Nonlinear Pricing under Asymmetric Competition with Complete Information*

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Abstract

Motivated by several recent antitrust cases, we study a strategic model of competition in intermediate-goods markets. Our model is a three-stage game with complete information in which a dominant firm offers a general tariff first and then a rival firm responds with a per-unit price, followed by a buyer making her decision to purchase from one or both firms. We characterize subgame perfect equilibria of the game and study the implications of the equilibrium outcome.

Our paper makes three main contributions. First, it provides a novel explanation for the prevalence of nonlinear pricing (a menu of offers conditional on volumes) under duopoly in the absence of private information: The dominant firm can use a menu of offers to constrain its rival’s choices and extract surplus from the buyer. Second, it shows that when the capacity of the rival firm is constrained, as compared to linear pricing schemes, the nonlinear pricing tariff adopted by the dominant firm reduces the price, sales, and profits of the rival.

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firm as well as the buyer’s surplus. In other words, nonlinear pricing may have antitrust implications in the sense that it can lead to partial foreclosure and harm consumer welfare. Third, we establish an equivalence between a subgame perfect equilibrium of the game and an optimal mechanism in a “virtual” principal-agent model with hidden action and hidden information. This involves treating the rival firm’s (an agent’s) price as its hidden action meanwhile letting the buyer (another agent) to report the rival firm’s price as her private information to the dominant firm (the principal). As a result of such an equivalence, we can apply mechanism design techniques to solve for subgame perfect equilibria of the game.

**Keywords:** Nonlinear Pricing, Asymmetric Competition, Capacity Constraint, Complete Information, Subgame Perfect Equilibrium, Principal-agent Model, and Partial Foreclosure.

**JEL Code:** L13, L42, K21

### 1 Introduction

Nonlinear pricing is often observed in intermediate-goods markets. It takes the form of various rebates and discounts conditional on volumes (or share of the volumes among competitors) purchased by a buyer. An example is all-units discount pricing scheme that lowers a buyer’s marginal price on every unit purchased when the buyer’s purchase exceeds or is equal to a pre-specified volume threshold. The adoption of such conditional rebates and discounts by dominant firms has become a prominent antitrust issue. Indeed, in a number of recent antitrust cases in the U.S., E.U., Canada, and China, a plaintiff (a government antitrust agency or a rival firm) alleged that a dominant firm used pricing schemes such as conditional rebates/discounts to its downstream buyers to fully or partially exclude its rival firm(s) and that such an exclusion had harmed competition and consumer welfare. Those antitrust cases share some common features: First, there is a firm that is considered as “dominant” in market share, capacity, product lines, profits, and so on. Second, there is one or several smaller firms (or recent entrants) that have limited capacity, narrower product
lines, or limited distribution channels. Third, the “dominant” firm typically offers more complex pricing schemes (e.g., rebates/discounts conditional on volumes) than its rival(s). What explains the observed practices of various nonlinear pricing schemes in intermediate-goods markets and what are the implications of those practices? The main objective of this paper is to provide an explanation for nonlinear pricing in the presence of asymmetric competition and in the absence of private information.

Motivated by recent antitrust cases, we study a stylized model of asymmetric competition. In the model, there are two firms, a dominant firm (Firm 1) and a rival firm (Firm 2). Both firms can produce a homogeneous product at constant marginal cost. However, the rival firm is capacity constrained. There is a representative downstream buyer who may purchase the product from one or both firms. We consider a three-stage game with complete information in which the dominant firm offers a general tariff first and then its rival firm responds with a per-unit price, followed by the buyer making her decision to purchase from one or both firms. We characterize subgame perfect equilibria of the game and study the implications of the equilibrium outcome.

Our model involves three kinds of asymmetries between the two firms. The first is concerned with pricing schemes: The dominant firm is able to make nonlinear tariff schedules, i.e., payments conditional on volumes, while the rival firm can only choose linear pricing schemes. This assumption appears to be consistent with the observations from the major antitrust cases, and is perhaps due to the fact that the dominant firm is more experienced in dealing with downstream buyers than new entrants to the market. The second asymmetry concerns the timing of the game: The dominant firm commits to offering tariffs before its rival. This might be related to the dominant firm’s bargaining power and its willingness to commit its offers when dealing with the buyer. Another asymmetry is about capacity levels of the firms. That is, relatively to the demand size the dominant firm has no capacity limit while its rival is capacity-constrained. Our analysis suggests that the asymmetry in capacity is not crucial for the equilibrium adoption of nonlinear pricing by the dominant firm, but is important for the results of partial foreclosure and harming the buyer welfare.
Our paper makes several major contributions. First, it provides a novel explanation for the prevalence of nonlinear pricing (a menu of offers conditional on volumes) under duopoly in the absence of private information: The dominant firm can use a menu of offers to constrain its rival’s choices and extract surplus from the buyer. Second, it shows that when the capacity of the rival firm is relatively small, as compared to linear pricing schemes, the nonlinear pricing tariff adopted by the dominant firm reduces the price, sales, and profits of the rival firm as well as the buyer’s surplus. In other words, nonlinear pricing in this context can lead to partial foreclosure and harm consumer welfare, which may have antitrust implications. Third, we establish an equivalence between a subgame perfect equilibrium of the game and an optimal mechanism in a “virtual” principal-agent model with hidden action and hidden information. This involves treating the rival firm’s (an agent’s) price as its hidden action meanwhile letting the buyer (another agent) to report the rival firm’s price as her private information to the dominant firm (the principal). As a result of such an equivalence, we can apply mechanism design techniques to characterize subgame perfect equilibria of the game. Other properties of the equilibrium tariffs are also discussed in the paper.

[Link to the literature to be added.]

The remainder of the paper is organized as follows. In Section 2 we set up our model of asymmetric competition in intermediate-goods markets. Section 3 establishes an equivalence between a subgame perfect equilibrium of the game and an optimal mechanism in a “virtual” principal-agent model with hidden action and hidden information. Section 4 characterizes the equilibrium outcome of the game. In particular, Subsection 3.2 describes the dominant firm’s optimization problem and establishes an equivalence between a subgame perfect equilibrium of the game and an optimal mechanism in a “virtual” principal-agent model. Subsection 4.1 characterizes the buyer’s incentive compatibility and individual rationality constraints. Subsection 4.3 provides a complete characterization of the equilibrium outcome. Other properties and implications of the equilibrium are discussed in Section 5. Section 6 contains concluding remarks.
2 Model

There are two firms, producing a homogeneous product, and one buyer. To capture a notion of dominance, we allow for a possible capacity asymmetry between two firms. In particular, firm 1, as a dominant firm, can produce any quantity at a unit cost $c \geq 0$. Firm 2, as a possibly smaller firm, has a capacity $k \in (0, \infty]$, up to which it can produce any quantity at the same unit cost $c$. If the buyer chooses to buy $Q \geq 0$ units from firm 1 and $q \in [0, k]$ units from firm 2, his payoff is the gross utility given by $u(Q + q)$, less the payments to the two firms.

The game is as follows. First, firm 1 offers a nonlinear tariff $\tau(\cdot)$, which specifies the payment $\tau(Q) \in \mathbb{R} \cup \{\infty\}$ that the buyer has to make if the buyer chooses to buy $Q \geq 0$ units from firm 1, with the restriction that $\tau(0) \leq 0$. Second, after observing $\tau(\cdot)$, firm 2 offers a unit price $p$ (up to $k$ units). Third, after observing $\tau(\cdot)$ and $p$, the buyer chooses the quantities he buys from each of the firms. Note that our game is a complete information and perfect information one. Our equilibrium concept is subgame-perfect equilibrium (SPE).

We say a tariff $\tau : \mathbb{R}_+ \to \mathbb{R} \cup \{\infty\}$ is regular if the subgame after firm 1 offers $\tau$ has some SPE. Accordingly, the set of feasible tariffs firm 1 can choose from is

$$
\mathcal{T} \equiv \{\tau \in \mathbb{R} \cup \{\infty\} : \tau \text{ is regular and } \tau(0) \leq 0\}.
$$

Note that firm 2 always makes non-positive profit if it chooses a unit price below $c$. Therefore, the set of feasible unit prices firm 2 can choose from is

$$
\mathcal{P} \equiv [c, \infty].
$$

A SPE is composed of a firm 1’s strategy $\tau^* \in \mathcal{T}$, a firm 2’s strategy $p^* : \mathcal{T} \to \mathcal{P}$, and a buyer’s strategy $q^* : \mathcal{T} \times \mathcal{P} \to \mathbb{R}_+ \times [0, k]$, such that

$$
q^*(\tau, p) \in \operatorname{argmax}_{(Q,q) \in \mathbb{R}_+ \times [0,k]} \{u(Q + q) - pq - \tau(Q)\} \quad \forall (\tau, p) \in \mathcal{T} \times \mathcal{P}, \quad (1)
$$

$^1\tau(Q) = \infty$ means that purchasing $Q$ units is not allowed.

$^2$By definition, if we allow firm 1 to choose an irregular tariff, the whole game has no SPE.
\[ p^*(\tau) \in \arg\max_{p \in P} \{(p - c) \cdot q^*_2(\tau, p)\} \quad \forall \tau \in \mathcal{T}, \quad (2) \]

\[ \tau^* \in \arg\max_{\tau \in \mathcal{T}} \{\tau(q^*_1(\tau, p^*(\tau))) - c \cdot q^*_1(\tau, p^*(\tau))\}. \quad (3) \]

We make the following two assumptions.

**Assumption 1.** \( u : \mathbb{R}_+ \to \mathbb{R} \) is twice continuously differentiable, satisfies \( u(0) = 0, \ u''(\cdot) < 0, \ u'(0) > c, \) and there exists a unique \( q^c > 0 \) such that \( u'(q^c) = c. \)

The quantity demanded by the buyer at per-unit price \( p \) is \( D(p) \equiv \arg\max_{q \geq 0} \{u(q) - pq\}, \) and the monopoly profit under linear pricing is \( \pi(p) \equiv (p - c)D(p). \) Assumption \([1]\) implies that \( D(\cdot) \) and \( \pi(\cdot) \) are continuously differentiable and \( D(\cdot) \) is strictly decreasing on \([c, u'(0)].\)

**Assumption 2.** The monopoly profit function \( \pi(\cdot) \) is strictly concave on \([c, u'(0)].\)

Assumption \([2]\) implies that there is a unique optimal monopoly price \( p^m \in (c, u'(0)) \) given by \( \pi'(p^m) = 0. \)

### 3 Equivalence between the SPE and a Principal-Agent Problem

If we were going to use the standard method, backward induction, to solve the SPE, we would need to solve the three players’ sequential rationality conditions, i.e., \([1]\), \([2]\), and \([3]\). However, those conditions are hard to solve directly. Specifically, it is impossible to pin down \( q^*(\tau, p) \) in \([1]\) without knowing \( \tau(\cdot) \), which can be anything and is to be determined. In the following, we shall transform our original problem into a one-principle-two-agents problem, which allows us to determine SPE outcomes.

#### 3.1 Buyer’s problem

In the last stage, given the two firms’ offers \( \tau \in \mathcal{T} \) and \( p \in \mathcal{P}, \) the buyer’s maximization problem in \([1]\) can be decomposed into two sub-stages: in the first sub-stage, for any
given $Q$, the buyer chooses the purchase $q$ from firm 2, i.e.,

$$V(Q,p) \equiv \max_{q \in [0,k]} \{u(Q + q) - pq\}, \quad (4)$$

and one may call $V(\cdot, \cdot)$ the conditional payoff of the buyer if he is endowed with $Q$ units and can buy at most $k$ more units at price $p$; in the second sub-stage, the buyer chooses the purchase $Q$ from firm 1, i.e.,

$$\max_{Q \geq 0} \{V(Q,p) - \tau(Q)\}.$$

For any closed interval $X \subset \mathbb{R}$ and any point $x \in \mathbb{R}$, let $\text{Proj}_X(x)$ denote the projection of $x$ on $X$, that is, $\arg\min_{y \in X} |y - x|$. Note that the first sub-stage maximization problem (4) has a unique maximizer

$$\text{Proj}_{[0,k]}(D(p) - Q) = \max \{\min \{D(p) - Q, k\}, 0\}.$$ 

### 3.2 Dominant firm’s mechanism-design problem

Let $\pi(Q, \cdot)$ denote firm 2’s profit function conditional on the buyer’s purchase from firm 1 being $Q$, i.e.,

$$\pi(Q, p) \equiv (p - c) \text{Proj}_{[0,k]}(D(p) - Q).$$

Now we are ready to formulate a mechanism-design problem that allows us to determine SPE outcomes. Observe that, every tariff $\tau \in \mathcal{T}$ firm 1 might offer induces a continuation subgame in which firm 2 and the buyer sequentially choose their actions. When choosing $\tau$, firm 1 understands that firm 2 and the buyer would play a SPE of the continuation subgame. Given $\tau$, the buyer would optimally choose some purchase $Q(p) \geq 0$ from firm 1, contingent on any possible price $p \in \mathcal{P}$ chosen by firm 2. The payment for this purchase is thus $\tau(Q(p)) \equiv T(p)$. Given that the buyer’s optimal purchase from firm 1 is $Q(p)$, and hence the optimal purchase from firm 2 is $\text{Proj}_{[0,k]}(D(p) - Q(p))$, firm 2 would optimally choose some price $\bar{p} \in \mathcal{P}$. 

7
In the spirit of revelation principle (imagining firm 1 asks the buyer to report firm 2’s price), solving SPE for the whole game is equivalent to solving the following constrained optimization problem (OP1), over a quantity function $Q : \mathcal{P} \to \mathbb{R}_+$, a payment function $T : \mathcal{P} \to \mathbb{R}$, and recommendation of firm 2’s price $\bar{p} \in \mathcal{P}$:

$$\max_{Q(p), T(p), \bar{p}} T(\bar{p}) - c \cdot Q(\bar{p})$$  \hspace{1cm} (OP1)

subject to

$$V(Q(p), p) - T(p) \geq V(Q(\bar{p}), p) - T(\bar{p}) \quad \forall p, \bar{p} \in \mathcal{P}$$  \hspace{1cm} (5)

$$V(Q(p), p) - T(p) \geq V(0, p) \quad \forall p \in \mathcal{P}$$  \hspace{1cm} (6)

$$\pi(Q(\bar{p}), \bar{p}) \geq \pi(Q(p), p) \quad \forall p \in \mathcal{P}.$$  \hspace{1cm} (7)

The equivalence is formalized by the following theorem.

**Theorem 1. (Equivalence)** Take any $Q^\ast : \mathcal{P} \to \mathbb{R}_+$, $T^\ast : \mathcal{P} \to \mathbb{R}$, and $\bar{p}^\ast \in \mathcal{P}$. $(Q^\ast(\cdot), T^\ast(\cdot), \bar{p}^\ast)$ is a solution of (OP1) if and only if there is a SPE $(\tau^\ast, p^\ast, q^\ast)$ such that

$$Q^\ast(p) = q_1^\ast(\tau^\ast, p) \quad \forall p \in \mathcal{P},$$  \hspace{1cm} (8)

$$\text{Proj}_{[0, k]}(D(p) - Q^\ast(p)) = q_2^\ast(\tau^\ast, p) \quad \forall p \in \mathcal{P},$$  \hspace{1cm} (9)

$$T^\ast(p) = \tau^\ast(Q^\ast(p)) \quad \forall p \in \mathcal{P},$$  \hspace{1cm} (10)

$$\bar{p}^\ast = p^\ast(\tau^\ast).$$  \hspace{1cm} (11)

**4 Equilibrium Characterization**

**4.1 Characterizing Constraints for Buyer**

This subsection characterizes the constraints for the Buyer: (5) and (6).

Recall that $V(Q, p) \equiv \max_{q \in [0,k]} \{ u(Q + q) - pq \}$. By Milgrom-Segal Envelope Theorem (Milgrom and Segal, 2002), $V(Q, \cdot)$ and $V(\cdot, p)$ are absolutely continuous.
on any compact interval, and
\[ V_p(Q, p) = - \text{Proj}_{[0,k]}(D(p) - Q), \]  \hspace{1cm} (12)\]
\[ V_Q(Q, p) = u'(\text{Proj}_{[Q,Q+k]}(D(p))) = \text{Proj}_{[u'(Q+k),u'(Q)]}(p). \] \hspace{1cm} (13)\]

It is easy to see that, $V_p$ and $V_Q$ are absolutely continuous on any compact rectangle, $V(\cdot, \cdot)$ is continuously differentiable, and satisfies weak increasing differences as follows:
\[ V_{Q\theta}(Q, p) = V_{pQ}(Q, p) = \begin{cases} 1 & \text{if } D(p) - k \leq Q \leq D(p) \\ 0 & \text{otherwise} \end{cases} . \]

**Lemma 1. (Buyer’s Constraints)** Take any $Q : P \to \mathbb{R}_+$ and $T : P \to \mathbb{R}$. $Q(\cdot), T(\cdot)$ satisfy (5) and (6) if and only if the following conditions hold:

\[ Q(p) \text{ is non-decreasing in } p \text{ on } \Phi \equiv \{ p \in P : D(p) - k \leq Q(p) \leq D(p) \} \]

\[ \forall p \in P, \ T(p) - T(c) = V(Q(p), p) - V(Q(c), c) - \int_c^p V_p(Q(t), t)dt. \] \hspace{1cm} (15)\]
\[ V(Q(c), c) - T(c) \geq V(0, c) \] \hspace{1cm} (16)\]

(14) is a weakened version of the standard monotonicity condition: monotonicity is required only on region $\Phi$. It is weakened because the increasing differences property of $V$ is not always strict. So in contrast to standard mechanism-design problem, we do allow $Q(\cdot)$ to be decreasing, but only in a particular way. Namely, whenever $p_1 < p_2$ and $Q(p_1) > Q(p_2)$, the rectangle $[Q(p_2), Q(p_1)] \times [p_1, p_2]$ must not intersect region $\Phi$. Such weakened monotonicity implies the following lemma.

**Corollary 1.** (14) implies $\text{Proj}_{[0,k]}(D(p) - Q(p))$ is non-increasing in $p$ on $P$.

(15) is the Envelope formula for payment in standard mechanism-design problem. (16) is a sufficient and necessary condition for (6), which is a result of $V(Q(p), p) -
Using (15), firm 1’s profit in equilibrium is

$$\Pi_1 = T(\bar{p}) - c \cdot Q(\bar{p})$$

$$= T(c) + V(Q(\bar{p}), \bar{p}) - V(Q(c), c) - \int_{c}^{\bar{p}} V_p(Q(t), t) dt - c \cdot Q(\bar{p}).$$

It follows that (16) must be binding, otherwise firm 1 can increase its profit by increasing $T(c)$, i.e.,

$$T(c) = V(Q(c), c) - V(0, c). \quad (17)$$

Hence, substituting (17) into firm 1’s profit, we have

$$\Pi_1 = V(Q(\bar{p}), \bar{p}) - V(0, c) - \int_{c}^{\bar{p}} V_p(Q(t), t) dt - c \cdot Q(\bar{p})$$

$$= V(Q(\bar{p}), c) - V(0, c) + V(Q(\bar{p}), \bar{p}) - V(Q(\bar{p}), c) - \int_{c}^{\bar{p}} V_p(Q(t), t) dt - c \cdot Q(\bar{p})$$

$$= \int_{0}^{q(\bar{p})} [V_Q(Q, c) - c] dQ + \int_{c}^{\bar{p}} [V_p(Q(\bar{p}), t) - V_p(Q(t), t)] dt \quad (18)$$

By Lemma 1 after substituting $T(p)$ from (15) and $T(c)$ from (17) (i.e., the binding (16)), (OP1) can be rewritten as

$$\max_{Q(p), \bar{p}} \int_{0}^{q(\bar{p})} [V_Q(Q, c) - c] dQ + \int_{c}^{\bar{p}} [V_p(Q(\bar{p}), t) - V_p(Q(t), t)] dt \quad (OP1')$$

subject to (14) and (7).

### 4.2 Constraints for Firm 2

To deal with constraint (7), now we take a close look at firm 2’s conditional profit $\pi(Q, p) = (p - c) \text{Proj}_{[0, k]}(D(p) - Q)$. From Assumption 2, $\pi(Q, \cdot)$ is concave on $\{p : \pi(Q, p) > 0\}$ for every $Q \geq 0$. Figure 1 shows firm 2’s iso-profit curves (or the level curves of $\pi(Q, p)$). If firm 2 does not have capacity constraint (i.e., $k \geq q^c$), firm
2’s iso-profit curves are the same as the level curves of $\pi(p) - (p - c)Q$, whose slopes are $(p - c)/(\pi'(p) - Q)$, as shown in Figure 1a. When firm 2 has capacity constraint (i.e., $k < q^e$), the iso-profit curves are flat when $Q < D(p) - k$, and coincide the level curves of $\pi(p) - (p - c)Q$ otherwise, as shown in Figure 1b. We denote firm 2’s profit at $\bar{p}$ as

$$\Pi_2 = (\bar{p} - c) \text{Proj}_{[0,k]}(D(\bar{p}) - Q(\bar{p})).$$

Then constraint (7) $\Pi_2 \geq \pi(Q(p), p), \forall p \in P$ is illustrated as the shaded areas in Figure 1. Note that the largest feasible $\Pi_2$ is $\pi(\max\{p^m, u'(k)\})$.

### 4.3 Equilibrium

By virtue of Theorem 1, we reduce our job of finding SPE to finding the solution to (OP1'). Our strategy of solving (OP1') is as follows. We decompose (OP1') into two stages: in the first stage, for any given $\Pi_2$, $Q(\cdot)$ and $\bar{p}$ are chosen contingent on $\Pi_2$; in the second stage, optimal $\Pi_2$ is chosen. Lemma 2 below solves the first stage for any feasible $\Pi_2 > 0$, and Lemma 3 below solves the second stage to pin down $\Pi_2$. 

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**Figure 1: Firm 2’s Iso-profit Curves**

(a) when $k \geq q^e$

(b) when $k < q^e$
Note firm 1’s profit, as given by (18), can be written as

$$\Pi_1 = \int_0^{Q(\bar{p})} [\text{Proj}_{[u'(Q+k),u'(Q)')}(-c)]dQ + \int_{\bar{p}}^{Q(\bar{p})} [\text{Proj}_{[0,k]}(D(p) - Q(p)) - \text{Proj}_{[0,k]}(D(p) - Q(\bar{p}))]dp \quad (\because (12) \text{and } (13))$$

$$= \int_0^{Q(\bar{p})} [\text{Proj}_{[u'(Q+k),u'(Q)')}(-c)]dQ + \int_{\bar{p}}^{Q(\bar{p})} [\text{Proj}_{[D(p)-k,D(p)\cap D(p)-k,D(p)]}(Q(\bar{p}) - \text{Proj}_{[D(p)-k,D(p)\cap D(p)-k,D(p)]}Q(p))]dp$$

(20)

Figure 2 shows the area of $\Pi_1$ given by (20) for an example of $Q(\cdot)$ and $\bar{p}$: Area A and Area B correspond to the first and the second integral in (20) respectively. It is worth noting that Figure 2 demonstrates that what really matters for $\Pi_1$ is the part of $Q(\cdot)$ in region $\Phi$. Hence, we denote the intersection point of $Q(p)$ and $\max\{D(p) - k, 0\}$ as $(Q_0, x_0)$, i.e.,

$$\max\{D(x_0) - k, 0\} = D(x_0) - \frac{\Pi_2}{x_0 - c}$$

(21)

$$Q_0 = \max\{D(x_0) - k, 0\}. \quad (22)$$

It can be seen from Figures 1 and 2 that, given a $\Pi_2$ and hence a firm 2’s iso-profit
Figure 3: Optimal $Q(p)$ contingent on $\Pi_2$

curve, in order to maximize $\Pi_1$ subject to (14) and (7), (i) the function $Q(\cdot)$ must lie on the iso-profit curve in region $\Phi$, (ii) the point $(Q(\bar{p}), \bar{p})$ must be chosen to be the most rightward point on the firm 2’s iso-profit curve. Lemma 2 below formalizes these claims. Figures 3a and 3b graphically show the partial solutions contingent on $\Pi_2$ for two examples when firm 2’s capacity is large and small, respectively.

**Lemma 2.** Contingent on any $\Pi_2 \in (0, \pi(\max\{p^m, u'(k)\}))$, there exist solutions $(Q(\cdot), \bar{p})$ of (OP1). Any such contingent solution satisfies

$$Q(p) = D(p) - \frac{\Pi_2}{p - c} \quad \forall p \in [x_0, \bar{p}],$$

(23)

and $\bar{p}$ is the unique solution of

$$\max\{D(\bar{p}) - k, \pi'(\bar{p})\} = D(\bar{p}) - \frac{\Pi_2}{\bar{p} - c},$$

(24)

where $x_0$ and $Q_0$ are given by (21) and (22), respectively.

To solve (OP1), it remains to pin down $\Pi_2$, which should be chosen to make the $\Pi_1$ area in Figure 3 as large as possible. The corresponding first-order condition can be simplified as (25) below. Once a solution $(Q(\cdot), \bar{p}, \Pi_2)$ of (OP1) is obtained, we can use the equivalence between (OP1) and (OP1) to obtain a solution $(Q(\cdot), T(\cdot), \bar{p})$.
Lemma 3. \( \text{[OP1]} \) has at least one solution. For any such solution \((Q(\cdot), \bar{p}, \Pi_2)\), \((\bar{p}, x_0)\) are determined by

\[
\bar{p} - c = e \cdot (x_0 - c) > 0, \\
(\bar{p} - c)(D(\bar{p}) - \pi'(\bar{p})) = (x_0 - c) \min \{D(x_0), k\},
\]

and

\[
\Pi_2 = (\bar{p} - c)(D(\bar{p}) - \pi'(\bar{p})), \\
\bar{Q} = \pi'(\bar{p}),
\]

\(Q_0\) is given by \(\text{[22]}\).

A solution of \(Q(\cdot)\) is given by

\[
Q(p) = \begin{cases} 
D(p) - \frac{\Pi_2}{p-c} & \text{if } p \in [x_0, \bar{p}] \\
Q_0 & \text{if } c \leq p < x_0 \\
\bar{Q} & \text{if } \bar{p} < p \leq u'(0)
\end{cases},
\]

and the \(T(\cdot)\) satisfies

\[
T(p) = u(Q_0 + k) - u(k) + \int_{x_0}^{p} t dQ(t) \quad \forall p \in [x_0, \bar{p}].
\]

Finally, we can use the equivalence between solving SPE of the original game and solving \(\text{[OP1]}\) or \(\text{[OP1]}\) established in Theorem 1 to characterize the equilibrium outcome of the original game. Figure 4 illustrates the features of an equilibrium tariff offered firm 1.

Theorem 2. (SPE) There exists at least one equilibrium. In any equilibrium, \((\Pi_2, \bar{p}, x_0, \bar{Q}, Q_0)\) solves \(\text{[22]}, \text{[25]}, \text{[28]}\). Firm 2 chooses \(p = \bar{p}\), and the buyer purchases \(\bar{Q}\) units and \(D(\bar{p}) - \bar{Q} < k\) units from firm 1 and firm 2 respectively.
An equilibrium tariff $\tau(\cdot)$ offered by firm 1 can be constructed as

$$\tau(Q) = \begin{cases} 
    u(Q_0 + k) - u(k) + \int_{Q_0}^{Q} x(\bar{Q})d\bar{Q} & \text{if } Q \in [Q_0, \bar{Q}] \\
    0 & \text{if } Q = 0 \\
    \infty & \text{otherwise}
\end{cases}, \quad (31)$$

where $x(\cdot)$ on $[Q_0, \bar{Q}]$ is the inverse of $Q(\cdot)$ on $[x_0, \bar{p}]$ given by (23).

[Intuition to be added: What’s wrong if firm 1 deletes all quantities but $Q$? Why strictly convex? Why stop at $Q_0$?]

Strictly speaking, equilibrium is never unique because $Q(p), T(p)$ for $p \notin [x_0, \bar{p}]$ and hence $\tau(Q)$ for $Q \notin [Q_0, \bar{Q}]$ are not unique. As demonstrated in Figure 2, only the part of $Q(p)$ for $p \in [x_0, \bar{p}]$ matters. In Theorem 2, equilibrium $\tau(Q)$ for $Q \notin [Q_0, \bar{Q}] \cup \{0\}$ can be anything large enough. However, those values do not affect the allocation.

We say the equilibrium is essentially unique if the equilibrium objects $\Pi_1, \Pi_2, \bar{p}, x_0, \bar{Q}, Q_0$ and hence $Q(p)$ for $p \in [x_0, \bar{p}]$ are unique. The following proposition provides a simple sufficient condition for the uniqueness.

**Proposition 1. (Uniqueness)** The equilibrium is essentially unique if one of the following two equivalent conditions is satisfied:

$$u'(q) - c \text{ is strictly log-concave in } q \text{ on } [0, q^e]; \quad (32)$$

$$-(p - c)D'(p) \text{ is strictly increasing in } p \text{ on } \mathcal{P}. \quad (33)$$

Note that $-(p - c)D'(p) = D(p) - \pi'(p)$. Thus, a graphical interpretation of condition (32) is that, for any $k$, the curve $\pi'(p) = Q$ (or $D(p) + (p - c)D'(p) = Q$) and the curve $D(p) - k = Q$ cross at most once, as shown in Figure 1.
5 Implications of the equilibrium

5.1 Other properties of the equilibrium

Corollary 2. (Increasing and Convex Tariff) In any equilibrium, firm 1’s tariff $\tau$ is strictly increasing and strictly convex on $[Q_0, \bar{Q}]$.

A typical equilibrium tariff is shown in Figure 4.

Interestingly, as stated in the following corollary, what firm 2 and the buyer jointly earn in equilibrium is equal to their joint outside option under the counterfactual situation that firm 2’s unit cost was raised to $x_0$.

Corollary 3. In any equilibrium,

$$\Pi_2 + BS = \int_{x_0}^{\infty} \min\{D(p), k\} dp$$

$$= u(\min\{D(x_0), k\}) - x_0 \cdot \min\{D(x_0), k\}$$

$$= u(D(x_0) - Q_0) - x_0 \cdot (D(x_0) - Q_0).$$

Corollary 4. There is a unique $\hat{k} \in (D(p^m), q^e)$ such that $Q_0 = 0$ in equilibrium if and only if $k \geq \hat{k}$. The set of equilibria is independent of $k$ on $[\hat{k}, \infty]$.

The comparative statics for $k$ are as follows.
(a) The equilibrium objects $\Pi_2, \bar{p}, x_0, \bar{p} - x_0$ (and also $D(\bar{p}) - \bar{Q}$ if we assume condition (33)) are increasing in $k$ on $(0, \hat{k}]$.

(b) The equilibrium objects $\Pi_1, \bar{Q}, Q_0, TS$ are decreasing in $k$ on $(0, \hat{k}]$.

(c) The equilibrium objects $\Pi_2 + BS$ and $BS$ are increasing in $k$ when $k$ is small, and are decreasing in $k$ when $k$ is close to but below $\hat{k}$.

5.2 Comparing with linear pricing

Consider a game that is similar to the one we presented in Section 2, except that firm 1 can only offer a unit price (linear pricing, or LP for short). First, firm 1 offers a unit price $p_1 \in \mathbb{R}_+$. Second, after observing $p_1$, firm 2 offers a unit price $p_2 \in \mathbb{R}_+$. Third, after observing $p_1$ and $p_2$, the buyer chooses the quantities $q_1 \in \mathbb{R}_+$ and $q_2 \in [0, k]$ he buys from firm 1 and from firm 2. Call it the LP vs LP game, and the game presented in Section 2 the NLP vs LP game. We use superscript “LP” to denote various variables for the LP vs LP game.

Proposition 2. (LP vs LP Equilibrium) Consider the LP vs LP game. If $k < q^e$, then there is a unique SPE outcome, in which both firms offer $\bar{p}$, where $\pi'(\bar{p}) = k$, and the buyer purchases $q_1^{LP} = D(\bar{p}) - k$ and $q_2^{LP} = k$ units from firm 1 and firm 2 respectively. If $k \geq q^e$, then there are multiple SPE outcome, in which the prevailing price can be any $\bar{p}^{LP} \in [c, p^m]$ (either $p_1 = p_2 = \bar{p}^{LP} \in [c, p^m]$ or $p_1 \geq p^m = p_2$) and firm 1 makes no sales.

Corollary 5. (Comparative Statics for LP vs LP) (a) $\bar{p}^{LP}, \Pi_1^{LP} + \Pi_2^{LP}, \Pi_1^{LP}$ (and also $q_1^{LP}$ if we assume condition (33)) are decreasing in $k$ on $(0, q^e)$.

(b) $TS^{LP}, q_2^{LP}, q_1^{LP} + q_2^{LP}, BS^{LP}, \Pi_2^{LP} + BS^{LP}$ are increasing in $k$ on $(0, q^e)$.

(c) $\Pi_2^{LP}$ is increasing in $k$ when $k$ is small, and is decreasing in $k$ when $k$ is close to but below $q^e$.

ADD INTERPRETATION

Proposition 3. (Comparison) Let $k \in (0, q^e)$ and compare any SPE outcome of the NLP vs LP game with the unique SPE outcome of the LP vs LP game.
(a) $D(\bar{p}) - Q < q_2^{LP} = k$, $\Pi_1 > \Pi_1^{LP}$, $\Pi_2 + BS < \Pi_2^{LP} + BS^{LP}$.
(b) $\bar{p} < \bar{p}^{LP}$, $D(\bar{p}) > D(\bar{p}^{LP}) = q_1^{LP} + q_2^{LP}$, $TS > TS^{LP}$, $\Pi_2 < \Pi_2^{LP}$ when $k$ is small, and the opposite is true when $k \in [\hat{k}, q^e)$.
(c) $BS < BS^{LP}$ when $k$ is small or $k \in [\hat{k}, q^e)$.
(d) $Q > q_1^{LP} = D(\bar{p}^{LP}) - k$ when $k$ is small or close to $q^e$.

[Intuition to be added: Why $\Pi_2$ gets higher under NLP than LP when $k$ is large? (softening competition.) Why $\Pi_2 > \Pi_1$ under NLP when $k$ is large? (Second-mover advantage.)]

5.3 A linear demand example

This subsection considers a linear demand example. Suppose that $u(q) = q - q^2/2$ and $c \in [0, 1)$. Then $D(p) = 1 - p$, $\pi(p) = (p - c)(1 - p)$, and $\pi'(p) = 1 + c - 2p$ for all $p \in P = [c, 1]$. Assumptions 1, 2 and the conditions in Proposition 1 are satisfied, so that the equilibrium is essentially unique.

Substituting (25) into (26), the latter becomes

$$p - c = \frac{1}{e} \cdot \min \left\{ 1 - c - \frac{p - c}{e}, k \right\}.$$ 

So

$$\bar{p} = c + \frac{1}{e} \cdot \min \{k, \hat{k}\},$$

where

$$\hat{k} = \frac{e^2(1 - c)}{1 + e^2}.$$ 

Other endogenous objects follow directly, and the solution is listed in Table 1.

As we claim generally in Corollary 1, all the above objects except $\Pi_2 + BS$ and $BS$ are monotone in $k$. When $k < \frac{e^2(1-c)}{2+e^2}$, $\Pi_2 + BS$ is increasing in $k$. When $\frac{e^2(1-c)}{2+e^2} < k < \hat{k}$, $\Pi_2 + BS$ is decreasing in $k$. When $k < \frac{e^2(1-c)}{4+e^2}$, $BS$ is increasing.
\[
\begin{align*}
\text{Pricing} & \quad x_0 & \quad Q_0 & \quad \bar{p} & \quad \bar{Q} \\
& \quad c + \frac{1}{e^2} \min\{k, \hat{k}\} & \quad \frac{1 + e^2}{e^2} \max\{\hat{k} - k, 0\} & \quad c + \frac{1}{e} \min\{k, \hat{k}\} & \quad 1 - c - \frac{2}{e} \min\{k, \hat{k}\}
\end{align*}
\]

\[
\begin{align*}
\text{Surplus} & \quad \Pi_1 & \quad \Pi_2 \\
& \quad \frac{(1-c)^2}{2(1+e^2)} + \frac{1+e^2}{2e^2}(\max\{\hat{k} - k, 0\})^2 & \quad \frac{1}{e^2}(\min\{k, \hat{k}\})^2 \\
& \quad (1 - c) \min\{k, \hat{k}\} - \frac{4+e^2}{2e^2}(\min\{k, \hat{k}\})^2 & \quad \frac{(1-c)^2}{2} - \frac{1}{2e^2}(\min\{k, \hat{k}\})^2
\end{align*}
\]

Table 1: Linear Demand Example

Figure 5: BS and $\Pi_2 + BS$ for Linear Demand Example
Figure 6: Dominant firm’s equilibrium tariff schedules and the corresponding chosen purchases under assumptions $D(p) = 1 - p$, $c = 0$, and $k = 0.9$ or $0.2$, when nonlinear pricing (NLP), quantity forcing (QF), or linear pricing (LP) is feasible to the dominant firm (the LP schedule is omitted in the right panel because its scale is far below that of the NLP schedule)

in $k$. When $\frac{e^2(1-c)}{4+e^2} < k < \hat{k}$, $BS$ is decreasing in $k$. Figure 5 demonstrates these non-monotone patterns.

Figure 6 and Table 2 show various equilibrium objects in the linear demand example, when nonlinear pricing (NLP), quantity forcing (QF) (i.e. offering a take-it-or-leave-it quantity-payment bundle), or linear pricing (LP) is feasible to the dominant firm.

6 Concluding remarks

Recall that our model involves three kinds of asymmetries between the two firms: (1) the dominant firm is able to make nonlinear tariff schedules, while the rival firm can only choose linear pricing schemes; (2) the dominant firm commits to offering tariffs before its rival; and (3) relatively to the demand size the dominant firm has no capacity limit while its rival is capacity-constrained. Our analysis above suggests that the asymmetry in capacity is not crucial for the equilibrium adoption of nonlinear pricing by the dominant firm, but is important for the results of partial foreclosure and harming the buyer welfare.
<table>
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<tr>
<th></th>
<th>$q_1$</th>
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<th>$\Pi_2$</th>
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<th>$TS$</th>
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<td>0.2829</td>
<td>0.4475</td>
</tr>
</tbody>
</table>

Table 2: Equilibrium outputs ($q_1, q_2$), profits ($\Pi_1, \Pi_2$), buyer’s surplus ($BS$), and total surplus ($TS$) under assumptions $D(p) = 1 - p, c = 0$, and $k = 0.9$ or $0.2$, when nonlinear pricing (NLP), quantity forcing (QF), or linear pricing (LP) is feasible to the dominant firm.

What would happen if we relax our assumptions about the asymmetry between the two firms by endogenizing the choices of timing and tariff options? One may consider a 4-stage extended game as follows. In Stage 0, each firm simultaneously decides whether to commit itself to use linear pricing. Any firm who makes this commitment can only offer a linear pricing scheme in later stages; and otherwise can more generally offer a nonlinear tariff schedule in later stages. In Stage 1, each firm can either offer a tariff (from the feasible set determined by its choice in stage 0), or wait until stage 2. In stage 2, any firm who chose waiting in stage 1 has to offer a tariff (again from the feasible set determined by its choice in stage 0). Lastly, in stage 3, the buyer chooses the quantities she purchases from the two firms. We can show that this extended game has a subgame perfect equilibrium with the following properties: only the rival firm commits itself to linear pricing in stage 0, the dominant firm offers a nonlinear tariff in stage 1, the rival firm waits in stage 1 and offers a linear tariff in stage 2, and their offers and the buyer’s choices are the same as those we characterized for our original 3-stage game. As a result, when both firms can choose their timing and pricing options the equilibrium outcome in the original 3-stage game remains to be part of the subgame perfect equilibrium outcome in the extended game. This demonstrates that our assumptions regarding the sequence
of the moves and asymmetry in tariff options are not crucial for our main results. The asymmetry in capacity between the firms allows the unconstrained firm to take advantage of a menu of tariff offers in order to restrict the choices of the constrained firm and extract surpluses from the buyer.
Appendix

The proof of Theorem 1 requires the following two lemmas.

Lemma A.1. For any \( Q : \mathcal{P} \to \mathbb{R}_+ \), \( T : \mathcal{P} \to \mathbb{R} \), and \( \tilde{p} \in \mathcal{P} \) that satisfy (5), (6), and (7), there is a \( \tau \in \mathcal{T} \) and a SPE of the subgame after firm 1 offers \( \tau \) such that

(i) in this SPE of the subgame, firm 2 chooses \( p = \tilde{p} \), and the buyer, contingent on any firm 2’s unit price \( p \in \mathcal{P} \), chooses to buy \( Q(p) \) and \( \text{Proj}[0,k](D(p) - Q(p)) \) units from firm 1 and firm 2 respectively, and

(ii) \( \tau(Q(p)) = T(p) \) for all \( p \in \mathcal{P} \).

Proof of Lemma A.1. Suppose that \( Q : \mathcal{P} \to \mathbb{R}_+ \), \( T : \mathcal{P} \to \mathbb{R} \), and \( \tilde{p} \in \mathcal{P} \) satisfy (5), (6), and (7). Define

\[
\tau(Q) = \begin{cases} 
T(p) & \text{if } \exists p \in \mathcal{P} \text{ s.t. } Q(p) = Q \\
0 & \text{if } Q = 0 \text{ and } \exists p \in \mathcal{P} \text{ s.t. } Q(p) = 0 \\
\infty & \text{otherwise}
\end{cases} \tag{A1}
\]

Note that (5) implies \( T(p) \leq T(\tilde{p}), \forall p, \tilde{p} \in \mathcal{P} \) whenever \( Q(p) = Q(\tilde{p}) \). So we must have \( T(p) = T(\tilde{p}), \forall p, \tilde{p} \in \mathcal{P} \) whenever \( Q(p) = Q(\tilde{p}) \). Hence, the above \( \tau \) is well defined. Clearly, (ii) holds. To see that \( \tau(0) \leq 0 \), note that if \( \exists p \in \mathcal{P} \) s.t. \( Q(p) = 0 \), then \( \tau(0) = 0 \); if \( \exists \hat{p} \in \mathcal{P} \) s.t \( Q(\hat{p}) = 0 \), then \( \tau(0) = T(\hat{p}) \leq 0 \), where the inequality follows from (6). Thus, \( \tau(0) \leq 0 \).

Given this \( \tau \) and any \( p \in \mathcal{P} \), (5) and (6) imply that a buyer’s optimal action is to buy \( Q(p) \) and \( \text{Proj}[0,k](D(p) - Q(p)) \) units from firm 1 and firm 2 respectively. Given \( \tau \) and that the buyer uses the above strategy, (7) implies that a firm 2’s optimal action is to choose \( p = \tilde{p} \). Therefore, the strategies in (i) constitute a SPE of the subgame after firm 1 offers \( \tau \). It follows that \( \tau \) is regular and hence \( \tau \in \mathcal{T} \). \( \blacksquare \)

Lemma A.2. For any \( \tau \in \mathcal{T} \) and any SPE of the subgame after firm 1 offers \( \tau \), if \( Q : \mathcal{P} \to \mathbb{R}_+ \), \( T : \mathcal{P} \to \mathbb{R} \), and \( \tilde{p} \in \mathcal{P} \) satisfy (i) and (ii) in Lemma A.1, then \( Q(\cdot), T(\cdot), \tilde{p} \) also satisfy (5), (6), and (7).

Proof of Lemma A.2. Take any \( \tau \in \mathcal{T} \) and any SPE of the subgame after firm 1 offers \( \tau \). Suppose that \( Q : \mathcal{P} \to \mathbb{R}_+ \), \( T : \mathcal{P} \to \mathbb{R} \), and \( \tilde{p} \in \mathcal{P} \) satisfy (i) and (ii) in
Lemma A.1. Since the strategies described in (i) constitute a SPE of the subgame after firm 1 offers \( \tau \), we have (7) and

\[
V(Q(p), p) - \tau(Q(p)) \geq V(Q, p) - \tau(Q) \quad \forall (Q, p) \in \mathbb{R}_+ \times \mathcal{P}.
\]  

(A2)

To see (5), take \( Q = Q(\bar{p}) \) for arbitrary \( \bar{p} \in \mathcal{P} \) in (A2) and use (ii). To see (6), take \( Q = 0 \) in (A2) and use \( \tau(0) \leq 0 \) and (ii).

**Proof of Theorem 1.** ("only if" part) Suppose that \((Q^*(\cdot), T^*(\cdot), \bar{p}^*)\) is a solution of (OP1). Then \((Q^*(\cdot), T^*(\cdot), \bar{p}^*)\) satisfy (5), (6), and (7). From Lemma A.1, there is a \( \tau^* \in \mathcal{T} \) (defined by (A1) with \( \tau(\cdot), Q(\cdot), T(\cdot) \) replaced by \( \tau^*(\cdot), Q^*(\cdot), T^*(\cdot) \)) such that (10) holds and a SPE \((p^*(\tau^*), q^*(\tau^*, \cdot))\) of the subgame after firm 1 offers \( \tau^* \) is described by (8), (9), and (11).

Next, we show that, \((\tau^*, p^*, q^*)\) is a SPE of the whole game.

In the subgame after firm 1 offers this \( \tau^* \), we let firm 2 and the buyer play the SPE \((p^*(\tau^*), q^*(\tau^*, \cdot))\), so that firm 1’s profit is \( T^*(\bar{p}^*) - c \cdot Q^*(\bar{p}^*) \). In the subgame after firm 1 offers any other \( \tau \in \mathcal{T} \setminus \{\tau^*\} \), we let firm 2 and the buyer play any SPE \((p^*(\tau), q^*(\tau, \cdot))\), which exists because every \( \tau \in \mathcal{T} \) is regular. By such constructions, \( p^*, q^* \) satisfy (1) and (2).

From Lemma A.2, the SPE outcome of the subgame after firm 1 offers an arbitrary \( \tau \in \mathcal{T} \) must be characterized by some \( Q(\cdot), T(\cdot), \bar{p} \) that satisfy (5), (6), and (7), and the associated firm 1’s profit is \( T(\bar{p}) - c \cdot Q(\bar{p}) \). Since \((Q^*(\cdot), T^*(\cdot), \bar{p}^*)\) is a solution of (OP1), firm 1 cannot make strictly higher profit than \( T^*(\bar{p}^*) - c \cdot Q^*(\bar{p}^*) \) by offering any \( \tau \in \mathcal{T} \). That is, \((\tau^*, p^*, q^*)\) satisfies (3) and hence is a SPE of the whole game.

("if" part) Suppose that \((\tau^*, p^*, q^*)\) is a SPE and \( Q^*(\cdot), T^*(\cdot), \bar{p}^* \) satisfy (8), (9), (10), and (11). From Lemma A.2, \( Q^*(\cdot), T^*(\cdot), \bar{p}^* \) satisfy (5), (6), and (7). Suppose, by way of contradiction, that \((Q^*(\cdot), T^*(\cdot), \bar{p}^*)\) is not a solution of (OP1). Then, there is some \((Q^0(\cdot), T^0(\cdot), \bar{p}^0)\) satisfying (5), (6), and (7), such that \( T^0(\bar{p}^0) - c \cdot Q^0(\bar{p}^0) > T^*(\bar{p}^*) - c \cdot Q^*(\bar{p}^*) \). We shall show that firm 1 then can offer a tariff in \( \mathcal{T} \) that guarantees itself a profit arbitrarily close to \( T^0(\bar{p}^0) - c \cdot Q^0(\bar{p}^0) \) in every SPE of the firm 2-buyer subgame that follows. Once this is proved, offering such a tariff is a firm 1’s profitable deviation in the SPE \((\tau^*, p^*, q^*)\), which is a contradiction.
To do that, we perturb the solution \((Q^0(\cdot), T^0(\cdot), \bar{p}^0)\) so that firm 2 would have to lower its price a bit more if it wishes to increase its sales by any given amount. We can keep \(\bar{p}^0\) unchanged and, for any \(\varepsilon > 0\), let

\[
Q_\varepsilon(p) = \begin{cases} 
Q^0(p) & \text{if } p \geq \bar{p}^0 \\
Q^0(\bar{p}^0) & \text{if } \bar{p}^0 - \varepsilon < p < \bar{p}^0 \\
Q^0(p + \varepsilon) & \text{if } p \leq \bar{p}^0 - \varepsilon,
\end{cases}
\]

and

\[
T_\varepsilon(p) = V(Q_\varepsilon(p), p) - V(0, c) - \int_c^p V_p(Q_\varepsilon(t), t) dt.
\]

Note that \((Q_\varepsilon(\cdot), T_\varepsilon(\cdot), \bar{p}^0)\) satisfies all the constraints of (OP1); the (7) constraint holds strictly at every \(p \neq \bar{p}^0\); the value of (OP1) evaluated at \((Q_\varepsilon(\cdot), T_\varepsilon(\cdot), \bar{p}^0)\) is arbitrarily close to the maximum value \(T^0(\bar{p}^0) - c \cdot Q^0(\bar{p}^0)\) when \(\varepsilon\) is made arbitrarily small.

Define \(\tau_\varepsilon(\cdot)\) by (A1) with \(Q(\cdot)\) and \(T(\cdot)\) replaced by \(Q_\varepsilon(\cdot)\) and \(T_\varepsilon(\cdot)\). Now, if firm 1 offers \(\tau_\varepsilon\), the best responses of the buyer and firm 2 are unique. In particular, firm 2 would surely offer \(\bar{p}^0\); the buyer would surely purchase \(Q_\varepsilon(\bar{p}^0)\) from firm 1; firm 1’s profit would surely be the value of (OP1) evaluated at \((Q_\varepsilon(\cdot), T_\varepsilon(\cdot), \bar{p}^0)\). Therefore, offering \(\tau_\varepsilon\) with small enough \(\varepsilon\) is a firm 1’s profitable deviation as desired. ■

**Proof of Lemma 7.** Let \(U(p) \equiv V(Q(p), p) - T(p)\). Then (5) can be written as

\[
U(p) - U(\bar{p}) \geq V(Q(\bar{p}), p) - V(Q(\bar{p}), \bar{p}) \quad \forall p, \bar{p} \in \mathcal{P},
\]

and (15) can be written as

\[
U(p) - U(c) = \int_c^p V_p(Q(t), t) dt \quad \forall p \in \mathcal{P}.
\]

*(Necessity part)* First, we show that (5) implies (14) and (15).
Suppose (5) is satisfied. Then (A3) implies that, for any \( p_1, p_2 \in \mathcal{P}, \)

\[
V(Q(p_1), p_2) - V(Q(p_1), p_1) \leq U(p_2) - U(p_1) \leq V(Q(p_2), p_2) - V(Q(p_2), p_1). \quad (A5)
\]

If (14) does not hold, then there exist \( p_1, p_2 \in \oplus \) such that \( p_1 < p_2 \) and \( Q(p_1) > Q(p_2). \)

Then we have \( D(p_1) > Q(p_2) \) and \( Q(p_1) > D(p_2) - k, \) where the first one follows from \( D(p_1) \geq Q(p_1) \) and the second one follows from \( Q(p_2) \geq D(p_2) - k. \) But then (A5) implies

\[
0 \geq [V(Q(p_1), p_2) - V(Q(p_1), p_1)] - [V(Q(p_2), p_2) - V(Q(p_2), p_1)]
\]

\[
= \int_{p_1}^{p_2} \int_{Q(p_2)}^{Q(p_1)} V_{pq}(Q,p)dQdp > 0,
\]

which is a contradiction. The last inequality holds because \( V_{pq} \geq 0 \) almost everywhere and, we have \((Q(p_2), p_1)\) is below \( Q = D(p) \) (: \( D(p_1) > Q(p_2) \)) and \((Q(p_1), p_2)\) is above \( Q = D(p) - k \) (: \( Q(p_1) > D(p_2) - k. \)), so there must exist a convex combination \((\hat{Q}, \hat{p})\) between \((Q(p_2), p_1)\) and \((Q(p_1), p_2)\) such that, for all \((Q, p)\) in some open neighborhood of \((\hat{Q}, \hat{p}),\) it holds that \( D(p) - k < Q < D(p) \) and hence \( V_{pq}(Q, p) = 1. \)

Therefore, (14) must hold.

Moreover, (A5) implies (A4). Therefore, (15) holds.

(Sufficiency part) Next, we show that (14) and (15) imply (5).

First, (14) implies that, for all \( p_1, p_2 \in \mathcal{P} \) with \( p_1 \leq p_2, \) we have

\[
\text{Proj}_{[0,k]}(D(p_2) - Q(p_1)) \geq \text{Proj}_{[0,k]}(D(p_2) - Q(p_2)), \quad (A6)
\]

\[
\text{Proj}_{[0,k]}(D(p_1) - Q(p_1)) \geq \text{Proj}_{[0,k]}(D(p_1) - Q(p_2)). \quad (A7)
\]

Indeed, \( p_1 \leq p_2 \) and (14) imply either (i) \( Q(p_1) \leq Q(p_2), \) or (ii) \( D(p_1) \leq Q(p_2), \) or (iii) \( Q(p_1) \leq D(p_2) - k. \) In case (i), clearly (A6) and (A7) hold. In case (ii), we have \( D(p_2) \leq D(p_1) \leq Q(p_2) \) so that the right-hand sides of (A6) and (A7) are 0. In case (iii), we have \( Q(p_1) + k \leq D(p_2) \leq D(p_1) \) so that the left-hand sides of (A6) and (A7) are \( k > 0. \) Therefore, (A6) and (A7) hold in each case.

Recall that (15) is equivalent to (A4). Therefore, for any \( p_1, p_2 \in \mathcal{P} \) (no matter
whether \( p_1 \leq p_2 \) or not), we have

\[
U(p_2) - U(p_1) = \int_{p_1}^{p_2} V_p(Q(p), p) dp
\]

\[
= - \int_{p_1}^{p_2} \text{Proj}_{[0,k]}(D(p) - Q(p)) dp
\]

\[
\geq - \int_{p_1}^{p_2} \text{Proj}_{[0,k]}(D(p) - Q(p_1)) dp \tag{\ref{A6}}
\]

\[
= \int_{p_1}^{p_2} V_p(Q(p_1), p) dp
\]

\[
= V(Q(p_1), p_2) - V(Q(p_1), p_1),
\]

which proves \( \text{(A3)} \) and hence \( \text{(5)} \).

Last, we establish the equivalence between \( \text{(6)} \) and \( \text{(16)} \).

Here we show that \( V(Q(p), p) - T(p) - V(0, p) \) is absolutely continuous and non-decreasing in \( p \) on \( P \). To see this, recall that \( V(Q, \cdot) \) is absolutely continuous and \( V_p(Q, p) = - \text{Proj}_{[0,k]}(D(p) - Q) \). \( \text{\ref{A15}} \), which is equivalent to \( \text{(A4)} \), implies

\[
U(p) - V(0, p) = U(c) + \int_{p}^{0} V_p(Q(t), t) dt - V(0, p).
\]

Thus \( \frac{d(U(p) - V(0, p))}{dp} = V_p(Q(p), p) - V_p(0, p) \geq 0 \) because \( V_{qp}(Q(p), p) \geq 0 \) for all \( p \). It follows that \( \text{(6)} \) is equivalent to \( \text{(16)} \).

\text{Proof of Corollary \ref{7}}. It is implied by \( \text{(A6)} \) in the proof of Lemma \ref{1}.

\text{Proof of Lemma \ref{2}}. Fix any \( \Pi_2 \in (0, \pi(\max\{P^m, u'(k)\})) \) and hence a firm 2's iso-profit curve in the \( Q-p \) space (see Figure 1). Note that \( \Pi_2 > 0 \) implies \( \bar{p} > c \) and \( D(\bar{p}) > Q(\bar{p}) \). Moreover, from Assumption \ref{2} firm 2's iso-profit curves are (horizontally) single-peaked, so each iso-profit curve has a unique most rightward point.

Here we prove \( (Q(\bar{p}), \bar{p}) \) must be the most rightward point (the unique horizontal peak) on the iso-profit curve by contradiction. Suppose that \( (Q(\bar{p}), \bar{p}) \) is not the horizontal peak on the iso-profit curve. Consider the case where \( (Q(\bar{p}), \bar{p}) \) lies on the
strictly decreasing portion of the iso-profit curve (which implies $D(\bar{p}) - Q(\bar{p}) < k$). Then, to satisfy (7), for small $\varepsilon > 0$, we have $Q(\bar{p} - \varepsilon) > Q(\bar{p})$. But then, from Lemma 1 (14) is violated. Now consider the case where $(Q(\bar{p}), \bar{p})$ lies on the non-decreasing portion of the iso-profit curve. Then, $\Pi_1$ can be raised by increasing both $Q(\bar{p})$ and $\bar{p}$ along the iso-profit curve toward the horizontal peak (see Figures 1 and 2). (24) follows immediately.

From Figure 3, it is easy to see $Q(\cdot)$ on $[x_0, \bar{p}]$ must coincide the iso-profit curve, which satisfies (14) and (7), otherwise $\Pi_1$ can be improved by shifting the part of $Q(\cdot)$ on $[x_0, \bar{p}]$ that does not match with the iso-profit curve toward the latter. Thus, we have (23).

**Proof of Lemma 3.** Lemma 2 has characterized the optimal $(Q(\cdot), \bar{p})$ and maximum $\Pi_1$ contingent on any $\Pi_2 \in (0, \pi(\max\{p^m, u'(k)\}))$. Clearly, the maximum $\Pi_1$ contingent on $\Pi_2 = 0$ is equal to the limiting contingent maximum $\Pi_1$ as $\Pi_2 \downarrow 0$ (which is equal to $\pi(\max\{q^e - k, 0\}) - c \cdot \max\{q^e - k, 0\}$), and the maximum $\Pi_1$ contingent on $\Pi_2 = \pi(\max\{p^m, u'(k)\})$ is equal to the limiting contingent maximum $\Pi_1$ as $\Pi_2 \uparrow \pi(\max\{p^m, u'(k)\})$ (which is equal to 0). After reducing the second stage (where $(Q(\cdot), \bar{p})$ is chosen contingent on $\Pi_2$), (OP1) has only one choice variable, $\Pi_2$, and the reduced objective function is continuous in $\Pi_2$ on $[0, \pi(\max\{p^m, u'(k)\})]$. Thus, (OP1) has at least one solution.

If $\Pi_2 = 0$, then the contingent maximum can be raised by increasing $\Pi_2$ (contemplating an upward-and-leftward shift of $Q(\cdot)$ to a higher firm 2’s iso-profit curve in Figure 3). Thus, at any optimum, $\Pi_2 > 0$. On the other hand, if $\Pi_2$ is $\pi(\max\{p^m, u'(k)\})$ or is so large that the contingent solution exhibits $D(\bar{p}) - \bar{Q} = k$, then the contingent maximum can be raised by decreasing $\Pi_2$ (contemplating a downward-and-rightward shift of $Q(\cdot)$ to a lower firm 2’s iso-profit curve in Figure 3 again). Thus, at any optimum, $0 < \Pi_2 < \pi(\max\{p^m, u'(k)\})$ and $D(\bar{p}) - \bar{Q} < k$. So (26)-(29) follow from Lemma 2.
Next, we show (25). From Figure 2 (20) can be rewritten as

\[ \Pi_1 = \int_0^{Q_0} u'(Q + k) dQ + x_0 \cdot (\bar{Q} - Q_0) + \int_{x_0}^{\bar{p}} (\bar{Q} - Q(p)) dp - c\bar{Q} \]

\[ = \int_0^{Q_0} [u'(Q + k) - x_0] dQ + (\bar{p} - c)\bar{Q} - \int_{x_0}^{\bar{p}} Q(p) dp \]

\[ = \int_{x_0}^{\infty} \max\{D(p) - k, 0\} dp + (\bar{p} - c)\bar{Q} - \int_{x_0}^{\bar{p}} \left[ D(p) - \frac{\Pi_2}{p - c} \right] dp (\because (22) and (23)) \]

\[ = \int_{x_0}^{\infty} \max\{D(p) - k, 0\} dp + TS(\bar{p}) - \Pi_2 - \int_{x_0}^{\infty} D(p) dp + \Pi_2 \cdot \ln \frac{\bar{p} - c}{x_0 - c} \]

\[ (\because (\bar{p} - c)\bar{Q} = TS(\bar{p}) - \Pi_2 - \int_{\bar{p}}^{\infty} D(p) dp) \]

\[ = TS(\bar{p}) - \int_{x_0}^{\infty} \min\{D(p), k\} dp + \left( \ln \frac{\bar{p} - c}{x_0 - c} - 1 \right) \Pi_2, \quad (A8) \]

where

\[ TS(\bar{p}) \equiv u(D(\bar{p})) - cD(\bar{p}) = \int_{\bar{p}}^{\infty} D(p) dp + (\bar{p} - c)D(\bar{p}) \quad (A9) \]

denotes the total surplus.

The partial derivatives of (A8) are

\[ \frac{\partial \Pi_1}{\partial \bar{p}} = (\bar{p} - c)D'(\bar{p}) + \frac{\Pi_2}{\bar{p} - c} = \pi'(\bar{p}) - \bar{Q} (\because (23)), \]

\[ \frac{\partial \Pi_1}{\partial x_0} = \min\{D(x_0), k\} - \frac{\Pi_2}{x_0 - c}, \]

\[ \frac{\partial \Pi_1}{\partial \Pi_2} = \ln \frac{\bar{p} - c}{x_0 - c} - 1. \]

Note that (26) (28) imply that \( \partial \Pi_1 / \partial \bar{p} = \partial \Pi_1 / \partial x_0 = 0 \). Therefore, the total derivative of (A8) with respect to \( \Pi_2 \) is

\[ \frac{d\Pi_1}{d\Pi_2} = \ln \frac{\bar{p} - c}{x_0 - c} - 1. \quad (A10) \]

Therefore, the first-order condition \( d\Pi_1 / d\Pi_2 = 0 \) implies (25).
Last, we derive (30). From (15) and (17),

\[ T(p) = V(Q(p), p) - V(0, c) - \int_c^p V_p(Q(t), t) dt \]

\[ = \int_0^{Q(p)} V_Q(Q, c) dQ + \int_c^p [V_p(Q(p), t) - V_p(Q(t), t)] dt \]

\[ = \int_0^{Q(p)} \text{Proj}_{u'(Q+k), u'(Q)}(c) dQ + \int_c^p \text{Proj}_{[0,k]}(D(t) - Q(t)) - \text{Proj}_{[0,k]}(D(t) - Q(p))] dt \]

\[ = \int_0^{Q_0} u'(Q + k)dQ + x_0 \cdot (Q(p) - Q_0) + \int_{x_0}^p (Q(p) - Q(t)) dt \]

\[ = u(Q_0 + k) - u(k) + \int_{x_0}^p t dQ(t) \]

where the third equality follows from (12) and (13), the fourth one follows from 

\[ \text{Proj}_{[u'(Q+k), u'(Q)]}(c) = u'(Q + k) \text{ for } Q < Q(p) \text{ when } p \in [x_0, \bar{p}], \]

\[ \text{Proj}_{[0,k]}(D(t) - Q(t)) - \text{Proj}_{[0,k]}(D(t) - Q(p))] = \begin{cases} 
0 & \text{if } c \leq t < u'(Q(p) + k) \\
Q(p) + k - D(t) & \text{if } u'(Q(p) + k) \leq t \leq x_0 \\
Q(p) - Q(t) & \text{if } x_0 < t \leq \bar{p} 
\end{cases} \]

\[ \int_{u'(Q(p)+k)}^{Q(p)+k} [Q(p) + k - D(t)] dt = \int_{Q_0}^{Q(p)} [x_0 - u'(Q + k)] dQ \text{ through integration by substitution, and the last equality follows from integration by part. } \]

**Proof of Theorem 2.** The results are from Lemma 3 and Theorem 1. (31) is derived from (30) through changing of variable: \( x(\cdot) \) for \( Q(\cdot) \).

Suppose that firm 1’s tariff \( \tau \) is given by (31). Then, firm 2’s profit would be \( \Pi_2 \) if it chooses any \( p \in [x_0, \bar{p}] \). One can see from Figure 3 that, firm 2’s profit would be lower than \( \Pi_2 \) if it chooses any \( p > \bar{p} \) (so that the buyer would still purchase \( \bar{Q} \) units from firm 1) or any \( p < x_0 \) (so that the buyer would purchase \( Q_0 \) units from firm 1).

**Proof of Proposition 1.** From Theorem 2, the equilibrium is essentially unique if and only if the solution of \( (\bar{p}, x_0) \), i.e., \( (25) \sim (26) \), is unique.
Using (25) to eliminate $x_0$ and dividing both sides of (26) by $\bar{p} - c$, (26) becomes

$$-(\bar{p} - c)D'(\bar{p}) = \frac{1}{c} \min \left\{ D \left( c + \frac{\bar{p} - c}{e} \right), k \right\}.$$  \hspace{1cm} (A11)

(A11) can be solved for $\bar{p}$. Therefore, the equilibrium is essentially unique if and only if (A11) has at most one solution. Under condition (33), the left-hand side of (A11) is strictly increasing in $\bar{p}$, and the right-hand side is non-increasing in $\bar{p}$. Therefore, (A11) has at most one solution under (32).

It is easy to see that (32) and (33) are equivalent, so the proposition follows. ■

**Proof of Corollary 2.** From (30) and $T(p) = \tau(Q(p))$, the first line of (31) must hold for any equilibrium $\tau$. Thus, $\tau'(\cdot) = x'(\cdot)$ on $[Q_0, \bar{Q}]$. Since $x'(\cdot)$ on $[Q_0, \bar{Q}]$ is the inverse of $Q'(\cdot)$ on $[x_0, \bar{p}]$, and the latter is positive and strictly increasing on $[Q_0, \bar{Q}]$. The corollary follows. ■

**Proof of Corollary 3.** In equilibrium, the total output is $D(\bar{p})$ and firm 2’s output is $D(\bar{p}) - \bar{Q} < k$. So the buyer’s surplus

$$BS \equiv u(D(\bar{p})) - \bar{p}(D(\bar{p}) - \bar{Q}) - \tau(\bar{Q}) = TS - \Pi_1 - \Pi_2$$

$$= TS - \Pi_1 - \Pi_2 \hspace{1cm} (\because (A9)) \hspace{1cm} (A12)$$

(25) and (A8) give a simple formula to compute $\Pi_1$, i.e.,

$$\Pi_1 = TS - \int_{x_0}^{\infty} \min\{D(p), k\} dp. \hspace{1cm} (A13)$$

So the corollary follows. ■

**Proof of Corollary 4.** Let $\hat{x}_0$ be the minimum equilibrium $x_0$ when $k = \infty$, given by (25) and (26) with $\min \{D(x_0), k\} = D(x_0)$ in (26). Define $\hat{k} \equiv D(\hat{x}_0)$. From Theorem 2, $\hat{k}$ satisfies the first two claims (see Figure 3).

The rest of the proof considers comparative statics for $k \in (0, \hat{k}]$. Following the proof of Lemma 3, we regard $\Pi_1, \bar{p}, \bar{Q}, x_0, Q_0, x(\cdot), BS, TS$ as functions of $\Pi_2$. Here we also regard them as functions of $k$. In particular, we write $\Pi_1(\Pi_2; k)$.

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Fix $\Pi_2$ and let $k$ increase on $(0, \hat{k}]$. Note that $Q_0 = \max\{D(x_0) - k, 0\} > 0$ before the increase, so that we have $D(x_0) > k$ before the increase. The $\bar{p}$ and $\bar{Q}$ determined by (27) and (28) do not change. The $x_0$, $Q_0$, and $\Pi_1$ determined by (26), (22), and (A8) decrease as $k$ increases (see Figure 3).

In equilibrium, $\Pi_1 = \max_{\Pi_2} \{\Pi_1(\Pi_2; k)\}$ decreases, because $\Pi_1(\cdot; k)$ shifts down as $k$ increases. From (A10), we see that $\partial \Pi_1(\Pi_2; k)/\partial \Pi_2$ increases, because $\bar{p}$ is unchanged whereas $x_0$ decreases when we fix $\Pi_2$ and let $k$ increase. In other words, $\Pi_1(\Pi_2; k)$ satisfies strict increasing differences. Therefore, the $\Pi_2$ that maximizes $\Pi_1$ must increase when $k$ increases. Then, from (27) and Assumption 2, $\bar{p}$ must increase, and hence $\bar{Q}$ decreases follows from (28). Then from (A9), $TS$ decreases. From (25), $x_0$ increases. From (22), $Q_0$ decreases. Also, $\bar{p} - x_0$ increases because (25) can be written as $\bar{p} - x_0 = (e - 1)(x_0 - c)$. The result for $D(\bar{p}) - \bar{Q}$ can be immediately seen from $D(\bar{p}) - \bar{Q} = -(\bar{p} - c)D'(\bar{p})$. This completes the proof of parts (a) and (b).

Last, we prove part (c). To see the first half of part (c), note that both $\Pi_2$ and $BS$ are positive and tend to zero as $k \to 0$. To see the second half of part (c), first note that, as shown above, we have $\min\{D(x_0), k\} = k$ when $k \leq \hat{k}$. From Proposition 34, $\Pi_2 + BS = u(k) - x_0 k$ whenever $k \leq \hat{k}$. Hence,

$$\frac{d(\Pi_2 + BS)}{dk}\bigg|_{k \nearrow \hat{k}} = u'(\hat{k}) - x_0 - \hat{k} \cdot \frac{dx_0}{dk}\bigg|_{k \nearrow \hat{k}} < 0.$$  

The last inequality follows from $u'(\hat{k}) - x_0 \leq u'(\hat{k}) - \hat{x}_0 = 0$ and $\frac{dx_0}{dk}\bigg|_{k \nearrow \hat{k}} > 0$. Therefore, $\Pi_2 + BS$ is decreasing in $k$ when $k$ is close to but below $\hat{k}$. This is true for $BS$ as well, because $\Pi_2$ is increasing in $k$.

**Proof of Proposition 2** Straightforward and omitted.

**Proof of Corollary 5** Straightforward and omitted.

**Proof of Proposition 3** In the proof of Lemma 3, we have shown $D(\bar{p}) - \bar{Q} < k$.  

Clearly, $\Pi_1 > \Pi_2^{LP}$ hold.

$$\Pi_2^{LP} + BS^{LP} = v(p^{LP}) + (p^{LP} - c)k$$

$$> \int_{c}^{\infty} \min \{D(p), k\} dp$$

$$\geq \int_{x_0}^{\infty} \min \{D(p), k\} dp (\because x_0 > c)$$

$$= \Pi_2 + BS (\because (34)).$$

This completes the proof of part (a).

Compare $\pi'(\bar{p}) = \bar{Q}$ with $\pi'(\bar{p}^{LP}) = k$ and note that $\bar{Q} > k$ when $k$ is small, and $\bar{Q} < k$ when $k \geq \hat{k}$ because $\bar{Q} < D(\bar{p}) < D(x_0) < D(\hat{x}_0) = \hat{k} \leq k$. It proves the result for $\bar{p}, \bar{p}^{LP}$. The results for $D(\bar{p}), D(\bar{p}^{LP})$ and $TS, TS^{LP}$ follows.

Clearly, both $\Pi_2$ and $\Pi_2^{LP}$ tend to zero as $k \searrow 0$. Since $\Pi_2^{LP} = (\bar{p}^{LP} - c)k$,

$$\frac{d\Pi_2^{LP}}{dk} \bigg|_{k \searrow 0} = \bar{p}^{LP}|_{k \searrow 0} - c = p^m - c > 0.$$ 

Since $\Pi_2 = (\bar{p} - c)(D(\bar{p}) - Q)$, and both $\bar{p} - c$ and $D(\bar{p}) - \bar{Q}$ tend to zero as $k \searrow 0$ (contemplating($\bar{Q}, \bar{p}$) moves along the curve $Q = \pi'(p)$ toward $(q^*, c)$ in Figure 3b),

$$\frac{d\Pi_2}{dk} \bigg|_{k \searrow 0} = 0.$$ 

It proves the result for $\Pi_2, \Pi_2^{LP}$ when $k$ is small.

When $k \in [\hat{k}, q^*)$, (26) implies $\Pi_2 = (\hat{x}_0 - c)k$, where $\hat{x}_0$ is (as in the proof of Corollary 4) the minimum equilibrium $x_0$ when $k = \infty$. Therefore, $\Pi_2^{LP} = (\bar{p}^{LP} - c)k < \Pi_2$ since $\bar{p}^{LP} < u'(k) \leq \hat{x}_0$. It proves the result for $\Pi_2, \Pi_2^{LP}$ when $k \in [\hat{k}, q^*)$. It completes the proof of part (b).

Compare $BS = TS - \Pi_1 - \Pi_2$ and $BS^{LP} = TS^{LP} - \Pi_1^{LP} - \Pi_2^{LP}$. When $k \in [\hat{k}, q^*)$, our previous results that $TS < TS^{LP}$, $\Pi_1 > \Pi_1^{LP}$, and $\Pi_2 > \Pi_2^{LP}$ together imply $BS < BS^{LP}$. As $k \searrow 0$, from (34), $BS$ tends to zero but $BS^{LP}$ is positive. Therefore, we also have $BS < BS^{LP}$ when $k$ is small. It completes the proof of part (c).
For any \( k \), \( \bar{Q} + k > D(\bar{p}) \) (see Figure 3). It, together with part (b), implies that
\[
\bar{Q} > D(\bar{p}^{LP}) - k
\]
when \( k \) is small. As \( k \uparrow q^e \), \( D(\bar{p}^{LP}) - k \) tends to zero and \( \bar{Q} \) tends to \( q^e > 0 \). It proves part (d).
References


