Asymmetric equilibria of the sequential second-price auction in the IPV model*

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Abstract
We revisit Milgrom and Weber’s model of sequential second-price auction where the units to sell are identical and the bidders are homogeneous, risk-neutral, and have demands that are single unit, independent, and private. With 2 units and 3 bidders, we characterize all equilibria that are robust to discounting in the sense that they are weak limits of strict equilibria for the discount factor tending towards 1. While in no robust equilibrium can the current price always be larger than the conditional expectation of the future price, there may exist robust equilibria with decreasing unconditional price expectations. We characterize the distributions of values for which such equilibria exist for 2 units and any number of bidders larger than 2. We offer some extensions of our results to more than 2 units.

Keywords: sequential second-price auction; robustness to discounting; declining price; Milgrom and Weber’s model; independent and private values; homogeneous bidders; asymmetric equilibria.

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1. Introduction

In a general model with bidders who are risk-neutral and homogeneous exante and who have single-unit demands and observe affiliated signals about
their unit values, Milgrom and Weber (2000) explicitly characterize the symmetric equilibria of sequential auctions where identical units are sold one at a time through first-price, second-price, and English auctions. Among other elegant results they find that the equilibrium price at the next auction does not deviate in expectation from the price at the current auction, that is, that the auction price is a martingale, when the values are independent and private, and that it may deviate upwards in expectation, that is, is a submartingale, otherwise. Spurred by the empirical evidence of nonconstant and often decreasing price expectations, later researchers have added to the model from Milgrom and Weber’s paper, henceforth MW, features that allow such price movements even with independent and private values. Some of these features are: risk aversion or, more generally, nonlinear utilities and with or without risk involved in losing; entry costs; exogenously or endogenously uncertain supply; multi-unit demands combined with decreasing marginal values, complementarities, option to buy more than one unit at the current price, or budget constraints; different items being auctioned with various assumptions on when bidders learn their values; costs of waiting; and nonstrategic bidders. In this paper, we go back to the source of this research by considering the sequential second-price auction, SSPA for short, within the simplest version of MW model where values are private and independent. However, we look at what other reasonable equilibria may exist alongside MW equilibrium.

In this basic model, all bids smaller than the bid MW equilibrium pre-

\[1\] It first appeared as a discussion paper in 1982. See also Weber (1983) and Mezzetti, Pekeč, and Testlin (2008).

\[2\] Obviously, while its expectation stays constant with such values, the price along MW equilibrium actually goes up and down. Price decreases may even be more probable than price increases. For example, with 2 units and 3 bidders, this will be the case if the value density is decreasing. Indeed, the median of the 2nd auction price conditional on the 1st auction price will then be smaller than the 1st auction price (which equals the expectation of the 2nd auction price).

\[3\] Some authors have also appealed to concepts from behavioral economics such as the anchoring effect.
scribes a bidder at any given auction different from the last are also optimal\textsuperscript{4}. But if slight discounting is introduced the symmetric equilibrium will be close to MW original equilibrium and, outside an event of probability zero, will have bidders submit strictly optimal bids. We consider only equilibria that are similarly robust to the delays and uncertainty inherent to many real life multi-stage procedures, in the sense that they are limits of strict equilibria with discounting when the discount factor tends towards one.

We first characterize all robust equilibria of the SSPA with 2 units and 3 bidders. We find that a robust equilibrium must be nondecreasing and can differ from MW equilibrium only according to “asymmetric components.” Over its domain interval, an asymmetric component has two bidders follow the same bidding function which may have up to two discontinuity points and may stay constant over one subinterval. The other bidder’s bidding function may be constant over one disjoint subinterval and may jump at one value from below to above his opponents’ bidding function (see Figures 1 a to d in Section 5, 3b in Section 6, and 4 in Section 7). The bidding functions are determined by a differential equation with initial condition where they all share the same range, an explicit expression where they do not, and an indifference condition at the jump of the bidding function followed by a single bidder.

Thus, there exist robust equilibria with bid atoms, probable ties, discontinuous bidding functions, and where no two bidders use the same strategy throughout (as the equality among at least two bidders’ bidding functions is only local).

Conditional on the current price $P_1$, the expectation $E(P_2|P_1)$ of the future price $P_2$ may be larger or smaller even within the same robust equilibrium. However, in no robust equilibrium is it always at most equal and sometimes smaller, that is, is the price sequence a supermartingale.

\textsuperscript{4}Similar indifference occurs at equilibria of models that embed an auction into a larger game. See the next section.
Our result about conditional price expectations notwithstanding, there do exist distributions $F$ of values such that the unconditional expectations $E(P_1), E(P_2)$ are strictly decreasing, that is, such that $E(P_1) > E(P_2)$, along some robust equilibria. We offer a complete and particularly simple characterization of these distributions: $F$ with derivative $f$ is one of them if and only if the “marginal revenue,” or “virtual value,” function $\tilde{\omega}(v) = v - (1 - F(v)) / (3f(v))$ of the distribution of the minimum $V_{(3,3)}$ of a random sample of 3 values is not everywhere nondecreasing.

For an arbitrary number of bidders, that the sequence of price expectations cannot decrease strictly if the marginal revenue function $\omega(v) = v - (1 - F(v)) / f(v)$ of $F$ is nondecreasing follows from the multi-unit extension of Myerson (1981)’s general revenue equivalence theorem. In fact, as the bidders who remain at the 2nd auction bid their values, the price $P_2$ is the 2nd highest value among these bidders’ and hence at least equal to the 3rd highest value among all bidders’. The 2nd auction price $P_2^{MW}$ along MW equilibrium is precisely equal to this 3rd highest value because the highest value bidder wins the 1st auction. Thus, we always have $P_2 \geq P_2^{MW}$. If $E(P_1) > E(P_2)$, the expected revenues $E(P_1) + E(P_2)$ to the seller would be strictly larger than the revenues $2EP_2^{MW}$ from MW equilibrium, where the price is a martingale. But this would contradict the optimality of MW equilibrium, a consequence of its ex-post efficiency and the extension of Myerson (1981)’s theorem.

While the condition that $\omega$ be somewhere strictly decreasing is only necessary for the existence of robust equilibria with strictly decreasing price expectations, the same condition on $\tilde{\omega}$ is necessary and sufficient when there are 3 bidders. It is also sufficient for the existence of such equilibria for any number $n > 2$ of bidders. Furthermore, for an arbitrary number of units, the condition on $\tilde{\omega}$ ensures the existence of publicly correlated robust

\[5\text{See Myerson (1981) and Bulow and Roberts (1989).} \]

\[6\text{For the extension, see, for example, Engelbrecht-Wiggans (1988) and Maskin and Riley (1989).} \]
equilibria where the unconditional price expectation goes down from at least one auction to the next.\footnote{If the winner’s bid and identity, and nothing else, are announced after each auction. The results for 2 units hold true for any announcement policy.}

We review some related literature in the next section, define the game and equilibrium concepts in Section 3, obtain general properties of the equilibria in Section 4, and introduce the asymmetric components in Section 5 in the 2-unit 3-bidder case. In Section 6 we state our characterization of the robust equilibria in this case and apply it to an example in Section 7. To characterize the robust equilibria we make the discount factor tend towards one in a characterization of the strict equilibria with discounting that applies outside a vanishing neighborhood of the lowest possible value. While the proof of our characterization is too long to include in the paper, we present its main steps and provide an overview of the intermediary results and informal explanations of their proofs in the online appendices (accessible through the link http://blebrun.info.yorku.ca/work-in-progress/).

We study the robust equilibrium allocations in Section 8, the conditional expectations of the equilibrium prices in Section 9, and their unconditional expectations in Section 10 for \( n = 3 \) and in Section 11 for \( n \geq 4 \). In Section 12, we find when some robust equilibria bring higher expected revenues than MW equilibrium. Section 13 contains some first extensions to any number of units. Section 14 concludes.

All complete formal statements and proofs that are missing from the paper itself or the online appendices can be found in the supplements to this paper (also available online) which we denote Supp. Mat. from here on.

## 2. Related literature

Surveys of the literature that “augments” MW’s model of sequential auctions can be found, for example, in the introductory section of Deltas and Kosmopolou (2004) and the survey by Trifunović (2014). A part of this
literature adds discounting of some sort. An uncertain number of units or “stochastic supply” can be modeled by a discount factor equal to the probability that a further auction will occur. When different objects are being auctioned, discounting can also model their declining values. For models with exogeneously stochastic supply, see Jeitsko (1999); with endogeneously uncertain supply, see Zeithammer (2009) and Rodriguez (2012); with declining values, see Beggs and Graddy (1997); and with discounting, see Bonet and Pesendorfer (2014). Kittsteiner, Nikutta, and Winter (2004) add a preference for the present more general than discounting. Here, we consider discounting only when checking the robustness of equilibria. Thus, all our robust equilibria are equilibria when bidders do not discount, so that decreasing price expectations can no longer be explained by bidders’ impatience.

In a model of a 2-unit SSPA with \( n \) bidders and 2-unit demands that extends those in Black and De Meza (1992), Katzman (1999) and Noussair (1995), Lamy (2015) characterizes all the pure symmetric equilibria that satisfy a list of mathematical assumptions (on the number of bid atoms and the continuity and continuous differentiability of the bid marginal distribution and the distribution of the value conditional on the bid). Here, we make only the more game-theoretical requirement that the equilibria be limits of strict equilibria with discounting and characterize all symmetric and asymmetric such robust equilibria in MW’s model with 3 bidders and single-unit demands.

While Lamy (2015) confirms the uniqueness of the symmetric equilibrium when \( n \geq 3 \), he describes a larger equilibrium multiplicity when \( n = 2 \) than in Black and De Meza (1992) and Katzman (1999). In this 2-bidder case, a unique equilibrium is the limit of symmetric equilibria when the probability that new bidders with very high values stay out of the 2nd auction tends towards one. As this probability plays the same role as our discount factor, Lamy (2015)’s concept of robustness is similar to our own.

The literature on sequential auctions had already noticed that a bidder’s
equilibrium bid is generally only one of many optimal bids. See, for example, Figure 1 in Katzman (1999) and Figure 2 in Lamy (2015). Here, we focus on the equilibria that become strict as soon as small discounting is introduced.

Asymmetric equilibria of sequential auctions had previously been obtained although mostly in asymmetric models. See, for example, de Frutos and Rosenthal (1998) and Krishna and Rosenthal (1996). Katzman (1999) does obtain an asymmetric equilibrium even in a symmetric model and shows that the unconditional expectations of the price may be decreasing if the model itself becomes sufficiently asymmetric. In our paper, although confining ourselves to a symmetric model–MW model–we find an infinity of asymmetric equilibria that are robust to symmetric perturbations of the common discount factor. When $n = 3$, we describe all such robust equilibria, prove that their prices are never supermartingales, and find a necessary and sufficient condition for the existence of a robust equilibrium, necessarily asymmetric, where the exante price expectations decrease. The condition stays sufficient whatever the number $n$ of bidders may be.

Because we also show when there exist robust equilibria of the SSPA that are more profitable to the seller than MW equilibrium, which is payoff-equivalent to the weakly dominant equilibrium of the Vickrey-Clarke-Groves, or VCG, mechanism, here the uniform 3rd-price auction, the paper also relates to others comparing the revenues from these auctions, such as Mezzetti, Saša Pekeč, and Testlin (2008). The possible existence of inefficient equilibria of single-unit auctions that improve upon the revenues from the efficient equilibrium of the VCG mechanism is found in Lebrun (2012), where even optimality\(^8\) in the presence of heterogeneous bidders may be achieved through an equilibrium of the English auction as long as resale among bidders is allowed.

3. The two-unit sequential second-price auction and equilibrium

\(^8\)Restricted to equilibria of mechanisms with probability one of sale. Unrestricted optimality may be achieved by introducing distribution-specific reserve prices.
Two identical units of a good are auctioned sequentially in two second-price sealed-bid auctions where the winner of the 1st auction and any bidder who did not take part in it may not enter the 2nd. Prior to the 1st auction, the $n \geq 3$ bidders’ monetary values for a unit are drawn independently from a nondegenerate interval $[c,d]$ according to a common probability distribution function $F$ that is continuous over the whole interval and twice-continuously differentiable over $(c,d)$, with a strictly positive first-order derivative $f$ over this semi-open interval. A bidder’s value is his private information and stays constant throughout both auctions. The reserve price is set at the lowest possible value $c$ and fair lotteries break ties. After the 1st auction, the seller may release publicly some information about its outcome and the bids that were submitted. We assume that he follows a given policy in this regard but do not restrict it in any way.

The payoff to a bidder from a particular auction is zero if he does not win and the difference between his value and the price he pays if he does. A bidder’s total payoff is the sum of his payoff from the 1st auction and the payoff from the 2nd auction discounted by the common factor $\delta \leq 1$. All bidders rank uncertain outcomes according to the expectations of their total payoffs, that is, have a neutral attitude towards risk.

We assume that every bidder submits an acceptable bid at or below his value in the 1st auction and, if he loses this auction, submits his value in the 2nd auction. We refer to this Bayesian game of sequential second-price auction as SSPA$(n;\delta)$ or simply SSPA$(n)$ if $\delta = 1$. A bidder’s strategy is a

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9 This assumption simplifies the presentation.

10 As we will focus on limits of equilibria that recommend truth bidding—the only weakly undominated strategy—in the 2nd auction and that are strict (almost everywhere) in the 1st auction, we assume from the start that bidders do not bid above values at the 1st auction. Indeed, submitting one’s value is at least as good as submitting a larger bid when the bidders’ behaviors at the 2nd auction do not depend on what happened previously.

Although not participating (or submitting a bid strictly below $c$) for a bidder with value $c$ is not weakly dominated, because $c$ occurs only with probability zero we make the further simplification that a bidder always participates when allowed.
probability distribution over 1st auction value-bid couples on or below the diagonal and whose marginal distribution over values agrees with $F$. An equilibrium of SSPA($n; \delta$) is a Bayesian Nash equilibrium that is described as an $n$-tuple of such strategies.

Thus, in any equilibrium, each bidder’s strategy must possess a distribution of bids conditional on values that is optimal for $F$-almost all values. If every bidder’s such conditional is strictly optimal for $F$-almost all values, we say that the equilibrium is strict. An equilibrium is pure if every bidder’s strategy is concentrated on the graph of some measurable bidding function.

An equilibrium of SSPA($n$) is robust, or more precisely, robust to discounting, if it is the weak limit of strict equilibria of SSPA($n, \delta^l$), for some strictly increasing sequence of discount factors $(\delta^l)_{l \geq 1}$ tending towards 1.12 We call such a sequence of strict equilibria an “approaching sequence.”

### 4. General properties of the equilibria

Throughout the paper, calligraphic capital letters refer to sets and bold capital letters to random variables. We denote realizations of random variables as lower case letters. Thus, we denote $\mathcal{N}$ the set of bidders $\{1, \ldots, n\}$, $V_k$ bidder $k$’s value viewed as a random variable and $v_k$ an realization of $V_k$. We denote $\lor$ and $\land$ the maximum and minimum operators.

Let $\sigma = (\sigma_1, \ldots, \sigma_n)$ be a strategy profile in SSPA($n; \delta$). We use the standard notation $\sigma_{-i}$ for the $(n-1)$-tuple of strategies of all bidders different

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11 Thus, it is a “distributional” strategy as in Milgrom and Weber (1985).

12 A priori, a limit of equilibria when the discount factor tends towards 1 may fail to be an equilibrium as some payoffs may not be continuous at the limit due to ties among highest bidders. Through “endogenous externalities,” such discontinuities may occur to payoffs after losing as well as after winning. Therefore, in addition to the existence of a convergent sequence of strict equilibria, we maintain the requirement that their limit be an equilibrium.
from some bidder $i \in \mathcal{N}$. When these bidders follow their strategies in $\sigma_{-i}$, there will ensue a joint probability distribution over their values $V_k$ and bids $B_k$, $k \neq i$. We denote $\pi_i (v_i, b_i; \sigma_{-i})$ or simply $\pi_i (v_i, b_i)$ the expectation according to this distribution of bidder $i$'s total payoff when his value is $v_i$ and his bid $b_i$ in the 1st auction. We use the notation $BR_i (v_i; \sigma_{-i})$ for his set of optimal replies within $[c, v_i]$.

If bidder $k \neq i$ happens to win the 1st auction with the bid $b_k$, bidder $i$'s realized payoff will be $\delta (v_i - v_i \wedge (\lor_{l \neq i,k} v_l))$, as if this latter bidder bought one unit in the 2nd auction at the price $v_i \wedge (\lor_{l \neq i,k} v_l)$. This is indeed the price bidder $i$ would effectively pay, as he will obtain some payoff from the 2nd auction only when his value is higher than the highest value $\lor_{l \neq i,k} v_l$ among the other remaining bidders’. Increasing his bid above $b_k$ would change bidder $i$’s payoff by $(1 - \delta) v_i + \delta v_i \wedge (\lor_{l \neq i,k} v_l) - b_k$, the difference between $v_i - b_k$ and $\delta (v_i - v_i \wedge (\lor_{l \neq i,k} v_l))$. We may therefore write the change $\Delta_i (v_i; b_i, b'_i) = \pi_i (v_i, b'_i) - \pi_i (v_i, b_i)$ in bidder $i$’s expected payoffs, for $b'_i > b_i$ as the expectation below:

$$\Delta_i (v_i; b_i, b'_i) = E \left( \sum_{k \neq i} (v_i \wedge (\lor_{l \neq i,k} V_l \lor W_\delta) - B_k) I (WIN (b'_i) = i; WIN (b_i) = k) \right), (1)$$

where: $WIN$ is the random variable equal to the identity of the winner at the 1st auction; $I (WIN (b'_i) = i; WIN (b_i) = k)$ is the indicator function of the event where $WIN$ changes from bidder $k$ to bidder $i$ if bidder $i$ changes his bid from $b_i$ to $b'_i$; and $W_\delta$ is the discrete random variable independent of all others that takes the value $c$ with probability $\delta$ and $d$ with probability $1 - \delta$. Although $W_\delta$ is mainly a convenient notation device, it formally expresses the alternative interpretation of the discount factor as the probability that the game continues past the current period.

Obviously, fixing $b_i$ at a particular bid (even a bid smaller than $c$, hence signalling nonparticipation) transforms $\Delta_i (v_i; b_i, .)$ into a payoff function
equivalent to $\pi_i(v_i;\cdot)$. From (1), the 1st auction then shares features with auctions with interdependent values—the net benefit from winning is a function $v_i \land (\lor_{i \neq j,k} V_i \lor W_j)$ of the signals other bidders observe—as well as auctions with externalities—this function itself depends on the identity ($k$) of the second-highest bidder\(^{13}\).

From (1), a bidder’s expected payoff has nondecreasing differences in his own value and bid. As we state in Lemma 1 below, the strategies in the strict equilibria of SSPA($n;\delta$) and robust equilibria of SSPA($n$) must be pure and nondecreasing\(^{14}\). The “induced set ordering” in (i) is as defined in Section 2.4 in Topkis (1998)\(^{15}\). The detailed proof of Lemma 1 is in Section 1 of Supp. Mat. In it as throughout the paper, by a “generalized inverse” $\alpha_i$ of a nondecreasing function $\beta_i$ from $[c,d]$ to itself we mean another such function such that $\alpha_i (\beta_i(v)^-) \leq v \leq \alpha_i (\beta_i(v)^+)$ and $\beta_i (\alpha_i(b)^-) \leq b \leq \beta_i (\alpha_i(b)^+)$, for all $v$ and $b$, where the superscript $-$ ($+$) to the argument of a function indicates a limit from the left (right)\(^{16}\).

**Lemma 1:** Let $\sigma$ be a strategy profile in SSPA($n;\delta$), with $0 < \delta \leq 1$.

(i) **Nondecreasing differences:** Then, for all $i$, $\pi_i(v_i,b_i;\sigma_{-i})$ has nondecreasing differences in $(v_i,b_i)$ and $\mathcal{BR}_i(v_i;\sigma_{-i})$ is nondecreasing in $v_i$ for the set ordering induced by the standard order on $[c,d]$.

(ii) **Monotonicity:** If $\sigma$ is a strict equilibrium, then, for all $i$, all

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\(^{13}\)While this similarity will be a helpful source of intuition in characterizing the equilibria with 3 bidders, for the general analysis of the equilibrium price paths (actually, even already in proving Lemma 2 (i), which we state in Online Appendix 1) it will be more useful to consider the 1st auction as the 1st stage of a two-stage incentive-compatible mechanism. Notice that, although the allocation is dynamic, the private information is not, that is, the mechanism is not as in Bergemann and Wambach (2015), for example, where players learn more private information as it goes along.

\(^{14}\)If a bidding strategy was not nondecreasing, there would be indifference between several bids over an interval of values and the equilibrium could not be strict. Monotonicity of robust equilibrium strategies follow by passing to the limit.

\(^{15}\)That is, $\mathcal{BR}_i(v_i;\sigma_{-i})$ is nondecreasing for this relation if $v_i \leq v'_i$, $b_i \in \mathcal{BR}_i(v_i;\sigma_{-i})$, and $b'_i \in \mathcal{BR}_i(v'_i;\sigma_{-i})$ imply $b_i \land b'_i \in \mathcal{BR}_i(v_i;\sigma_{-i})$ and $b_i \lor b'_i \in \mathcal{BR}_i(v'_i;\sigma_{-i})$.

\(^{16}\)We adopt the convention, $\alpha_i(c^-) = \beta_i(c^-) = c$ and $\alpha_i(d^+) = \beta_i(d^+) = d$. 

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selections\textsuperscript{17} from $\text{BR}_i(v_i; \sigma_{-i})$ are nondecreasing in $v_i$.

(iii) **Alternative definition of robustness:** Assume $\sigma$ is an equilibrium of SSPA($n$). Then $\sigma$ is robust if and only if there exist a strictly increasing sequence of discount factors $(\delta^t)_{t \geq 1} \rightarrow 1$, a sequence $(\sigma^t)_{t \geq 1}$ of strict equilibria of $(\text{SSPA}(n; \delta^t))_{t \geq 1}$, a sequence of profiles of nondecreasing functions $(\beta^t)_{t \geq 1} = ((\beta^t_1, ..., \beta^t_n))_{t \geq 1}$ that are best selections in $(\sigma^t)_{t \geq 1}$, and a profile $(\beta_1, ..., \beta_n)$ of nondecreasing functions such that, for all $i$, $\sigma_i$ is the probability distribution of the random vector $(V_i, \beta_i(v_i))$ and $\beta_i^t(v_i) \rightarrow \beta_i(v_i)$ at all continuity point $v_i$ of $\beta_i$\textsuperscript{18}.

From Lemma 1 (iii) (an immediate consequence of a technical result on weak convergence, see Supp. Mat.), instead of the weak convergence of the distributional strategies of the sequence of strict equilibria approaching a robust equilibrium, we could as well have required the “weak” pointwise convergence, that is, the convergence at the continuity points of the limits of the bidding functions. From here on, we will describe any strict or robust equilibrium as a profile $\beta = (\beta_1, ..., \beta_n)$ of nondecreasing bidding functions that are almost everywhere selections from the best reply correspondences.

We list more general properties of the strict and robust equilibria in Lemma 2 in Online Appendix \textsuperscript{19}.

5. **Asymmetric components of SSPA(3)**

\textsuperscript{17} A measurable function $\beta_i$ is a selection from $\text{BR}_i(v_i; \sigma_{-i})$ if $\beta_i(v_i) \in \text{BR}_i(v_i; \sigma_{-i})$ for all $v_i$ such that $\text{BR}_i(v_i; \sigma_{-i}) \neq \emptyset$.

\textsuperscript{18} Equivalently, we may require only convergence almost everywhere.

\textsuperscript{19} Among these properties are “generalized” equilibrium conditions that must hold true at all values of any bidder, even those where an optimal bid may not exist (because of other bidders’ bid atoms), and expressions for the one-sided limits with respect to the own bids of the bidders’ payoffs. Also, in any strict equilibrium with discounting, each bidder’s bids must belong to the range of at least one other bidder. We give an explicit expression for the bidding function any two bidders must follow at equilibrium if they are the only two to bid within some range, for the bid submitted by all bidders and at the same value by at least $n-1$ bidders, and for the maximum of the highest bid support. We also rule out ties in strict equilibria with discounting and a certain type of ties at $c$ in robust equilibria.
Starting from this section and until Section 10, we assume \( n = 3 \) bidders. The letter \( V \) denotes a generic bidder’s value, that is, a random variable distributed according to \( F \). The following conditional expectations play a prominent role in our results.

**Definition 1:** The effective-price function\(^{20}\): \( e \) is the real-valued function defined over \([c,d]^2 \times (0,1]\) as follows:

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e(x,y;\delta) = E(x \wedge (V \vee W_\delta) \mid V \leq y).
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We simplify the notation \( e(x,x;\delta) \) for \((1-\delta)x + \delta E(V \mid V \leq x)\) to \( e(x;\delta) \). We often drop \( \delta \) from the arguments if it is equal to 1. Thus, \( e(x,y;\delta) = (1-\delta)x + \delta e(x,y) \), \( e(x,y) = E(x \wedge V \mid V \leq y) \), and \( e(x) = E(V \mid V \leq x) \).

We may interpret \( e(x,y;\delta) \) as the effective price a bidder with value \( x \) would expect to pay if the second stage of SSPA(3;\( \delta \)) occurred with probability \( \delta \) and he faced in it a competitor with value below \( y \). In fact, from a bidder’s point of view, not receiving a unit, because the auction did not occur or it did occur and he lost it, is equivalent to receiving one unit and paying his value for it.

When \( \delta = 1 \), it is simply the expected price \( e(x,y) \) in a second-price auction between two bidders: one with his value equal to \( x \) and the other with his value smaller than \( y \). Thus, if \( c < x < y \), permuting \( x \) and \( y \) in \( e(x,y) \) decreases it strictly as the bidder with value \( y \) becomes the sure winner and pays the other bidder’s value which is strictly smaller than \( x \) and whose expectation is \( e(y,x) = e(x) \). However, as \( e(x,y;\delta) \) includes the term \((1-\delta)x\), this remains true with discounting only if \( x \) is bounded away from \( c \) and \( \delta \) is large enough. Because of this simple fact, which we state formally in Lemma 3 below, our characterization of the strict equilibria of SSPA(3;\( \delta \))

\(^{20}\)We have shortened the more correct phrase “expected-effective-price function.”
applies outside some neighborhood of $c^{21}$.

From its definition, $e(y, x) = e(x)$ if $y \geq x$ and one can express $e(x, y)$ in this case as, for example, $\frac{1}{F(y)} \int_c^x (w - x) f(w) dw + x$. Differentiating this last expression, Lemma 3 follows easily from the definitions of the functions $\gamma$ and $\bar{\delta}$.

**Lemma 3: Properties of the effective-price function $e$:** Let $\gamma$ and $\bar{\delta}$ be the nondecreasing and nonincreasing continuous functions defined over $(c, d)$ as follows: $\gamma(b) = \min_{x, y \in [b, d]} f(y) \int_c^x (x - w) f(w)dw / F(y)^2 > 0$; $\bar{\delta}(b) = 1 / (1 + \gamma(b)) < 1$.

(i): If $c < x < y < d$, then: $\frac{\partial}{\partial x} e(x, y) \leq 1$; and $\frac{\partial}{\partial y} e(x, y) = f(y) \int_c^x (x - w) f(w) dw / F(y)^2 > 0$.

(ii): If $c < b \leq x < y \leq d$ and $b \leq z < z' \leq d$, then: $e(x, z') \geq e(x, z) + \gamma(b) (z' - z)$; $e(x, y; \delta) \geq e(y, x; \bar{\delta}) + \{\gamma(b) \bar{\delta} - (1 - \bar{\delta})\} (y - x)$; and $e(x, y; \delta') > e(y, x; \delta')$; for all $\delta$ in $(0, 1]$ and all $\delta'$ in $(\bar{\delta}(b), 1]$.

With no discounting, MW equilibrium has all bidders follow the bidding function $e(v)$. It is robust to discounting as it is the limit of the symmetric strict equilibrium with bidding function $e(v; \delta)$ for $\delta \to < 1$. We will prove that the other robust equilibria are pure equilibria that depart locally from MW equilibrium according to “asymmetric components,” which we now define$^{22}$. According to this definition, any “degenerate” component

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$^{21}$This will nevertheless be sufficient to fully characterize all robust equilibria of SSPA(3) as letting $\delta$ tend towards 1 makes this neighborhood vanish.

$^{22}$MW equilibrium is actually the only symmetric equilibrium among all equilibria, not only among all robust equilibria. In fact, there cannot exist a symmetric equilibrium with a bid atom. Otherwise, with strictly positive probability a tie would involve at least 3 bidders, which cannot occur at an equilibrium (see Footnote 3 in the proof in Supp. Mat of Lemma 2 (v) from Online Appendix 1). Next, if there existed a symmetric equilibrium without bid atom that was nondecreasing, there would exist $v_i < v_i'$ and $b_i > b_i'$ such that $b_i$ would be a best response at $v_i$, $b_i'$ at $v_i'$, and where bids are submitted with strictly positive probability within $(b_i', b_i)$ for values larger than $v_i$. From (1), the
$AC (j; \mu, y, z, \eta)$ with $\mu = \eta$ agrees with MW equilibrium on its single-value domain.

**Definition 2: Asymmetric components of SSPA(3):** Let $j \in \mathcal{N}$, and $\mu, y, z, \eta$ be such that $c \leq \mu \leq y \leq z \leq \eta \leq d$. An asymmetric component $AC (j; \mu, y, z, \eta)$ is the triple $(\beta_1, \beta_2, \beta_3)$ of bidding functions defined over $[\mu, \eta]$ such that, for all $k \neq j$:

\[
\begin{align*}
\beta_j(v) &= e(\mu), \text{ if } \mu \leq v \leq y; \\
&= e(v, \varphi(v)), \text{ if } y < v \leq \eta; \quad \beta_k(v) = e(\mu), \text{ if } v = \mu; \\
&= e(v, y), \text{ if } \mu < v \leq y; \\
&= e(y), \text{ if } y < v \leq z; \\
&= e(\varphi^{-1}(v), v), \text{ if } z < v \leq \eta;
\end{align*}
\]

where:

(i) $\varphi$ is a continuous function over $[y, \eta]$ that solves over $(y, \eta)$ the differential equation $DE1$ with domain $\{(v, \varphi) \in (c, d)^2 | \varphi > v\}$ and the initial condition $IC1$ at $\eta$ below:

\[
DE1: \frac{d \ln F(\varphi(v))}{d \ln F(v)} = \frac{e(\varphi(v)) - e(v, \varphi(v))}{e(v, \varphi(v)) - e(v)}; \quad IC1: \varphi(\eta) = \eta;
\]

(ii) both following conditions are satisfied if $y < d$:

\[
C1: \text{ if } c < y, \quad \int_{\mu}^{z} (e(y, w) - e(w, y)) dF^2(w) = 0; \\
C2: \quad z = \varphi(y).
\]

Theorem 1 below ensures the general existence and uniqueness of some asymmetric components. It follows from our results about the equations $DE1$ and $C1$ (see Remark A1 in Online Appendix 4) such as our removal of the singularity of $DE1$ at $IC1$, where its denominator vanishes. The formal difference $\Delta_i (v_i; b'_i, b_i)$ would increase strictly if $v_i$ increased to $v'_i$. But then $b'_i$ could not be a best response at $v'_i$. 

15
proof is in Section 5 of Supp. Mat.

Theorem 1: Existence and uniqueness of asymmetric components of SSPA(3): Let $j$ be an element of $N$.

(i): If $AC(j; \mu, y, z, \eta)$ exists, it is unique.

(ii): For all $\mu \in (c, d)$, there exists $y' \in (\mu, d)$ with the following property: for all $y \in (\mu, y')$, $AC(j; \mu, y, z, \eta)$ exists for some $y < z, \eta < d$. Moreover, $z, \eta \rightarrow \mu$ if $y \rightarrow \mu$.

(iii): For all $\mu \in (c, d)$, $AC(j; \mu, d, d, d)$ exists.

(iv): For all $\eta \in (c, d)$, $AC(j; c, c, c, \eta)$ exists.

(v): For all $\mu, \eta$ such that $c < \mu < \eta \leq d$, $AC(j; \mu, \mu, \mu, \eta)$ does not exist.

(vi): (vi.1): If $AC(j; \mu, y, z, \eta)$ exists and $c = \mu = y$, then $z = c$.

(vi.2): If $AC(j; \mu, y, z, \eta)$ exists and $c < \mu, y < \eta$, then $\mu < y < z < \eta$.

(vi.3): If $AC(j; \mu, y, z, \eta)$ exists and $\mu < y = z = \eta$, then $y = z = \eta = d$.

The component $AC(j; \mu, y, z, \eta)$ specifies over $[\mu, \eta]$ a common bidding function $\beta_k$ for bidders $k \neq j$ and a different bidding function $\beta_j$ for bidder $j$. When, as depicted in Figure 1a, $c < \mu < y < d$, the function $\beta_j$ stays constant at $e(\mu)$ from $\mu$ to $y$, jumps up to $e(y, z)$ at $y$, from which it is then equal to the strictly increasing and differentiable function $e(. \varphi(.))$. From an initial jump at $\mu$, the other bidders’ bidding function $\beta_k$ is strictly increasing, continuous, and equal to $e(., y)$ until $y$, then takes the constant value $e(y)$ up to $z$, where it jumps to the same bid $e(y, z)$ as bidder $j$’s $\beta_j$ at $y$. From there on, it coincides with the differentiable and strictly increasing function $e(\varphi^{-1}(.), .)$ and remains strictly below $\beta_j$, except at $\eta$ where it is equal. The function $\varphi$ links the values of bidders $j$ and $k$ at which they submit the same bid, that is, $\beta_j(v) = \beta_k(\varphi(v))$, for all $v$ in $(y, \eta)$, or, equivalently, $\varphi = \alpha_k \beta_j$ over this interval. In the online appendices and in Supp. Mat., we refer to $\beta_k$’s discontinuity at $\mu$ as a type-A discontinuity, $\beta_j$’s discontinuity at $y$ as a type-B discontinuity, and $\beta_k$’s discontinuity at $z$ as a type-C discontinuity.
One can understand the equations in Definition 2 from the standard necessary first-order condition for optimality in a second-price auction: that the bid should be equal to the bidder’s expected benefit from winning, here the effective future price he would save, conditional on barely beating the highest competing bid. Thus, when bidder \( j \) with value \( v \) bids \( \beta_j(v) \) in the range \( (e(y,z), e(\eta)) \) where all three bidders bid, it must equal \( e(v, \varphi(v)) \), as both other bidders follow the bidding function \( \beta_k \). \(^{23}\)

On the other hand, the benefit from winning to a bidder \( k \neq j \) with value \( \varphi(v) \), hence who also submits \( \beta_j(v) = e(v, \varphi(v)) \), depends on the identity of the runner up with the same bid: it is \( e(\varphi(v)) \), larger than \( e(v, \varphi(v)) \), if he just beats bidder \( j \) with value \( v \), and the smaller \( e(\varphi(v), v) = e(v) \), if he just beats bidder \( l \neq j, k \) with the same value \( \varphi(v) \). The bid must then be the weighted average of these benefits with the weights in the same ratio as the likelihoods \( F(\varphi(v)) \frac{d}{dv} F(v) \) and \( F(v) \frac{d}{dv} F(\varphi(v)) \) of these two events. Equivalently, the distances \( e(\varphi(v)) - e(v, \varphi(v)) \) and \( e(v, \varphi(v)) - e(v) \) between the benefits and the bid must be in the inverse ratio \( \frac{d\ln F(\varphi(v))}{d\ln F(v)} \) of these likelihoods, that is, equation DE1 must hold.

The same first-order condition gives \( \beta_k(v) = e(v, y) \) where the bidders different from \( j \) are the only ones to bid in the same range, that is, for \( v \) in \( (\mu, z) \). \(^{24}\) Bidder \( k \) with value \( v \in (y, z) \) is indifferent between losing and winning when he ties with the other bidder different from \( j \), something that occurs with strictly positive probability.

Because \( e(y, w) - e(w, y) \) is equal to \( e(w) - e(w, y) \) for \( w < y \) and \( (y - e(y))(F(w) - F(y))/F(w) \) for \( w > y \), Condition C1 can also be written as follows:

\[
\int_{\mu}^{y} (e(w) - e(w, y)) dF(w)^2 + (y - e(y)) (F(z) - F(y))^2 = 0.
\]

\(^{23}\)See Lemma 2 (ix) in Online Appendix 1.

\(^{24}\)See Lemma 2 (viii) in Online Appendix 1.

\(^{25}\)As then \( e(y, w) = (e(y) F(y) + y (F(w) - F(y))) / F(w) \) and \( e(w, y) = e(y) \).
It expresses the indifference of bidder \( j \) with value \( y \) between the two extremities of his bid jump\(^{26} \). The first term in the LHS is the change in this bidder's payoff due to moving his bid from the lower extremity to just below \( e(y) \), as passing the highest bid \( \beta_k(w) = e(w,y) \) from the other bidders contributes \( e(y,w) - \beta_k(w) = e(w) - e(w,y) \) to his net payoff. Going above \( e(y) \) to reach the upper extremity of the jump does not contribute (as \( e(y,y) - e(y) = 0 \)) when \( e(y) \) is the strictly higher competing bid and contributes \( y - e(y) \) when both other bidders tie at \( e(y) \) and hence have values between \( y \) and \( z \). This explains the second term.

A component of any other type may be regarded as a limiting case of the component in Figure 1a. In \( AC(j;\mu,d,d,d) \) (Figure 1b), \( \beta_j \) is constant to the right of \( \mu \) and continuous everywhere. Discontinuity of \( \beta_k \) occurs only at \( \mu \) and only if \( \mu > c \), for all \( k \neq j \). In our proofs, we keep referring to it as a type-A discontinuity.

In \( AC(j;c,y,z,\eta) \) with \( c < y < z \) (Figure 1c), there remain only the discontinuity of \( \beta_j \) at \( y \) and the discontinuity of \( \beta_k \) at \( z \), to which we still refer in our proofs as type-B and -C discontinuities. While Theorem 1 does not guarantee the existence of components of this type, they do exist in some examples, such as when \( F \) is the uniform distribution (see Section 7).

Finally, from Theorem 1 (iv) and (v), \( AC(j;c,c,c,\eta) \) is the only type of asymmetric components with no discontinuity. In it, \( \beta_j \) is strictly above \( \beta_k \) over the interior \((c,\eta)\) of the domain (Figure 1d).

\(^{26}\)See the “generalized equilibrium conditions” in Lemma 2 (vi) in Online Appendix 1.
Figure 1a: Asymmetric component $AC(1; \mu, y, z, \eta)$: type-A discontinuity at $\mu$; type-B at $y$; type-C at $z$.

Figure 1b: Asymmetric component $AC(1; \mu, d, d, d)$: type-A discontinuity at $\mu$. 

19
Figure 1c: Asymmetric component AC(2; c, y, z, η): type-B discontinuity at y; type-C at z.

Figure 1d: Asymmetric component AC(3; c, c, c, η)

6. All robust equilibria of SSPA(3)

From Theorem 2 and its corollary below, in the neighborhood of almost any value, a robust equilibrium agrees either with some asymmetric component or with MW equilibrium. Inversely, any collection of asymmetric components that is extended according to MW equilibrium provides a robust equilibrium. In Definition 3 below, we ensure consistency among the different asymmetric components by requiring that their domains not overlap except possibly at their extremities.
Definition 3: Extension of a nonoverlapping collection of asymmetric components: Let $\beta$ be a triple of nondecreasing strategies and $A = (AC(k^s; \mu^s, y^s, z^s, \eta^s))_{s \in S}$ an indexed set of asymmetric components.

(i): $A$ is nonoverlapping if $[\mu^s, \eta^s] \cap [\mu^{s'}, \eta^{s'}] \subseteq \{\mu^s, \eta^s\}$, for all $s, s' \in S$ with $s \neq s'$.

(ii): $A$ extends into $\beta$, which we denote $\beta = \text{ext}(A)$, if it is nonoverlapping and, possibly after changing $\beta$ over a set of values of probability zero:

(ii.1): $\beta = AC(k^s; \mu^s, y^s, z^s, \eta^s)$ over $[\mu^s, \eta^s]$, for all $s \in S$;

(ii.2): $\beta(v) = (e(v), e(v), e(v))$ for all $v \notin \bigcup_{s \in S} [\mu^s, \eta^s]$.

Theorem 2: Characterization of the robust equilibria of SSPA(3):

(i): Every robust equilibrium of SSPA(3) is an extension of some nonoverlapping collection of components.

(ii): Assume $\beta = \text{ext} \{AC(k^s; \mu^s, y^s, z^s, \eta^s)\}_{s \in S}$. Then:

(ii.1): $\beta$ is an equilibrium of SSPA(3).

(ii.2): If $\mu^s > c$, for all $s \in S$, $\beta$ is a robust equilibrium of SSPA(3).

(ii.3): If there exists $\pi > 0$ such that $1/F$ is strictly convex over $(c, c + \pi)$, $\beta$ is a robust equilibrium of SSPA(3).

Corollary 1: If $1/F$ is strictly convex over a nondegenerate interval with lower extremity $c$, the sets of robust equilibria of SSPA(3) and of extensions of nonoverlapping components are identical.

For our proofs to go through despite the singularity of our differential equations at the lowest value $c$, we need the additional assumption in Theorem 2 (ii.3) of strict convexity of $1/F$ in the vicinity of $c$. This is equivalent to the ratio $f/F^2$ being strictly decreasing there and is therefore implied by a reverse hazard rate $f/F$ coming down strictly monotonically from infinity at $c$, that is, local log-concavity, which in turn follows, for example, from the

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27 We do not need this assumption to remove the singularities away from $c$. 
strict positivity of \( f(c) \) and the existence (hence finiteness) of \( f'(c) \). Take \( c < \mu < y < z < d \) that satisfy C1 and the inequality \( e(z) < y \). Consider the profile of nondecreasing functions \( \beta \) that agree with MW equilibrium outside \([\mu, z]\) and such that \( \beta_1 \) takes the constant value \( e(\mu) \) over \((\mu, y)\) and the constant value \( e(z) \) over \((y, z)\) and \( \beta_2 \) and \( \beta_3 \) are both equal to \( e(., y) \) over \((\mu, z)\) (see Figure 2). Such strategy profiles are equilibria where no bidders bid above value. Yet, none is robust to discounting as none fits the characterization in Theorem 2. For this and other examples of equilibria that do not pass our robustness test, see Online Appendix 5.

Figure 2: An equilibrium (if C1 is satisfied) that is not robust.

The proof of Theorem 2 consists in obtaining a characterization of the strict equilibria with discounting, applying it to the members of the approaching sequence of any given robust equilibrium without discounting (to prove

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28 Convexity of \( 1/F \) is \((-1)\)-concavity of \( F \) and is then weaker than \( 0 \)-concavity, that is, log-concavity, of \( F \). The couple of assumptions \( f(c) > 0 \) and \( f'(c) < +\infty \) imply \( \rho \)-concavity of \( F \) close to \( c \), for all \( \rho < 1 \). See Ewerhart (2013).

29 From Theorem 1 (ii), \( y \) and \( z \) satisfying those conditions exist for all \( \mu \) in \((c,d)\).

30 In any approaching sequence of strict equilibria with discounting, bids close to \( \beta_1(y^+) = e(z) \) would be too high to be strictly optimal for values slightly larger than \( y \). In fact, as bidders 2 and 3 would also bid in the same neighbourhood for values close to \( z \), bidder 1’s effective price at the 2nd auction upon barely losing the 1st would be close to \( e(y,z) \), which is strictly smaller than \( e(z) \).
the necessity of the characterization in Theorem 2), and using it to construct an approaching sequence of any extension of asymmetric components (to prove the sufficiency). Characterizing the strict equilibria with discounting involves ruling out bid atoms, finding what types of discontinuities may occur, proving differentiability where continuity holds, obtaining properties of the differential equations that ensue, and deriving consequences of these properties, such as the fact that no bid can be submitted by the three bidders at three different values\textsuperscript{31}. For a given discount factor, our characterization of the strict equilibria is partial as it only applies to bids above some bound strictly above $c$. Nevertheless, the characterization of the robust equilibria with no discounting it implies is complete because the lower bound on bids tends towards $c$ as the discount factor grows to 1, as illustrated in Figures 3a and b. Online Appendix 2 presents the main steps of the proof of Theorem 2. We explain informally the intermediary results and their proofs in Online Appendices 3 and 4. The supplements to the paper (Supp. Mat.) include all complete technical statements and proofs.

\textsuperscript{31}Assuming the bidding functions are nondecreasing everywhere, and not only almost everywhere.
Figure 3b: The robust equilibrium that is the limit at $\delta = 1$ of the approaching sequence.

7. An example

Assume $F$ is the uniform distribution over $[0,1]$. For all $x \leq x'$ in this interval, one easily finds the explicit expression $e(x, x') = \frac{x(2x' - x)}{2x'}$ and MW equilibrium strategy is then $e(v) = v/2$, as is well known. Thus the bidding functions in the definition of $AC(j; \mu, y, z, \eta)$ are as follows, for all $k \neq j$:

$$\beta_j(v) = \begin{cases} \mu/2, & \text{if } \mu \leq v \leq y; \\ \frac{v(2\varphi(v) - v)}{2\varphi(v)}, & \text{if } y < v \leq \eta; \end{cases} \quad \beta_k(v) = \begin{cases} \mu/2, & \text{if } v = \mu; \\ \frac{v(2y - v)}{2y}, & \text{if } \mu < v \leq y; \\ y/2, & \text{if } y < v \leq z; \\ \frac{\varphi^{-1}(v)(2v - \varphi^{-1}(v))}{2v}, & \text{if } z < v \leq \eta. \end{cases}$$

Thus, in any component $AC(j; \mu, 1, 1, 1)$, bidder $j$ submits the constant bid $\mu/2$ and the other bidders follow the bidding function $v(2 - v)/2$ over $(\mu, 1]$.

Assume $y < 1$. The numerator and denominator in the RHS of DE1 are $e(\varphi(v)) - e(v, \varphi(v)) = \frac{(\varphi(v) - v)^2}{2\varphi(v)}$ and $e(v, \varphi(v)) - e(v) = \frac{v(\varphi(v) - v)}{2\varphi(v)}$, respectively. Dividing both by $\varphi(v) - v$, which should be strictly positive over $(y, \eta)$, DE1 simplifies to $\frac{d\ln F(\varphi(v))}{d\ln F(v)} = \frac{\varphi(v) - v}{v}$ or still (2) below$^{32}$:

$^{32}$A similar removal of the singularity at IC1, where $\varphi(\eta) = \eta$, extends to the general case (see Online Subappendix 3.4).
\[
\frac{d\varphi (v)}{dv} = \frac{\varphi (v)(\varphi (v) - v)}{v^2}. \tag{2}
\]

Elementary calculations\(^{33}\) prove that the only solution of (2) and IC1 and its inverse are as follows:

\[
\varphi (v) = \frac{2v}{1 + (v/\eta)^2}; \quad \varphi^{-1} (w) = \frac{1 - (1 - (w/\eta)^2)^{1/2}}{w/\eta^2};
\]

for all \(0 \leq v \leq \eta\) and \(0 < w \leq \eta\). Thus, C2 is equivalent to (3) below:

\[
z = \frac{2y}{1 + (y/\eta)^2}. \tag{3}
\]

Replacing \(\varphi (v)\) and \(\varphi^{-1} (v)\) by their values in the formulas above for \(\beta_j\) and \(\beta_k, k \neq j\), one can also obtain explicit expressions for these bidding functions. For example, in \(AC (j; 0, 0, 0, \eta)\):

\[
\begin{align*}
\beta_j (v) / \eta &= \frac{(v/\eta)(3 - (v/\eta)^2)}{4} \\
\beta_k (v) / \eta &= \frac{(v/\eta)^2 - 2\left(1 - (v/\eta)^2\right)^{3/2}}{2(v/\eta)},
\end{align*}
\]

for all \(k \neq j\).

When \(c < y < d\), through an easy computation one finds that C1 is equivalent to:

\[
z = y + \left\{ \frac{y^2}{6} - \frac{\mu^3 y/3 - \mu^4/4}{y^2/2} \right\}^{1/2}. \tag{4}
\]

When \(\mu = 0\), the condition reduces to \(z = y \left(1 + 6^{-1/2}\right)\). With C2 or (3) above, it implies \(\eta = (6^{1/2} + 1) y / (6^{1/2} - 1)\). As \(\eta\) cannot exceed 1, there exists a component \(AC (j; 0, y, z, \eta)\) for all \(y \leq (6^{1/2} - 1) / (6^{1/2} + 1) \approx 0.4202^{34}\). In any such component, \(\beta_j\) and \(\beta_k, k \neq j\), jump at \(y\) and \(z\) from

\(^{33}\)See Online Appendix 6.

\(^{34}\)As we mentioned in Section 5, Theorem 1 does not guarantee the existence of such
0 and \( y/2 \), respectively, to the same bid \( \beta_j (y) = \frac{y(2z-y)}{2z} = \frac{6^{1/2} + 2}{2(6^{1/2} + 1)} y \).

If \( \mu > 0 \), it is convenient to define \( k \) as the ratio \( y/\mu > 1 \). Solving for \( y \) and \( z \) the system of conditions (C1, C2), or equivalently (3,4), and appealing to the definition of \( k \), we find:

\[
y = h(k) \eta \quad (5); \quad \mu = h(k) \eta \quad (6);
\]

where the factor \( h(k) \) is the following function of \( k \):

\[
h(k) = \left\{ \frac{6^{1/2} k^2 - (k^4 - 4k + 3)^{1/2}}{6^{1/2} k^2 + (k^4 - 4k + 3)^{1/2}} \right\}^{1/2}.
\]

One can then construct robust equilibria with an infinity of nonoverlapping components. For example, to construct an infinity of components whose domains tend from below towards some \( u \) in \((0,1]\), it suffices to start from a sequence \((\eta^s)_{s \geq 1}\) strictly increasing towards \( u \) and a sequence \((\tilde{k}^s)_{s \geq 1}\) strictly decreasing towards 1 and such that \( \eta^s < \frac{h(\tilde{k}^{s+1})}{k^{s+1}} \eta^{s+1} \). Because \( \frac{h(k)}{k} \to 1 \) when \( k \to 1 \), such sequences exist. Then, the sequence \((AC(j^s; \mu^s, y^s, z^s, \eta^s))_{s \geq 1}\) where \( \mu^s, y^s, z^s \) are computed from \( \tilde{k}^s \) and \( \eta^s \) according to (5, 6, 3) and \( j^s \in \mathcal{N} \) will be nonoverlapping as \( \eta^l < \mu^{l+1} \), for all \( l \), and all \( \mu^l, y^l, z^l, \eta^l \) will tend towards \( u \). Figure 4 depicts a similar equilibrium.

\(35\) (4) then becomes \( z = y \left( 1 + 6^{-1/2} \left\{ 1 + \frac{3-4k}{k^4} \right\}^{1/2} \right) \).

\(36\) The function \( h \) is defined and takes values strictly below 1 for all \( k > 1 \).
8. Equilibrium allocations

For $\eta > c$, it is helpful to denote $\varphi (\cdot ; \eta)$ the unique\(^{37}\) solution of DE1 and IC1 and $\varphi^{-1} (\cdot ; \eta)$ its inverse.

If the highest value bidder does not win the 1st auction he will win the 2nd, where both remaining bidders bid their values. Consequently, the lowest value bidder can win only at the 1st auction and while following a bidding function strictly higher than the other bidders’. From our characterization of the robust equilibria, this does not occur unless his value belongs to the part $(y, \eta)$ of some asymmetric component $AC (k; \mu, y, z, \eta)$, in which he is the “singled-out” bidder $k$. In this case, he will outbid his competitors as soon as the maximum $\lor_{j \neq k} v_j$ of their values is below $z$ and otherwise if his value $v_k$ is larger than $\varphi^{-1} (\lor_{j \neq k} v_j; \eta)$. Thus, the allocation is efficient when the equilibrium is MW equilibrium or has only one asymmetric component and this component is of the kind $AC (k; \mu, d, d, d)$ (see Theorem 1 (vi.3)).

Obviously, our description above of the allocation may not hold true on a set of values of probability zero. We have Corollary 2 below\(^{38}\). In it, note

\(^{37}\)By Lemma 18 in Supp. Mat.

\(^{38}\)See Supp. Mat. for a more formal proof.
that $\varphi^{-1}(z^s \lor \vee_{j \neq k^s} V_j; \eta^s) = y^s$ if $\vee_{j \neq k^s} V_j < z^s$.

**Corollary 2: Equilibrium allocations:** Let $\beta$ be a robust equilibrium of SSPA(3) and $(AC (k^s; \mu^s, y^s, z^s, \eta^s))_{s \in S}$ be the collection of nondegenerate asymmetric components it extends. Then, $\beta$ does not allocate the units efficiently if and only if, up to a zero probability event:

$$\varphi^{-1}(z^s \lor \vee_{j \neq k^s} V_j; \eta^s) < V_{k^s} < \land_{j \neq k^s} V_j < \vee_{j \neq k^s} V_j < \eta^s,$$

for some $s \in S$. Furthermore, the allocation is almost surely efficient if and only if $S = \emptyset$ or $S = \{s\}$ and $\mu^s < y^s = z^s = \eta^s = d$.

There is a sense in which the components $AC (i; c, c, c, d)$, $i \in N$, are the most inefficient robust equilibria. Conditional on the extension of components $(AC (k^s; \mu^s, y^s, z^s, \eta^s))_{s \in S}$ allocating the units inefficiently and $V_h$ belonging to some $(y^s, \eta^s)$, with probability one $AC (i; c, c, c, d)$ will also allocate inefficiently once the bidders are relabeled such that bidder $k^s$ becomes bidder $i$. This follows from the previous lemma and the inequality $\varphi^{-1}(z^s \lor \vee_{j \neq k^s} V_j; d) \leq \varphi^{-1}(z^s \lor \vee_{j \neq k^s} V_j; \eta^s)$, a consequence of the inequality $\varphi(., d) \geq \varphi(., \eta^s)$\(^{39}\). As the joint distribution of the values is symmetric, the expectation of the total surplus is lower in $AC (i; c, c, c, d)$\(^{40}\).

**9. Conditional price expectations**

Following Milgrom and Weber (2000), we now consider the expected future price conditional on the current price.

\(^{39}\)See Lemma 18 (iv,3) in Supp. Mat.

\(^{40}\)Whatever bidder $i$’s value is, $AC (i; c, c, c, d)$ gives a unit to bidder $i$ with the highest probability among all robust equilibria. Using standard arguments from the theory of incentive compatible mechanisms, this equilibrium gives bidder $i$ the highest interim and exante expected payoffs and, if the value hazard rate $f (v) / (1 - F (v))$ is nondecreasing, it produces the highest exante expectation of the sum of all bidders’ payoffs.

Similarly, the extension of a component $AC (i; c, c, c, \eta)$ gives bidder $i$ higher expected payoffs and may be called more inefficient than any $\text{ext} \langle (AC (k^s; \mu^s, y^s, z^s, \eta^s))_{s \in S} \rangle$, with $\eta^s \leq \eta$ for all $s$. Under nondecreasing hazard rate, it also gives bidders a higher expected total payoff.
The bid a bidder submits at the 1st auction is the expectation of the price he will pay implicitly or explicitly at the 2nd auction conditional on barely losing the 1st auction, where the price would then be almost equal to his bid. In MW equilibrium, bidding is symmetric and the future price will be the same whomever he loses to. Consequently, the expectation of the future price equals the current one. However, along an asymmetric equilibrium the effective price a bidder will pay depends on which bidder will remain at the 2nd auction and hence which competitor he nearly ties with at the 1st auction. While averaging to find his 1st auction bid, the bidder weighs the effective price by the likelihood of the identity of this “near” competitor. It is then no surprise that the future price averaged rather over all, large and small, winning margins may differ from the current price, even given a specific highest or 2nd highest bidder\textsuperscript{41}.

In Corollary 3 below, we condition the price $P_2$ at the 2nd auction on the price $P_1$, the identity $\text{WIN}$ of the winner, and the identity of the second highest bidder at the 1st auction.

For example, assume as in Corollary 3 (ii) that $P_1 = p$ belongs to the upper convex part $(e(y^s,z^s), e(\eta^s))$ of the bid range of the component $AC(k^s, \mu^s, y^s, z^s, \eta^s)$, with $y^s < z^s < \eta^s$, all bidding functions fill while continuous. When bidder $k^s$ is the 1st auction runner up and bids $p$, the expectation of $P_2$ equals $p$, no matter who among the other two bidders wins the 1st auction. Indeed, these two bidders follow a common bidding function when bidding close to $p$.

On the other hand, $p$ is an average between two different expectations of $P_2$ when a bidder $j \neq k^s$ is the 1st auction runner up after bidding $p$: the expectation when bidder $k^s$ wins and the expectation when the other bidder different from $k^s$ wins. Because of the locally more aggressive bidding

\textsuperscript{41} Another reason it may differ when there are more than 3 bidders is that the effective future price paid by the 2nd highest bidder from the 1st auction may differ from the 2nd auction actual price. In fact, his value may be lower than the second highest value among the remaining bidders, in which case he will not win the 2nd auction nor set its price.
from bidder \( k^s \), the former expectation is larger than the latter. Thus, the expectations of \( P_2 \) conditional on the identity of the winner will be different from 1st auction price \( p \): larger if it is \( k^s \) and smaller if not. The formal proofs of all results in this section are in Supp. Mat.

**Corollary 3: Expectation of future price conditional on current price and identities of winner and runner-up:** Let \( \beta \) be a robust equilibrium such that \( \beta = \text{ext} \langle (AC(k^s; \mu^s, y^s, z^s, \eta^s))_{s \in S} \rangle \). For all \( i', i'' \in \mathcal{N} \) such that \( i' \neq i'' \) and \( \Pr (\text{WIN} = i'; P_1 = B_{i''}) > 0 \), let the function \( E_{i,i''}(p) \) of \( p \) be a conditional expectation \( E(P_2|\text{WIN} = i'; P_1 = B_{i''} = p) \). Then, for all such \( i', i'' \), there exists a Borel set \( \mathcal{P}_{i',i''} \) such that \( \Pr (P_1 \in \mathcal{P}_{i',i''}|\text{WIN} = i'; P_1 = B_{i''}) = 0 \) and \( E_{i,i''}(p) \) agrees outside \( \mathcal{P}_{i',i''} \) with the relevant parts of the statements below:

(i): if \( p \notin \cup_{s \in S} (e(\mu^s), e(\eta^s)) \) or \( p \in [e(\mu^s), e(\mu^s)] \) for some \( s \in \mathcal{S} \) such that \( y_s = d \): \( E_{i,j}(p) = p \), for all \( j \neq i \in \mathcal{N} \);

(ii): if \( p \in (e(y^s), z^s), e(\eta^s)) \), for some \( s \in \mathcal{S} \) such that \( y^s < d \), and \( \mathcal{N} = \{k^s, j, h\}: E_{k^s,j}(p) > p = E_{j,k^s}(p) > E_{j,h}(p) \);

(iii): if \( p \in (y^s), \) for some \( s \in \mathcal{S} \) such that \( c < y^s < d \), and \( \mathcal{N} = \{k^s, j, h\}: E_{k^s,j}(p) > p = E_{j,h}(p) \);

(iv): if \( p \in (e(\mu^s,y^s), e(y^s)) \), for some \( s \in \mathcal{S} \) such that \( c < y_s \), and \( \mathcal{N} = \{k^s, j, h\}: E_{j,h}(p) = p > E_{k^s,h}(p) \).

While still conditioning on the current price, we now average the future price over all possible winners and runner ups. As an example, assume again the current price \( p \) belongs to the part \( (e(y^s,z^s), e(\eta^s)) \) of the range \( AC(k^s; \mu^s, y^s, z^s, \eta^s) \), with \( y^s < z^s < \eta^s \). Thus, \( p = \beta_{k^s}(v) \) or, equivalently, \( p = e(v, \varphi(v; \eta^s)) \) for some \( v \) in \((y^s, \eta^s)\). In expectation: the price does not change if bidder \( k^s \) has set the price as the runner up of the 1st auction; goes up by \( e(\varphi(v; \eta^s)) - e(v, \varphi(v; \eta^s)) \) if bidder \( k^s \) has won the 1st auction and another bidder has set the price; goes down by \( e(v, \varphi(v; \eta^s)) - e(v) \) if bidder \( k^s \) has lost the auction and another bidder has set the price. The
expected future price will fall short of \( p \) if the product of the likelihood ratio 
\[
\frac{(1-F(v))2F(\varphi(v;\eta^s)) - F(\varphi(v;\eta^s))}{F(v)2(1-F(\varphi(v;\eta^s)))} 
\]
"bid hazard rate" \( \frac{d}{dv} F(v) / (1 - F(v)) \) at \( \beta_{k^s}(v) \) is larger than any bidder \( j \neq k^s \)'s \( \frac{d}{dv} F(\varphi(v;\eta^s)) / (1 - F(\varphi(v;\eta^s))) \) or, equivalently, if:
\[
\frac{d}{dv} \frac{1 - F(\varphi(v;\eta^s))}{1 - F(v)} > 0.
\]
If \( \eta^s < d \), the ratio \( (1 - F(\varphi(v;\eta^s))) / (1 - F(v)) \) must have a strictly positive slope at some points close to \( \eta^s \), as it is equal to 1 at \( \eta^s \) and strictly smaller to the left of it. Thus, in expectation, the price must go down from some current prices close to \( e(\eta^s) \). The arrows in Figure 5 below indicate the location of the expected future price relative to the current price in the cases we review in Corollary 4 below\(^{42}\).

\(^{42}\)For a formal proof, see Supp. Mat.
Figure 5: The location of the expected future price relative to the current price.

**Corollary 4:** Expectation of future price conditional on current price only: Let $\beta$ be a robust equilibrium of the SSPA(3) such that $\beta = \text{ext} (\langle AC(k^s; \mu^s, y^s, z^s, \eta^s) \rangle_{s \in S})$, where all asymmetric components are nondegenerate. Let $E(P_2 | P_1)$ be a conditional expectation. Then:

(i): there exists a Borel set $\mathcal{P}'$ such that $\Pr (P_1 \in \mathcal{P}') = 0$ and for all $p \notin \mathcal{P}'$ and $s \in S$ such that $y^s, z^s < \eta^s$:

(i.1): if $p = e(v, \varphi(v; \eta^s)) \in (e(y^s, z^s), e(\eta^s))$: $E(P_2 | P_1 = p) < (>) p$

if and only if $\frac{\frac{d}{dv} 1 - F(v; \eta^s)}{1 - F(v)} > (<) 0$;

(i.2): if $\mu^s < y^s < z^s$:

(i.2.1): if $p = e(y^s)$: $\Pr (P_1 = p) > 0$ and $E(P_2 | P_1 = p) > p$;

(i.2.2): if $p \in (e(\mu^s, y^s), e(y^s))$: $E(P_2 | P_1 = p) < p$;

(ii): if $\eta^s < d$: for all $\varepsilon > 0$ there exists a measurable set $\mathcal{P}'_\varepsilon \subseteq (e(\eta^s) - \varepsilon, e(\eta^s))$ such that $\Pr (P_1 \in \mathcal{P}'_\varepsilon) > 0$ and, for all $p$ in $\mathcal{P}'_\varepsilon$, $E(P_2 | P_1 = p) < p$.

\[ ^{43} \text{Or, equivalently, } y^s < d. \text{ See Theorem 1 (Section 5).} \]

\[ ^{44} \text{p here is } \beta_{k^s}(v). \]

\[ ^{45} \text{Or, equivalently, } c < y^s < d. \text{ See Theorem 1 (Section 5).} \]
(iii): if \( y^* = c \): for all \( \varepsilon > 0 \) there exists a measurable set \( \mathcal{P}_\varepsilon^\prime \subseteq (c, c + \varepsilon) \) such that \( \Pr (\mathbf{P}_1 \in \mathcal{P}_\varepsilon^\prime) > 0 \) and, for all \( p \) in \( \mathcal{P}_\varepsilon^\prime \), \( E (\mathbf{P}_2 | \mathbf{P}_1 = p) > p \).

As an example, consider the robust equilibrium \( \beta = \text{ext} \langle AC \ (1; 0, 0, 0, \eta) \rangle \) when \( 0 < \eta \leq 1 \) and \( F \) is the uniform distribution over \([0, 1]\). Then, \( E (\mathbf{P}_2 | \mathbf{P}_1 = p) = p \) for almost all \( p \) above \( e (\eta) \). Obviously, \( \frac{1 - F (v; \eta)}{1 - F (v)} = \frac{1 - \varphi (v; \eta)}{1 - v} \), for \( v \) in \((0, \eta)\). Substituting its value from Section 7 to \( \varphi (v; \eta) \), differentiating, and rearranging, we find: \( \frac{d}{dv} \frac{1 - F (v; \eta)}{1 - F (v)} > (\eta) > 0 \) if and only if \( 0 < (\eta) < \frac{\eta^4 + 4\eta^2 - 1}{4\eta^3} \), where \( \bar{v} \) is the ratio \( v/\eta \). Simple calculations show that the function \( \frac{\eta^4 + 4\eta^2 - 1}{4\eta^3} \) is nonnegative only over \([\bar{v}, 1]\), where \( \bar{v} = (5^{1/2} - 2)^{1/2} \approx 0.485 \), and strictly increases over this interval from 0 at \( \bar{v} \) to 1 at 1. Thus, if the couple \((\eta, v/\eta)\) is above the graph in Figure 6 below of the inverse of this function, we have \( E (\mathbf{P}_2 | \mathbf{P}_1 = p) < p \) where \( p = \beta_1 (v) \). Below this graph, the inequality is reversed. In particular, \( E (\mathbf{P}_2 | \mathbf{P}_1 = p) > p \) for almost all \( p \) when \( \eta = 1^{46} \).

\[
\bar{v} = v/\eta
\]

Figure 6: Comparison between current and expected future prices for the continuous asymmetric robust equilibria \( \text{ext} \langle AC \ (1; 0, 0, 0, \eta) \rangle \) in the uniform case.

\(^{46}\)The computations are particularly simple in this case. From \( \varphi (v; 1) = 2v/ (1 + v^2) \), the ratio \( \frac{1 - F (v; 1)}{1 - F (v)} \) is \( \frac{1 - \varphi (v; 1)}{1 + v^2} \) (if \( v < 1 \)), whose derivative is everywhere strictly negative.
We call the couple \((P_1, P_2)\) a martingale if there exists a conditional expectation \(E(P_2|P_1)\) such that \(E(P_2|P_1 = p) = p\), for all \(p\).\(^{47}\) We say that it is a supermartingale if \(E(P_2) < E(P_1)\) and \(E(P_2|P_1 = p) \leq p\) for all \(p\); and that it is a submartingale if \(E(P_2) > E(P_1)\) and \(E(P_2|P_1 = p) \geq p\) for all \(p\).\(^{48}\)

From Corollary 2, a robust equilibrium must count at least one inefficient component, that is, a nondegenerate component \(AC(k^s; x^s, y^s, z^s, \eta^s)\) where \(y^s < d\), for the price to be a supermartingale. If \(\mu^s > c\) and hence \(\mu^s < y^s\) (Theorem 1 (vi), Section 5), the expected future price is actually larger than a current price equal to the bid atom \(c(y^s)\) of the bidders \(j \neq k^s\) (Corollary 4 (i.2.1)). And the price goes up in expectation from small current prices when \(\mu^s = c\) (Corollary 4 (iii)). Thus, at no robust equilibrium is the price a supermartingale. This and other properties of the price sequence are gathered in the corollary below.

**Corollary 5: Stochastic process properties of the auction price:**

Let \(\beta\) be a robust equilibrium of SSPA(3).

- \((P_1, P_2)\) is a martingale if and only if \(\beta\) is either \(\beta = (\emptyset)\), that is, \(\beta\) is MW equilibrium, or \(\beta = \text{ext } AC(i; \mu, d, d, d)\), for some \(i \in N\) and \(\mu < d\);
- \((P_1, P_2)\) is not a supermartingale;
- If \(\beta \neq \text{ext } AC(i; c, c, c, d)\), for all \(i \in N\), \((P_1, P_2)\) is not a submartingale.

Appealing to the revenue equivalence theorem as we did in the intro-

\(^{47}\)In probability theory, the term refers rather to an infinite sequence. By replicating \(P_2\) infinitely often following \(P_1\), we would obtain a martingale in this sense.

\(^{48}\)Contrary to some of the literature, this last inequality may not be strict everywhere. Nevertheless, it will be strict for a set of values of \(P_1\) of strictly positive probability.

\(^{49}\)These concepts are mutually exclusive, that is, for example, a supermartingale is not a martingale.
duction (Section 1), one can show that if the equilibrium is efficient the unconditional price expectation is constant across auctions. From Corollary 5 (ii) above and Corollary 2 (Section 7), we obtain that the sequence of prices is actually a martingale for all efficient equilibria and only those. From Corollary 5 (iii), only for the most inefficient equilibria, $AC(i; c, c, c, d), 1 \leq i \leq 3$, can the sequence of prices $(P_1, P_2)$ be a submartingale. We saw just after Corollary 4 that it actually is when $F$ is the uniform distribution over $[0, 1]$. We explore further in the next section the link between the equilibrium allocation and the sequence of prices.

10. Unconditional price expectations

In this section, we show that the necessary and sufficient condition for the unconditional price expectation to decrease strictly in some robust equilibria, that is, for $E(P_2) < E(P_1)$, is that $\tilde{\omega}$ below decrease strictly for some values:

$$\tilde{\omega}(v) = v - \frac{1 - F(v)}{3f(v)}.$$  

Denoting $V_{(k,n)}$ the $k$th highest value out of a sample of size $n$, the function $\tilde{\omega}$ is the marginal revenue function of the distribution $G = 1 - (1 - F)^3$ of $V_{(3,3)}$. That $\tilde{\omega}$, rather than the marginal revenue function $\omega(v) = v - (1 - F(v)) / v$ of $F$, not be nondecreasing everywhere is a stronger requirement, as $\tilde{\omega}'(v) < 0$ implies $\omega'(v) < 0$, for any $v > c$.

Proving the necessity of the condition proceeds by expressing the expected difference $E(P_1) - E(P_2)$ as the difference between the total revenue $E(P_1 + P_2)$, which, by the revenue equivalence theorem, is the expected sum of the marginal revenues of $F$ at the 1st and 2nd auction winners’ values, and the expectation $E(2P_2)$ of twice the future price, that is, of $2V_{(3,3)}$ when the allocation is efficient and of $2V_{(2,3)}$ when it is not. As MW equilibrium price is a martingale, this expected difference would vanish if the allocation were always efficient and hence the future price
always $V_{(3,3)}$. Consequently, the expected difference in prices is the expectation restricted to where the allocation is inefficient of the difference $(\omega(V_{(3,3)}) - \omega(V_{(2,3)})) - (2V_{(2,3)} - 2V_{(3,3)})$ between the two changes due to inefficiency: in marginal revenues and in 2nd auction price. This double difference can be arranged as $(\omega(V_{(3,3)}) + 2V_{(3,3)}) - (\omega(V_{(2,3)}) + 2V_{(2,3)})$, that is, $3\left(\bar{\omega}(V_{(3,3)}) - \bar{\omega}(V_{(2,3)})\right)$. As it would be nonpositive if $\bar{\omega}$ was non-decreasing, the expectation of the future price could not be the smaller one in this case.

To prove the sufficiency of the condition, we construct a robust equilibrium that departs from efficiency only in an interval where the function $\bar{\omega}$ is strictly decreasing. Then, as in this interval $3\left(\bar{\omega}(V_{(3,3)}) - \bar{\omega}(V_{(2,3)})\right)$ is strictly positive, we will have $E(P_1) > E(P_2)$. The construction is possible thanks to Theorem 1 (ii) (Section 5) which ensures the existence of inefficient components inside any neighborhood of any value $v > \bar{\omega}$. Such components are of the type $AC(j; \mu, y, z, \eta)$ with $c < \mu < y < z < \eta$ and are therefore discontinuous. If we consider only value distributions $F$ such that $\bar{\omega}$ is quasi-convex (i.e. U shaped graph) and $1/F$ is strictly convex near $c$, then Theorem 3 below can be proved using continuous equilibria that extend single components of the type $AC(j; c, c, c, \eta)$. For example, among all power distributions $F(v) = v^a$ over $[0,1]$ with $a > 0$ robust equilibria with strictly decreasing price expectations exist only when $a < 1$, in which case there exist continuous such equilibria.

The formal proof of Theorem 3 is in Online Appendix 7.

**Theorem 3: Characterization of the value distributions that al-**
low decreasing equilibrium price expectations: The two following statements are equivalent:

(i): The marginal revenue function of the distribution \( G = 1 - (1 - F)^3 \) is not everywhere nondecreasing\(^{53}\).

(ii): There exists a robust equilibrium \( \beta \) of SSPA(3) where \( E(P_1) > E(P_2) \).\(^{54}\)

11. Extension to \( n \geq 3 \) bidders

Our numbering of equivalent statements that started in Theorem 3 in the previous section continues in Theorem 4 below\(^{55}\).

**Theorem 4: Extension of Theorem 3 to an arbitrary number of bidders:** (i) and (ii) in Theorem 3 (Section 10) are equivalent to (iii) below:

(iii): For all number \( n \geq 3 \) of bidders, there exists a robust equilibrium \( \beta \) of SSPA(n) where \( E(P_1) > E(P_2) \).

That (iii) implies (i) already follows from the 3-bidder case and Theorem 3. What Theorem 4 adds to Theorem 3 is that (i) implies the existence of an equilibrium with decreasing price expectations even for \( n > 3 \) bidders. The proof of this new result is similar to the proof for \( n = 3 \) bidders. Reasoning as in the proof of Theorem 3, we see that the difference between the expected current and future prices is the expectation where the allocation is inefficient of \( (\omega (V_{(L,n)}) - \omega (V_{(2,n)})) - (2V_{(2,n)} - 2V_{(3,n)}) \), where the L’s highest value

\(^{53}\)This is equivalent to requiring that \( 1/(1 - F)^3 \) not be everywhere convex (see McAfee and McMillan, 2007), that is, that \( 1 - F \) not be everywhere \((-3)\)-concave (see Ewerhart, 2013).

\(^{54}\)As any such equilibrium is inefficient and brings more revenues to the seller, the sum of the bidders’ expected payoffs will obviously be smaller than in MW equilibrium. Nevertheless, the bidder who follows a different bidding function in the only asymmetric component of the equilibrium we construct in our proof wins a unit more often and hence obtains a higher payoff than in MW equilibrium. Thus, inefficiency and decreasing price expectations may result from some bidders’ securing high payoffs.

\(^{55}\)And in Theorem 5 in the following section.
$V_{(L,n)}$ is the inefficient winner’s value. While $L = 3$ in the 3 bidder case, $L$ is at least 3 and may be strictly larger in this more general setting. As obviously $V_{(3,n)} \geq V_{(L,n)}$, the double difference is at least $(\omega (V_{(L,n)}) + 2V_{(L,n)}) - (\omega (V_{(2,n)}) + 2V_{(2,n)})$, that is, $3(\omega (V_{(L,n)}) - \omega (V_{(2,n)}))$. It will be strictly positive again and we will have $E(P_1) > E(P_2)$ if the inefficiency is confined to an interval of values where $\omega$ decreases strictly.

To extend the construction in the proof of Theorem 3 of a robust equilibrium that is inefficient only in a given neighborhood of some value, we must extend the definition of an asymmetric component $AC(j; \mu, y, z, \eta)$ with $\mu > c$. As for $n = 3$ bidders, all bidders but bidder $j$ follow the same bidding function and bidder $j$’s bid range over $(y, \eta)$ is the same as the other bidders’ over $(z, \eta)$. For example, if $\varphi$ denotes the function that sends bidder $j$’s value to the other bidders’ value where they would submit the same bid, bidder $j$’s $\beta_j$ is as follows over $(y, \eta)$:

$$\beta_j(v) = E\left(v \wedge V_{(1,n-2)}|V_{(1,n-2)} < \varphi(v)\right).$$

The first-order condition for the optimality of submitting $\beta_j(v)$ by any other bidder with value $\varphi(v)$ gives (once divided by $F(\varphi(v))^{n-2} F(v)$ and where the previous expression has been substituted to $\beta_j(v)$) the following differential equation:

$$\frac{d \ln F(\varphi(v))}{d \ln F(v)} = \frac{\frac{1}{n-2}}{\frac{E(\varphi(v) \wedge V_{(1,n-2)}|V_{(1,n-2)} < \varphi(v))}{E(v \wedge V_{(1,n-2)}|V_{(1,n-2)} < \varphi(v))} - \frac{E(\varphi(v) \wedge (V \vee V_{(1,n-3)})|V < v; V_{(1,n-3)} < \varphi(v))}{E(v \wedge V_{(1,n-2)}|V_{(1,n-2)} < \varphi(v))}}.$$ (DE2)

Contrary to DE1 in the 3-bidder case, (DE2) above with $n \geq 4$ is not defined over all arguments $v < \varphi$ away from $c$. Despite such difficulties, our construction in the proof of Theorem 3 extends to any number $n \geq 4$ of
bidders and proves Theorem 4. We refer the reader to Online Appendix 8 for the main steps of the proofs and to Supp. Mat. for all the technical details.

12. Expected revenues

As MW equilibrium is ex post efficient, no mechanism can sell all units with certainty and bring strictly higher expected revenues to the seller if the marginal revenue function $\omega$ of $F$ is nondecreasing. As soon as $\omega$ is not everywhere nondecreasing, starting from the construction of an asymmetric component on whose domain $\omega$ decreases strictly and proceeding then exactly as in the proof of Theorem 4 one obtains a robust equilibrium that tilts the allocation towards bidders with strictly higher marginal revenues. We obtain Theorem 5 below.

**Theorem 5: Expected revenues:** The two following statements are equivalent:

(i): The marginal revenue function of the distribution $F$ is not everywhere nondecreasing.

(ii): For all number $n \geq 3$ of bidders, there exists a robust equilibrium that brings the seller strictly higher expected revenues than MW equilibrium.

We may of course substitute in Theorem 5 the equilibrium in weakly dominant strategies of the Vickrey-Clarke-Groves, or VCG, mechanism to MW equilibrium as both are efficient. Here the VCG mechanism is the

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56 In particular such asymmetric components $AC(j; \mu, y, z, \eta)$ exist with arbitrarily small domains and extend into robust equilibria.

57 Even at a revenue improving equilibrium as in (ii), if the marginal revenue function $\tilde{\omega}$ never decreases Theorems 3 and 4 imply $EP_1 \leq EP_2 > EP_{1MW} = EP_{2MW}$. In any robust equilibrium with strictly decreasing price expectation as in Theorem 4 (iii), $EP_1 > EP_2 > EP_{2MW} = EP_{1MW}$ and the same equilibrium will also bring higher revenues than MW equilibrium. From Theorem 5 and the proof of Theorem 3, when $\tilde{\omega}$ is constant and therefore $\omega$ decreases strictly over some interval any equilibrium of SSPA(3) constructed from an inefficient asymmetric component inside this interval will be such that $EP_1 = EP_2 > EP_{1MW} = EP_{2MW}$. 39
uniform third-price auction, the single auction where each of the two highest
bidders receives one unit and pays the same price equal to the third highest
bid.

13. An extension to $k \geq 2$ units and $n \geq k + 1$ bidders

In this section, we assume that $k \geq 2$ units are sold one at a time through
a sequence of $k$ second-price auctions to $n \geq k + 1$ bidders. No bidder may
participate in an auction without having participated in all previous ones.
Contrary to the 2-unit case, we specify a particular announcement policy:
after each auction and before the next, the identity of the winner, if there
is one, his bid, and no other information are released. As in the previous
sections, from any auction the auction winner alone earns a payoff, which is
the difference between his value and the auction price. All bidders discount
their payoffs according to the same discount factor $\delta \leq 1$. We denote this
game SSPA($k, n; \delta$) and simply SSPA($k, n$) when $\delta = 1$.

We allow the bidders to correlate their bids by observing before the first
auction and between any two consecutive ones the outcomes of endogeneous
public lotteries. The probabilities of the lottery outcomes may depend only
on the public information previously revealed, that is, the ordered list of the
previous winners and their bids and the previous lottery outcomes$^{58}$. A
bidder’s bidding strategy specifies (in a measurable way) a bid probability
distribution at every auction where he is still active and for every possible
history he has observed, i.e. his own value and the previous winners’ iden-
tities and bids and the previous lottery outcomes. We consider again only
strategies that recommend to submit acceptable bids not exceeding one’s
value at all auctions and to submit one’s value at the last auction$^{59}$.

$^{58}$Our result holds as long as lotteries are allowed between the $k - 2^{th}$ and $k - 1^{th}$
auctions.

$^{59}$Given the announcement policy in this section, these strategies are the weakly un-
dominated strategies. See Remark A3 in Online Appendix 9.
According to our equilibrium concept, the bidders’ beliefs should be specified along with the lotteries and the bidding strategies and the following requirements should be met: that everywhere along equilibrium paths the bidders’ beliefs be modeled as common products of independent distributions; that beliefs and strategies be consistent; and that, for almost all\textsuperscript{60} his own values and after any history he has observed, a bidder’s bidding strategy be an optimal response to the other bidders’ strategies given his beliefs. We call such an equilibrium a \textit{publicly correlated equilibrium}.

A publicly correlated \textit{strict} equilibrium is a publicly correlated equilibrium such that any bidder’s bid is strictly optimal after every history where no deviation occurred and for almost all his values\textsuperscript{61}. A publicly correlated equilibrium $\beta$ of SSPA($k, n$) is \textit{robust} if there exists a sequence of publicly correlated strict equilibria of SSPA($k, n; \delta^l$) where the discount factor $\delta^l$ increases strictly towards 1 such that, along every path consistent with $\beta$, a bidder’s bidding strategy\textsuperscript{62} at any auction is the weak limit of the strategies in the strict equilibria.

Given a publicly correlated equilibrium, $P_t$ denotes the price at the $t^{th}$ auction assuming all bidders follow their strategies throughout. From Theorem 6 below, under the same assumption on the distribution of $V(3; 3)$ as in Theorem 3 there exist robust publicly correlated equilibria where the unconditional price expectation decreases strictly from at least one auction to the next.

\textbf{Theorem 6: Extension of Theorem 3 to arbitrary numbers of units and bidders:} \textit{(i), (ii) in Theorem 3 and (iii) in Theorem 4 are equivalent to (iv) below:}

\textbf{(iv):} For all number $k \geq 2$ of units and $n \geq k + 1$ of bidders, there exists a publicly correlated robust equilibrium $\beta$ of SSPA($k, n$) such that, for

\textsuperscript{60}For the initial value distribution $F$.

\textsuperscript{61}For the common beliefs about the distribution of values.

\textsuperscript{62}Viewed as a probability distribution over own values and bids.
some $1 \leq l \leq k - 1$, $E(P_l) > E(P_{l+1})$ and, for some conditional expectations $E(P_l|.)$ and $E(P_{l+1}|.)$ and all public history $h_l$ leading to the $l^{th}$ auction, $E(P_l|h_l) \geq E(P_{l+1}|h_l)$.

Appealing to Theorem 3 in the case $k = 2$ and $n = 3$, (iv) implies (i). We prove the reverse implication by constructing, under the assumption that $\bar{\omega}$ is not everywhere nondecreasing, a robust publicly correlated equilibrium $\beta$ along the path of which the price expectation decreases strictly from the next to last auction to the last auction, that is, such that $E(P_{k-1}) > E(P_k)$. The main steps of the construction are as follows: first, construct asymmetric inefficient equilibria as in Theorem 4 of the penultimate auction where the value interval is truncated near the upper extremity $d$ and that are such that $E(P_{k-1}) > E(P_k)$; second, through lotteries just before this auction, obtain symmetric combinations of such equilibria; third, proceeding backwards from the penultimate auction to the first, construct symmetric increasing equilibria$^{63}$. The proof can be found in Online Appendix 9.

As MW equilibrium is efficient also in SSPA($k, n$), we obtain Theorem 7 below through constructions similar to those in the proof of Theorem 6.

**Theorem 7: Expected revenues:** (i) and (ii) in Theorem 5 are equivalent to (iii) below:

(iii): For all number $k \geq 2$ of units and $n \geq k+1$ of bidders, there exists a robust publicly correlated equilibrium $\beta$ that brings strictly higher expected revenues than MW equilibrium$^{64}$.

$^{63}$Strictly increasing bidding functions in the $(k - 2)^{th}$ auction can be constructed once such symmetric mixtures of asymmetric equilibria have been defined in the $(k - 1)^{th}$ auction if the asymmetric equilibria allocate the units inefficiently only where the value density function $f$ decreases strictly. This follows from an intermediary result about general incentive compatible mechanisms (Lemma 27 in Online Appendix 9, with proof in Supp. Mat.). As $f$ decreases if $\bar{\omega}$ or $\omega$ decreases, the equilibria we use in the proof of Theorem 4 and Theorem 6 are such asymmetric equilibria.

$^{64}$And hence than the weakly dominant equilibrium of the VCG mechanism, here the uniform $(k + 1)^{th}$ price auction.
14. Conclusion

We reexamined Milgrom and Weber’s simplest model of sequential second-price auction: the model with single-unit demands, risk-neutrality, and independently and identically distributed values. In the case of 2 units and 3 bidders, we completely characterized all equilibria that become strict when small discounting is introduced. These robust equilibria are the combinations of Milgrom and Weber’s equilibrium with asymmetric components where bidders follow different monotonic bidding functions, in general discontinuous and with flat portions. Some levels of the current price are below the expectations of the future price in those equilibria that allocate the units inefficiently. If the equilibrium while being inefficient is not one of the most inefficient among all equilibria, there will also be current prices that are above the expected future price.

We showed that decreasing price expectations are compatible with robust equilibrium behavior. We characterized all those probability distributions of the bidders’ values for which robust equilibria with decreasing price expectations exist. Such robust equilibria also exist for those same distributions whatever the number of bidders may be.

We proved that if some mechanisms sell all units for higher total payments than Milgrom and Weber’s equilibrium then so do some inefficient robust equilibria of the sequential auction.

We obtained some first extensions of our results to arbitrary numbers of units.

References


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