Estimating Macroeconomic Models of Financial Crises: An Endogenous Regime Switching Approach *

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Abstract

We develop a novel approach to specifying, solving and estimating Dynamic Structural General Equilibrium (DSGE) models of financial crises. We first propose a new specification of the standard Kiyotaki-Moore type collateral constraint where the movement from the unconstrained state of the world to constrained state is a stochastic function of the endogenous leverage ratio in the model. This specification results in an endogenous regime switching model. We then develop perturbation methods to solve this model. Using the second order solution of the model, we design an algorithm to estimate the parameters of the model with full-information Bayesian methods. Applying the framework to Mexico we find that the model’s estimated crisis regime probabilities correspond closely with narrative dates for Sudden Stops in Mexico. Our results shows that fluctuations in the non-crisis regime of the model are driven primarily by real shocks, while leverage shocks are the prime driver in the crisis regime. This provides the first structural estimates of financial shocks consistent with the reduced form literature which finds that financial/credit shocks only matter in periods of high financial stress.

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1 Introduction

In response to the Financial Crisis a large literature has emerged to model the impact of financial frictions. Much of this literature has highlighted the importance of collateral constraints in amplifying shocks and providing a theoretical justification for policy interventions. However, due to computational complexity this literature largely eschews formal econometric analysis of these models and the actual sequence of shocks that historically has driven crisis episodes. That is, a Smets and Wouters (2007) style evaluation of this class of models, an evaluation that is needed for implementation of policy recommendations, has not been done. In this paper we bridge the econometric evaluation of DSGE models in the spirit of Smets and Wouters (2007) with the collateral constraint models emphasized in the recent theoretical financial frictions literature.

This paper makes contributions to four areas of the literature. First, we propose a new specification of the standard Kiyotaki and Moore (1997) type collateral constraint where the movement from the unconstrained state of the world to constrained state is a stochastic function of the endogenous leverage ratio in the model. Our model is such that as leverage rises, the probability of the constraint binding increases, but there is no specific leverage ratio where the constraint must bind. This specification results in an endogenous regime switching model. Our second contribution is to develop perturbation methods to solve endogenous regime switching models rapidly and to higher orders. Third, using the second order solution of the model, we design an algorithm to estimate the parameters of the model with full-information Bayesian methods, which has previously only been done for first order solutions of exogenous switching models. Our fourth contribution is to apply the framework to Mexico and provide the first formal econometric analysis of this class of models.

The model is estimated from 1981 to 2016 using data for Mexico. Our results reveal three novel empirical findings. First, we find that the probability of a crisis is an increasing function of leverage, but also that there is range of leverage ratios where a crisis is likely to occur. Second, the model provides estimated crisis regime probabilities which correspond closely with narrative dates for Sudden Stops in Mexico. Third, our results shows that fluctuations in the non-crisis regime of the model are driven primarily by the usual real shocks (TFP, preference and terms of trade). In the crisis regime we find that leverage shocks are the prime driver of economic fluctuations. Our results then, provide the first structural estimates of financial shocks consistent with the reduced form literature which finds that financial/credit shocks only matter in periods of high financial stress.

The core of the new methodology is an endogenous regime switching approach to modeling financial crises. In the model there are two regimes, one a crisis regime, the second a
regime for normal economic times. A crisis regime is a regime where an occasionally binding borrowing constraint binds (e.g. Mendoza, 2010) determined by economic variables in the economy. Likewise, the switch back to normal times is based on economic fundamentals. In our model the probability of moving to the crisis regime where the borrowing constraint binds is a logistic function of the debt to output ratio. This ratio in turn, is a function of endogenous state variables, exogenous shocks and control variables. Agents in the economy know of this probability and how debt, output and other choices map into the probability of moving in or out of the crisis state. That is, it is a rational expectations solution of the model. Our solution then ensures that decisions made in the normal state fully incorporate how those decision affect the probability of moving into the crisis state as well how the economy will operate in a crisis (i.e the decision rules in this crisis).

The approach we develop allows us to capture all of the salient features one would want in an empirical model of financial crises. First, it captures the non-linear nature of a crisis: the crisis state can have very different properties/parameters from the normal state. Second, we solve the regime switching model using perturbation methods and a second order solution. This means that we can capture the change in decision rules as risk changes in a crisis. Third, since our solution method is perturbation based we can handle a number of state variables and many shocks. That is, we are less constrained than current non-linear methods in terms of the size of the model. Fourth, the speed of the solution method means that we can use non-linear filters to calculate the likelihood function of the model for a full Bayesian estimation of he relevant shocks and frictions that are fundamental to models of financial crises.

In the literature on Markov-switching DSGE models this paper is most closely related to Foerster et al. (2016), who develop perturbation methods to solve exogenous regime switching models working directly with the non-linear model. This differs from the Markov-Switching linear rational expectations (MSLRE) literature which starts with with a system of linear rational expectations equations and imposing Markov Switching after linearizing the model (e.g. Leeper and Zha, 2003; Davig and Leeper, 2007; Farmer et al., 2011). Since our model structural has a regime switching in the fundamental equations of the model, this is the natural approach. It has the added benefit that the MSDGE model can be solved to higher orders, where the MSLRE model of course is restricted to first order solutions. We find that the second order solution is critical for endogenous switching models to differ from exogenous switching models in the decision rules.

There is a small literature emerging to solve endogenous regime switching models. Davig and Leeper (2008), Davig et al. (2010), and Alpanda and Ueberfeldt (2016) all consider endogenous regime switching, but employ computationally costly global solution meth-
ods that eliminate the possibility for likelihood-based estimation. Lind (2014) develops a regime-switching perturbation approach for approximating non-linear models, but the approach requires repeatedly refining the points of approximation and hence is not suitable for estimation purposes. Most closely related to our approach is the method developed by Barthlemy and Marx (2017), but who consider a class of models with regime-dependent steady states that our framework does not satisfy. In contrast, our extension of the Foerster et al. (2016) perturbation approach is well suited for solving a model of crises where regime-dependent steady states may not be relevant given the relatively short-lived nature of crises, and is fast enough to allow for likelihood-based estimation.

The application of the methodology is most closely related to the literature that has built on the seminal work of Mendoza (2010). This literature has studied the normative properties of model economies with endogenous financial crises (also labeled sudden stops or credit crunches). Some examples include Bianchi (2011) who uses an endowment version of such an economy and finds that the competitive equilibrium always entails more borrowing relative to the constrained social planner allocation, and that a prudential tax on debt (i.e., a prudential capital control) can replicate the social planner allocation. Benigno et al. (2013) show that in a production economy agents can actually borrow too little relative to what is socially optimal. Benigno et al. (2016) compare alternative tax instruments chosen by a Ramsey planner in the same economy analyzed by Bianchi (2011) and find that taxes on consumption (i.e., real exchange rate interventions) dominate capital controls as a policy tool because they can achieve the unconstrained allocation while capital controls can achieve only the constrained efficient one. Cespedes et al. (2016) compare the transmission mechanism of alternative policy interventions in a similar model. Jeanne and Korinek (2010) and Bianchi and Mendoza (2010) analyze models in which the price externality arises because agents fail to internalize the effect of their decisions on an asset price. Korinek and Mendoza (2013) provide a thorough review of the models, questions and results from this large literature. They conclude by stating that an important future step for this literature is the “development of numerical methods that combine the strengths of global solution methods in describing non-linear dynamics with the power of perturbation methods in dealing with a large number of variables so as to analyze sudden stops in even richer macroeconomic models”. This is exactly what the methodology developed in this paper delivers, and allows us to empirically evaluate this class of models with the potential to return to these normative questions in future work.

There are many possible applications of our approach to other classes of models. For example, Bocola (2015) builds and estimates a model of sovereign default. His estimation procedure is to first estimate the model outside of the crisis period, using a solution tech-
nique that assumes a crisis will not occur. Conditional on those parameter estimates a crisis probability that is exogenous is appended to the model. Our approach allows one to estimate model parameters fully incorporating the possibility of a crisis outside of the crisis period, and allowing for that crisis to be a function of the economic decisions. The methods here also apply to the literature on the zero-lower bound on interest rates. \(^1\)

The rest of the paper is organized as follows. Section 2 describes the model and introduces the new formulation for the collateral constraint. Section 3 develops the perturbation solution methodology for endogenous regime switching models. Section 4 describes our procedure for estimating the regime switching models using full information Bayesian Methods. Section 5 contains the empirical results and Section 6 concludes.

## 2 The Model

The model is a small, open, production economy with an occasionally binding collateral constraint subject to productivity, preference, income, interest rate, terms of trade, and financial shocks. The restriction on access to international credit markets that we specify depends on key endogenous variables of the model, including borrowing, capital, and its price. Capital and debt choices respond to exogenous shocks in the model and affect leverage. Leverage in turn affects the probability of a binding constraint. Because of the occasionally binding collateral constraint, this framework can potentially account for both normal business cycles as well as key aspects of financial crises in emerging market economies (Mendoza, 2010).

### 2.1 The Borrowing Constraint

The collateral constraint limits total debt to a fraction of the market value of physical capital (i.e. it is a limit on leverage). As in Mendoza (2010), Kiyotaki and Moore (1997), Aiyagari and Gertler (1999), and Kocherlakota (2000), among others, the collateral constraint is not derived from an optimal credit contract, but imposed directly on the economy. However, the borrowing constraint may result from limited enforcement problems preventing lenders from collecting more than a fraction of the value of the collateral. When the constraint binds, the model produces endogenous risk premia over the world interest rate at which borrowers

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\(^1\)Guerrieri and Iacoviello (2015) develop a set of procedures called OccBin to solve models with occasionally binding constraints. OccBin is a certainty equivalent solution method which requires agents to know precisely how long a regime (the one you are not currently in) will apply if there are no shocks, making it functionally quite similar to perfect foresight methods. These methods rule out precautionary effects, which are important for the model in this paper.
borrowers would agree to contracts while satisfying it. Like in the specifications in the literature above, when the constraint binds, debt is limited to a fraction of the market value of the capital stock. Here we follow Mendoza (2010) and include also working capital in the borrowing constraint to pin down well behave supply response of the economy during financial crises.

While our constraint is the same as in the quantitative financial friction literature above, we propose a new specification of the occasionally binding nature that is more tractable and has appealing empirical properties. We model the occasionally binding nature of the constraint as an endogenous regime switching process. Thus, there is one regime in which the constraint binds (a crisis regime), and one in which it does not (a normal regime). And the probabilities to switch from one regime to the other is assumed to depend on key endogenous variables in the model in a logistic manner.

In particular, the probability to switch from the normal regime to the crisis regime is assumed to be a logistic function of the distance between actual borrowing and the borrowing limit. Therefore it is affected by all endogenous variables that enter the credit constraint. The probability to switch from a crisis regime to the normal one, instead, is assumed to be a function of the borrowing multiplier. As a result, a key feature of this specification is that the probability of

For instance, being in a binding regime depends on leverage and borrowing, without ever actually reaching a binding state. This captures a key finding of the empirical literature on financial crises, which documented extensively how the likelihood of a financial crisis raises with leverage, without requiring a crisis necessarily to occur.

Most importantly, agents in the model, know that higher leverage and borrowing levels (and hence lower collateral) increases the probability of switching to a constrained regime (or vice versa. This preserves the interaction in agents’ behavior between the two regimes that gives rise to precautionary behaviors and distinguishes this class of models from those in which financial frictions are always binding.

From an empirical perspective, we also note that not having a given leverage ratio that trigger a crisis event (i.e., the collateral constraint binding), but rather leverage affecting the likelihood of the constraint binding in a smooth manner adds element of realism to the model.

From a computational perspective, the key advantage of writing the occasionally binding constraint model as an endogenous regime switching model is that we can take the model to the data using likelihood based estimation methods.
2.2 Representative Household-Firm

There is a representative household that maximizes the following utility function

$$ U \equiv \mathbb{E}_0 \sum_{t=0}^{\infty} \left\{ v_t \beta^t \frac{1}{1 - \rho} \left( C_t - \frac{H_t}{\omega} \right)^{1-\rho} \right\}, \quad (1) $$

with $C_t$ denoting the individual consumption and $H_t$ the individual supply of labor. The elasticity of labor supply is $\omega$, while $\rho$ is the coefficient of relative risk aversion. The variable $v_t$ represents an exogenous and stochastic preference shock. Households choose consumption, labor, capital, intermediate inputs, and holdings of real, one-period international bonds maximizing utility subject to the budget constraint

$$ C_t + I_t = A_t K_{t-1}^{\eta} H_t^\alpha V_t^{1-\alpha - \eta} - P_t V_t - \phi r_t (W_t H_t + P_t V_t) + S_t - \frac{1}{(1 + r_t)} B_t + B_{t-1}. \quad (2) $$

The first term of the right hand side of equation is the production function. Goods are produced with capital ($K_{t-1}$), labor ($H_t$) and imported intermediate goods ($V_t$). $P_t$ is the price of intermediate imports, which follows a stochastic process specified below. This shock is interpreted as a terms of trade shock. $B_t$ is a one-period international bond with net interest rate $r_t$. The interest rate is exogenous and equal to a stochastic process specified below. The $\phi r_t$ term is the working capital constraint, and says that a fraction of both wages and intermediate goods must be paid for with borrowed funds. The price of labor and capital are given by $w_t$ and $q_t$, both of which are endogenous variables, but taken as given by the household. We allow also for exogenous spending shock represented by the variable $S_t$. Gross investment $I_t$ is subject to adjustment costs as a function of net investment:

$$ I_t = \delta K_{t-1} + (K_t - K_{t-1}) \left( 1 + \frac{t}{2} \left( \frac{K_t - K_{t-1}}{K_{t-1}} \right) \right). \quad (3) $$

Households face a regime specific collateral constraint, where the regimes are denoted by $s_t \in \{0, 1\}$. When $s_t = 1$, the constraint binds strictly, and total borrowing is equal to a fraction of the value of collateral

$$ \frac{1}{(1 + r_t)} B_t - \phi (1 + r_t) (W_t H_t + P_t V_t) = -\kappa_t q_t K_t \quad (4) $$

On the left hand side of this equation we have total debt and and working capital loans. The presence of the binding constraint limits both international borrowing (hence consumption smoothing) as well as borrowing to pay for intermediate inputs. This latter limit constraints output, which will cause the constraint to bind even tighter, resulting in additional capital
outflows in the next period. In this regime, as the quantity and value of capital fluctuates, the amount of borrowing will also fluctuate. When \( s_t = 0 \), the constraint is slack and the value of the collateral is enough for international lenders to finance all the desired borrowing levels. Thus,

\[
\frac{1}{(1 + r_t)} B_t - \phi (1 + r_t) (W_tH_t + P_tV_t)
\]

is unconstrained by \( \kappa_t q_t K_t \) in the current period.

In order to specify how the economy changes regimes it is useful to first define the notion of "borrowing cushion" as the distance of the borrowing from the debt limit in (5):

\[
B^*_t = \frac{1}{(1 + r_t)} B_t - \phi (1 + r_t) (W_tH_t + P_tV_t) + \kappa_t q_t K_t.
\]

(6)

When the borrowing cushion is small then the constraint is close to binding. In this case, the leverage ratio is high because borrowing relative to the value of the collateral is high.

In regime \( s_t = 0 \), when the constraint is not binding, the probability that it binds the next period depends on the value of debt relative to the credit limit in (6) in a logistic way. That is, the transition probability from regime 0 to regime 1 is a function of all endogenous variables in \( B^*_t \):

\[
Pr (s_{t+1} = 1 | s_t = 0) = \frac{\exp (-\gamma_0 B^*_t)}{1 + \exp (-\gamma_0 B^*_t)}.
\]

(7)

The parameter \( \gamma_0 \) controls how the likelihood of hitting the debt limit is linked to the borrowing cushion. For small values of this parameter, the cushion has little impact on the probability of a transition to the binding regime. For large values of this parameter, the probability of a crisis moves rapidly from 0 to 1 as \( B^*_t \) approaches 0.

In regime 1, when the constraint is binding, the Lagrange multiplier associated with the constraint is non-zero. Denoting the multiplier as \( \lambda_t \), the transition probability from the binding regime to the non-binding regime is given by:

\[
Pr (s_{t+1} = 0 | s_t = 1) = \frac{\exp (-\gamma_1 \lambda_t)}{1 + \exp (-\gamma_1 \lambda_t)}.
\]

(8)

This expression implies that as the multiplier approaches 0, the probability of transitioning back to the non-binding state rises.

The logistic function can be given a structural interpretation by adding a stochastic monitoring (or enforcement) shock \( \epsilon^M_t \) to the standard borrowing constraint used in the literature:

\[
\frac{1}{(1 + r_t)} B_t - \phi (1 + r_t) (W_tH_t + P_tV_t) = -\kappa q_t K_t + \epsilon^M_t
\]
This shock has two interpretations, based on its sign. When the shocks is negative, the LHS is greater than the value of collateral, but the lender monitors and decides to impose the borrowing constraint. When the shock is positive, the LHS is then less than the value of collateral, but the constraint does not bind because the lender does not monitor. We assume that the distribution of \( \epsilon^M_t \) is such that when borrowing is much less than the value of collateral the probability of drawing a monitoring shock that leads to a binding constraint is 0. When borrowing exceeds the value of collateral by a large amount the probability of drawing a monitoring shock is such that the probability the lender audits goes to 1. The logistic function satisfies these assumptions, though other function do as well.\(^2\)

The equation for the country specific interest rate is given by:

\[
r_t = r^* + (\psi_r + \sigma_r \varepsilon_{r,t}) \left( e^{B_t - B_t} - 1 \right) + \sigma_w \varepsilon_{w,t} \tag{9}
\]

The interest rate has an exogenous component and a debt elastic component, which pins down a well defined steady state in the non-binding regime. Following Garcia-Cicco et al. (2010), we consider also a shock to the the debt elasticity interpreted as a country specific risk premium shock, \( \varepsilon_{r,t} \). The shock \( \varepsilon_{w,t} \) is interpreted as a world interest rate shock, as it is not related to the countries debt level or other domestic factors.

The model therefore has five exogenous processes and seven shocks. The three exogenous processes are

\[
\log A_t = (1 - \rho_A(s_t)) a^*(s_t) + \rho_A(s_t) \log A_{t-1} + \sigma_A(s_t) \varepsilon_{A,t} \tag{10}
\]

\[
\log S_t = (1 - \rho_S(s_t)) s^*(s_t) + \rho_A(s_t) \log S_{t-1} + \sigma_S(s_t) \varepsilon_{S,t} \tag{11}
\]

\[
\log v_t = \rho_v(s_t) \log v_{t-1} + \sigma_v(s_t) \varepsilon_{v,t} \tag{12}
\]

\[
\log P_t = (1 - \rho_P(s_t)) p^*(s_t) + \rho_P(s_t) \log P_{t-1} + \sigma_P(s_t) \varepsilon_{P,t} \tag{13}
\]

\[
\kappa_t = (1 - \rho_A(s_t)) \kappa^*(s_t) + \rho_A(s_t) \kappa_{t-1} + \sigma_A(s_t) \varepsilon_{\kappa,t} \tag{14}
\]

Note here that the intercept are regime-dependent. This induce switches in their mean values in the binding regime.

\(^2\)The logistic function is also used by Kumhof et al. (2015) to model theoretically the transition to a default regime in their model.
2.3 First Order Conditions

Households maximize (1) subject to (2) and (4) and (5) by choosing $C_t$, $B_t$, $K_t$, $V_t$ and $H_t$. The first-order conditions of this problem are the following:

\[ v_t \left(C_t - \frac{H_t^\omega}{\omega}\right)^{-\rho} = \mu_t \]  
(15)

\[(1 - \alpha - \eta) A_t K_t^n H_t^\alpha V_t^{-\alpha - \eta} = P_t \left(1 + \phi r_t + \frac{\lambda_t}{\mu_t} \phi (1 + r_t)\right) \]  
(16)

\[\alpha A_t K_t^n H_t^{1-\alpha-\eta} V_t^{1-\alpha-\eta} = \phi W_t \left(r_t + \frac{\lambda_t}{\mu_t} (1 + r_t)\right) + H_t^{\omega-1} \]  
(17)

\[\mu_t = \lambda_t + \beta (1 + r_t) E_t \mu_{t+1} \]  
(18)

\[E_t \mu_{t+1} \beta \left(1 - \delta + \left(\frac{1}{2} \left(\frac{K_{t+1}}{K_t}\right)^2 - \frac{\iota}{2}\right) + \eta A_t K_t^{\eta-1} H_t^{\alpha-\eta} V_t^{1-\eta-\alpha}\right) = \mu_t \left(1 - \iota + \iota \left(\frac{K_t}{K_{t-1}}\right)\right) - \lambda_t K_t q_t \]  
(19)

The market prices for capital and labor are

\[q_t = 1 + \iota \left(\frac{K_t - K_{t-1}}{K_{t-1}}\right) \]  
(20)

\[W_t = H_t^{\omega-1} \]  
(21)

The last two conditions are the budget and the complementary slackness conditions

\[C_t + I_t = A_t K_t^n H_t^\alpha V_t^{1-\alpha-\eta} + S_t - P_t V_t - \phi r_t (W_t H_t + P_t V_t) - \frac{1}{(1 + r_t)} B_t + B_{t-1} \]  
(22)

\[B_t^\iota \lambda_t = 0 . \]  
(23)

This last equation is key in our model. It combines information on the borrowing constraint in both regimes (4 and 5) as well on the switching between regimes 0 and 1 (7, 8). In regime 0, the multiplier is strictly positive and the borrowing cushion is equal to 0. In regime 1 the borrowing cushion is positive, but the constraint is not binding so the multiplier is 0. The switch between regimes is then governed by the analogous of the traditional complementary slackness condition, which here is controlled by (7, 8) and hence remains differentiable in its support.

We note here that from (18), when the collateral constraint binds, the small open
economy faces an endogenous premium on debt measured as

\[ \frac{\mu_t}{\beta E_t \mu_{t+1}} = \frac{\lambda_t}{\beta E_t \mu_{t+1}} + (1 + r_t) \]

\[ = \frac{\lambda_t}{\beta E_t \mu_{t+1}} + \left(1 + r^* + (\psi_r + \sigma_r \varepsilon_{r,t}) (e^{B_t - B_{t-1}} - 1) + \sigma_w \varepsilon_{w,t}\right) \]

in which \( B_t \) is equal to the borrowing limit specified as in (4). From the previous expressions that are two endogenous components that determine the external financing premium. One component depends on the constraint being binding (\( \lambda_t > 0 \)), the second component depends on the fact that the interest rate is debt-elastic.

### 2.4 Competitive Equilibrium

A competitive equilibrium in our framework is a sequence of quantities \( \{K_t, B_t, C_t, H_t, V_t, I_t, A_t, \kappa_t, Y_t, \lambda_t, \mu_t, B^*_t\} \) and prices \( \{P_t, r_t, q_t, w_t\} \) that satisfy the household’s first order conditions (13)-(17), the market prices for capital and labor (18)-(19), the market clearing condition (20), the definition of the borrowing cushion (6), the slackness condition (21), and the exogenous processes (9)-(12).

### 3 Solving the Endogenous Switching Model

The key insight for mapping the model presented above into an endogenous regime-switching framework is to modify the slackness condition (23) so that the relevant variables are zero only in the relevant state. In particular, in the normal regime \( (s_t = 0) \), the borrowing constraint does not bind and \( \lambda_t = 0 \). In the crisis regime \( (s_t = 1) \), on the other hand, the borrowing constraint binds and \( B^*_t = 0 \).

To capture this feature in a regime switching framework, we introduce two state-dependent variables \( \varphi (s_t) \) and \( \nu (s_t) \), and re-write (23) as

\[ \varphi (s_t) B^*_{ss} + \nu (s_t) (B^*_t - B^*_{ss}) + (1 - \varphi (s_t)) \lambda_{ss} + (1 - \nu (s_t)) (\lambda_t - \lambda_{ss}) = 0. \] (24)

In this modified slackness condition, \( \varphi (0) = \nu (0) = 0 \) when \( s_t = 0 \), and so the equation simplifies to \( \lambda_t = 0 \). While \( \varphi (1) = \nu (1) = 1 \) when \( s_t = 1 \), so that the equation simplifies to \( B^*_t = 0 \). This representation helps to preserve information in our perturbation approximation, since at first order, the above implies \( d\lambda_t = 0 \) for \( s_t = 0 \), and \( dB^*_t = 0 \) for \( s_t = 1 \), meaning that both variables are constant in the respective regimes.
3.1 Deterministic Steady State

Given the modified slackness condition (24), our perturbation method builds second-order Taylor expansions of the decision rules of the model equilibrium around a non-stochastic steady state. Defining a non-stochastic steady state in an endogenous regime-switching framework, however, is not trivial.

**Definition:** A steady state in our framework can be defined as a state in which ensues when all shocks are zero \((\varepsilon_{A,t} = \varepsilon_{P,t} = \varepsilon_{w,t} = \varepsilon_{r,t} = 0)\) for all \(t\), and the regime switching variables \(\varphi(s_t), a^*(s_t), p^*(s_t)\), and \(\kappa^*(s_t)\) are at their ergodic means across regimes associated with the steady state transition matrix:

\[
P_{ss} = \begin{bmatrix} p_{00,ss} & p_{01,ss} \\ p_{10,ss} & p_{11,ss} \end{bmatrix} = \begin{bmatrix} 1 - \frac{\exp(-\gamma_{0,1}B_{ss}^r)}{1+\exp(-\gamma_{0,1}B_{ss}^r)} & \frac{\exp(-\gamma_{0,1}B_{ss}^r)}{1+\exp(-\gamma_{0,1}B_{ss}^r)} \\ \frac{\exp(-\gamma_{1,1}\lambda_{ss})}{1+\exp(-\gamma_{1,1}\lambda_{ss})} & 1 - \frac{\exp(-\gamma_{1,1}\lambda_{ss})}{1+\exp(-\gamma_{1,1}\lambda_{ss})} \end{bmatrix}.
\]

Note here that, since this matrix also depends on the steady state level of debt and multiplier, which in turn depend upon the ergodic means of the regime-switching variables, such state is the solution of a fixed point problem, which we describe in the Appendix.

The model has regime specific parameters that can affect the steady state of the economy in that regime. Namely, the switching parameters \(\varphi(s_t), \beta(s_t), a(s_t), \text{ and } p(s_t)\) affect the level of the economy and matter for steady state calculations. Let \(\xi = [\xi_0, \xi_1]\) denote the ergodic vector of \(P_{ss}\). Then define the ergodic means of the switching parameters as

\[
\bar{\varphi} = \xi_0\varphi(0) + \xi_1\varphi(1) \\
\bar{\beta} = \xi_0\beta(0) + \xi_1\beta(1) \\
\bar{a} = \xi_0a(0) + \xi_1a(1) \\
\bar{p} = \xi_0p(0) + \xi_1p(1). 
\]

The steady state of the economy depends on these ergodic means and satisfies the following equations in appendix.

In order to avoid circularity in finding the steady state, which in turn depends on the steady state of the transition probabilities, we first calibrate the steady state probabilities
and then back out the associated parameters of the transition function. That is we assume

\[
\begin{align*}
\gamma_{0,0} &= \log \left( \frac{1}{p_{00,ss}} - 1 \right) + \gamma_{0,1} B_{ss}^* \\
\gamma_{1,0} &= \log \left( \frac{1}{p_{11,ss}} - 1 \right) - \gamma_{1,1} \lambda_{ss}
\end{align*}
\]

and we calibrate \( p_{00,ss} \) and \( p_{11,ss} \). We then estimate the \( \gamma \).

The following table shows the steady state values for the variables in steady state. Note that these are the deterministic steady states associated with each model.

### 3.2 Second order approximation

Armed with the steady state of the endogenous regime-switching economy, we then construct a second-order approximations to the decision rules by taking derivatives of the equilibrium conditions. We relegate details of these derivations to the Appendix, but here we summarize.

For perturbation, we take the stacked equilibrium conditions \( \mathbb{F}(x_{t-1}, \epsilon_t, \chi_t) \), and differentiate with respect to \( (x_{t-1}, \epsilon_t, \chi_t) \). In general, regime-switching models, the first-order derivative with respect to \( x_{t-1} \) produces a complicated polynomial system denoted

\[
\mathbb{F}_x(x_{ss}, 0, 0) = 0.
\]

Often this system needs to be solved via Gröbner bases, which finds all possible solutions in order to check them for stability. In our case, all the regime switching parameters show up in the steady state, and we write \( \theta_t = \bar{\theta} + \chi_t \bar{\theta}(s_t) \) so the steady state can be solved. This is the Partition Principle of Foerster et al. (2016). Given these parameters, the regime switching in \( \mathbb{F}_x(x_{ss}, 0, 0) \) disappears and simplifies to the standard no-switching case that can be solved via a generalized eigenvalue procedure.

After solving the eigenvalue problem, the other systems to solve are

\[
\begin{align*}
\mathbb{F}_\epsilon(x_{ss}, 0, 0) &= 0 \\
\mathbb{F}_\chi(x_{ss}, 0, 0) &= 0
\end{align*}
\]

and second order systems of the form (can apply equality of cross-partials)

\[
\mathbb{F}_{ij}(x_{ss}, 0, 0) = 0, \ i,j \in \{x, \epsilon, \chi\}.
\]
Recall the decision rules have the form

\[ x_t = h_s_t(x_{t-1}, \varepsilon_t, \chi) \]

\[ y_t = g_s_t(x_{t-1}, \varepsilon_t, \chi) \]

and so the second-order approximation takes the form

\[ x_t \approx x_t + H_s^{(1)} S_t + \frac{1}{2} H_s^{(2)} (S_t \otimes S_t) \]

\[ y_t \approx y_t + G_s^{(1)} S_t + \frac{1}{2} G_s^{(2)} (S_t \otimes S_t) \]

where \( S_t = \left[ (x_{t-1} - x_{ss})' \varepsilon_t' 1 \right]' \).

4 Estimating the Endogenous Switching Model

Our estimation procedure is Bayesian and relies on the second-order solution of the model. The derivation of the state space representation used to construct the likelihood is reported in Appendix. We summarize the procedure and discuss calibrated parameters and priors for the estimated ones are discussed below.

Description To be added

4.1 Procedure

4.2 Calibrated Parameters

The calibration of the parameters that we do not estimate follows Mendoza (2010). Consider first the steady state of the model in the case in which the non-binding regime occurs. We normalize \( a(0) = p(0) = 1 \). Mendoza targets an annualized real rate of interest of 8.57\%. In the regime where the constraint does not bind, the steady state interest rate is \( r_{ss} = r^* = \frac{1}{\beta} - 1 \), and the debt level is \( B_{ss} = \bar{B} \). Setting \( \beta = 0.97959 \) yields \( r^* = 0.0208352 \), which matches the target annualized rate. Mendoza also targets a debt-to-output ratio of \(-0.86\), which requires \( \bar{B} = -1.7517 \).

\[ \bar{B} = \left( \frac{\bar{B}}{Y} \right)_{ss} \Omega_w \Omega_k (\Omega_k \Omega_{1-a-\eta})^{\frac{\varepsilon}{(1-\omega)}} \]
Table 1: Calibrated Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Discount Factor</td>
<td>$\beta = 0.97959$</td>
</tr>
<tr>
<td>Risk Aversion</td>
<td>$\rho = 2$</td>
</tr>
<tr>
<td>Labor Share</td>
<td>$\alpha = 0.592$</td>
</tr>
<tr>
<td>Capital Share</td>
<td>$\eta = 0.306$</td>
</tr>
<tr>
<td>Wage Elasticity of Labor Supply</td>
<td>$\omega = 1.846$</td>
</tr>
<tr>
<td>Capital Depreciation (8.8% Annually)</td>
<td>$\delta = 0.022766$</td>
</tr>
<tr>
<td>Interest Rate Intercept</td>
<td>$r^* = 0.0208352$</td>
</tr>
<tr>
<td>Interest Rate Elasticity</td>
<td>$\psi_r = 0.05$</td>
</tr>
<tr>
<td>Neutral Debt Level</td>
<td>$\bar{B} = -1.7517$</td>
</tr>
<tr>
<td>Mean of TFP Process, Normal Regime</td>
<td>$a(0) = 0$</td>
</tr>
<tr>
<td>Mean of Import Price Process, Normal Regime</td>
<td>$p(0) = 0$</td>
</tr>
<tr>
<td>Mean of Leverage Process, Normal Regime</td>
<td>$\kappa(0) = 0.15$</td>
</tr>
<tr>
<td>Persistence of TFP Process, Crisis Regime</td>
<td>$\rho_A(1) = 0$</td>
</tr>
<tr>
<td>Persistence of Import Price Process, Crisis Regime</td>
<td>$\rho_P(1) = 0$</td>
</tr>
</tbody>
</table>

Now consider the steady state of the model in the case when only the binding regime occurs. In line with Mendoza’s estimates on the Mexican sudden stop, we set $a(1) = -0.005$ and $p(1) = 0.005$, which, combined with $\rho_a$ and $\rho_p$, lead to a roughly 5% decrease in TFP and a 5% increase in import prices. We set the interest rate elasticity $\psi_r = 0.001$, which implies that the real rate is increasing in debt. The following Table summarizes the parameterization:

Empirical Results to Be Added

5 Conclusions
References


A Details of the Solution Method

This Appendix gives detail about two aspects of the solution method. First, the definition and solution for the steady state of the endogenous regime-switching economy. Second, the perturbation method that generates Taylor expansions to the solution of the economy around the steady state.

A.1 Steady State

The model has two features that make defining the steady state non-standard. First, as is common in regime-switching models, the switching parameters \( \varphi(s_t), a^*(s_t), p^*(s_t), \) and \( \kappa^*(s_t) \) all affect the level of the economy directly, and will thus matter for steady state calculations. Solution methods such as Foerster et al. (2016) define the steady state by using the ergodic means of these parameters across regimes. However, in our case the transition matrix \( P \) is endogenous, making the ergodic distribution problematic, since it depend on economic variables that in turn depend on the ergodic means. Our solution method for the steady state proceeds in two steps.

A.1.1 Step 1: Find Variables given \( P_{ss} \)

To find the steady state, we first assume that \( P_{ss} \) is known. Let \( \xi = [\xi_0; \xi_1] \) denote the ergodic vector of \( P_{ss} \). Then define the ergodic means of the switching parameters as

\[
\bar{\varphi} = \xi_0 \varphi(0) + \xi_1 \varphi(1) \tag{A.1}
\]

\[
\bar{a}^* = \xi_0 a^*(0) + \xi_1 a^*(1) \tag{A.2}
\]

\[
\bar{p}^* = \xi_0 p^*(0) + \xi_1 p^*(1) \tag{A.3}
\]

\[
\bar{\zeta}_0 = \xi_0 \zeta_0(0) + \xi_1 \zeta_0(1) \tag{A.4}
\]

\[
\bar{\zeta}_1 = \xi_0 \zeta_1(0) + \xi_1 \zeta_1(1) \tag{A.5}
\]

The steady state of the economy depends on these ergodic means, and satisfies the following equations

\[
\left(C_{ss} - \frac{H_{ss}^\omega}{\omega}\right)^{-\rho} = \mu_{ss} \tag{A.6}
\]

\[
(1 - \alpha - \eta) A_{ss} K_{ss}^\eta H_{ss}^a V_{ss}^{-\alpha-\eta} = P_{ss} \left(1 + \phi r_{ss} + \frac{\lambda_{ss}}{\mu_{ss}} \phi (1 + r_{ss})\right) \tag{A.7}
\]
\[
\alpha A_{ss}K_{ss}^\eta H_{ss}^{\alpha-1}V_{ss}^{1-\alpha-\eta} = \phi W_{ss} \left( r_{ss} + \frac{\lambda_{ss}}{\mu_{ss}} (1 + r_{ss}) \right) + H_{ss}^{\omega-1} \tag{A.8}
\]

\[
\mu_{ss} = \lambda_{ss} + \beta \left(1 + \frac{r_{ss}}{1 + \tau B_{ss}}\right) \mu_{ss} \tag{A.9}
\]

\[
\mu_{ss} \beta \left(1 - \delta + \left( \frac{1}{2} \left( \frac{K_{ss}}{K_{ss}} \right)^2 - \frac{1}{2} \right) + \eta A_{ss} K_{ss}^{\eta-1} H_{ss}^{\alpha} V_{ss}^{1-\eta-\alpha} \right) = \mu_{ss} \left(1 - t + t \left( \frac{K_{ss}}{K_{ss}} \right) \right) - \lambda_{ss} \kappa q_{ss} \tag{A.10}
\]

\[
q_{ss} = 1 + t \left( \frac{K_{ss} - K_{ss}}{K_{ss}} \right) \tag{A.11}
\]

\[
W_{ss} = H_{ss}^{\omega-1} \tag{A.12}
\]

\[
C_{ss} + I_{ss} = A_{ss}K_{ss}^\eta H_{ss}^{\alpha} V_{ss}^{1-\alpha-\eta} - P_{ss}V_{ss} - \phi r_{ss} (W_{ss}H_{ss} + P_{ss}V_{ss}) - \frac{1 + \tau B_{ss}}{1 + r_{ss}} B_{ss} + B_{ss} - T_{ss} \tag{A.13}
\]

\[
I_{ss} = \delta K_{ss} + (K_{ss} - K_{ss}) \left(1 + t \left( \frac{K_{ss}}{K_{ss}} \right) \right) \tag{A.14}
\]

\[
B_{ss}^* = \frac{1 + \tau B_{ss}}{1 + r_{ss}} B_{ss} - \phi (1 + r_{ss}) (W_{ss}H_{ss} + P_{ss}V_{ss}) + \kappa q_{ss} K_{ss} \tag{A.15}
\]

\[
\phi B_{ss}^* + (1 - \phi) \lambda_{ss} = 0 \tag{A.16}
\]

\[
T_{ss} = \bar{T}_{ss}B_{ss} \tag{A.17}
\]

\[
\tau^B_{ss} = \check{\tau}_0 + \check{\tau}_1 \left( \frac{B_{ss}}{Y_{ss}} \right) \tag{A.18}
\]

\[
r_{ss} = r^* + \psi_r \left( e^{\bar{B}_{ss}} - 1 \right) \tag{A.19}
\]

\[
\log A_{ss} = (1 - \rho_A (s_t)) \bar{a}^* + \rho_A (s_t) \log A_{ss} \tag{A.20}
\]

\[
\log P_{ss} = (1 - \rho_P (s_t)) \bar{p}^* + \rho_P (s_t) \log P_{ss} \tag{A.21}
\]

\[
k_{ss} = K_{ss} \tag{A.22}
\]

\[
Y_{ss} = A_{ss}K_{ss}^\eta H_{ss}^{\alpha} V_{ss}^{1-\alpha-\eta} \tag{A.23}
\]

\[
\Phi^b_{ss} = \frac{B_{ss}}{Y_{ss}} \tag{A.24}
\]

We can partially solve for some of these directly

\[
A_{ss} = \exp \bar{a}^* \tag{A.25}
\]

\[
P_{ss} = \exp \bar{p}^* \tag{A.26}
\]
Suppose know $r_{ss}$ and $\tau_{ss}$

$$q_{ss} = 1 \quad (A.27)$$

$$\Omega_v = \frac{P_{ss} \left( 1 + \phi r_{ss} + \phi (1 + r_{ss}) \left( 1 - \beta \frac{(1 + r_{ss})}{(1 + \tau_{ss})} \right) \right)}{(1 - \alpha - \eta)} \quad (A.28)$$

$$\Omega_h = \frac{1 + \phi \left( r_{ss} + (1 + r_{ss}) \left( 1 - \beta \frac{(1 + r_{ss})}{(1 + \tau_{ss})} \right) \right)}{\alpha} \quad (A.29)$$

$$\Omega_k = \frac{1}{\eta} \left( \frac{1 - \beta \frac{(1 + r_{ss})}{(1 + \tau_{ss})}}{\beta - 1 + \delta} \right) \quad (A.30)$$

$$H_{ss} = \left( \frac{A_{ss}}{\Omega_k^\alpha \Omega_h^\beta \Omega_v^{1-\alpha-\eta}} \right)^{1/\omega(\omega-\gamma)} \quad (A.31)$$

$$Y_{ss} = \Omega_h H_{ss}^{\omega} \quad (A.32)$$

$$V_{ss} = \frac{\Omega_h}{\Omega_k} H_{ss}^{\omega} \quad (A.33)$$

$$K_{ss} = \frac{\Omega_h}{\Omega_k} H_{ss}^{\omega} \quad (A.34)$$

$$W_{ss} = H_{ss}^{\omega-1} \quad (A.35)$$

$$I_{ss} = \delta K_{ss} \quad (A.36)$$

$$k_{ss} = K_{ss} \quad (A.37)$$

$$B_{ss} = \bar{B} - \log \left( 1 + \frac{r_{ss} - r^*}{\psi_r} \right) \quad (A.38)$$

$$C_{ss} = Y_{ss} - I_{ss} - P_{ss} V_{ss} - \phi r_{ss} \left( W_{ss} H_{ss} + P_{ss} V_{ss} \right) - \frac{(1 + \tau_{ss}^B)}{(1 + r_{ss})} B_{ss} + B_{ss} - T_{ss} \quad (A.39)$$

$$C_{ss} = Y_{ss} - (1 + \phi r_{ss}) P_{ss} V_{ss} - \delta K_{ss} - \phi r_{ss} W_{ss} H_{ss} + \left( 1 - \frac{(1 + \tau_{ss}^B)}{(1 + r_{ss})} - \tau_{ss}^B \right) B_{ss} \quad (A.40)$$

$$\mu_{ss} = \left( C_{ss} - \frac{H_{ss}^{\omega}}{\omega} \right)^{-\rho} \quad (A.41)$$

$$\lambda_{ss} = \mu_{ss} \left( 1 - \beta \frac{(1 + r_{ss})}{(1 + \tau_{ss}^B)} \right) \quad (A.42)$$

$$T_{ss} = \tau_{ss}^B B_{ss} \quad (A.43)$$
\[ B_{ss}^* = \frac{(1 + \tau_{ss}^B)}{(1 + \tau_{ss})}B_{ss} - \phi \left(1 + r_{ss}\right) \left(W_{ss}H_{ss} + P_{ss}V_{ss}\right) + \kappa K_{ss} \quad (A.44) \]

\[ \Phi_{by}^{ss} = \frac{B_{ss}}{Y_{ss}} \quad (A.45) \]

and then \( r_{ss} \) and \( \tau_B \) solve

\[ \tau_{ss}^B = \bar{\zeta}_0 + \bar{\zeta}_1 \left(\frac{B_{ss}}{Y_{ss}}\right) \quad (A.46) \]

\[ \bar{\varphi}B_{ss}^* + (1 - \bar{\varphi}) \lambda_{ss} = 0 \quad (A.47) \]

### A.1.2 Steady State Solution, Step 2: Check \( P_{ss} \)

Given the variables \( B_{ss}^* \) and \( \lambda_{ss} \), have a new value

\[ P_{ss} = \begin{bmatrix} p_{00,ss} & p_{01,ss} \\ p_{10,ss} & p_{11,ss} \end{bmatrix} = \begin{bmatrix} 1 - \frac{\exp(\gamma_{00} - \gamma_{01} B_{ss}^*)}{1 + \exp(\gamma_{00} - \gamma_{01} B_{ss}^*)} & \frac{\exp(\gamma_{00} - \gamma_{01} B_{ss}^*)}{1 + \exp(\gamma_{00} - \gamma_{01} B_{ss}^*)} \\ \frac{\exp(\gamma_{10} - \gamma_{11} \lambda_{ss})}{1 + \exp(\gamma_{10} - \gamma_{11} \lambda_{ss})} & 1 - \frac{\exp(\gamma_{10} - \gamma_{11} \lambda_{ss})}{1 + \exp(\gamma_{10} - \gamma_{11} \lambda_{ss})} \end{bmatrix}, \quad (A.48) \]

which can be checked against the guess in step 1. The steady state solves \( \| P_{ss}^{(i+1)} - P_{ss}^{(i)} \| < \text{tolerance} \) for successive iterations \( i \).

### A.2 Perturbation

#### A.2.1 Equilibrium Conditions

The 19 equilibrium conditions are written as

\[ \mathbb{E}_t f (y_{t+1}, y_t, x_{t-1}, x_t, \varepsilon_{t+1}, \varepsilon_t, \Theta_{t+1}, \Theta_t) = 0 \quad (A.49) \]

where the variables are separated into the predetermined variables \( x_{t-1} \) and the non-predetermined variables \( y_t \). The variables are 4 predetermined variables

\[ x_{t-1} = [K_{t-1}, B_{t-1}, A_{t-1}, P_{t-1}] \quad (A.50) \]

and 15 non-predetermined variables

\[ y_t = [C_t, H_t, V_t, I_t, k_t, q_t, W_t, \mu_t, \lambda_t, B_t^*, \tau_t^B, T_t, Y_t, \Phi_{by}^t] \quad (A.51) \]

with 4 shocks

\[ \varepsilon_t = [\varepsilon_{r,t}, \varepsilon_{m,t}, \varepsilon_{A,t}, \varepsilon_{P,t}] \quad (A.52) \]
and 6 switching variables

\[ \theta_t = [\varphi(s_t), a^*(s_t), p^*(s_t), \zeta_0(s_t), \zeta_1(s_t), \gamma(s_t), \rho_A(s_t), \rho_P(s_t)] . \] (A.53)

These variables are partitioned into some that affect the steady state, \( \theta_{1,t} \), and some that do not, \( \theta_{2,t} \). The partition in this case is

\[ \theta_{1,t} = [\varphi(s_t), a^*(s_t), p^*(s_t), \zeta_0(s_t), \zeta_1(s_t)] \] (A.54)

\[ \theta_{2,t} = [\gamma(s_t), \rho_A(s_t), \rho_P(s_t)] \] (A.55)

For solving the model, the functional forms are

\[ \theta_{1,t+1} = \bar{\theta}_1 + \chi \hat{\theta}_1(s_{t+1}) \] (A.56)

\[ \theta_{1,t} = \bar{\theta}_1 + \chi \hat{\theta}_1(s_t) \] (A.57)

\[ \theta_{2,t+1} = \theta_2(s_{t+1}) \] (A.58)

\[ \theta_{2,t} = \theta_2(s_t) \] (A.59)

\[ x_t = h_{st}(x_{t-1}, \varepsilon_t, \chi) \] (A.60)

\[ y_t = g_{st}(x_{t-1}, \varepsilon_t, \chi) \] (A.61)

\[ y_{t+1} = g_{st+1}(x_t, \chi \varepsilon_{t+1}, \chi) \] (A.62)

\[ p_{st,t+1} = \pi_{st,t+1}(y_t) \] (A.63)

Using these in the equilibrium conditions and being more explicit about the expectation operator, given \((x_{t-1}, \varepsilon_t, \chi)\) and \(s_t\), the

\[ F_{st}(x_{t-1}, \varepsilon_t, \chi) = \int \sum_{s'=0}^{1} \pi_{st,s'}(g_{st}(x_{t-1}, \varepsilon_t, \chi)) f \left( \begin{array}{c} g_{st+1}(h_{st+1}(x_{t-1}, \varepsilon_t, \chi), \chi \varepsilon', \chi), \\ g_{st}(x_{t-1}, \varepsilon_t, \chi), \\ h_{st}(x_{t-1}, \varepsilon_t, \chi), \\ x_{t-1}, \chi \varepsilon', \varepsilon_t, \\ \bar{\theta} + \chi \hat{\theta}(s'), \bar{\theta} + \chi \hat{\theta}(s_t) \end{array} \right) d \mu \varepsilon' = 0 \] (A.64)

Stacking these conditions for each regime produces

\[ \mathbb{F}(x_{t-1}, \varepsilon_t, \chi) = \begin{bmatrix} F_{st=1}(x_{t-1}, \varepsilon_t, \chi) \\ F_{st=2}(x_{t-1}, \varepsilon_t, \chi) \end{bmatrix} \] (A.65)
A.2.2 Generating Approximations

For perturbation, we take the stacked equilibrium conditions $F(x_{t-1}, \epsilon_t, \chi_t)$, and differentiate with respect to $(x_{t-1}, \epsilon_t, \chi_t)$. In general regime-switching models, the first-order derivative with respect to $x_{t-1}$ produces a complicated polynomial system denoted

$$F_x(x_{ss}, 0, 0) = 0. \quad \text{(A.66)}$$

Often this system needs to be solved via Gröbner bases, which finds all possible solutions in order to check them for stability. In our case with endogenous probabilities, the standard stability checks fail, so we will focus on finding a single solution and ignore the possibility of indeterminacy from multiple solutions. In this case, we guess at a set of policy functions for regime $s_t = 1$, which collapses the equilibrium conditions $F_x(x_{ss}, 0, 0; s_t = 0)$ into a fixed-regime eigenvalue problem, and solve for the policy functions for $s_t = 0$. Then, using this solution as guesses, we solve for regime $s_t = 0$ under the fixed-regime eigenvalue problem, and iterate on this procedure to convergence.

After solving the iterative eigenvalue problems, the other systems to solve are

$$F_\epsilon (x_{ss}, 0, 0) = 0 \quad \text{(A.67)}$$

$$F_\chi (x_{ss}, 0, 0) = 0 \quad \text{(A.68)}$$

and second order systems of the form (can apply equality of cross-partials)

$$F_{ij} (x_{ss}, 0, 0) = 0, \ i, j \in \{x, \epsilon, \chi\}. \quad \text{(A.69)}$$

Recall the decision rules have the form

$$x_t = h_{s_t} (x_{t-1}, \epsilon_t, \chi_t) \quad \text{(A.70)}$$

$$y_t = g_{s_t} (x_{t-1}, \epsilon_t, \chi_t) \quad \text{(A.71)}$$

and so the second-order approximation takes the form

$$x_t \approx x_{ss} + H^{(1)}_{s_t} S_t + \frac{1}{2} H^{(2)}_{s_t} (S_t \otimes S_t) \quad \text{(A.72)}$$

$$y_t \approx y_{ss} + G^{(1)}_{s_t} S_t + \frac{1}{2} G^{(2)}_{s_t} (S_t \otimes S_t) \quad \text{(A.73)}$$

where $S_t = \begin{bmatrix} (x_{t-1} - x_{ss})' & \epsilon_t' & 1 \end{bmatrix}'$. 

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\section*{B State Space Representation}

For likelihood estimation, the state space representation is

\[ X_t = \mathcal{H}_{st}(X_{t-1}, \epsilon_t) \]

\[ y_t = \mathcal{G}_{st}(X_t, U_t), \]

where \( y_t \) is the vector of observables variables:

\[ y_t = \left[ \begin{array}{c} \Delta y_t \\ \Delta c_t \\ \Delta i_t \\ r_t \end{array} \right]'. \]

Given \( s_t \) and \( \epsilon_t \), we can construct a first order approximation to \( \Delta y_t \) by

\[ \Delta y_t = y_t - y_{t-1} = G^{(1)}_{st} \left[ \hat{x}'_{t-1} \epsilon_t 1 \right]' - y_{t-1} \]

and the first order approximation to \( x_t \) is

\[ x_t = x_{ss} + H^{(1)}_{st} \left[ \hat{x}'_{t-1} \epsilon_t 1 \right]' \]

Therefore, the state equation is

\[ X_t = \begin{bmatrix} x_t \\ y_t \\ \Delta y_t \end{bmatrix} = \begin{bmatrix} x_{ss} + H^{(1)}_{st} \left[ \hat{x}'_{t-1} \epsilon_t 1 \right]' \\ y_{ss} + G^{(1)}_{st} \left[ \hat{x}'_{t-1} \epsilon_t 1 \right]' \\ G^{(1)}_{st} \left[ \hat{x}'_{t-1} \epsilon_t 1 \right]' - y_{t-1} \end{bmatrix} \]

and the observation equation is

\[ y_t = \begin{bmatrix} \Delta y_t \\ \Delta c_t \\ \Delta i_t \\ r_t \end{bmatrix} = D \begin{bmatrix} \hat{x}_t \\ y_t \\ \Delta y_t \end{bmatrix} + U_t \]
where $D$ denotes a selection matrix. Therefore, in matrix form, we have:

$$
\begin{bmatrix}
    x_t \\
    y_t \\
    \Delta y_t
\end{bmatrix} =
\begin{bmatrix}
    x_{ss} + H^{(1)}_{\chi,st} \\
    y_{ss} + G^{(1)}_{\chi,st} \\
    G^{(1)}_{\chi,\delta_t}
\end{bmatrix} +
\begin{bmatrix}
    H^{(1)}_{x,\delta_t} & 0 & 0 \\
    G^{(1)}_{x,\delta_t} & 0 & 0 \\
    G^{(1)}_{x,\delta_t} & -I & 0
\end{bmatrix}
\begin{bmatrix}
    \hat{x}_{t-1} \\
    y_{t-1} \\
    \Delta y_{t-1}
\end{bmatrix} +
\begin{bmatrix}
    H^{(1)}_{\epsilon,\delta_t} \\
    G^{(1)}_{\epsilon,\delta_t} \\
    G^{(1)}_{\epsilon,\delta_t}
\end{bmatrix} \epsilon_t
$$

and

$$
\begin{bmatrix}
    \Delta y_t \\
    \Delta c_t \\
    \Delta i_t \\
    r_t
\end{bmatrix} = S \Delta y_t + \mathcal{U}_t
$$

which can be denoted as

$$
X_t = A_{st} x_{t-1} + C_{st} \epsilon_t
$$

$$
Y_t = DX_t + E \mathcal{U}_t
$$