Wealth Dynamics and Asset Prices with Recursive Preferences and Heterogeneous Beliefs

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Abstract
The market selection hypothesis by Alchian (1950) and Friedman (1953) implies that traders with incorrect beliefs will drop out of the market in the long-run and thus do not have any impact on prices in financial markets. In this paper I study the market selection hypothesis in an economy with heterogeneous beliefs and recursive preferences. To model heterogeneous beliefs we follow Kurz (1994) and restrict the beliefs to the subset of rational beliefs. Furthermore, households ability to borrow is limited by a collateral constraint and hence markets are incomplete. Numerical results indicate that agents with non-rational expectations do survive and thus have impact on asset prices.

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1 Introduction

The empirical literature on financial markets reports several paradoxes which cannot be explained by standard models such as the excess volatility puzzle (Shiller (1981)), the equity premium puzzle (Mehra and Prescott (1985)). The literature has identified several ways to explain these paradoxes, these possibilities include incomplete markets, non-standard preferences or heterogeneous beliefs. However, even though heterogeneous expectations have some success in explaining these paradoxes, the market selection hypothesis as formulated by Alchian (1950) and Friedman (1953) cast some doubts whether heterogeneous expectations are a suitable explanation for some of these paradoxes. They argue that in competitive markets agents with incorrect expectations make investment mistakes and lose their wealth. As their wealth depletes they drop out of the market and only investors with correct beliefs survive. Hence, investors with wrong expectations should not have any impact on prices in the long-run.

A first formal study on the market selection hypothesis was carried out by De Long et al. (1990) who showed in a partial equilibrium setting that irrational noise traders may not only survive in the long-run but can also dominate the market. Blume and Easley (2006) argue that with complete markets and time-and state separable preferences households with incorrect beliefs may not survive in the long run and thus seem to confirm the market selection hypothesis. A different picture emerges when we move from an economy with complete markets to economies without a full set of Arrow-Debreu securities. In numerical examples Cao (2013) and Cogley, Sargent, and Tsyrennikov (2014) show that less informed agents may not only survive in the long-run but may also dominate the market. These sharp differences in predictions on the survival of the less informed agent arises from the fact that due to the lack of Arrow-Debreu securities better informed agents are not able to exploit the less

1Other papers studying the survival of agents with complete markets and separable preferences include Blume and Easley (1992), Sandroni (2000), Kogan et al. (2009), Fedyk, Heyerdahl-Larsen, and Walden (2013), Cvitanic and Malamud (2010) and Cvitanić and Malamud (2011)

2Other papers studying economies with incomplete markets and CRRA-preferences, such as Coury and Sciubba (2012), come to similar conclusion
informed agents. These differences in results imply that market structure, i.e. the set of tradable assets and attainable portfolios, is an important factor in determining survival of agents.

Easley and Yang (2015) and Borovicka (2015) extend the analysis to economies with recursive preferences and complete markets. They show that households with incorrect beliefs still survive in the long run. Yet, even though agents survive in economies with a complete set of Arrow-Debreu securities one would expect that moving from complete to incomplete markets affects dynamics of financial wealth and hence equilibrium properties of the economy. Thus, several questions are unanswered in the literature: What is the impact of heterogeneous beliefs on survival in an economy with incomplete markets when agents have recursive preferences and heterogeneous beliefs? What is its impact on asset prices?

To answer these questions, we study a simple asset pricing economy with heterogeneous agents. Agents live infinitely who disagree about the probability of aggregate endowment shocks. Furthermore, they are endowed with recursive preferences that allow us to disentangle the effects of risk aversion and elasticity of intertemporal substitution on wealth dynamics and prices in the economy.

To model heterogeneous beliefs, we follow Kurz (1994) and restrict the set of possible beliefs to the subset of rational beliefs. Compared to Rational Expectations, Rational Beliefs has weaker requirements on the knowledge of the agents. While under rational expectations agents know the true underlying data-generating process, under rational beliefs agents do not know the true underlying data-generating process but use the empirical distribution to form their beliefs. If the economy is not stationary (or at least agents belief that it is not stationary), then there exists many valid theories.

Simulation results indicate that not only the elasticity of intertemporal affects the distribution of wealth in the economy but also the market structure.

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3There is also a more technical difference between our paper and Borovicka (2015). Under the rational beliefs principle non-convergence of beliefs stems from the fact that households assume non-stationarity while Borovicka (2015) assumes that beliefs are not equivalent, i.e. agents disagree on the null-sets of the underlying probability space, to ensure non-convergence of beliefs.
In particular, if the market for bonds is shut down agents cannot borrow and thus the elasticity of intertemporal has no effects on the distribution of financial wealth. If agents are instead allowed to borrow the financial wealth is affected by the precautionary savings motive, i.e. agents who don’t believe that the empirical distribution is the true distribution and a low elasticity of intertemporal substitution lose more wealth in the long-run than agents with a high elasticity of intertemporal substitution.

The rest of the paper is structured as follows: In section 2 we outline the model and section 3 discusses the existence of a rational belief equilibrium. In section 4 we discuss numerical results regarding survival and asset prices Section 5 concludes the paper.

2 The Model

Consider an endowment, a single consumption good economy in infinite horizon. Time runs from $t = 0$ to $\infty$. There are $H$ types of consumers:

$$h \in \mathcal{H} = \{1, 2, ..., H\}$$

in the economy. These consumers might differ in many dimensions including their preferences and their endowment of final good $e^h$. The consumers might also differ in their initial endowment of a real asset that pays off real dividends. However, in this paper we focus on the heterogeneity of beliefs over the evolution of the exogeneous states of the economy. There $S$ possible exogeneous states:

$$s \in \mathcal{S} = \{1, 2, ..., S\}.$$  

The state captures both aggregate uncertainty (e.g. dividends) and idiosyncratic shocks. The evolution of the economy is captured by the realizations of the shocks over time: $s^t = (s_0, s_1, ..., s_t)$. We assume that the shocks follow a markov-process with the transition probabilities $\pi(s, s')$.

**Real asset:** There is one real asset in the economy that pays off state-
dependent real dividends \( d(s_t) \). Agents can purchase \( \theta^h_t = \theta^h(s^t) \) units of
the assets, which can also be used as collateral for borrowing. The ex-dividend
price of the asset in history \( s^t \) is denoted by \( q_t = q(s^t) \). Consumers are also not
allowed to short sell the asset. Furthermore, the total supply of the real asset
is \( 1 \).

**Bond:** In addition to purchasing real assets, consumers can also borrow
subject to collateral constraints. The agents borrow by selling \( b^h_t = b^h(s^t) \) units
of a one-period bond which pays one unit of the consumption good in the next
period at price \( p_t = p(s^t) \) and use their holdings of the real asset as collateral.
In particular, we consider a collateral constraint of the following form:

\[
b^h_t + (1 - m)\theta^h_t \min_{s^t+1|s^t} (q_{t+1} + d_{t+1}).
\]

Here, \( m \) can be interpreted as the margin requirement.

**Consumers:** In each state \( s^t \), each consumer is endowed with some endow-
ment \( e^h_t = e^h(s_t) \) units of the consumption good. The aggregate endowment in
the economy is \( \bar{e}_t = \sum_{h \in H} e^h_t + d_t \) and the growth rate is denoted by \( g_t = \frac{\bar{e}_t}{\bar{e}_{t-1}} \).
Furthermore, we assume that they have recursive preferences as in Epstein and
Zin (1989) and Weil (1990) which are an intertemporal generalization of Kreps
and Porteus (1978). With recursive preferences the temporal resolution of un-
certainty matters and preferences are not separable over time.

Consumers take the sequence of prices \( \{p_t, q_t\} \) as given and maximize the
following recursive utility function:

\[
U^h_t = \left(1 - \beta \right) \left( e^h_t \right)^{\rho^h} + \beta E Q^h_t \left( U^{h}_{t+1} \left( 1 - \gamma^h \right) \left| \mathcal{F}_t \right. \right)^{\frac{\rho^h}{1 - \gamma^h}},
\]

with \( \beta \) as the subjective discount factor, \( \gamma^h \) as the coefficient of relative risk
aversion, and the Elasticity of Intertemporal Substitution \( \psi^h \geq 0 \). The parameter \( \rho^h \) is defined as \( \rho^h := (1 - \gamma^h) / (1 - \frac{1}{\psi^h}) \). And \( Q^h_t \) represents the subjective
(effective) beliefs of agent \( h \) subject to the information set \( \mathcal{F}_t \). The maximization
problem is subject to the intertemporal budget constraint:

\[ c^h_t + q_t \theta^h_t + p_t b^h_t \leq c^h_{t+1} + (q_{t+1} + d_{t+1}) \theta^h_{t+1}, \]  

(3)

the short-sale constraint on the real asset

\[ \theta^h_t \geq 0, \]  

(4)

and the margin constraint

\[ b^h_t + (1 - m) \theta^h_t \min_{s \in S_{t+1} \mid s_1} (q_{t+1} + d_{t+1}) \geq 0. \]  

(5)

Beliefs: In the construction of beliefs we follow Kurz and Schneider (1996) and introduce a random variable \( n^h_t \) called a generating variable. Now, agent \( h \) forms a belief on \( ((S \times \mathcal{N})^\infty, \mathcal{B}((S \times \mathcal{N})^\infty)) \), with the state-space of \( n^h_t \) denoted as \( \mathcal{N}^h = \{0, 1\} \). And \( (S \times \mathcal{N})^\infty \) generates the Borel \( \sigma \)-field \( \mathcal{B}((S \times \mathcal{N})^\infty) \).

**Assumption 1.** The marginal distribution for \( n^h_t \) with respect to \( Q^h_t \) is i.i.d. with \( Q^h_t(n^h_t = 1) = \mu^h \).

Because of the construction of beliefs in our model the state space is expanded and includes the effective beliefs \( Q^h_t \) of the consumers. Denote the exogeneous state-space without heterogeneous beliefs as \( \hat{S} \) with a typical element \( \hat{s} \in \hat{S} \). We define the expanded state \( S \) space as

\[ s = (\hat{s}_t, (n^h_t)_{h \in \mathcal{H}}) \quad \forall t. \]  

(6)

### 3 Equilibrium

In this section we prove the existence of rational belief equilibrium. The proof consists of two steps. In the first step we show that the equilibrium is a stable and ergodic dynamical system by proving the existence of a markov equilibrium. The existence of a Markov equilibrium implies the existence of a dynamical system with an ergodic measure (see Duffie et al. (1994)). To prove the
existence of the Markov equilibrium we adopt the proof of Dou and Verdelhan (2015)\(^4\). In the second step, we show the existence of a stationary measure for the equilibrium. The existence of a stationary measure then implies the existence of a rational belief equilibrium which we define below.

### 3.1 The Definition of Equilibrium

In addition to the first order conditions of the consumer’s problems, the following market clearing conditions hold at the equilibrium, i.e.

\[
\begin{align*}
\sum_{h \in \mathcal{H}} \theta_t^h &= 1, \\
\sum_{h \in \mathcal{H}} b_t^h &= 0, \\
\sum_{h \in \mathcal{H}} c_t^h &= d_t + \sum_{h \in \mathcal{H}} c_t^h.
\end{align*}
\]

To describe the state space and the equilibrium of the economy, we follow Duffie et al. (1994). They have shown that if the exogeneous dynamics can be described by a finite-valued time-homogeneous markov chain, then there exists a competitive equilibrium in which endogeneous variables are a function of a finite number of endogeneous variables as well as the exogeneous states. Furthermore, the endogeneous variables follow a time-homogeneous markov process with an ergodic measure which is called a recursive Markov equilibrium. In our model the endogeneous state variable is the distribution of financial. In the terminology of Duffie et al. (1994) we have a wealth-recursive Markov equilibrium. Duffie et al. (1994) show the a Markov equilibrium is a competitive equilibrium under general regularity conditions while Kubler and Schmedders (2003) show the same for mild regularity conditions. The endogeneous variables in the economy are \((\theta_t^h, b_t^h)_{h \in \mathcal{H}}, p_t\) and \(q_t\) as well as consumption \(c_t^h\), the Lagrange-multipliers \((\mu_t^{hc})\) for the collateral constraint and Lagrange-multipliers \(\mu_t^{hs}\) for the short-sale constraints. Hence,

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\(^4\)Ma (1993) shows the existence of equilibria for a larger class of recursive preferences. The specification used here are a special case, however it violates several assumptions made my Ma (1993) to prove existence.
we define the state space \( \mathcal{Y} \) for possible vectors of the choice variables, i.e. 
\(((\theta^h_t, b^h_t))_{h \in H}, (c^h_t)_{h \in H}, (\mu^h_{tc}, \mu^h_{ts})_{h \in H}, (p_t, q_t))\), at a point in time as follows:
\[
\mathcal{Y} := \left\{ y \in \mathbb{R}^H \times \mathbb{R}^H \times \mathbb{R}^{2H} \times \mathbb{R}^2 \mid \sum_{h \in H} \theta^h_t = 1, \sum_{h \in H} b^h_t = 0, \right.
\]
\[
(\mu^{hs} \theta^h_t = 0)_{h \in H}, (\mu^{hb} (b^h_t + (1 - m) \theta^h_t \min_{s \in S^{t+1}} (q_{t+1} + d_{t+1})) = 0) \right\}.
\]

The financial wealth of the agents in the economy is given by
\[
w^h_t = \frac{\theta^h_{t-1} (q_t + d_t) + b^h_{t-1}}{q_t + d_t}.
\]

The space of endogeneous variables \( \mathcal{Z} \) is a closed subset of \( \mathbb{R}^{5+2H+|S|} \). The space of both exogeneous and endogeneous variables is \( \mathcal{Z} \equiv \mathcal{Y} \times \mathcal{S} \). Let \( \hat{\mathcal{Z}} \equiv [0, 1] \times \mathcal{Y} \times \mathcal{S} \). The expectations correspondence is denoted by
\[
\Phi : \mathcal{Z} \Rightarrow ([0, 1] \times \mathcal{Y})^{\mathcal{S}}
\]

We then define a wealth-recursive markov equilibrium as follows:

**Definition 1.** A wealth-recursive markov equilibrium consists of a nonempty valued ‘policy correspondence’ \( \Pi : [0, 1] \times \mathcal{S} \times \mathbb{R}_+ \Rightarrow \mathcal{Y} \), where \( \mathcal{Y} \) is the space of endogeneous policy-variables defined in (10) and a transition map \( \Omega : [0, 1] \times \mathcal{S} \rightarrow [0, 1]^{\mathcal{S}} \) such that for any given \((w, s) \in [0, 1] \times \mathcal{S}\) with \((w^+(s^+)) = \Omega(w, s)\), it holds that \(\forall y \in \Pi(w, s) \) and \(\forall y^+(s^+) \in \Pi(w^+(s^+))\),
\[
(w^+(s^+), y^+(s^+)) \in \Phi(w, y, s).
\]

### 3.2 The Existence of a Stationary Measure

Before showing the existence of the markov equilibrium we have to prove for any truncated economy the existence of a compact set that covers all endogeneous and exogeneous variables in the equilibrium in every period:

**Lemma 1.** Assume, that the elasticity of intertemporal substitution \( \psi \) is greater than
Furthermore, assume there exist some lower bounds for the endowment \( w_m \) and dividends \( d_m \) such that \( d_t \geq d_m \) and \( e_t^h \geq w_m \) for all \( \forall t = 1, ..., T \) and \( h \in \mathcal{H} \) for all finite \( T \geq 1 \). There exists an equilibrium for the truncated economy in which prices, portfolio holdings and consumption lie lie in a compact set \( \mathcal{Y}^* \subset \mathcal{Y} \).

**Proof.** see Appendix C.

The proof of lemma 1 proceeds in two steps. In the first step we show that there exists an upper bound for prices. In the second step we show that there exists a fixed point. We prove by contradiction that there exists a finite upper bound for prices as follows. If prices for bonds or stocks are above some finite upper bound, then there we can shift some wealth from the consumers portfolio to current consumption. This feasible strategy results in more consumption today but less consumption in all subsequent periods. However, this new strategy is preferred over the old strategy thus violating the fact that households maximize their expected utility.

The second step of the proof is standard and relies on the usual fixed-point arguments.

The result of lemma 1 can now be used to proof the following theorem:

**Theorem 1.** There exists a wealth-recursive Markov equilibrium in the economy with heterogeneous agents and recursive utility.

**Proof.** see Appendix D

3.3 The Rational Belief Equilibrium

Before we define a rational beliefs equilibrium we define what actually constitutes a rational beliefs:

**Definition 2.** A sequence of effective beliefs \( \{Q_t^h\}_{t=0}^\infty \) constitutes a rational belief if it induces a stationary measure that is equivalent to the one induced by the true probability measure \( \Pi \).

This definition states that rational beliefs are compatible with the empirical data which makes it impossible to reject a rational belief by examining the
data. However, rational beliefs still allow for mistakes as the definition does not require the belief to the true probability. It is important to note the rational beliefs principle rules out fixed (or dogmatic) beliefs, unless they believe that the empirical distribution is the true distribution.

With the definition of rational belief, we can define the Rational Belief Equilibrium as follows:

**Definition 3.** A Rational Belief Equilibrium is an equilibrium that is characterised as an equilibrium transition that has an ergodic measure, and in which the sequence of each consumer’s effective beliefs constitutes a rational belief.

Let $X$ denote the state-space of $(p_t, q_t, d_t, (e_t)_{b \in H})$ for all $t$ and $X^\infty$ the state space for the entire sequence. The Borel $\sigma$ field generated by $X^\infty$ will be denoted as $B(X^\infty)$. The true stochastic process of the economy is described by a stochastic dynamic system $(X^\infty, B(X^\infty), T, \Pi)$, where $T$ denotes the shift-transformation and $\Pi$ the true probability measure. To include the beliefs of the agents we expand the probability space and include the sequence of effective beliefs $(n^h_t)_{t=1}^\infty$.

The extended state-space over the whole sequence is given by $(X \times N^h)^\infty$. One should note that the extended state-space is specific to an individual agent and does not include the beliefs of other agents. This is due to the fact that agents do not know the beliefs of other agents. Given an extended state-space $(X \times N^h)^\infty$ we define the Borel $\sigma$ algebra $B((X \times N^h)^\infty)$ with its probability measure $\hat{\Pi}^h$. The probability measure $\hat{\Pi}^h$ has the property that the marginal measure on $X^\infty$ is $\Pi$ and that for $(N^h)^\infty$ is $\bar{\mu}^h$ and the marginal measure for $n^h_t$ is $\mu^h$.

Before we state the Conditional Stability Theorem, we introduce some important notation.

Let $\Pi^h_k$ denote the conditional probability of $\hat{\Pi}^h$ given a particular sequence of effective beliefs $k \in (N^h)^\infty$:

$$\hat{\Pi}^h_k(\cdot) : (N^h)^\infty \times B(X^\infty) \mapsto [0, 1]$$  

(13)

The shift transformation $T$ is defined as $x_{t+1} = Tx_t$.  

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For each $A \in \mathcal{B}(X^\infty)$, $\hat{\Pi}^h_k$ is a measurable function of $k$ and for each $k$, $\hat{\Pi}^h_k(\cdot)$ is a probability on $(N^\infty, \mathcal{B}(X^\infty))$. For $A \in \mathcal{B}(X^\infty)$ and $B \in \mathcal{B}((N^h)^\infty)$, we have

$$\hat{\Pi}^h(A \times B) = \int_{k \in B} \hat{\Pi}^h_k(A) \hat{\mu}^h(dk).$$

(14)

Also, as we noted above,

$$\Pi(A) = \hat{\Pi}^h(A \times (N^h)^\infty), \forall A \in \mathcal{B}(X^\infty)$$

$$\hat{\mu}^h(B) = \hat{\Pi}(X^\infty \times B), \forall B \in \mathcal{B}((N^h)^\infty).$$

(15)

If $(\Omega^h, \mathcal{B}^h, T, \hat{\Pi}^h)$ is a stable dynamical system with a stationary measure $m^{\hat{\Pi}^h}$, we define the two marginal measures of $m^{\hat{\Pi}^h}$ as follows:

$$m(A) := m^{\hat{\Pi}^h}(A \times (N^h)^\infty), \forall A \in \mathcal{B}(X^\infty)$$

(17)

$$m_Q^h(B) := m^{\hat{\Pi}}(X^\infty \times B), \forall B \in \mathcal{B}((N^h)^\infty).$$

(18)

Also, let $\hat{m}_k$ denote the stationary measure of $\hat{\Pi}^h_k$, which is a measure on $(X^\infty, \mathcal{B}(X^\infty))$. Given the construction of the dynamical system, we have the following theorem:

**Theorem 2.** (Conditional Stability Theorem, Kurz and Schneider (1996)).

Let $(\Omega^h, \mathcal{B}^h, T, \hat{\Pi}^h)$ be a stable and ergodic dynamical system. Then,

1. $(X^\infty, \mathcal{B}^\infty, T, \hat{\Pi}^h_k)$ is stable and ergodic for $\hat{\Pi}^h$ a.a. $k$.

2. $\hat{m}^h_k$ is independent of $k$, $m^h_k = m = \Pi$.

3. If $(X^\infty, \mathcal{B}(X^\infty), T, \hat{\Pi}^h_k)$ is stationary, then the stationary measure of $\hat{\Pi}^h_k$ is $\Pi$. That is

$$\hat{m}^h_k = m = \Pi.$$
4 Quantitative Analysis

In this section we focus on the quantitative analysis of the model. In section 4.1 we discuss how to apply the structure for rational beliefs as outlined in the previous section into a simulation framework and the parameterization of the model. Section 4.2 discusses the results regarding the survival of agents and section 4.3 discusses the Asset-pricing implications.

4.1 The Simulation Model

To solve the model it is convenient to normalize the variables. In particular we normalize all variables with respect to aggregate output. We have now the following normalized budget constraint

\[
    c^h_t + q_t \theta^h_t + p_t b^h_t \leq c^h_t + \frac{b^h_{t-1}}{g_t} + (q_t + d_t) \theta^h_{t-1},
\]

which implies the following definition for the normalized financial wealth

\[
    \omega^h_t = \frac{\theta^h_{t-1}(q_t + d_t) + \frac{b^h_{t-1}}{g_t}}{q_t + d_t}.
\]

The agents’ objective function changes now to

\[
    U^h_t = \left( 1 - \beta \right) \left( c^h_t \right)^{1-\gamma^h_t} + \beta \mathbb{E}_{Q^h_t} \left[ \left( U^h_{t+1} g^h_{t+1} \right)^{1-\gamma^h_t} \left| F^h_t \right. \right]^{\frac{1}{1-\gamma^h_t}},
\]

We assume that there are only 2 agents in the economy, that is, \( H = 2 \). We also assume that there are two growth states, i.e. \( g_t \in \{g, \bar{g}\} \). The empirical distribution \( \{g_t\} \) follows a markov-process:

\[
    \Psi = \begin{bmatrix}
        \phi & 1 - \phi \\
        1 - \phi & \phi
    \end{bmatrix}.
\]
Then, the stationary transition probability matrix has to satisfy the following conditions:

- the empirical distribution for the process $g_t$ is specified by transition probability matrix $\Psi$.
- the marginal distribution for $n^h_t$ is i.i.d with frequency of $\{n^h_t = 1\} = \alpha^h$.

Here, we use a specification similar to Kurz and Motoles (2001) as we know that the beliefs are compatible with the stationary distribution and it can generate large fluctuations. We assume that the $8 \times 8$ matrix $\Gamma$ has the following structure:

$$\Gamma = \begin{bmatrix} \phi A & (1 - \phi)A \\ (1 - \phi)A & \phi A \end{bmatrix}$$

(23)

$A$ is a $4 \times 4$ matrix defined by 6 parameters $(a^1, a^2, a, b)$ and $a = (a^1, a^2, a^3, a^4)$ as follows:

$$A = \begin{bmatrix} a^1 & a^1 - a^1 & a^2 - a^1 & 1 + a^1 - a^1 - a^2 \\ a^2 & a^1 - a^2 & a^2 - a^2 & 1 + a^2 - a^1 - a^2 \\ a^3 & a^1 - a^3 & a^2 - a^3 & 1 + a^3 - a^1 - a^2 \\ a^4 & a^1 - a^4 & a^2 - a^4 & 1 + a^4 - a^1 - a^2 \end{bmatrix}$$

(24)

We also have to specify the transition probability matrices that represent the beliefs of the agents. As noted above, agent $h \in \{1, 2\}$ in period $t$ uses $F^h_1$ when his generating variable is $n^1_t = 1$ and $F^h_2$ when his generating variable is $n^1_t = 0$. The rationality of belief condition implies that

$$\alpha^h F^h_2 + (1 - \alpha^h) F^h_2 = \Gamma.$$  

(25)

Thus to fully pin down a traders’ belief we only have to specify $F^h_1$ while $F^h_2$ can be inferred from $\Gamma$ and $F^h_1$. The matrix $F^h_1$ is parametrized by $\eta^h$ as follows:

$$F^h_1(\eta^h) = \begin{bmatrix} \phi \eta^h A & (1 - \eta^h \phi) A \\ (1 - \phi) \eta^h A & (1 - (1 - \phi) \eta^h) A \end{bmatrix}$$

(26)
From the above equation one can see that if $\eta^h > 1$ a trader places more weight on the growth states, i.e. he is overly optimistic that the economy grows when his beliefs are given by $F^h$. Furthermore, the larger the $\eta^h$ implies a more optimistic trader. Furthermore, parameter $\alpha^h$ determines the frequency of optimistic beliefs, when $\alpha^h = 0.5$ then optimistic and pessimistic have the same frequency while $\alpha^h > 0.5$ implies that a trader is more often optimistic then pessimistic. This has also implications for pessimistic beliefs. In particular if $\eta^h > 1$ and $\alpha^h > 0.5$ then beliefs are more asymmetrically distributed to satisfy the rationality condition.

For the beliefs of the agents we follow Kurz and Motolese (2001) and set $(a_1, a_2, a_3, a_4) = (0.5, 0.14, 0.14, 0.14)$. Furthermore, we assume that $\alpha^1 = \alpha^2 = \alpha = 0.57$. As our focus is not only on differences in beliefs but also on how much traders deviate from the The maximum value for $\eta$ is $1/0.57 \approx 1.7$ and we will examine several different cases of $\eta$. To focus on the survival aspect, we consider the case that agent 2 believes that the empirical distribution is the true distribution, i.e. $\eta^2 = 0$, while agent 1 does not believe that the empirical distribution is the true distribution. In particular we consider $\eta^1 \in \{1.2, 1.4, 1.6\}$.

Following Mehra and Prescott (1985) we consider the following transition probability matrix for $\Psi$:

$$
\Psi = \begin{bmatrix}
0.43 & 0.57 \\
0.57 & 0.43 
\end{bmatrix},
$$

(27)

and set $\bar{g} = 1.054$ and $g = 0.982$.

Our choices for preferences follow the literature. We set the time-preference parameter to $\beta = 0.96$, the coefficient of relative risk-aversion is set to $\gamma = 1.5$ which is standard in the literature. On the other hand, for the value of the EIS there is a bigger range of estimates. Some authors estimate a rather low value for the EIS, for example Hall (1988) estimates a value much smaller than 1, while several asset pricing models (e.g., Collin-Dufresne, Johannes, and Lochstoer (2014) or Bansal and Yaron (2004)) have used a EIS greater than 1. An EIS greater than 1 is needed to capture the negative correlation between consumption volatility and the price/dividend-ratio. For the baseline model
we set the elasticity of intertemporal substitution for both agents to $\psi = 1.5$, a value which is in line with the asset pricing literature.

The model is solved using a policy function iteration and the details of the solution algorithm are outlined in appendix B.

4.2 Survival of Agents

4.2.1 Wealth Distribution

We follow the literature and define survival in terms of financial wealth. We say that a household becomes extinct if it’s financial wealth share converges to zero and survives if it doesn’t converge to zero. Furthermore, a household dominates the market if it’s financial wealth converges to 1. Formally, a household $h$ becomes extinct if

$$\lim_{t \to \infty} \omega^h_t = 0 \text{ almost surely,}$$

and survives if doesn’t become extinct. And a household $h$ dominates the market if

$$\lim_{t \to \infty} \omega^h_t = 1 \text{ almost surely.}$$

All results are obtained from simulations. We simulate the economy for 100 periods (years) and the number of simulations is $N=50000$.

We first turn our attention to the case of $m = 1.00$, i.e. agents cannot borrow. In this case, the budget constraint for both agents reduces to

$$c^h_t + q^h_t \theta^h_t \leq c^h_t + (q_t + d_t) \theta^h_{t-1}.$$  \hspace{1cm} (30)

Kehoe and Levine (2001) refer to this situation as liquidity constrained and the financial wealth share is equivalent to the position in the risky asset. As agents cannot borrow, financial frictions do not play and the only driver of the results are aggregate growth risk as well as risks stemming from differences of opinion.

In figure (1) we show the results of the simulation exercise. The three panels
Figure 1: This graphs shows the dynamics of the wealth distribution of Agent 1 over 100 years. The quantiles of the wealth distribution are 0.01, 0.1, 0.5, 0.9, 0.99. The initial wealth share is $\omega^1 = 0.5$. The coefficient of relative risk-aversion is $\gamma = 1.5$ and the Elasticity of Intertemporal substitution is $\psi = 1.5$. The margin requirement is $m = 1.00$.

show the quantiles of the financial wealth distribution for different levels of disagreement, i.e. different values of $\eta^1$. One can see that for all three possible values of $\eta^1$ that after 100 years the median financial wealth share is at $\omega^1_{100} = 0.5$, i.e. both agents survive in the long-run but neither starts to dominate the market. Furthermore, the distribution is symmetric around the median wealth share. The graphs also show that with a higher disagreement the wealth distribution becomes more dispersed.

The intuition behind this result is rather simple. If agent 1 is overly optimistic about next periods returns on holding the asset she is keen to increase her position in the risky asset. However, as she cannot borrow she has to pay the asset by using dividend as well as labor income which would reduce today’s consumption which reduces her willingness to buy the asset. The willingness to buy the asset is increased if agents disagree more, i.e. greater $\eta^1$. This results in a larger dispersion of financial wealth.

For the case of $m = 0.50$ we make several observations. First, with increasing diversity in beliefs the distribution of financial wealth becomes more narrow. Second, in contrast to the case of $m = 1.00$ the median wealth distribution drops below 0.5. Third, after some first gains or losses the quantiles of
Figure 2: This graphs shows the dynamics of the wealth distribution of Agent 1 over 100 years. The quantiles of the wealth distribution are 0.01, 0.1, 0.5, 0.9, 0.99. The initial wealthshare is $\omega^1 = 0.5$. The coefficient of relative risk-aversion is $\gamma = 1.5$ and the Elasticity of Intertemporal substitution is $\psi = 1.5$. The margin requirement is $m = 0.50$.

the wealth distribution are stable.

4.2.2 The Role of the Elasticity of Intertemporal Substitution

In order to understand the importance of the EIS, i.e. the desire to smooth consumption, on the results we keep $\eta^1$ fixed at $\eta^1 = 1.6$ and consider now three different case of $\psi$. In particular, we consider the cases $\psi \in \{0.35, 1/\gamma, 1.5\}$. Figure (3) shows the dynamics of the median financial wealth over 100 years when $m = 1$ and $m = 0.50$. If there is no borrowing, the effects of changes in the EIS on the dynamics of financial wealth are negligible. This is in contrast to the case when borrowing is allowed where the elasticity of intertemporal substitution clearly impacts the dynamics. In particular, we see that a lower EIS implies a lower median financial wealth.

This implies that the main driver of the results is the effect the EIS has on the composition on the portfolio. If there is no borrowing in the economy, the only reason to invest in the asset is the expected return. The ability to borrow changes the dynamics drastically as an additional motive to hold the risky asset comes into play. Agents now hold the risky asset in order to borrow. In order to further understand the importance of the EIS we look at the policy
functions. Given the fact that agent 2 believes that the empirical distribution is the true distribution we have 4 possible states to look at.

In figures (4) the log bond-price is shown as a function of the financial wealth share of the agents. Obviously when agent 1’s wealth share rises above 0.5 then he starts to dominate the market and her influence on prices becomes bigger. In particular we can see that when agent 1 is optimistic the interest rate is larger than in pessimistic states as in optimistic states agent 1 prefers to invest into the risky asset as he subjectively believes that there is a high rate of return. This implies that she is also more keen to borrow from agent 2, hence in equilibrium the interest rate has to rise to equilibriate the supply and demand for bonds.

This effect is exacerbated when the EIS is low as a low EIS implies a low desire to smooth consumption and hence lower precautionary savings. Thus in equilibrium interest rate decreases in optimistic states when the EIS increases. This can also be seen from figure (5), i.e. with a higher EIS agents save more.

This in turn affects the long-run distribution of financial wealth in the economy, i.e. agents with a lower EIS will have a lower long-run financial wealth as
Figure 4: This figure shows the log bond price as a function of the financial wealth share for different values of $\Psi$. The beliefs of agent are set to $\eta^1 = 1.6$ and the collateral constraint is set to $m = 0.5$.

Figure 5: This figure shows the bond holdings as a function of the financial wealth share for different values of $\Psi$. The beliefs of agent are set to $\eta^1 = 1.6$ and the collateral constraint is set to $m = 0.5$. 
they invest more aggressively in the risky asset and save less and thus are more hit from the malinvestment when the wrong state occurs. Consequently, if the EIS is lower than the median financial wealth is also at a lower level.

4.3 Equilibrium Asset Prices

In the previous section we have seen that the margin requirement $m$ and the elasticity of intertemporal substitution are key determinants for the endogenous distribution of financial wealth as they affect portfolio and savings decisions of households which in turn affects equilibrium properties of the economy. In this section we further examine the asset pricing properties of the economy.

4.3.1 Volatility

We are now turning to the properties of equilibrium asset prices. In particular, we are looking at the unconditional volatility of asset prices and interest rates. Figure (6) shows the volatility of normalized equity prices $q_t$ and interest rate $r_t = \frac{1}{p_t} - 1$ for three different values of $\eta^1$.

While volatility of the normalized and interest rate is always higher with
more disagreement and is thus consistent with the literature, we find that relaxing the margin requirement $m$ has non-monotonic effects on the volatility of the normalized equity prices.

As already noted, in the case $m = 1.00$ there are only two risk factors: Changes in aggregate endowment as well as belief. As no agent is driven out prices are determined by the euler-equations of both agents. Thus, larger disagreements about the aggregate growth implies a higher volatility.

If agents are allowed to borrow, i.e. $m < 1$, the margin requirements adds additional risk. As long as the margin constraint is not binding, i.e. wealth shares do not approach the upper or lower limit, a negative shock to prices has no effects as agents are not forced to rewind their position. If, on the other hand, the margin constraint is binding a negative shock to prices triggers the fire-sale dynamics of prices.

4.3.2 Collateral Premium
Following Fostel and Geanakoplos (2008), we define the collateral premium as follows:

$$CV^h = E \left[ \frac{\mu^{hc}(1-m)^0\min(q^+ + d^+)}{q\mu^{hb}} \right]$$ (31)

The interpretation of the collateral value is straightforward and is simply the fraction of the assets value that is due to its use as collateral. If agents cannot borrow, i.e. \( m = 1 \), then there is no value added through its use as collateral. Furthermore, as the Lagrange-multiplier \( \mu^{hc} \) is positive, the collateral premium is greater than zero.

Figure (7) shows the collateral premium of the risky asset for the three cases of beliefs as well as the three cases of EIS. We can see that, in general, more disagreement leads to a higher collateral premium, regardless of the EIS. Similarly, a larger EIS results in a higher collateral premium.

5 Conclusion

In this paper we studied the impact of market completeness and elasticity of intertemporal substitution on survival and the impact on asset prices in an economy with heterogeneous beliefs. Simulation results have shown that agents who do not believe that the empirical distribution is the true underlying distribution survive on the long-run. Additionally the evolution of financial wealth in the economy depends on the ability to borrow as well as the elasticity of intertemporal substitution.

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A Derivation of the first order conditions

For ease of notation, we drop the reference to a household $h$. The maximization problem of the agent can be written as the following Lagrangian:

$$L = \left( 1 - \beta \right) \left( c_t \right)^{1-\gamma} + \beta \left[ EQ_t \left[ (U_{t+1}g_{t+1})^{1-\gamma} | F_t \right] \right]^{\frac{\rho}{\gamma}}$$

$$-\mu^h_t \left( c_t + \theta_t q_t + b_t p_t - \theta_{t-1} (q_t + d_t) - \frac{b_{t-1}}{g_t} - e_t \right) - \mu^s_t \theta_t$$

$$-\mu^c_t \left( b_t + (1 - m) \theta_t \min_{s^{t+1} \mid s^t} (q_{t+1} + d_{t+1}) \right).$$

The lagrange multiplier with respect to the budget constraint is denoted by $\mu^b_t$, for the short-sale constraint $\mu^s_t$ and for the collateral constraint $\mu^c_t$. Taking now the derivative with respect to consumption and rearranging yields

$$\frac{\partial L}{\partial c_t} = (U_t)^{\psi-1} c_t^{\psi-1} = \mu^c_t.$$  (33)

The derivative with respect to asset purchases is

$$\frac{\partial L}{\partial \theta_t} = (U_t)^{\psi-1} \beta EQ_t \left[ \left( U_{t+1}g_{t+1} \right)^{1-\gamma} \right]^{\frac{1-\rho}{\gamma}} \left\{ (U_{t+1}g_{t+1})^{-\gamma} g_{t+1} \frac{\partial U_{t+1}}{\partial \theta_t} \right\}$$

$$-\mu^h_t q_t - \mu^s_t - \mu^c_t (1 - m) \min_{s^{t+1} \mid s^t} (q_{t+1} + d_{t+1}),$$

and because of the envelope theorem the derivative of $U_{t+1}$ with respect to $\theta_t$ is given by

$$\frac{\partial U_{t+1}}{\partial \theta_t} = \frac{\partial U_{t+1}}{\partial c_{t+1}} \frac{\partial c_{t+1}}{\partial \theta_t} = (U_{t+1})^{\psi-1} (1 - \beta) (c_{t+1}^h)^{-\psi-1} (q_{t+1} + d_{t+1}).$$  (35)
Combining the last two equations we get

\[ q_t \mu_t^b = (U_t)^{\psi^{-1}} \beta E_Q t \left[ (U_{t+1} g_{t+1})^{1-\gamma} \right]^{\frac{1-\rho}{\rho}} E_Q t \left[ (U_{t+1})^{\psi^{-1}-\gamma} g_{t+1}^{1-\gamma} (1 - \beta) \left( c_{t+1}^h \right)^{-\psi^{-1}} (q_{t+1} + d_{t+1}) \right] + \mu^s + \mu^c (1 - m) \min_{s_{t+1}|s^t} (q_{t+1} + d_{t+1}). \]  

The first order conditions for bond holdings can be derived similarly, i.e.

\[ p_t \mu_t^b = (U_t)^{\psi^{-1}} \beta E_Q t \left[ (U_{t+1} g_{t+1})^{1-\gamma} \right]^{\frac{1-\rho}{\rho}} E_Q t \left[ (U_{t+1})^{\psi^{-1}-\gamma} g_{t+1}^{1-\gamma} (1 - \beta) \left( c_{t+1}^h \right)^{-\psi^{-1}} \right] + \mu^c. \]

**B Numerical Algorithm**

To solve for the stationary equilibrium we use a time-iteration algorithm. The algorithm proceeds as follows:

**Step 0:** Set an error-tolerance \( \epsilon \) and form a grid \( M \) over \([0, 1]\), Set an initial guess \( f^0 \) for policy and price functions.

**Step 1:** Given a set of policy and price functions \( f^{n-1} \), we obtain a new set of policies and prices \( f^n \) by solving the system of equilibrium conditions, the law of motion for the wealth share and optimality conditions (40)-(48) for each gridpoint \((\omega, s) \in M \times S\). As the short-sale constraint as well as the margin requirement are not always binding, the Lagrange-Multipliers \( \mu^{hc} \) and \( \mu^{hs} \) are not differentiable at edge-cases. Hence, the system of equation is not differentiable. To circumvent the problem we use the Garcia-Zangwill trick (Zangwill and Garcia (1981)) and replace the lagrange multiplier \( \mu^{hs} \) and \( \mu^{hc} \) with \( \mu^{hs+} = \max\{0, \mu^{hs}\}^2, \mu^{hs-} = \max\{0, -\mu^{hs}\}^2, \mu^{hc+} = \max\{0, \mu^{hc}\}^2, \mu^{hc-} = \max\{0, -\mu^{hc}\}^2 \).
max\{0, -\mu^h_{hc}\}^2$. Thus, the system of equations is now as follows:

\[
q_n(U^h_n)^{\psi^{-1}}(c^h_n)^{-\psi^{-1}} = (U^h_n)^{\psi^{-1}} \beta E_Q^h \left[ \left( \frac{U^h_{n-1}(\omega^+, s^+)}{c^h_{n-1}(\omega^+, s^+)} \right)^{1-\gamma^h} g(s^+) \right] \frac{1-\mu^h}{\rho^h} \tag{40}
\]

\[
P_n(U^h_n)^{\psi^{-1}}(c^h_n)^{-\psi^{-1}} = (U^h_n)^{\psi^{-1}} \beta E_Q^h \left[ \left( \frac{U^h_{n-1}(\omega^+, s^+)}{c^h_{n-1}(\omega^+, s^+)} \right)^{1-\gamma^h} g(s^+) \right] \frac{1-\mu^h}{\rho^h} + \mu^h_{hc} \tag{41}
\]

\[
c^h_n = e^h + \omega^h(q_n + d) - \theta^h q_n - b^h p_n \tag{42}
\]

\[
b_1^n + b_2^n = 0 \tag{43}
\]

\[
\theta^1 + \theta^2 = 1 \tag{44}
\]

\[
c_1^n + c_2^n = 1 \tag{45}
\]

\[
\omega^h_{n+} = \frac{\theta^h(q_{n-1}(\omega^+, s^+) + d(s^+) + \frac{b^h_n}{g(s^+)})}{q_{n-1}(\omega^+, s^+) + d(s^+)} \tag{46}
\]

\[
\mu^h_{ns-} = \theta^h_n \tag{47}
\]

\[
\mu^h_{hc-} = \left( \frac{b^h_n + \theta^h_n(1 - m) \min_{s^+}(q_{n-1}(\omega^+, s^+) + d(s^+))}{1 - \beta} \right) \tag{48}
\]

Here, equations (40) and (41) are the first order conditions for asset and bond holdings respectively. Equation (42) is the budget constraint while equations (43)-(45) are the equilibrium conditions, equation (46) is the dynamics for wealthshare and equations (47) and (48) are the modified complementary slackness conditions.

To solve for equilibrium prices, in addition to next periods prices, only next
periods consumption and Value-function are needed and not portfolio choices. Thus, we do not need to interpolate next periods portfolio choices.

Step 2: Prices and policy functions are updated until $\|f^n - f^{n-1}\| < \epsilon$.

In our application, the grid $M$ has 101 equidistant points and $\epsilon$ is set to $10^{-4}$ and the algorithm is implemented in Matlab. To solve the system of nonlinear equations (40)-(48) one can use fsolve, however fmincon was usually faster and more robust.

C Proof of Lemma 1

For the proof of lemma 1, we need to make an auxiliary assumption. In particular, we follow Magill and Quinzii (1994) and assume that consumers are uniformly impatient:

**Assumption 2.** For any $h \in \mathcal{H}$, there exists a $0 \leq \delta < 1$ and $K > 0$ such that for every $s^t \in S$,

$$
\left( c^h_-(s^t), c^h(s^t) + Ke^h_t, \delta c^h_+(s^t) \right) \succ \left( c^h_-(s^t), c^h(s^t), c^h_+(s^t) \right)
$$

(49)

Furthermore, we transform the preferences. In particular, let $V^h_{t} \equiv \frac{U^h_1 - \psi^{-1}}{1 - \psi^{-1}}$ and rewrite the utility function as follows:

$$
V^h_{h,t} = \frac{(c^h_1)^{1 - \psi^{-1}}}{1 - \psi^{-1}} + \beta E^h_\mathcal{Q}^h_{t} \left[ V^h_{h,t+1} \right]^{1/\rho^h}.
$$

(50)

First, we show that the price of the bond has an upper bound. If a consumption plan $c$ can be supported by an initial wealth $W$, then the consumption plan $\delta c$ can be supported by the initial wealth $\delta W$ for any constant $\delta \in (0,1)$. Thus, we know that for each agent $h \in \mathcal{H}$, there exist $K > 0$ and $0 \leq \delta < 1$ such that for every $s^t \in S$,

$$
\left( c^h_-(s^t), c^h(s^t) + Ke^h_t, \delta W(s^{t+1}) \right) \succ^h \left( c^h_-(s^t), c^h(s^t), W(s^{t+1}) \right)
$$

(51)

for all current consumption satisfying $c^h_1 \leq \sum_{h \in \mathcal{H}} e^h_t + d_t$ for all $s^t \in S$ and
wealth in the beginning of the next period satisfying \( W^h(s^{t+1}) \leq q_{t+1} + d_{t+1} \) for all \( s^t \in S \). Let \( \bar{p} = \frac{K_e}{w_m(1 - \delta)} \) and we are going to show that the bond price \( p_t \) cannot be higher than \( \bar{p} \) by contradiction. Suppose in equilibrium there is a node \( s^{t} \) such \( p_t > \bar{p} \). At this node, there must be one agent who has a positive net-position in the bond, and thus her wealth in the next node \( s^{t+1} \) must be at least \( w_m \). Without loss of generality, we assume that this is agent 1. If agent 1 sells now \( w_m(1 - \delta) \) units of the bond she could gain at least \( K_e \) of proceeds and then use the proceeds to buy the consumption good which are consumed in \( s^t \). Her budget constraint is still satisfied, yet her wealth in \( s^{t+1} \) is still at least \( \delta W(s^{t+1}) \). By uniform impatience, the new plan is preferred to the old plan and her budget constraint is still satisfied. Thus violates the assumption that the old plan is optimal.

We now turn to equity prices. For all \( T \geq 1 \), agent h’s value function at each node \( s^t \) is bounded from above by \( \frac{e^{1-\psi^{-1}}}{1-\psi^{-1}} + U_h(\bar{e}, \bar{e}, ...) \), where \( U_h(\bar{e}, \bar{e}, ...) \) denotes the value function for the consumption plan of consuming constant \( \bar{e} \) over the infinite horizon tree. It can be shown that \( U_h(\bar{e}, \bar{e}, ...) = \frac{1}{1-\beta} e^{1-\psi^{-1}}. \) Therefore, we have

\[
U_h(c^h(s^t), c^h(s^{t+1})) \leq \frac{2}{1-\beta} \frac{e^{1-\psi^{-1}}}{1-\psi^{-1}}. \quad (52)
\]

Because \( \psi > 1 \), there exists a large \( K \) such that for all \( h \in H \)

\[
\frac{K^{1-\psi^{-1}}}{1-\psi^{-1}} \geq \frac{2}{1-\beta} \frac{e^{1-\psi^{-1}}}{1-\psi^{-1}}. \quad (53)
\]

Thus, by uniform impatience we know that for each agent \( h \in H \) there exists a \( K > 0 \) such that

\[
c^h(s^t) + K > h(c^h(s^t), W^h(s^{t+1})). \quad (54)
\]

for all current consumption satisfying \( c^h(s^t) \leq 1 \) and wealth in the beginning of the next period satisfying \( W^h(s^{t+1}) \leq d(s_{t+1}) + q(s^{t+1}) \) for all \( s^t \in S \). We define the constant \( \bar{Q} = 4|H|\max\{2K, Q\bar{D}\bar{e}\} \) and show by contradiction that
equity prices are bounded from above by this constant. Suppose there exists a node $s^t$ such that $q_t > \bar{Q}$ in a truncated equilibrium. There must be one agent in the economy whose holdings of the real asset is at least $1/|H|$ in equilibrium. Without loss of generality, we assume it is agent 1. If agent 1 sells $1/(2|H|)$ and consumes the proceeds for $K$ units of the consumption good, then the new plan is strictly preferred to the old plan but it still satisfied the budget constraint. Which is a contradiction to the assumption of optimality.

Thus, we have shown that for all $T \geq 1$ equilibria lie within a bounded rectangular area, denoted as $Y^*$.

To prove existence of the equilibrium it is useful to normalize the prices. In particular, we assume that prices belong to a simplex such that

$$p^c_t + \tilde{p}_t + \tilde{q}_t = 1, \forall t,$$

with $p^c_t$ as the price of the consumption good while we define $\tilde{p}_t := p^c_t p_t$ and $\tilde{q}_t = p^c_t q_t$. Then the budget set of agent $h$ in period $t$ consists now of the following equation

$$p^c_t c^h_t + \tilde{q}_t \theta^h_t + \tilde{p}_t b^h_t \leq p^c_t c^h_{t-1} + b^h_{t-1} + (\tilde{q}_t + d_t) \theta^h_{t-1}.$$

For every finite-horizon economy, each agents’ budget set is compact-valued, convex-valued and continuous correspondence of $p^c_t$, $\tilde{p}_t$, and $\tilde{q}_t$. Thus, each agent’s demand correspondence for the consumption good, the bond and the asset is non-empty, compact-valued and upper-hemicontinuous.

We define now the following aggregate demand-correspondence, for every $y \in Y$ in a finite horizon economy:

$$z(p^c, \tilde{p}, \tilde{q}_t) = (z_c(p^c, \tilde{p}, \tilde{q}_t), z_b(p^c, \tilde{p}, \tilde{q}_t), z_\theta(p^c, \tilde{p}, \tilde{q}_t)) = \left( \sum_{h \in H} c^h_t - \sum_{h \in H} e^h_t - d_t, \sum_{h \in H} b^h_t, \sum_{h \in H} \theta^h_t - 1 \right).$$

Define now the following optimisation problem

$$\max_{p^c, \tilde{p}, \tilde{q}_t} p^c z_c + \tilde{p} z_b, \tilde{q} z_\theta \text{ subject to } p^c + \tilde{p} + \tilde{q} = 1.$$
Then the set of the optimal \((p^c, \tilde{p}, \tilde{q})\) for this optimisation problem is nonempty, compact, and upper-hemicontinuous correspondence.

Next we show that \(z^* = 0\) and \((p^c^*, \tilde{p}_t, \tilde{q}_t)\) by contradiction. Suppose there is a positive excess demand in some market at \(t = 0\). Furthermore, suppose the excess demand is in the good market. Then the solution to the above maximization problem is \((p^c^*, \tilde{p}_t, \tilde{q}_t) = (1, 0, 0)\). This implies a strictly positive value for the objective function. However, for every \(t\) the summation of the budget constraints over the agents yields

\[
p_t^c \left( \sum_{h \in \mathcal{H}} c_t^h - \sum_{h \in \mathcal{H}} e_t^h - d_t \right) + \tilde{q}_t \sum_{h \in \mathcal{H}} (\theta_t^H - 1) + \tilde{p}_t \sum_{h \in \mathcal{H}} b_h^t = 0
\]

given that the market clearing conditions are satisfied in the previous period and indeed it is the case in the initial period because by assumption \(\sum_{h \in \mathcal{H}} \theta_0^h = 1\) and \(\sum_{h \in \mathcal{H}} b_0^h = 0\). Clearly, a strictly positive objective function is a contradiction to this. A similar argument holds if the excess demand is in the bond market or the market for the real asset.

Now, suppose there is a negative excess demand in some market at \(t = 0\). Suppose the most negative one is in the good market. Then it is optimal to set \(p^c = 0\). However, this results in a positive excess demand for the consumption good for all agents, which contradicts the assumption of a negative excess demand in the goods market. Similar arguments hold for negative excess demands for the risk-free asset and the real asset. In particular, if there is a negative excess demand in the bond market, it is optimal to set \(\tilde{p} = 0\). Which results in a large \(b^h\), which again is a contradiction. In the market for the real asset we have the same, i.e. if there is a large negative excess demand in the market for the real asset, it is optimal to set \(\tilde{q}\) to 0 which contradicts again the assumption of a large negative excess demand as there would be a large positive excess demand.

Thus, \(z^* = 0\) and \((p^c^*, \tilde{q}, \tilde{p}) > 0\). Similar arguments hold for every \(t\) with the assumption that \(z_{t-1}^* = 0\) and \((p_{t-1}^c, \tilde{q}_{t-1}, \tilde{p}_{t-1}) > 0\).
D Proof of Theorem 1

For any compact set \( \mathcal{K} \subset \mathcal{Y} \), and a policy correspondence \( Y : S \times [0, 1] \rightarrow \mathcal{K} \) we define an operator \( O_{\mathcal{K}} \), that maps the policy correspondence \( Y : S : [0, 1] \rightarrow \mathcal{K} \) to another policy correspondence \( O_{\mathcal{K}}(Y) \) such that for all \( s \in S \) and \( w \in [0, 1] \)

\[
O_{\mathcal{K}}(Y)(s, w) = \left\{ y \in \mathcal{K} : \exists \left( (\bar{w}_1, \bar{y}_1), ..., (\bar{w}_{|S|}, \bar{y}_{|S|}) \right) \in \Phi(w, y, s) \text{ s.t. } \bar{y}_s \in Y(\bar{w}_s, \bar{y}_s) \right\}.
\]

This correspondence computes the endogeneous variables \( y \in \mathcal{Y} \) given the state variables \((w, s)\) in the current period and the periods equilibrium endogeneous variables \(((\bar{w}_1, \bar{y}_1), ..., (\bar{w}_{|S|}, \bar{y}_{|S|}))\).

Define the constant correspondence \( Y^0 \) by \( Y^0(w, y) \equiv \mathcal{Y}^* \) for all \( w \in [0, 1] \) and all \( y \in \mathcal{Y} \). Given a correspondence \( Y^n \) we define \( Y^{n+1} \) recursively as \( Y^{n+1} = O_{\mathcal{Y}^*} \). Because of Lemma 1, for each \( n \) the set \( Y^n \) is nonempty. Next, we show by induction that \( Y^n \) is closed for each \( n \). It is obvious that \( Y^0 \) is closed. Suppose \( Y^n \) is closed, then \( Y^{n+1} = O_{\mathcal{Y}^*}(Y^n) \) is also closed, because the graph of \( \Phi \) is closed and the graph of \( Y^n \) is closed. For each \( n \), we have \( Y^{n+1} \subset Y^n \). By definition, we have \( Y^1 \subset Y^0 \). Suppose that \( Y^{n+1} \subset Y^n \), then we have \( Y^{n+2} \subset Y^{n+1} \), because by definition we have \( O_{\mathcal{Y}^*}(Y^n) \subset O_{\mathcal{Y}^*}(Y^{n+1}) \).

We define the correspondence \( Y^* \) such that for all \((w, s) \in [0, 1] \times S\)

\[
Y^*(w, s) \equiv \cap_{n=0}^{\infty} Y^n(w, s).
\]

Because for each \((w, s) \in [0, 1] \times S\), the sequence of sets \( \{Y^n(w, s)\} \) are compact, nested, and nonempty, thus \( Y^*(w, s) \) is a closed and nonempty set \( Y^*(w, s) \) is policy correspondence in recursive Markov equilibria and the definition of \( O_{\mathcal{Y}^*} \) implies the existence of a transition for the recursive Markov equilibrium.