) to (2) with \( A = a^*(W) \), \( \delta = \delta^*(W_-) \), \( \beta = \phi(A, N_W, \delta)T_W \) and \( M \equiv 0 \) remains on \( \mathcal{C} \) until an endpoint of \( \mathcal{C} \) is reached or an event occurs.

**Proof.** Fix \( w \) in the relative interior of \( \mathcal{C} \) and choose \( \eta > 0 \) small enough such that \( N_w^\top N_v > 0 \) for all \( v \in \mathcal{C} \cap B_\eta(w) \), where \( B_\eta(w) \) denotes the closed ball around \( w \) with radius \( \eta \). On \( B_\eta(w) \), \( \partial \mathcal{W} \) admits a local parametrization \( f \) in the direction \( N_w \). For any \( v \in B_\epsilon(w) \), define the orthogonal projection \( \hat{v} = T_w^\top v \) onto the tangent, where \( T_w \) is the vector obtained by rotating \( N_w \) by 90° in clockwise direction. Denote by \( \pi(v) = (\hat{v}, f(\hat{v})) \) the projection of \( v \in B_\eta(w) \) onto \( \partial \mathcal{W} \) in the direction \( N_w \).

Let \((W, A, \beta, \delta, Z, (J^y)_{y \in Y}, M)\) be a weak solution to (2) with initial condition \( W_0 = w \) such that \( M \equiv 0 \) and for all \( t \geq 0 \), \( A_t = a^*(\pi(W_t)) \), \( \delta_t(y) = \delta^*(\pi(W_{t^-})) \), \( y \) for every \( y \in Y \) and \( \beta_t = T_t\phi_t \) on \([0, \tau]\), where we abbreviated \( N_t = N_{\pi(W_t)} \), \( T_t = T_{\pi(W_t)} \) and \( \phi_t = \phi(A_t, N_t, \delta_t) \), and define \( \tau := \sigma \wedge \inf\{t \geq 0 \mid W_t \not\in B_\eta(w)\} \) for \( \sigma := \inf\{t \geq 0 \mid \Delta J^y \neq 0 \text{ for any } y \in Y\} \). Then, the solution satisfies (b) in Lemma 4.1 up to time \( \tau \). Indeed, \( \delta \in \Psi_{a^*}^\epsilon(\pi(W), N, r, \mathcal{D}) \) a.e. by construction. Since
the maximizer of a measurable function is measurable and \( \pi \) is measurable, \( A, \beta \) and \( \delta \) are all predictable. Moreover, because \( \delta^* \) is bounded due to the condition that \( W + r\delta^*(y) \in \mathcal{W} \) for every \( y \in Y \) and \( \phi \) is a Lipschitz-continuous function of \( \delta^* \), they are both square-integrable.

We measure the distance of \( W \) to \( \mathcal{C} \) by \( D_t = N^TW_t - f(\hat{W}_t) \). Note that \( f \) is differentiable by assumption and \((-f'(\hat{W}_t), 1) = \ell_t N_t \), where \( \ell_t := \|(-f'(\hat{W}_t), 1)\| \). Since \( f \) is locally convex it is second order differentiable at almost every point by Alexandrov’s theorem. In particular, \( f' \) has Radon-Nikodým derivative \( f''(\hat{W}_t) = -\kappa(\pi(W_t))\ell_t^3 \). It follows from the Meyer-Itô formula (see Theorem 19.5 in Kallenberg [2]) that

\[
dD_t = r\ell_t N_t^T(W_t - g(A_t) - \delta_t \lambda(A_t)) \, dt + r\ell_t N_t^T T_t \phi_t \left( dZ_t - \mu(A_t) \, dt \right) \\
+ r\ell_t \sum_{y \in \mathcal{Y}} N_{t-}^\top \delta^* (\pi(W_t), y) \, dJ^y_t - \frac{1}{2} f''(\hat{W}_{t-}) \, d[\hat{W}]_t,
\]

The volatility term is zero because \( N^T T = 0 \). Note that on \([0, \sigma]\), \( \Delta J^y \equiv 0 \) for any \( y \in Y \) implies that \([W] = \langle \hat{W} \rangle \). Using (11) and the fact that \( N_w^T N_t = T_w^T T_t = \ell_t^{-1} \), we obtain that on \([0, \tau)\),

\[
dD_t = r\ell_t N_t^T(W_t - g(A_t) - \delta_t \lambda(A_t)) \, dt + \frac{1}{2} \kappa(\pi(W_t)) \ell_t^3 \left| T_w^T T_t \right|^2 |\phi_t|^2 \, dt = rD_t \, dt,
\]

where we used \( N_t^T(W_t - \pi(W_t)) = N_t^T N_w D_t = \ell_t^{-1} D_t \) in the second equality. It follows that \( D_t = D_0 e^{\alpha t} \), which is identically zero because \( D_0 = 0 \). On \( \{ \tau < \sigma \} \) we can repeat this procedure and concatenate the solutions to obtain a solution to (2) that remains on \( \mathcal{C} \) until either an accident occurs or an endpoint of \( \mathcal{C} \) is reached. Let \( \rho \) denote the hitting time of an endpoint of \( \mathcal{C} \). Then \( D_0 = 0 \) on \([0, \rho \wedge \sigma)\) implies that \( \pi(W) = W \) and hence \( \delta \in \Psi_A(W, N_w, r, \mathcal{D}) \).

**Corollary C.2.** For \( \varepsilon \geq 0 \) and \( \mathcal{D} \) of class \( B \), let \( \mathcal{C} \) be a \( C^1 \) solution to (10) with positive curvature throughout. Then any payoff in the relative interior is attainable by the continuation value \( W \) of an enforceable strategy profile with \( \delta(y) \in \mathcal{D}(W) \) such that \( W \) remains in \( \mathcal{C} \) until either an endpoint of \( \mathcal{C} \) is reached or an event occurs.

**Proof.** For any \( w \in \mathcal{C} \), let \( a^*(w) \) and \( \delta^*(w) \) denote the maximizers in (10). Since \( \mathcal{C} \) is assumed to have positive curvature throughout, the maximization in (10) is not taken over empty sets. By Corollary B.6, the maximizers are attained.

The following two lemmas establish that locally, \( \partial \mathcal{B}(r, \mathcal{W}) \) coincides with a solution to (5) at almost every point outside \( \mathcal{G}(r, \mathcal{W}) \). Lemma C.3 states that it is impossible for a solution to (5) to cut through \( \mathcal{B}(r, \mathcal{W}) \). For a convex curve \( \mathcal{C} \), let \( \mathcal{N}_\mathcal{C} := \{(w, N) \in \mathcal{C} \times S^1 \mid N^T(w - v) \geq 0 \ \forall \ v \in \mathcal{C} \} \) denote its outward normal bundle.
Lemma C.3. Let \( \varepsilon > 0 \) and let \( w \in \partial B_\varepsilon(r, \mathcal{W}) \) with outward normal \( N' \). Define the projection \( \pi : B_{\varepsilon/2}(w) \rightarrow \partial B_\varepsilon(r, \mathcal{W}) \) onto \( \partial B_\varepsilon(r, \mathcal{W}) \) in the direction of \( N' \) and set

\[
\mathcal{D}(w) := \{ \delta \in \mathbb{R}^2 \mid \exists \kappa \in [0,1] \text{ such that } \kappa w + (1 - \kappa)\pi(w) + r\delta \in \mathcal{W} \}. \tag{12}
\]

It is impossible for a \( C^1 \) solution \( \mathcal{C} \) to (10) with \((0, \mathcal{D})\) oriented by \( v \mapsto N_v \) to simultaneously satisfy

(i) \( \mathcal{C} \cap B_{\varepsilon}(r, \mathcal{W}) \subseteq B_{\varepsilon/2}(w) \),

(ii) \( \inf_{v \in \mathcal{C}} N_v^\top N' > 0 \),

(iii) \( \mathcal{N}_\varepsilon \cap (\Gamma^0(r, \mathcal{D}) \cup \mathcal{P}) = \emptyset \),

(iv) for any \( a \in \mathcal{A} \), \( \mathcal{N}_\varepsilon \cap \partial E_a^0(r, \mathcal{D}) = \emptyset \),

(v) there exists \( v_0 \in \mathcal{C} \) such that \( v_0 + \eta N' \in B_{\varepsilon}(r, \mathcal{W}) \) for some \( \eta > 0 \).

Proof. Suppose towards a contradiction that there exists such a curve \( \mathcal{C} \). Observe that \( \mathcal{D} \) is of class B, hence it follows from Conditions 3 and 4 as well as Proposition B.7 that \( \mathcal{C} \) is \( C^2 \) at almost every point. By Condition 2, there exists a local parametrization \( f \) of \( \mathcal{C} \) in the direction \( N' \). Define the orthogonal projection \( \hat{v} = T'^\top v \) onto the tangent for any \( v \in B_{\varepsilon/2}(w) \), where \( T' \) is the counterclockwise rotation of \( N' \) by \( 90^\circ \). Denote by \( \hat{\pi}(v) = (\hat{v}, f(\hat{v})) \) the projection of \( v \in B_{\varepsilon/2}(w) \) onto \( \mathcal{C} \) in the direction \( N' \).

By definition of \( B_\varepsilon(r, \mathcal{W}) \), there exists a solution \((W, A, \beta, \delta, M, Z, (J^y)_{y \in Y})\) to (2) with \( W_0 = v_0 + \eta N' \) such that on \([0, \sigma)\), \((\beta, \delta)\) enforces \( A \), \( W + r\delta(y) \in \mathcal{W} \) and \( N^\top \delta(y) \leq 0 \) for every normal vector \( N \) to \( \partial B_\varepsilon(r, \mathcal{W}) \) at \( v \in \partial B_\varepsilon(r, \mathcal{W}) \cap B_\varepsilon(w) \) and every \( y \in Y \). Define the stopping time \( \tau_1 := \inf\{t \geq 0 \mid W_t \notin B_{\varepsilon/2}(w)\} \). Condition 1 together with convexity implies that \( \mathcal{C} \) intersects \( \partial B_\varepsilon(r, \mathcal{W}) \) at two points \( v_L, v_R \). Since any two points in \( B_{\varepsilon/2}(w) \) are within distance \( \varepsilon \) of each other, it follows that \( N^\top \delta(y) \leq 0 \) on \([0, \tau_1]\) for every normal vector \( N \) to \( \partial B_\varepsilon(r, \mathcal{W}) \) between \( v_L \) and \( v_R \).

Since \( \mathcal{C} \) cuts through \( B_\varepsilon(r, \mathcal{W}) \) by Condition 5, convexity implies that the set of normal vectors to \( \mathcal{C} \) between \( v_L \) and \( v_R \) is a subset of the normal vectors to \( B_\varepsilon(r, \mathcal{W}) \) between these two points. Therefore, \( N^\top \delta(y) \leq 0 \) also for any normal vector \( N \) to \( \mathcal{C} \) between \( v_L \) and \( v_R \) on \([0, \tau_1]\).

Suppose first that \( \mathcal{N}_\varepsilon \subseteq E_0^a(r, \mathcal{D}) \) for some \( a \in \mathcal{A} \), i.e., \( \mathcal{C} \) is a non-trivial solution to (10). Let \( \bar{N}_t := N \bar{t}(W_t) \) and \( T_t := T \bar{t}(W_t) \) and observe that these projections are well defined on \([0, \tau_1]\). We measure the distance of \( W \) to \( \mathcal{C} \) by \( D_t = N_t^\top W_t - f(\bar{W}_t) \). Denote \( \ell_t := 1/T_t^\top T' \) and \( \gamma_t := \ell_t N_t^\top T' \) for the sake of brevity and observe that \( \bar{\gamma} := \sup_{w \in \mathcal{C}} N_w^\top T'/T_w^\top T' < \infty \) by Condition 5. Then, similarly as in Footnote 3 of Hashimoto [K], it follows from Itô’s formula that

\[
D_t \geq D_0 + \int_0^t \zeta_s \, ds + \int_0^t \xi_s \left(dZ_s - \mu(A_s) \, ds\right) + \sum_{y \in Y} \int_0^t \rho_s(y) dJ^y_s + \bar{M}_t,
\]
where
\[ ζ_t = rℓ_t \left( N^T_t (W_t - g(A_t) - δ_t \lambda(A_t)) + \frac{r}{2} ι(W_t) \right) N^T_t β_t + \gamma_t N^T_t β_t \right)^2 \]
\[ = rD_t + rℓ_t \left( N^T_t (π(W_t) - g(A_t) - δ_t \lambda(A_t)) + \frac{r}{2} ι(W_t) \right) N^T_t β_t + \gamma_t N^T_t β_t \right)^2, \]

\[ ξ_t = rℓ_t N^T_t β_t, \quad ρ_t(y) = rℓ_t - δ_t(y) \] and \( M_t = \int_0^t rℓ_t - δ_t(y) \, dM_t \). Define the stopping time \( τ_2 := \inf \{ t \geq 0 \mid D_t \leq 0 \} \) and observe that \( τ_2 ≤ τ_1 \) a.s., since Condition 2 implies that \( v_L + η N'' ∈ B_ε(r, W) \) for any \( η > 0 \) and similarly for \( v_R \). We will show that there exists an equivalent probability measure \( R \) such that the drift rate of \( D_t \) is bounded from below by \( rD_t \). Then, \( D_t \) becomes arbitrarily large with positive \( R \)-probability, and hence positive \( Q^4 \)-probability. Because it may take arbitrarily long until an accident arrives, this leads to a contradiction because \( V \) is bounded.

Let \( Ξ_1 \) denote the set where \( N^T(π(W) - g(A) - δλ(A)) ≥ 0 \). On \( Ξ_1 \), \( ξ_t ≥ rD_t \), hence there is no need to change the probability measure. It follows from Condition 2 that \( β ≠ 0 \) on \( Ξ_1 \). Let \( Ξ_2 ⊆ Ξ_1 \) be the set where \( N_c ≥ E^0_A(r, D) \), i.e., \( Ψ^0_A(π(W), N, r, D) ≠ 0 \). Set
\[ \hat{δ} ∈ \arg \min \left[ \left\| \hat{δ}^1 - δ^1 \right\| + \left\| \hat{δ}^2 - δ^2 \right\| \right]. \]
then (10) implies that
\[ ζ ≥ rD - rℓN^T(\hat{δ} - \hat{δ})λ(A) - rℓN^T(g(A) + \hat{δ}λ(A) - \hat{π}(W)) \left( 1 - \frac{\left\| T^T β \right\|^2 - γ \left\| N^T β \right\|^2}{\left\| φ(a, N, δ) \right\|^2} \right). \]

Denote \( Λ := \max_{y∈Y} \sum_{y∈Y} λ(y|a) \) and observe that \( N^T(g(A) + \hat{δ}λ(A) - \hat{π}(W)) \) is uniformly bounded above by the constant \( K_1 := \text{diam} V + \sup(W - V)Λ < ∞ \). The condition that \( W + rδ(y) ∈ W \) implies that \( δ(y) ∈ D(W) \) on \( [0, τ_2] \) for every \( y ∈ Y \). Since \( N^Tδ(y) ≤ 0 \) holds by the choice of \( ε \), Lemma ?? asserts the existence of constants \( K_2, \tilde{Ψ} \) such that
\[ ζ ≥ rD - rℓA K_2 \left\| N^T β \right\| - rℓK_1 \frac{2K_2 + 2θ}{\tilde{Ψ}} \left\| N^T β \right\| =: rD_t - K_3 \| ξ_t \|. \]

On the set \( Ξ_1 \cap Ξ_2 \), condition 3 implies that \( N_c \) is bounded away from \( E^0_A(r, D) ∪ P \) by virtue of Corollary B.6. Lemma ?? thus implies that \( \left\| N^T β \right\| ≥ K_4 \) for some constant \( K_4 \) and hence
\[ ξ_t ≥ rD_t - rℓK_1 ≥ rD_t - \frac{K_1}{K_4} \| ξ_t \|. \]

Let \( T := \min(t ≥ 0 \mid D_0(1 + rt)/2 ≥ \sup_{w∈V} N^T w - f(w)) \) and observe that \( T \) is deterministic. We define a density process \( L \) on \([0, T]\) by setting
\[ \frac{dL_t}{L_t} = 1 + Z_t + \sum_{y∈Y} \left( \frac{1}{λ(y|A_{t-})} - 1 \right) J_t^y, \]
where
\[ \psi_t = K_3 \frac{\xi_t}{\|\xi_t\|} 1_{\xi_t} + \frac{K_1}{K_4} \frac{\xi_t}{\|\xi_t\|}^2 1_{\xi_t, Y_1 \not\subseteq Y_2}. \]

Because \( \int_0^T \|\psi_t\|^2 \, dt < \infty \) \( Q_1^A \)-a.s., it follows from Girsanov’s theorem that \( L \) defines a probability measure \( R \) equivalent to \( Q_1^A \) on \( \mathcal{F}_T \) such that \( dZ_t' = dZ_t - \psi_t \, dt \) is an \( R \)-Brownian motion on \([0, T] \), for every \( y \in Y \), \( J^y \) has intensity 1 and \( \tilde{M}_t \) is an \( R \)-martingale because it is orthogonal to \( L \). Then
\[
D_t \geq D_0 + \int_0^t r D_s \, ds + \int_0^t \xi_s^\top dZ_s' + \tilde{M}_t + \sum_{y \in Y} \int_0^t \rho_s(y) \, dJ^y_s. \tag{13}
\]

Since \( W \) is bounded, \( \int_0^t \xi_s^\top dZ_s \) is a \( BMO(Q^A) \)-martingale. Therefore, \( \int_0^t \xi_s^\top dZ_s' \) is a \( BMO(R) \)-martingale by Theorem 3.6 in Kazamaki \([10]\). Define the stopping time \( \tau_3 := \inf\{t \geq 0 \mid D_t \leq D_0(1 + rt)/2\} \) and observe that \( \tau_3 \leq \tau_2 \wedge T \). It follows from (13) that
\[
D_{\tau_3} - \frac{D_0}{2} (1 + r \tau_3) \geq \frac{D_0}{2} + F_{\tau_3} + \sum_{y \in Y} \int_0^{\tau_3} \rho_s(y) \, dJ^y_s,
\]
where \( F_t = \int_0^t \xi_s^\top dZ_s' + \tilde{M}_t \) is an \( R \)-martingale starting at 0. Define the \( R \)-martingale \( G_t := e^{[Y^\top, 1_{\{t < 3\}}) \text{d}t} \) and observe that \( G \) is orthogonal to \( F \). Because \( \tau_3 \leq T \) a.s.,
\[
0 \geq \mathbb{E}_R \left[ \left( D_{\tau_3} - \frac{D_0}{2} (1 + r \tau_3) \right) 1_{\{T < 3\}} \right] \geq \mathbb{E}_R \left[ \frac{D_0}{2} 1_{\{T < 3\}} + F_{\tau_3} 1_{\{T < 3\}} \right]
\]
\[
= \frac{D_0}{2} R(T < 3) + e^{-[Y^\top, 3]_T} \mathbb{E}_R[F_{\tau_3} G_T] > 0,
\]
where the last inequality follows from the optional stopping theorem and because \( R \) is equivalent to \( Q_1^A \). This is a contradiction.

Suppose now that \( \mathcal{N}_C \subseteq \bigcap_{\alpha \in A} F_\alpha^0 (r, D)^c \), i.e., \( C \) is a straight line segment. Let \( D \) denote the distance of \( W \) to \( C \) in the direction of the normal vector \( \mathcal{N}_C \) of \( C \). Condition 2 makes it possible to apply Lemma \( \text{??} \), hence any \((\beta, \delta)\) enforcing \( A \) it follows that \( \|N^\top \beta\| \geq K \) for some constant \( K \). Similarly as before, the drift of \( D_t \) is thus bounded from below by \( r D_t - K_1/K r \ell_t \|N^\top \beta_t\| \). Therefore, there exists an equivalent probability measure under which \( D \) grows arbitrarily large with positive probability, a contradiction.

\[ \square \]

**Lemma C.4.** Fix \( w \in \mathcal{B}(r, W) \setminus \mathcal{G}(r, W) \) with outward normal \( N \), where \( \text{??} \) is locally Lipschitz continuous. Then \( \partial \mathcal{B}(r, W) \) coincides with a solution to \( \text{??} \) in a neighbourhood of \( (w, N) \).
Proof. We first show that a solution to (10) with \( N \) sufficiently small neighbourhood of \((w, N)\) can be made arbitrarily small by choosing small \( \eta \) and suppose towards a contradiction that \( C \) fails to be locally Lipschitz continuous at \( w \). Note that \( N \) is a solution with positive curvature. We may assume without loss of generality that this happens to the right of \( w \). By convexity of \( B(r, W) \), hence for \( \eta > 0 \) small enough, \( C \) is a solution to (5) with initial conditions \((w, \eta N, \eta N)\) for \( \eta > 0 \) sufficiently small. Specifically:

- If \( \partial B(r, W) \) is not \( C^1 \) at \( w \), we obtain \( C' \) as a solution to (5) with initial conditions \((w - \eta N, \eta N)\) for \( \eta > 0 \) sufficiently small.

  - If \( \partial B(r, W) \) is \( C^1 \) at \( w \), we obtain \( C' \) for initial conditions \((w, N')\), where \( N' \) is a slight rotation of \( N \) as illustrated in the left panel of Figure 10.

Because the set where (10) fails to be locally Lipschitz continuous is closed by Corollary B.6 and Proposition B.7, a small enough perturbation satisfies \( \mathcal{N}_C \cap \Gamma_a^0(r, D) = \emptyset \) for every \( a \in A \) and either \( \mathcal{N}_C^a \subseteq E_a^0(r, D) \) or \( \mathcal{N}_C \cap (E_a^0(r, D) \cup \mathcal{P}) = \emptyset \) for any \( a \in A \), that is, \( C' \) satisfies conditions 3–5 of Lemma C.3. By choosing \( \varepsilon \) and \( \eta \) or \( N' \) suitably, we can get conditions 1 and 2 to hold as well, and hence \( C' \) is impossible due to Lemma C.3. We conclude that \( \partial B(r, W) \) is \( C^1 \) where (10) is locally Lipschitz continuous and that a solution to (10) cannot escape \( \text{cl} B(r, W) \).

Suppose towards a contradiction that \( C \) falls into the interior of \( B(r, W) \) in a neighbourhood of \((w, N)\), that is, there exists \( v \in C \cap B(r, W) \) arbitrarily close to \( w \). By convexity of \( B(r, W) \), this is not possible if \( C \) is a trivial solution to (10), hence \( C \) is a solution with positive curvature. We may assume without loss of generality that this happens to the right of \( w \) as illustrated in Figure 11. Let \( v \) be close enough to \( w \) such that (10) with \((0, D)\) is Lipschitz continuous on an open neighbourhood of \( \mathcal{N}_C := \{(\tilde{w}, N_{\tilde{w}}) \mid \tilde{w} \in C \text{ between } w \text{ and } v\} \). Let \( \delta > 0 \) such that the closed ball \( B_\delta(v) \) is contained in the interior of \( B(r, W) \). Then for \( \varepsilon > 0 \) small enough, \( B_\delta(v) \subseteq \text{int} B_\varepsilon(r, W) \) and it follows from Remark B.I that there exists \( \eta(\varepsilon) \) such that for any \( \tilde{w} \in C \), \( N_{\tilde{w}} \in B_{\eta(\varepsilon)}(N_{\tilde{w}}) \) for any \( \tilde{v} \in C \cap B_\varepsilon(\tilde{w}) \). Note that \( \eta(\varepsilon) \) can be made arbitrarily small by choosing small \( \varepsilon \), hence for \( \varepsilon \) sufficiently small, (10) with \((\eta(\varepsilon), D)\) is Lipschitz continuous on an open neighbourhood of \( \mathcal{N}_C \). For \( \zeta > 0 \) to
be chosen later, let \( \mathcal{W}_\varepsilon := \{ w \in \mathcal{W} \mid d(w, \partial \mathcal{W}) \leq \varepsilon \} \), where \( d(w, \partial \mathcal{W}) \) denotes the minimal distance of \( d \) from \( \partial \mathcal{W} \). Set
\[
\mathcal{D}_\varepsilon(w) := \{ \delta \in \mathbb{R}^2 \mid \exists \kappa \in [0, 1] \text{ such that } \kappa w + (1 - \kappa)\pi(w) + r\delta \in \mathcal{W}_\varepsilon \},
\]
where \( \pi \) is the projection onto \( \partial \mathcal{B}(r, \mathcal{W}) \) in the direction \( N \). Observe that for \( \varepsilon \) sufficiently small, \( \mathcal{D}_\varepsilon \) with \( (\eta(\varepsilon), \mathcal{D}_\varepsilon) \) is Lipschitz continuous in a neighbourhood of \( \mathcal{N}_\varepsilon \), hence it admits a unique solution \( \mathcal{C}_\varepsilon \). Choose now \( \varepsilon \) and \( \zeta \) small enough such that Lemma ?? asserts the existence of \( v' \in \mathcal{C}_\varepsilon \cap \partial \mathcal{B}(r) \).

Because \( \mathcal{C}_\varepsilon \) is continuous in initial conditions, a solution \( \mathcal{C}'_\varepsilon \) to \( \mathcal{D}_\varepsilon \) with \( (\eta(\varepsilon), \mathcal{D}_\varepsilon) \) for a slight rotation \( N' \) of \( N \) reaches a neighbourhood of \( v' \) in \( \mathcal{B}_\varepsilon(r, \mathcal{W}) \). As illustrated in Figure 11, \( \mathcal{C}'_\varepsilon \) will escape \( \text{cl} \mathcal{B}(r, \mathcal{W}) \) to the right of \( w \) and enter \( \mathcal{B}_\varepsilon(r, \mathcal{W}) \) to the left of \( w \). Thus, for \( N' \) close enough to \( N \), there exist \( v_L, v_R \in \mathcal{C}'_\varepsilon \cap \mathcal{B}_\varepsilon(r, \mathcal{W}) \), such that \( ||\bar{w} - \pi(\bar{w})|| \leq \zeta \) for all \( \bar{w} \in \mathcal{C}'_\varepsilon \). By Corollary ?? for any \( w' \in \mathcal{C}'_\varepsilon \) there exists a solution to \( \mathcal{C}'_\varepsilon \) with \( W_0 = w' \) such that \( \delta \in \Psi^{(\varepsilon)}_{\mathcal{A}}(W, N, r, \mathcal{D}_\varepsilon) \) on \( [0, \sigma] \) and \( W \in \mathcal{C}'_\varepsilon \) until it reaches an end point of \( \mathcal{C}'_\varepsilon \) or an event occurs. Let \( \tau := \inf \{ t \geq 0 \mid W_t \in \{ v_L, v_R \} \} \) and observe that \( W_\tau \in \mathcal{B}_\varepsilon(r, \mathcal{W}) \) on \( \{ \tau < \sigma \} \). The condition that \( \delta(y) \in \mathcal{D}_\varepsilon(W) \) a.e. for every \( y \in \mathcal{Y} \) implies that \( x + r\delta(y) \in \mathcal{W}_\varepsilon \) for some \( x \) between \( W_t \) and \( \pi(W_t) \). On \( [0, \tau \wedge \sigma] \) it holds that \( ||W_t - x|| \leq \zeta \) and hence \( \delta \in \Psi^{(\varepsilon)}_{\mathcal{A}}(W, N_W, r, \mathcal{D}) \). Because \( W_\tau \in \mathcal{B}_\varepsilon(r, \mathcal{W}) \) on \( \{ \tau < \sigma \} \), by definition of \( \mathcal{B}_\varepsilon(r, \mathcal{W}) \) there exists a solution \( (\bar{W}, \bar{A}, \bar{\beta}, \bar{\delta}) \) with \( W_0 = W_\tau \) such that on \( [0, \sigma] \), \( (\bar{\beta}, \bar{\delta}) \) enforces \( \bar{A}, \bar{\delta}(y) \in \mathcal{D}(\bar{W}) \) and \( N^T \bar{\delta}(y) \leq 0 \) for every normal vector \( N \) at \( \partial \mathcal{B}_\varepsilon(r, \mathcal{W}) \) sufficiently close to \( \bar{W} \). Therefore, a concatenation of \( (\bar{W}, \bar{A}, \bar{\beta}, \bar{\delta}) \) with \( \bar{W}, \bar{A}, \bar{\beta}, \bar{\delta} \) satisfies the same properties, which shows that \( \text{co} \mathcal{C}'_\varepsilon \cup \mathcal{B}_\varepsilon(r, \mathcal{W}) \subseteq \mathcal{B}_\varepsilon(r, \mathcal{W}) \) by maximality of \( \mathcal{B}_\varepsilon(r, \mathcal{W}) \).

Finally, because \( \mathcal{C}'_\varepsilon \) is Lipschitz continuous almost everywhere, we need to show that \( \partial \mathcal{B}(r, \mathcal{W}) \) is \( C^1 \) to grant uniqueness of the solution. By convexity, \( \mathcal{B}(r, \mathcal{W}) \) cannot have inward corners, and it will follow with another escaping argument that it cannot have outward corners outside of \( \mathcal{G}(r, \mathcal{W}) \) either.

**Lemma C.5.** \( \partial \mathcal{B}(r, \mathcal{W}) \setminus \mathcal{G}(r, \mathcal{W}) \) is \( C^1 \) where \( \mathcal{C}'_\varepsilon \) fails to be Lipschitz continuous. Moreover, outside of \( \mathcal{P} \), the set of all points in \( \partial \mathcal{B}(r, \mathcal{W}) \setminus \mathcal{G}(r, \mathcal{W}) \), where \( \mathcal{C}'_\varepsilon \) fails to be Lipschitz continuous, has relative measure 0.
Proof. We first check that $\partial B(r, W) \setminus G(r, W)$ is $C^1$. Because of Lemma C.4 it is enough to verify this property at payoffs $w$ in $\partial B(r, W) \setminus G(r, W)$ where (5) fails to be locally Lipschitz continuous. Observe that $w \notin G(r, W)$ implies that $(w, N) \notin \Gamma^0(r, D)$ for any outward normal $N \in N_w(B(r, W))$, where $\mathcal{D}(w) = (W - w)/r$. Since $\Gamma^0(r, D)$ is closed by Corollary B.6, it follows that $(v, N) \notin \Gamma^0(r, D)$ for any $N \in N_w(B(r, W))$ and any $v$ in a sufficiently small open neighbourhood $U$ of $w$. Because the boundary of a Borel set has measure 0, it follows that for an arbitrarily small neighbourhood $U$ of $w$, there exists $(v, N)$ in the set $U \cap \text{int} \, B(r, W) \times \text{int} \, N_w(B(r, W)) \setminus \{\pm e_1\}$ of positive measure, for which (5) is locally Lipschitz continuous.

For sufficiently small $\varepsilon > 0$, let $(w_{\varepsilon}, N') \in N_{B_{\varepsilon}(r, W)}$ be such a pair. Because the points where (5) fails to be Lipschitz continuous is closed, (5) is also Lipschitz continuous in a small neighbourhood $U_{\varepsilon}$ of $(w_{\varepsilon}, N')$. Let $D_{\varepsilon}$ be defined as in (12) with respect to $N'$. By the definition of $D_{\varepsilon}$ it follows that $(v, N) \notin \bigcup_{a \in A} \partial E^0_a(r, D_{\varepsilon}) \cup \Gamma^0(r, D_{\varepsilon}) \cup \mathcal{P}$ for any $(v, N) \in U_{\varepsilon}$. Choose such a pair $(v, N)$ close enough to $(w_{\varepsilon}, N')$ and let $C$ be a solution to (10) with $(0, D_{\varepsilon})$ starting in $(v, N)$. We have shown that conditions 3 and 4 of Lemma C.3 are satisfied on $U_{\varepsilon}$ for any $\varepsilon > 0$. Since $B(r, W)$ has a corner at $w$, the curvature of $\partial B_{\varepsilon}(r, W)$ becomes arbitrarily large, hence for $\varepsilon$ small enough we can get $C$ to cut through $\text{int} \, B_{\varepsilon}(r, W)$, i.e., we can get conditions 1, 2 and 5 to hold for $(v, N)$ sufficiently close to $(w_{\varepsilon}, N')$. Such a curve $C$ is impossible by Lemma C.3, hence the first statement follows.

For the second statement, suppose that there exists $C \subseteq B(r, W) \setminus G(r, W)$ of positive length. By shortening the line segment we may assume that $\mathcal{N}_C \subseteq \mathcal{P}$ or $\mathcal{N}_{\text{int} \, C} \cap \mathcal{P} = \emptyset$. Suppose towards a contradiction that $\mathcal{N}_{\text{int} \, C} \cap \mathcal{P} = \emptyset$. Then Proposition B.7 shows that $(w, N)$ enters and leaves $E^0_a(r, D)$ of the maximizing action profile $a$ at almost every $(w, N) \in \mathcal{N}_C$. Because $A$ is finite we may assume that this is the same action profile. This implies that $\mathcal{N}_C \subseteq \partial E^0_a(r, D)$ and hence $\mathcal{N}_{\text{int} \, C} \subseteq E^0_a(r, D)$ by Corollary B.6, a contradiction.

The characterization of $\partial B(r, W)$ is concluded by showing that any $C^1$ segment in $G(r, W)$ must lie on a straight line through $g(a)$ for some action profile $a$.

Lemma C.6. Let $W$ have non-empty interior and let $w \in G(r, W)$ with outward normal vector $N$. Then $(w, N) \in \Gamma^0_a(r, D)$ implies that $N^\top (g(a) - w) = 0$.

Proof. It is sufficient to verify the statement for extremal normal vectors, as this implies that $g(a) = w$ if $B(r, W)$ has a corner at $w$. Let $N'$ be an extremal normal vector at $w$ and let $a \in A$ with $(w, N') \in \Gamma^0_a(r, D)$. Then there exists $\delta$ such that $(a, \delta)$ decomposes $w$. Suppose first that $B(r, W)$ has empty interior. By convexity of $B(r, W)$ it follows that $B(r, W)$ is a straight line segment and hence $-N'$ is an outward normal vector as well. Since the jumps are directed inwards, this implies that $N'^\top \delta(y) = 0$ for every $y \in Y$ and hence $N'^\top (g(a) - w) = 0$. Suppose, therefore, that $B(r, W)$ has non-empty interior. Then, by assumption on $W$, $H(N') \cap \mathcal{D}(w)$ has
non-empty interior as well. Thus, for any \((a, \delta)\) decomposing \(w\), there exists a small perturbation \(\delta\) of \(\delta\) with \(\hat{\delta}(y) \in \text{int}(H(N') \cap D(w))\) for every \(y \in Y\).

Suppose first that \(N'\) is not a coordinate direction and that \(w\) is decomposed by \((a, \delta_0)\) with \(N'^T(g(a) + \delta_0 \lambda(a) - w) > 0\). Since \(B(r, \mathcal{W})\) has non-empty interior, there exists \(d \in \mathbb{R}^{2 \times |Y|}\) such that \(\delta(y) := \delta_0(y) + \eta d(y) \in \text{int}(D(w) \cap H(N'))\) for every \(y \in Y\) and \(\eta > 0\) sufficiently small. Therefore, for small \(\eta > 0\), there exists a neighbourhood \(U(\eta)\) of \((w, N')\) such that for every \((v, N) \in U(\eta)\) it holds that \(\delta(\eta) \in \text{int}(D(v) \cap H(N))\). By making \(U(\eta)\) small enough such that \(U(\eta) \cap \mathcal{P} = \emptyset\), Lemma 5.7 implies the existence of \(\phi(a, N, \delta(\eta)) \in \mathbb{R}^d\) such that \(\langle \phi(a, N, \delta(\eta)), \delta(\eta) \rangle\) restricted-enforces \(a\) in the direction \(N\) at \(v\) for every \((v, N) \in U(\eta)\). Observe that \(\|\phi(a, N, \delta(\eta))\|\) can be made arbitrarily small by choosing \(\eta\) small enough. Since \(B(r, \mathcal{W})\) is convex, \(\partial B(r, \mathcal{W})\) is one-sided \(C^1\) at \(w\), hence it is possible to choose \(\eta\) sufficiently small such that a solution \(\mathcal{C}\) to

\[
\kappa(v) = \frac{2N_v^T(g(a) + \delta(\eta) \lambda(a) - v)}{r \|\phi(a, N_v, \delta(\eta))\|^2}
\]

with initial state \((\tilde{w}, N')\) for \(\tilde{w}\) close enough to \(w\) intersects \(\partial B(r, \mathcal{W})\) on both sides of \(\tilde{w}\) with \(N_\mathcal{C} \subseteq \text{int} U(\eta)\). Fix now \(\varepsilon > 0\) small enough such that \(N_{\mathcal{C}} \cap B_{\varepsilon}(\partial B_{\varepsilon}) \cap N_{B_{r}(r, \mathcal{W})}\) is contained in the interior of \(U(\eta)\), where \(B_{\varepsilon} := \text{co} \mathcal{C} \cup B_{\varepsilon}(r, \mathcal{W})\). We will now show that \(B_{\varepsilon}\) is \((\varepsilon, r, \mathcal{W})\)-admissible, which contradicts maximality of \(B_{\varepsilon}(r, \mathcal{W})\). Indeed, by Lemma \([C.1]\) any payoff on \(\mathcal{C}\) is attainable by a solution to (2) that is continuous on \([0, \sigma]\) until an endpoint is reached. Therefore, the time required for that solution to reach \(\mathcal{C} \cap \partial B_{\varepsilon}(\partial B_{\varepsilon}')\) from any point in \(\mathcal{C} \cap B_{\varepsilon}(\partial B_{\varepsilon}')\) is bounded from below by the times needed to get there from \(\mathcal{C} \cap \partial B_{\varepsilon}/2(\partial B_{\varepsilon}')\), which is strictly positive. Therefore, \(\mathcal{C}\) is locally \((\varepsilon, r, \mathcal{W})\)-admissible satisfying the conditions in Lemma \([A.2]\). Since \(B_{\varepsilon}(r, \mathcal{W})\) is \((\varepsilon, r, \mathcal{W})\)-admissible, Lemma \([A.2]\) implies that \(B_{\varepsilon}'\) is \((\varepsilon, r, \mathcal{W})\)-admissible as well.

Observe that a contradiction can be obtained in the same way if \(w\) is decomposed by \((a, \delta_0)\) with \(N'^T(g(a) + \delta_0 \lambda(a) - w) = 0\) and \(N'^T \delta_0(y) < 0\) for at least one \(y \in Y\). Indeed, this implies that \(N'^T(g(a) - w) > 0\) and hence for a suitable choice of \(d\) it follows that \(N'^T(g(a) + \delta(\eta) \lambda(a) - w) > 0\) and \(\delta(\eta) \in \text{int} D(w) \cap H(N')\) for all \(\eta > 0\) small enough. If, on the other hand, \(N'^T \delta(y) = 0\) for every \(y \in Y\) and every \(\delta\) decomposing \(w\), then \(N'^T(g(a) - w) = 0\).

Finally, let \(N' = \pm \varepsilon_i\) be a coordinate direction. By Assumption \([3]\) there exists an action profile \(\tilde{a} \in \mathcal{A}\) with the unique best response property for player \(i\) such that \(N'^T(g(\tilde{a}) - w) \geq 0\). Moreover, due to Lemma \([5.7]\) \(\tilde{a}\) is restricted-enforceable in the direction \(N'\) by \((\beta, 0)\). Suppose towards a contradiction that \(N'^T(g(\tilde{a}) - w) \neq 0\). Then it follows from the unique best response-property that \((\tilde{\phi}(\tilde{a}, N_v, 0), 0)\) restricted-enforces \(a\) in the direction \(N_v\) sufficiently close to \(N'\), where \(\tilde{\phi}(\tilde{a}, N_v, 0) := \beta/T_{v,-i}\). Moreover, for this choice of \(\tilde{\phi}\), (14) is locally Lipschitz continuous in a neighbourhood of coordinate directions. Therefore, a solution \(\mathcal{C}\) to (14) with initial conditions \((\tilde{w}, N')\)
for \( \tilde{w} \) sufficiently close to \( w \) intersects \( \partial \mathcal{B}(r, \mathcal{W}) \) on both sides of \( \tilde{w} \). The proof is completed by showing that \( \text{co} \mathcal{C} \cup \mathcal{B}_\varepsilon \mathcal{B}(r, \mathcal{W}) \) is \((\varepsilon, r, \mathcal{W})\)-admissible in the same way as before, thereby contradicting maximality of \( \mathcal{B}_\varepsilon(r, \mathcal{W}) \).

\[ \square \]

**Corollary C.7.** Any \( C^1 \) segment in \( \mathcal{G}(r, \mathcal{W}) \) is a straight line segment whose infinite continuation goes through \( g(a) \) for some \( a \in \mathcal{A} \).

**Corollary C.8.** All corners of \( \mathcal{B}(r, \mathcal{W}) \) are contained in \( \mathcal{V}^N \).

\[ \text{D Closedness of } \mathcal{B}(r, \mathcal{W}) \]

This appendix shows that under Assumptions [1, 3], the set \( \mathcal{B}(r, \mathcal{W}) \) is closed. It also contains the proof of Proposition 5.6 based on the auxiliary results in Appendix C and this appendix. Corollary C.8 implies that all corners of \( \mathcal{B}(r, \mathcal{W}) \) are contained in \( \mathcal{B}(r, \mathcal{W}) \). If the boundary between two corners has strictly positive curvature throughout, then it is contained in \( \mathcal{B}(r, \mathcal{W}) \) by Corollary C.2. It thus remains to verify points where the boundary changes from a curve to a straight line segment.

**Lemma D.1.** Let \( w \in \partial \mathcal{B}(r, \mathcal{W}) \setminus \mathcal{V}^N \) be a point where \( \partial \mathcal{B}(r, \mathcal{W}) \) changes from a curved solution \( \mathcal{C} \) to \( \mathcal{C}(5) \) to a straight line segment \( L \) in a differentiable way. Then \( w \) is in \( \mathcal{B}(r, \mathcal{W}) \) if the other end points of \( \mathcal{C} \) and \( L \) are contained in \( \mathcal{B}(r, \mathcal{W}) \).

**Proof.** Let \( \varepsilon > 0 \) be small enough such that the maximizer \( a^* \) of \( [5] \) does not change in \( B_\varepsilon(w) \cap (\mathcal{C} \setminus \{w\}) \) and such that \( B_\varepsilon(w) \cap \partial \mathcal{B}(r, \mathcal{W}) \) admits a parametrization \( f \) in the direction \( N_w \). Note that this implies \( T_w^\top (g(a^*) - w) > 0 \), where \( T_w \) denotes the tangent vector that points towards \( \mathcal{C} \). As usual, let \( \hat{v} := T_w^\top v \) denote the projection onto the tangent and let \( \pi(v) = (\hat{v}, f(\hat{v})) \) denote the projection onto \( \partial \mathcal{B}(r, \mathcal{W}) \).

Suppose first that \( (w, N_w) \in \Gamma_{a^*}(r, \mathcal{D}) \). Lemma C.6 implies \( N_w^\top (g(a^*) - w) = 0 \) and hence \( a^* \) is enforced by \( 0, \delta \) with \( N_w^\top \delta(y) = 0 \) for every \( y \in Y \). Suppose towards a contradiction that \( T_w^\top \delta \lambda(a) \leq 0 \). Then Assumption 2(ii) implies that \( (0, \delta) \) enforces \( a^* \) as well for \( \tilde{\delta} \) with \( \tilde{\delta}(y) = -\|\tilde{\delta}(y)\|T_w \) for every \( y \in Y \). That is, every jump points away from \( g(a^*) \), hence \( (0, \delta) \in \Xi_{a^*}(v, N_v, r, \mathcal{D}) \) for \( v \in \mathcal{C} \) sufficiently close to \( w \). This is a contradiction to \( \mathcal{C} \cap \mathcal{G}(r, \mathcal{W}) \subseteq \{w\} \). It follows that \( T_w^\top \delta \lambda(a) > 0 \) and hence Assumption 2(ii) implies that \( a^* \) is also enforced by \( (0, \tilde{\delta}) \) with \( \tilde{\delta}(y) = \|\tilde{\delta}(y)\|T_w \). As a result, a solution \( W \) to \( [2] \) with \( A \equiv a^*, \delta = \tilde{\delta}, \beta = 0, M = 0 \) remains on \( L \) with \( W + r\tilde{\delta}(y) \in \mathcal{W} \cap H(W, N_w) \) until a jump occurs or the other endpoint of \( L \) is reached. It follows that \( w \in \mathcal{B}(r, \mathcal{W}) \) also in that case.

Suppose next that \( (w, N_w) \notin \Gamma_{a^*}(r, \mathcal{D}) \). For a non-coordinate direction \( N_w \), Lemma B.5 implies that \( \Psi_{a^*}(w, N, r, \mathcal{D}) \neq \emptyset \). If \( N_w \) is coordinate, Lemma B.8 implies that \( a^* \) has the unique best response property. Indeed, \( \kappa_a(v, N_v) \to 0 \) as \( v \in \mathcal{C} \) approaches \( w \) for any \( a \) that is not enforceable on a coordinate direction. Hence, such an action profile is dominated by a profile \( \hat{a} \) with the best-response property in
a neighbourhood of \((w, N_w)\). Such an action profile exists with \(N_w^\top (g(\hat{a}) - w) \geq 0\) due to Assumption 3.(ii). We conclude that \(\Psi_{a^*}(w, N, r, D) \neq \emptyset\). Suppose towards a contradiction that \(N_w^\top (g(a^*) - w) = 0\). Then, \(\kappa_{a^*}(v, N_v) \to 0\) as \(v \to w\) and \(\kappa_{a^*}(w, N_w) = 0\). As a result, \((C \cap B_z(w)) \cup \{w\}\) is a solution to \((2)\) that remains on \(W\) and Corollary C.8 shows that all corners are contained in the set of static Nash payoffs. Finally, \(M = a^*, \beta_t = T_t\phi_t\) and \(D\)

\[
\delta_t(y) = \begin{cases} \delta_s(\pi(W_{t-}), y) - \frac{1}{\tau} D_t N_w, & W_t \geq 0, \\ 0, & W_t < 0, \end{cases}
\]

for every \(y \in Y\) on \([0, \sigma \wedge \tau]\), where \(N_t = N_{\pi(W_t)}\), \(T_t = T_{\pi(W_t)}\), \(\phi_t = \phi(a^*, N_t, \delta_t)\), \(D_t := N_w^\top W_t - f(\hat{W}_t)\), \(\tau := \inf\{t \geq 0 \mid W_t \not\in B_z(w)\}\) and \(\sigma\) is the first jump time of any of the processes \((J^y)_{y \in Y}\). Observe that \(\phi\) is well-defined by Lemma 5.7.

Let \(\rho := \inf\{t \geq 0 \mid D_t > 0\}\). We will show that \(\rho \geq \tau \wedge \sigma\) a.s. Indeed, on \([0, \tau \wedge \sigma]\), \(W\) and \(D\) are continuous, hence \(D\) has to reach 0 before it crosses it. Since \(\delta = \delta^s\) on \(\{D = 0\} \cap \{W \geq 0\}\), it follows in the same way as in the proof of Lemma C.1 that \(D\) remains 0 after reaching it until either \(W \not\in B_z(w)\) or \(W < 0\). On \(\{W < 0\}\), the drift is strictly inward pointing and the volatility tangential, hence \(D\) is strictly decreasing. It follows that \(D_t \leq 0\) on \([0, \tau \wedge \sigma]\) and hence \(\rho \geq \tau \wedge \sigma\). In particular, \(W\) remains in \(B(r, W)\). By construction, \(W_\sigma \in W\) on \(\{\sigma < \tau\}\) and the jumps are parallel to the boundary if \(W\) is on the boundary. On the set \(\{\tau < \sigma\}\), it follows that \(W_\tau\) is either in \(C \cap B_z(w)\) or in the interior of \(B(r, W)\), since \(D\) is strictly decreasing on \(\{W < 0\}\). If \(W_\tau \in \text{int} B(r, W)\), concatenate the solution with a solution to \((2)\) attaining \(W_\tau\) that remains in \(B(r, W)\) until time \(\tau\) and jumps into \(W\). If \(W_\tau \in C \cap B_z(w)\), Lemma C.1 implies that there exists a solution to \((2)\) that remains on \(C\) until either an end points is reached or an event occurs. Since the other end point is in \(B(r, W)\) by assumption, repeating the same procedure yields \(w \in B(r, W)\).

**Lemma D.2.** \(B(r, W)\) is closed.

**Proof.** By public randomization, a straight line segment is contained in \(B(r, W)\) if both of its end points are contained in \(B(r, W)\). Similarly, Lemma C.1 shows that curved parts of \(\partial B(r, W)\) are contained in \(B(r, W)\) if its end points are. These end points can be either corners of \(B(r, W)\) or points where a curved solution to \((3)\) turns into a straight line segment. Lemma D.1 implies that the closure of \(B(r, W)\) is \((0, r, W)\)-admissible, hence it is contained in \(B(r, W)\) by maximality.

**Proof of Proposition 5.6.** Lemmas C.4 and C.5 imply that \(\partial B(r, W) \setminus G(r, W)\) is a \(C^1\) solution to \((5)\). It follows from Corollary C.7 that \(G(r, W)\) has the desired properties and Corollary C.8 shows that all corners are contained in the set of static Nash payoffs. Finally, \(B(r, W)\) is closed by Lemma D.2.

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