Constrained-Efficient Partnership Agreement

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Introduction

How to divide profit is a frequently asked question by partnership members (Deeb (2014), Schall (2016), Spolsky (2011)).

I study the problem of designing a partnership agreement that

- divides profit
- recommends effort contributions.
Timing, Strategies, and Preferences

(Private) cost parameters: \((\theta_1, \theta_2) \in \Theta^2 \quad \Theta \equiv \{\theta_L, \theta_H}\)

- Agents report their cost parameters \(\hat{\theta} \equiv (\hat{\theta}_1, \hat{\theta}_2)\).
- A partnership agreement \((s, w, \sigma)\) assigns
  - share allocation \(s(\hat{\theta}) \in \Delta^1 \equiv \{(r_i)_{i \in N} \in \mathbb{R}^2_+ | \sum_i r_i = 1\}\),
  - wages \(w(\hat{\theta}) \in \mathbb{R}^2_+\),
  - effort contributions \(\sigma(\hat{\theta}) \in \mathbb{R}^2_+\).

- Agents contribute efforts \(e \equiv (e_1, e_2) \in \mathbb{R}^2_+\).
- Agent \(i\)'s payoff:
  \[
  s_i(\hat{\theta}) \left[ e_i + e_{-i} - w_i(\hat{\theta}) - w_{-i}(\hat{\theta}) \right] + w_i(\hat{\theta}) - c(e_i, \theta_i). 
  \]
  profit + cost of effort
Agent $i$’s payoff:

$$s_i(\hat{\theta}) \left[ e_i + e_{-i} - w_i(\hat{\theta}) - w_{-i}(\hat{\theta}) \right] + w_i(\hat{\theta}) - c(e_i, \theta_i).$$

**Remarks**

- Share allocation $s(\hat{\theta})$ affects efforts $(e_1, e_2)$.
- Efforts $(e_1, e_2)$ affect valuations of share assignments.
- Valuations are interdependent.
- Wages $w(\hat{\theta})$ affect reports $\hat{\theta}$, not efforts $(e_1, e_2)$.
- Adverse selection and moral hazard are present.
Desiderata

Define \( u_i(r, e, \theta_i) \equiv r_i(e_i + e_{-i}) - c(e_i, \theta_i) \)

\[
\nu_i(r, m, e, \theta_i) \equiv r_i[(e_i + e_{-i}) - m_i - m_{-i}] + m_i - c(e_i, \theta_i)
\]

\[
\forall i \quad \forall r \in \Delta^1 \quad \forall m \in \mathbb{R}^2_+ \quad \forall e \in \mathbb{R}^2_+ \quad \forall \theta_i \in \Theta.
\]

\((s, w, \sigma)\) is efficient if

\[
\forall \theta \in \Theta^2 \quad [s(\theta), \sigma(\theta)] \in \arg\max_{(r, e) \in \Delta^1 \times \mathbb{R}^2_+} \sum_i u_i(r, e, \theta_i)
\]

\((s, w, \sigma)\) is incentive compatible if

\[
\forall i \quad \forall \theta \in \Theta^2 \quad \forall \theta_i' \in \Theta \quad \forall e_i \in \mathbb{R}_+
\]

\[
\nu_i[s(\theta), w(\theta), \sigma(\theta), \theta_i] \geq \nu_i[s(\theta'_i, \theta_{-i}), w(\theta'_i, \theta_{-i}), e_i, \sigma_{-i}(\theta'_i, \theta_{-i}), \theta_i]
\]
Theorem 1 (Holmstrom, 1982)

\( \not\exists (s, w, \sigma) \) satisfying \textbf{efficiency} & \textbf{incentive compatibility}

Fix \( \theta \in \Theta^2 \).

\[
\Delta^1 \times \mathbb{R}_+^2 \\
\mathcal{IC}(\theta) = \{(r, e) \mid e_i \in \arg \max_{e_i' \in \mathbb{R}_+} u_i(r, e_i', e_{-i}, \theta_i) \ \forall i\} \\
(r^*, e^*) \in \arg \max_{(r, e) \in \Delta^1 \times \mathbb{R}_+^2} \sum_i u_i(r, e, \theta_i)
\]

\( (s, w, \sigma) \) is \textbf{constrained-efficient} if

\[
\forall \theta \in \Theta^2 \quad \quad \quad [s(\theta), \sigma(\theta)] \in \arg \max_{(r, e) \in \mathcal{IC}(\theta)} \sum_i u_i(r, e, \theta_i)
\]
(s, w, σ) satisfies individual rationality if
\[ \forall i \quad \forall \theta \in \Theta^2 \quad v_i[s(\theta), w(\theta), \sigma(\theta), \theta_i] \geq 0 \]

**Equal division rule.** (s^e, w^e, σ^e)

\[ s^e(\theta) = \left(\frac{1}{2}, \frac{1}{2}\right) \quad w^e(\theta) = (0, 0) \quad [s^e(\theta), \sigma^e(\theta)] \in IC(\theta) \quad \forall \theta \in \Theta^2. \]

(s, w, σ) Pareto dominates (s^e, w^e, σ^e) if

- \[ v_i[s(\theta), w(\theta), \sigma(\theta), \theta_i] \geq v_i[s^e(\theta), w^e(\theta), \sigma^e(\theta), \theta_i] \quad \forall i \quad \forall \theta \in \Theta^2, \]
- the inequality is strict for some \((i, \theta) \in N \times \Theta^2.\)
## Assumptions

### Assumption 1
- \( c \in C^3 \)
- \( c(e_i, \theta_i) \) is increasing in \( e_i \) and \( \theta_i \)
- \( c(e_i, \theta_i) \) is quadratic and strictly convex w.r.t. \( e_i \)

### Assumption 2
- \( c(0, \theta_i) = 0 \)
- \( \frac{\partial c(0, \theta_i)}{\partial e_i} = 0 \)
- \( \lim_{e_i \to \infty} \frac{\partial c(e_i, \theta_i)}{\partial e_i} = \infty \quad \forall \theta_i \in \Theta \)
- \( \frac{\partial^2 c}{\partial e_i \partial \theta_i} > 0 \)
- \( \frac{\partial^3 c}{\partial^2 e_i \partial \theta_i} > 0 \)
Result

Theorem 2

\[ \exists (s^*, w^*, \sigma^*) \text{ that} \]

- satisfies
  - constrained efficiency
  - incentive compatibility
  - individual rationality
- Pareto dominates \((s^e, w^e, \sigma^e)\)
The Wage Rule

Proposition 1

A constrained-efficient partial agreement \((s^*, \sigma^*)\) exists.

Define

\[
\bar{u}_i(r, e_{-i}, \theta_i) \equiv \max_{e'_i \in \mathbb{R}^+} r_i(e'_i + e_{-i}) - c_i(e'_i, \theta_i)
\]

\[
\quad \forall i, \forall r \in \Delta^1, \forall e_{-i} \in \mathbb{R}^+, \forall \theta_i \in \Theta
\]

\[
\beta_i(\theta_L) \equiv
\]

\[
\bar{u}_i[s^*(\theta_L, \theta_L), \sigma_{-i}^*(\theta_L, \theta_L), \theta_H] - \bar{u}_i[s^*(\theta_H, \theta_L), \sigma_{-i}^*(\theta_H, \theta_L), \theta_H]
\]

\[
\quad \forall i.
\]

Suppose \(\beta_i(\theta_L) > 0\).
Let $w^*_i(\theta_H, \theta_L) = \beta_i(\theta_L) \times [s^*_i(\theta_H, \theta_L)]^{-1} > 0$

$w^*_i(\theta_L, \theta_H) = w^*_i(\theta_L, \theta_L) = w^*_i(\theta_H, \theta_H) = 0 \quad \forall i.$

- $(s^*, w^*, \sigma^*)$ satisfies constrained efficiency, incentive compatibility, & individual rationality

- $A^* \equiv (s^*, w^*, \sigma^*)$ Pareto dominates $A^e \equiv (s^e, w^e, \sigma^e)$

If true cost parameters are $(\theta_H, \theta_L)$,

- agent 1 is indifferent between $A^*$ and $A^e$
- agent 2 strictly prefers $A^*$. 

The Wage Rule (cont.)
Conclusion

Under suitable assumptions, I design a partnership agreement that

- satisfies
  - constrained efficiency
  - incentive compatibility
  - individual rationality
- Pareto dominates equal division rule
- does not depend on distributions of cost parameters or agents’ beliefs.
Future Work

- Design a voting game to implement optimal profit distributions.
- Extend the analysis for an environment with multiple agents, multiple types, and multiple periods:
  - current efforts affect future marginal benefits of efforts
  - some agent(s) might leave the partnership due to some preference shocks
  - profit shares need to be re-allocated and departing agent(s) need to be paid.