Are agent-based simulations robust?  
The wholesale electricity trading case*

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Abstract
Agent-based computational economics is becoming widely used in practice and this paper explores the consistency of some of its standard techniques. As a particular case, we focus on prevailing wholesale electricity trading simulation methods. We include different supply and demand representations and propose the Experience Weighted Attractions method to include several behavioural algorithms. We compare the results across assumptions and to economic theory predictions. The match is good under best-response and reinforcement learning but not under fictitious play. The simulations perform well under flat and upward slopping supply bidding, and also for plausible demand elasticity assumptions. Learning is influenced by the number of bids per plant and the initial conditions. The overall conclusion is that agent-based simulation assumptions are far from innocuous. We link their performance to underlying features, and identify those that are better suited to wholesale electricity markets.

Keywords: Agent-based computational economics, electricity, market design, experience-weighted attraction (EWA), learning, supply functions, demand aggregation, initial beliefs.

1 Introduction

Firms and regulators make increasingly use of simulation techniques, as part of an “engineering” approach in which game theory, experimental and computational economics complement each other (Roth, 2002). Markets created with the aid of simulations are the U.S. National Resident Matching Program (Roth and Peranson, 1997), radio spectrum auctions (Rothkopf et al., 1998), kidney transplants (Roth et al. 2004), school choice (Abdulkadiroglu et al., forth.) and the electricity trading arrangements in many countries. Corporate examples include eBay, Google, IBM and Unilever.

Agent-based computational economics (ACE) is becoming an important part of those market simulation efforts. In the ACE paradigm, markets are modelled as dynamic systems of interacting,
often boundedly rational, agents. ACE has been used to model situations as diverse as competitive strategy (e.g. Denrell, 2004), innovation (e.g. Adner and Levinthal, 2001), financial markets (e.g. Noe et al. 2003; Pouget, 2007), business organisation (e.g. Rivkin and Siggelkow, 2001), or electricity trading.

One of the advantages of ACE is that the models are tailored to fit closely each industry. However, this is a disadvantage when it comes to understanding the factors driving the results. ACE models are often not comparable to each other (Fagiolo et al., 2006) and, probably as a consequence, ACE is struggling to reach its full potential. Leombruni et al. (2006) reveal that only eight papers among over 40,000 published in a list of well-regarded economics journals use it, and argue that this is due to the difficulty in interpreting and generalising the simulation outcomes and the lack of consensus on the techniques suitable for each situation.

This paper explores the theoretical reliability of agent-based simulation techniques. As a particular case, we focus on wholesale electricity markets. ACE is prominent in electricity because wholesale electricity markets are huge and especially awkward for standard methods. They feature imperfect competition, very low demand elasticity, discontinuously convex supply functions, high-frequency repeated trading, heterogeneous agents and high potential for collusion (Wilson, 2002).¹ Thus, the ACE electricity literature documents several models commissioned by large energy players (e.g. Gaz de France, E.ON, Shell, and UK’s Competition Commission) as well as some calibrated to the US market, like the “Electricity Market Complex Adaptive System” (EMCAS) (Macal and North, 2005) and the “Agent-based Modeling of Electricity Systems” (AMES) (Sun and Tesfatsion, 2007).

Despite the large electricity simulations literature, there is little consensus on what techniques are most appropriate (see Marks, 2006; Weidlich and Veit, 2008 for surveys). First, there are models with inelastic and elastic demands. Second, several papers use stepwise schedules to model the supply part of the market. In others, sellers bid linearly increasing functions. Third, some papers assume reinforcement learning while others use more complex forms of agent behaviour like fictitious play or best-response. Finally, the question of whether initial conditions influence the results remains unresolved.

We adopt a stylised setting and include flat, stepwise and linear supply function bids, and linear demands with a wide range of elasticity specifications. Agents’ behavior is governed by Camerer and Ho (1999)’s Experience Weighted Attraction algorithm (EWA). EWA has reinforcement learning, fictitious play and best-response as particular cases and allows for the specification of different initial conditions.

We investigate whether the assumptions affect simulation outcomes and how they compare to simple, empirically-supported, theoretical results. Specifically, we cast light on whether the simulations are consistent with the standard claim of pivotal dynamics determining the relationship between competition and prices. A firm is pivotal if the quantity demanded exceeds the sum of production capacities of all other firms and, as a result, it is necessary to fulfil demand. There is wide consensus on the importance of pivotal dynamics in spot electricity markets (see e.g. Rothkopf, 2002, for a discussion). Bushnell ESCRIURE REF . All firms are pivotal in settings with just few of them. In those featuring many firms, none is pivotal. We predict that prices will be high under

¹See Tesfatsion and Judd (2006) and Leigh Tesfatsion’s website for extensive information sources about ACE (http://www.econ.iastate.edu/tesfatsi/ace.htm).
monopoly, decrease with competition, drastically change at a pivotal dynamics "threshold", and approach marginal costs beyond that point.

The results are good under flat and supply function bidding, and several plausible demand elasticities. However, we show that not all simulations exhibit the breaking points. The performance of fictitious play is poor, and clearly outperformed by the best-response and reinforcement learning treatments. Moreover, initial conditions have an influence on the simulations’ performance, and learning is more difficult with multi-step bids. The results call into question a large part of the extant ACE electricity research and can potentially enhance the practical implementation of these techniques.

This paper is part of a new literature examining the consistency of ACE in various market design settings (e.g. Fagiolo et al., 2007; Leombruni et al., 2007; Marks, 2007; and Midgley et al., 2007). In the electricity industry, we are only aware of two related working papers, both of them recent: Li et al. (2009) check the robustness to several reinforcement learning parameters, elasticity, and price caps in the AMES model, and Kimbrough and Murphy (2009) compare step and supply function bidding. The question of which models best fit real market data is complementary and deserves future attention. Our approach is focused on theoretical reliability, is stylised and includes comparisons of supply bidding, demand and behavioural algorithms.

The remainder of the paper is organised as follows: in part 2 we discuss the literature. In part 3, we document the simulation implementations. In part 4 we introduce the theoretical prediction. Part 5 includes the comparative simulation results and we conclude in part 6. All proofs are in the Appendix.

2 Agent-based electricity modelling alternatives

The three main sets of assumptions in ACE electricity models are the representation of supply and demand, behavioural rules and initial conditions. This section is a survey of the choices made in previous work. In Table 1 we classify some of the most relevant papers.

2.1 Supply bidding

Bertrand and Bertrand with capacity constraints are generally not considered suitable in the ACE electricity literature because they do not fit the uniform pricing prevalent in power pools. Cournot quantity bidding is sometimes used as an alternative (e.g. Bunn and Oliveira, 2007 and 2008; Veit et al., 2006). However, a recurrent argument is that Cournot is also unsuitable because in real pools generators are allowed to submit multiple flat bids for sections of their capacity. Hence, most papers use either von der Fehr and Harbord (1993)’s stepwise auctions, or Green and Newbery’s (1992) adaptation of the “supply function” equilibrium due to Klemperer and Meyer (1989) –SF.

In the stepwise approach, the market is a sealed-bid, multiple-unit auction. Generators simultaneously submit single prices at which they are willing to supply sections of their capacity. An independent auctioneer ranks the bids according to their offer prices, intersects the demand and supply and determines the system marginal price.
The stepwise literature includes both per plant and overall firm bidding models. Stepwise auction papers with one bid per generator are more parsimonious and comparable to the theoretical literature. Nicolaisen et. al. (2001) and Richter and Sheblé (1998) create models similar to those of auction theory to study the structure and efficiency of electricity markets. Closer to industrial organisation, Rupérez Micola and Bunn (2008) and Rupérez Micola et al. (2008) examine how horizontal and vertical integration influence the firms’ ability to exert market power. Nanduri and Das (2007) add a simple electricity network. Bagnall and Smith (2005) study how their model replicates human behaviour in the England and Wales market.

Other papers allow one bid per plant rather than per firm. Bower and Bunn (2000, 2001) simulate the transition between the England and Wales’ pool and the New Electricity Trading Arrangements (NETA) discriminatory auction and how this could affect market prices. Bunn and Martoccia (2005) also replicate the UK market and Bower et al. (2001) focus on Germany. Only García et al. (2005) and Banal and Rupérez Micola (2009) include abstract models with multiple stepwise bids per plant.

Inherent inflexibilities in the operation of nuclear assets (e.g. safety concerns, very low marginal costs and high start-up and loss of volume costs) prompt generators to submit flat schedules at very low prices. However, the assumption is quite restrictive in comparison to most bid-based electricity markets, where they can submit many bid steps per unit. Multi-bidding leads to the well-known “hockey stick” shape of the supply curve, with base-load plants submitting flat schedules and peak-load generators offering steeper step functions. Accordingly, a number of simulations include several bids per plant. For example, Day and Bunn (2001) and Bunn and Day (2009) develop detailed models of the England and Wales pool between 1990 and 2001. Bunn and Oliveira (2001, 2003) look into the related effect of NETA’s introduction and test whether the incumbents could influence prices. However, these models are often computationally cumbersome due to two reasons. First, the algorithm’s operations grow with the number of bids. Second, agents’ coordination is more difficult, which complicates learning and convergence to a steady state.

The SF approach approximates actual bids with increasing supply functions relating quantities and prices. This is a compromise between providing realism and simplifying the simulation mechanics. Banal and Rupérez Micola (2009) include a SF model with two stepwise bids per firm. Cincotti et al (2005) study the effect of market microstructure and costs on prices and Visudhiphan and Ilic (1999) focus on dynamic learning. Day and Bunn (2001, 2009) propose an even more flexible approach in which firms submit several SF sections per plant.

However, the presence of multiple Nash equilibria complicates the comparison of ACE and equilibrium results. For example, the SF model has little predictive value if the range of variation in demand is small because almost anything between the Cournot and the competitive solution can be supported in equilibrium (see Bolle 1992). Further, the solution is undefined if there is no short-run demand elasticity (von der Fehr and Harbord, 1998). Similarly, there are often many non-Pareto ranked equilibria in stepwise auction settings (for discussions, see e.g. von der Fehr and Harbord, 1993; Crawford et al., 2006).²

²Economic theory offers refinements to single out unique supply function equilibria. Klemperer and Meyer (1989) show that if outcomes with infinite demand occur with positive probability and there are no capacity constraints, then there is a unique SF equilibrium. Green and Newbery (1992) focus on the highest profit equilibrium. Baldick and Hogan (2002) rule out unstable equilibria and add a price cap and capacity constraints. Newbery (1991) considers entry and assumes bid-coordination. More recently, Holmberg (2007, 2008) shows that a unique SF equilibrium exists
2.2 Demand representation

The current literature mostly represents demands with double-sided call auctions or fixed aggregated curves.

First, double-sided call auctions consist of supply and demand bidding and common valuations bounded between marginal cost and redemption values. Bids represent the price at which firms are willing to sell and buy all their capacity in a double version of the stepwise auction setting. Examples include the papers by Richter and Sheblé (1998), Nicolaisen et. al. (2001), Bunn and Oliveira (2001, 2003) and Rupérez Micola and Bunn (2008).

However, many ACE simulations use aggregate demands. The literature often models short-run electricity demand as inelastic, in part due to the lack of real-time metering systems (e.g. Stoft, 2002). Examples include Bagnall and Smith (2005), Bunn and Martoccia (2005), Cincotti et al (2005), García et al. (2005), Nanduri and Das (2007), Rupérez Micola et al. (2008) and Sun and Tesfatsion (2007).

At first glance, this may seem like an appropriate representation of reality, but there are several reasons why relaxing that assumption can add value. First, markets have some level of bid-in demand, or implicit elasticity provided through the actions of system operators who may take out-of-market actions to effectively reduce demand when prices rise. Second, most volume is traded outside of balancing markets, either in exchanges or bilaterally. Third, financial derivatives increase demand elasticity. Finally, inelastic demand models tend to present a large number of non-Pareto ranked pure strategy equilibria. Papers with elastic demands include several by Derek Bunn and coauthors (e.g. Bower and Bunn, 2000, 2001; Bower et al., 2001; Bunn and Oliveira, 2007, 2008; Day and Bunn, 2001), and also Veit et al. (2006).

To our knowledge, only Banal and Rupérez Micola (2009), Li et al. (2009) and Visudhiphan and Ilic (1999) use both elastic and inelastic demands. Still, they do not seek to explicitly explore the implications of the elasticity assumption.

2.3 Behavioural algorithm

ACE models require rules to govern agent behaviour. One of their main intends is to realistically represent human decision-making, and its proponents frequently argue that existing deductive mechanisms often do poorly in experiments (Camerer and Ho, 1999, Roth and Erev, 1995; 1998: and van Huyck et al., 1990 are regularly used to support this claim). The electricity ACE literature is based on adaptive learning algorithms mainly derived from psychology.

Some previous work uses reinforcement learning (RL). In RL, agents tend to repeat actions that led to positive outcomes and avoid those that were detrimental. Several papers have used modified versions of the Roth and Erev (1995) algorithm, e.g. Banal and Rupérez Micola (2009), Li et al. (2009), Nanduri and Das (2007), Nicolaisen et al. (2001), Rupérez Micola and Bunn (2008), Rupérez Micola et al. (2008), Sun and Tesfatsion (2007) and Veit et al. (2006). It is based on the law of effect, whereby actions that result in more positive consequences are more likely to be repeated in the future, and on the law of practice, whereby learning curves tend to be steep initially and then flatter. These are robust properties observed in the literature on human learning. One of RL’s main strengths is that one does not need to make assumptions on the information that players with binding capacity constraints.
have about each other's strategies, history of play and the payoff structure. This is consistent with the fact that, in many cases, electricity traders cannot observe one another's current strategies, and only imperfectly infer them from volatile prices. However, RL might be too simplistic to fully capture the strategic opportunities available to humans (Erev et al., 2007; Ert and Erev, 2007).

It is likely that players in real electricity markets attempt to engage in more sophisticated behaviour like best response to the their competitors' actions. There are two main types of best response algorithms: fictitious play (FP) and "Cournot" best response (BR).\footnote{The term "Cournot" in this context does not refer to quantity bidding but to the classic tâtonnement best-response process leading to market equilibrium. To avoid confusion, given that we do not use quantity bidding in our simulations, we refer to this behavioural algorithm as "best response" (BR).} In FP (Brown, 1951), each player assumes that her opponents play stationary, possibly mixed, strategies. In each round, the player best responds to his opponent’s empirical frequency of play. Electricity studies using FP include those by Bunn and Oliveira (2001, 2003) and García et al. (2005). BR implies that the player only responds to her opponents move in the directly precedent period. BR papers include those by Bunn and Oliveira (2007, 2008) and Day and Bunn (2001) and Bunn and Day (2009).\footnote{In addition, there are a number of papers who depart in varying degrees from those models. Bower and Bunn (2000, 2001), Bower et al. (2001) and Bunn and Martoccia (2005) all use an adjustment based on locally improving/not deteriorating the profit and of a desired level of market share, and Cincotti et al (2005) uses a different version of reinforcement learning. Bagnall and Smith (2005) use hierarchical classifier systems and Richter and Sheblé (1998) use genetic algorithms.}

To our knowledge, there is no research on whether the results obtained with RL, FP and BR differ substantially in the electricity context.\footnote{Pouget (2007)'s analysis is similar to ours, but focused on financial markets. A different aspect is the parameter choice within a learning paradigm. Banal-Estanol and Rupérez Micola (2009) test the robustness of their results to nine reinforcement learning parameter combinations and, more systematically, Li et al. (2009) use intensive parameter sweeps to determine suitable settings for two potentially critical learning parameters.}

Finally, many papers do not provide details about initial conditions. In those that do, the standard approach is to use a uniform initial probability distribution for all elements of the action space. Examples include Rupérez Micola and Bunn (2008), Rupérez Micola et al. (2008) and Banal-Estanol and Rupérez Micola (2009). We are not aware of any papers explicitly exploring the impact of alternative starting conditions.

3 Modelling specifications

Our model incorporates key features of electricity markets in the short-run. Although it could be easily extended to become more complex, it is stylised to facilitate the exposition, as well as the comparison between theoretical predictions and simulation results. We first present the market structure and trading rules that form our framework. Then, we describe the alternative parameter implementations. We have chosen parameter combinations so that they yield many of the literature’s models as particular cases.

3.1 Market structure and trading rules

Let there be $n$ symmetric generators, $i = 1, ..., n$. Denoting the constant market capacity as $K$, firms’ capacities are $k_n = K/n$. For a given $k$, $n$ parametrises the degree of competition, as
individual capacities decrease with it. All generators have constant marginal production costs, $c$, up to capacity.

Prices are bounded between marginal costs and $\Psi$, with $\Psi$ being the maximum “reasonable” price cap (e.g. Lin et al., 2009). This can be understood as a limit triggering regulatory intervention or the cost of alternative, expensive, load fuels to which the system administrator could switch at short notice. It also reflects high cost back-up power generation facilities owned by many industrial users. There are no grid constraints. Although relevant in the long term, we do not deal with capacity expansion, long-term contracts, ancillary and capacity payments.

Trading takes place through a compulsory, uniform-price auction. Suppliers simultaneously submit individual schedules. An independent auctioneer adds them horizontally and creates an ad hoc market supply function, $S_m(q)$. Then, she intersects $S_m(q)$ with the demand, $Q(p)$, and determines the uniform price $\hat{p}$. Finally, she assigns individual quantities, $q_i$, to the bidders. Profits for each firm are

$$\pi_i = (\hat{p} - c) \cdot q_i \quad \text{for } i = 1, ..., n.$$ (1)

### 3.2 Demand representations

Market demand can be price-sensitive but it is assumed to always lead to system overcapacity, i.e. $Q(p) < K$ for all $p$. In the inelastic case, $Q(p)$ is equal to a constant quantity $\bar{Q}$ for any price between zero and $\Psi$, i.e. a vertical line at $\bar{Q}$, $Q(p) \equiv \bar{Q}$. We rotate this curve to obtain linear functions with different elasticities at the same point. We denote the vertical rotation axis as $v$ ($0 \leq v \leq \Psi$) and the deviation to the left of $\bar{Q}$ at the price cap level as $u$ ($0 \leq u \leq \bar{Q}$). Thus, all demand curves are linear, pass through $(\bar{Q} - u, \Psi)$ and $(\bar{Q}, v)$ and can be written as

$$Q(p) \equiv \bar{Q} - \frac{u}{(\Psi - v)}(p - v).$$

We use alternative values of $\bar{Q}$ and $u$ to accommodate different demand levels and elasticities. Figure 1 shows an example with $\bar{Q} = 80$, $\Psi = 200$, $v = 100$ and $u = 0, 5, 10$. The three demands are $Q(p) = 80$ ($u = 0$, in purple), $Q(p) = 80 - \frac{5}{100}(p - 100)$ ($u = 5$, in magenta), and $Q(p) = 80 - \frac{10}{100}(p - 100)$ ($u = 10$, in red).

<<Figure 1: Examples of stepwise bidding and supply functions>>

### 3.3 Supply representations

Supply schedules vary along two dimensions. First, firms submit either flat bids (“stepwise bidding”) or increasing supply schedules (“SF bidding”). Second, in line with other papers (e.g. Day and Bunn, 2001, and Hobbs and Pang, 2007), we consider the implications of allowing multi-step schedules. To that purpose, we divide each firm’s capacity into $m$ equally-sized capacity bins, $k_n/m$. Firms submit flat bids or supply functions for each segment.\(^6\)

\(^6\)The models by Day and Bunn (2001 and 2009) are more elaborated because agents submit fixed slope and changing intercept, fixed intercept and changing slope, and changing both slope and intercept functions in a non-continuous supply function setting.
**Stepwise bidding**  The feasible price offer domain is approximated by a discrete grid. Generators choose among $S$ possible bids, equally spaced between $c$ and $\Psi$, at which they are willing to supply each bin’s capacity. That is, the set of possible bids is

$$S_1(q) \equiv \{c + s (\Psi - c) / S \mid s = 1, \ldots, S\}.$$  

Each corresponds to an “action $s$”. Bids generated from lower actions are closer to $c$, i.e. more competitive. The individual bin schedules are flat.

**Supply function bidding**  Supply schedules are non-decreasing. For each bin, the generators choose among $S$ possible angles, $s = 1, \ldots, S$, equally spaced between zero and ninety degrees (or $\pi/2$ radians). The schedules consist of the set of linear curves from the coordinates $(0,c)$ until $(k_n/m, b(s))$, capped at $\Psi$. Formally,

$$S_2(q) \equiv \left\{ \min(c + \frac{(b(s) - c)}{k_n/m} q, \Psi) \mid s = 1, \ldots, S \right\},$$

where

$$b(s) = c + \frac{\sin(s (\pi/2) / S)}{\cos(s (\pi/2) / S)} \frac{k_n}{m}.$$  

The angle of the plant’s supply schedule, $s$, is the “action” and therefore the choice variable. Schedules generated from lower actions are more competitive because they are flatter. The supply function for the lowest action ($s = 1$) is flat at $c$. The supply function when $s = S$ is the result of capping a vertical linear function at the origin. Schedules generated from high actions become flat at $\Psi$. Note that the amounts sold by each firm are always strictly positive.

**Market clearing**  Appendix A includes a formal derivation of the market clearing process for each treatment. The auctioneer sets $\hat{p}$ by intersecting the demand and supply functions. Under stepwise bidding, she gives full capacity to bids below $\hat{p}$; the remaining capacity to those equal to $\hat{p}$ (in case of a tie, the selling bin is selected randomly); and zero sales to the bids above $\hat{p}$. Under SF bidding, she assigns full capacity to the parts of each schedule below $\hat{p}$. Parts above $\hat{p}$ receive nothing. The two panels in Figure 1 show hypothetical bidding examples with $n = 2$, $m = 1$, $K = 100$, $\Psi = 200$, and $c = 0$. The market supply function (black line) is the horizontal addition of the individual functions (blue and green).

**3.4 Agent behaviour representation**

We adopt the Experience-Weighted Attraction (EWA) adaptive learning mechanism put forward by Camerer and Ho (1999). This is a general learning model that nests RL, FP and BR as special cases. EWA assumes that each action has a numerical attraction to determine its selection probability. In each round, generators submit supply schedules according to bin-specific probability distributions. Once the auctioneer determines the price and individual quantities, the attractions

\[^7\text{In the simulations we add a marginal amount to the angle of the denominator to avoid the indeterminacy when } s = S.\]
are adjusted with the behavioural rule and mapped into probabilities. This process is repeated until the simulation converges.

We now describe how agents use experience to update the attractions, and how these lead to choice probabilities. Then, we specify the initial attractions and the convergence definition.

**Updating rules and choice probabilities**  Each action $s$ for bin $j$ in generator $i$ has an “attraction” $A^j_{i,s}(t) > 0$ after period $t (\geq 1)$. Attractions are updated with

$$A^j_{i,s}(t) = \frac{\phi N(t - 1)A^j_{i,s}(t - 1) + \left[\delta + (1 - \delta)I(s, r^j_i)\right] \pi_i(s, r^{-j}_i)}{N(t)},$$

(2)

where $r^j_i$ and $r^{-j}_i$ denote the action taken in period $t$ by bin $j$ and the others. $I(x, y)$ is an indicator function with value 1 if $x = y$ and 0 if $x \neq y$. $N(t) = \rho N(t - 1) + 1$. $N(0) = 0$, represents the number of “observation-equivalents” of past experience. The EWA parameters $\delta$, $\phi$, and $\rho$ denote the weight placed on foregone payoffs, a discount factor to depreciate previous attractions, and a discount factor that weights the impact of previous against future experience.

When $\delta = 0$ and $\rho = 0$, EWA behaves like in a widely used class of RL models (e.g. Roth-Erev, 1995). RL models are generally based on the law of effect, whereby actions that result in more positive consequences are more likely to be repeated in the future, and on the law of practice, whereby learning curves tend to be steep initially and then flatter. When $\delta = 1$, and $\rho = \phi$, EWA is equivalent to the standard weighted belief-based models. In particular, it produces BR when $\rho = 0$ and FP when $\rho = 1$. In BR dynamics, players actions are determined by the best response to what her opponents did in the immediately preceding period, so that only the most recent observation counts. In FP, each player best responds to the empirical frequency of play of his opponent since the beginning of the game, and all observations count equally.

The algorithm linearly maps $A^j_{i,s}(t)$ into action choice probabilities. The probability of an action in the next period is its attraction divided by the sum of attractions for all actions,

$$P^j_{i,s}(t + 1) = \frac{A^j_{i,s}(t)}{\sum_{k=1}^{S} A^j_{i,k}(t)},$$

(3)

**Prior beliefs and initial conditions**  The probabilities in the first period, $P^j_{i,s}(1)$, are generated from prior values of the attractions, $A^j_{i,s}(0)$. These values can be thought of as reflecting pre-game experience. We consider four representative prior belief assumptions: the opponents (i) use a uniform distribution over their actions, (ii) choose the lowest possible action ($s = 1$) or (iii) choose the highest possible bid ($s = S$), (iv) choose the mid-point of the range, $S/2$. $A^j_{i,s}(0)$ is the hypothetical profit that each action $s$ would render if the prior beliefs were correct.

<<<Figure 2: Initial probability distributions>>>

Figure 2 reports the impact of prior beliefs on the $P^j_i$ initial probabilities, with one and twelve agents (stepwise bidding, $m = 1$, $j = 1$ $K = 100$, $\Psi = 200$, and $c = 0$). If an agent $i$ believes that her opponents will bid $\Psi$, she gives the same probability to any bid below that value and sell its full capacity at $\Psi$. Hence the initial propensities of actions below $S$ are equal to the maximum
profits, i.e. $A^{1}_{i,s}(0) = \Psi k_n$, and have the same initial probability, $P^{1}_{i,s}(1) \approx \frac{1}{S-1}$ for all $s < S$.

Similarly, when agents assume the others will bid in the middle of the distribution, they randomise in their lower half. In contrast, if an agent believes her opponents will bid the minimum price ($s = 1$), she should also choose it because any other bid would be out-of-the-money and earn her zero profits. Therefore, the initial propensities are concentrated on $s = 1$ with $A^{1}_{i,1}(0) = \Psi k_n/n$ and $P^{1}_{i,1}(1) = 1$. Finally, when players assume that the others follow a uniform distribution, they bid more competitively than the diagonal to sell with high probability their full capacity at the bid set by an opponent.

To our knowledge, the literature only includes uniform initial probability distributions for all elements of the action space, which implies that agents believe that their opponents will bid $\Psi$. In this paper, we use that assumption as a reference and study whether the results are sensitive to the alternatives.

**Convergence** Following the theoretical literature on learning (see Fudenberg and Levine, 1998, for an overview), we define convergence in terms of strategy profiles. A simulation run has converged if the maximum attainable per-period change in the probability of playing any strategy is below a (small) threshold.

**Definition 1** For a given $\tau$ (small), a simulation run has converged to a mixed strategy profile $z$ in period $t$ if for any potential action profile $a$ in time $t+1$, the probability distribution adjustment of any action $s$ of any bin $j$ of any generator $i$ is such that

$$|P^j_{i,s}(t+1) - P^j_{i,s}(t)| < \tau. \quad (4)$$

The simulation price is computed from the agents’ mixed strategy profile $z$.

In practice, we select the action with the lowest probability in period $t$. Then, we compute the hypothetical probability that would result from assigning maximum profits to it and minimum profits to all other actions. The simulation has not converged as long as the difference between present and future probabilities is higher than $\tau$. It has converged when it is lower. The smaller $\tau$, the more stringent the threshold and the higher the necessary $t$.

Once there is convergence, one can calculate expected end-of-simulation prices from the individual probability distributions. Note that convergence is compatible with the survival of several feasible trading actions, as in mixed strategies. Price volatility may not be equal to zero even if there is a steady state. Moreover, EWA bidding depends on the stochastic process and, as a result, simulation runs for the same parameters might lead to different end prices, i.e. the standard deviation of mean prices across simulations is not necessarily zero.

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8To be precise,

$$P^{1}_{i,s}(1) = \frac{\Psi k_n}{\Psi k_n/n + (S-1)\Psi k_n} = \frac{n}{1 + n(S-1)} \text{ for } s < S$$

and

$$P^{1}_{i,\Psi}(1) = \frac{\Psi k_n/n}{\Psi k_n/n + (S-1)\Psi k_n} = \frac{1}{1 + n(S-1)}.$$  

If the number of firms is high, then $P^{1}_{i,s}(1) \approx \frac{1}{S-1}$ and $P^{1}_{i,\Psi}(1) \approx 0$. 

10
4 Theoretical predictions

Our setting often presents a manifold of non-Pareto ranked Nash equilibria that depends on the number of “pivotal” plants (see e.g. Genc and Reynolds, 2005; Entriken and Wan, 2005; Perekhodtsev et al., 2002, and Banal-Estanol and Ruperez Micola, 2009). A firm is pivotal if it is necessary to satisfy the quantity in demand. In the inelastic case, the definition is straightforward as the demand is constant.

**Definition 2** A firm $i$ is **pivotal** for a given level of demand $Q'$ if this level exceeds the sum of production capacities of all other firms, $\Sigma_{j \neq i} k_n < Q'$.

Pivotal dynamics are simple in symmetric settings. In markets with few firms, they are all pivotal. In those featuring many firms, none is pivotal. In elastic cases, quantities depend on the supply bids and one has to define pivotality for any exogenous demand level. We next define the level of competition at which the number of pivotal firms changes for their highest and lowest demand levels.

**Definition 3** A level of competition $\hat{n}$ is a lower switching point if (i) all firms are pivotal at the minimum demand ($Q' = Q(\Psi)$) for $n < \hat{n}$ and (ii) none of them is for $n \geq \hat{n}$. A level of competition $\hat{n}$ is an upper switching point if (i) all firms are pivotal at the maximum demand ($Q' = Q(c)$) for $n < \hat{n}$ and (ii) none of them is for $n \geq \hat{n}$.

If demand is inelastic, the upper and lower switching points coincide and, for simplicity, we call them “the switching point”, $\hat{n}$. For example, if $Q(p) = 80$ for any $p$ and $K = 100$, the switching point is $\hat{n} = 5$ because if $n < 5$, $(n - 1)k_n < 80$, and all firms are necessary to fulfill demand but no firm is pivotal when $n \geq 5$, $(n - 1)k_n \geq 80$.

In the elastic case, the upper and lower switching points are different. For instance, the minimum and maximum demands are $Q(200) = 70$ and $Q(0) = 90$ if $\Psi = 200$, $v = 100$, $\overline{Q} = 80$, $u = 10$, $c = 0$ and $K = 100$. There is a lower switching point at $\hat{n} = 4$ since $(n - 1)k_n < 70 = Q(200)$ for $n < 4$ and $(n - 1)k_n \geq 70 = Q(200)$ for $n \geq 4$. The upper switching point is $\hat{n} = 10$ because $(n - 1)k_n < 90 = Q(0)$ for $n < 10$ and $(n - 1)k_n \geq 90 = Q(0)$ for $n \geq 10$. Now, the equilibria.

**Proposition 4** (a) For each $K$, $Q(p)$, there exists a unique upper switching point, $\hat{n} = K / (K - Q(c))$.

(b) Moreover, (i) if the number of firms is lower than this threshold ($n < \hat{n}$), then

$$b_i = p_m^* \text{ for any } i \text{ is part of any equilibrium, with the equilibrium price } p^* = p_m^*,$$

where $p_m^*$ is the monopoly price on the residual demand curve, i.e.

$$p_m^* = \arg \max_{0 \leq p \leq \Psi} \{ (p - c) [Q(p) - (n - 1)k_n] \};$$

(ii) if the number of firms is higher than this threshold ($n \geq \hat{n}$), then

$$b_i = c \text{ for any } i \text{ is an equilibrium with the equilibrium price } p^* = c.$$
If \( n = 1 < \hat{n}^u \) the unique equilibrium yields the monopoly price. If \( n > \hat{n}^u \), it yields competitive prices. If \( 1 < n < \hat{n}^u \), there are multiple pure strategy equilibria with many payoff-equivalent actions as part of each of them.\(^9\) Consider, for example, the case of two firms, one bin and stepwise bidding. A generator bidding \( \Psi \) and the other bidding close to \( c \), or vice versa, are both equilibria in which one obtains a low profit and the other gets the maximum. The situation is similar to the standard "battle of the sexes" game. Experimental evidence shows that coordination in this game can be low (Cooper et al., 1990), and especially difficult due to its payoff asymmetry (Crawford et al., 2008).

The next corollary shows that a higher \( n \) not only decreases \( p_m^* \) but also increases equilibrium profit asymmetries. This will further reduce the agents’ incentive to set prices, and induce them to submit lower bids.

**Corollary 5** As the number of firms \( (n) \) increases,

(a) the equilibrium price \( (p^*) \) decreases

(b) the relative profits of a price-setting firm with respect to a non-price setting firm \( ((p_m^* - c) [Q(p_m^*) - (n - 1)k_n] / [(p_m^* - c)k_n]) \) decrease.

The proposition and corollary allow us to sketch predictions about the effect of competition on prices. Monopoly prices should be equal to \( \Psi \) as this leads to maximum profits. Further, prices should decrease in \( n \) as long as the firms are pivotal at the minimum demand, \( n < \hat{n}^u \), due to growing profit asymmetries. Moreover, prices should drastically decrease at the upper switching point \( (\hat{n}^u) \) because the unique prediction is that prices are equal to marginal costs for high \( n \) levels.

**Summary 6** The dynamics pre- and post-switching point result in nonlinearities in the influence of \( n \) on prices. Pre \( \hat{n}^u \), prices decrease with \( n \) from the monopoly price, \( p_m^* \). Post \( \hat{n}^u \), prices are drastically reduced to \( c \).

Note that the prediction is not specific to any of the supply, demand, and behavioural assumptions that we explore in this paper. Further, there is wide consensus on the importance of pivotal dynamics in real electricity markets (for a discussion on the role of pivotal dynamics, see e.g. Rothkopf, 2002). In other words, the prediction is so standard that any combination of agent-based modelling assumptions aiming to reproduce spot electricity markets should probably be able to fulfil it.

## 5 Simulations

### 5.1 Parameters

We allow the number of firms to vary from one to twelve, \( n \in [1, 12] \). Total capacity is \( K = 100 \), so individual capacities decrease from \( k_1 = 100 \) to \( k_{12} = 8.33 \). The price ceiling is \( \Psi = 200 \), with a grid of \( S = 50 \) possible actions. Marginal costs are set to zero, \( c = 0 \).

To clarify the exposition, we focus in turn on the demand, supply and behaviour assumptions. We use \( Q = 80 \), \( u = 0 \), stepwise bidding, \( m = 1 \), RL and a uniform initial distribution as the reference. We perform 50 simulations for each specification.

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\(^9\)There are also many mixed strategy Nash equilibria.
For the demand assumptions, we fix the supply representation (stepwise bidding and \( m = 1 \)) and the behavioural algorithm (RL and uniform initial distribution). We include expected demand levels \( Q = \{80, 85, 90\} \) and elasticity parameters \( u = \{0, 5, 10\} \). The data set includes \( 3 \cdot 3 \cdot 12 \cdot 50 = 5,400 \) observations.

When we focus on the supply side, we fix the demand specification (\( \bar{Q} = 80 \) and \( u = 0 \)) and the behavioural algorithm (RL and uniform initial distribution). We perform simulations for stepwise and supply function bidding, and for one, two and three bins. The resulting data includes \( 2 \cdot 3 \cdot 12 \cdot 50 = 3,600 \) observations.

For the behavioural analysis, we fix the supply and demand specifications (stepwise, \( m = 1 \), \( \bar{Q} = 80 \) and \( u = 0 \)). We perform simulations for BR, RL and FP, in which agents believe their opponents will bid \( 0, \Psi, \frac{\Psi}{2} \) and randomly. There are \( 3 \cdot 4 \cdot 12 \cdot 50 = 7,200 \) observations in the data set.

### 5.2 Simulation results

In Figures 3, 4 and 5 we compare the results for the different demand, supply and behavioural specifications. Each panel plots the long-run average price (\( \bar{p} \)) and two standard deviations across the 50 simulation runs for each \( n \). Since the theoretical price predictions are corner solutions, the intervals sometimes exceed the simulation boundaries. The upper left panel in each figure corresponds to the reference specification (\( \bar{Q} = 80, u = 0 \), stepwise bidding, \( m = 1 \), RL and a uniform initial distribution).

**Demand specifications**  Figure 3 compares the simulations for demand levels \( \bar{Q} = \{80, 85, 90\} \) in columns, and elasticities \( u = \{0, 5, 10\} \) in rows. The supply and behavioural assumptions are one-bin stepwise bidding, and RL with uniform initial distribution.

<<Figure 3: The influence of demand specifications on prices>>

As predicted, monopoly prices are close to \( \Psi \) in all instances and the relationship between \( n \) and \( \bar{p} \) is decreasing. In the inelastic cases, the switching points are \( \bar{n} = \{5, 6, 10\} \) when \( \bar{Q} = \{80, 85, 90\} \), respectively. \( \bar{p} \) drops rapidly in \( n \) but flattens out when \( n > 5 \), \( n > 6 \) and \( n > 10 \). More competition has a small effect beyond those values but \( \bar{p} \) remains clearly above \( c \) even under tight capacity (\( \bar{Q} = 90 \), third column), where \( \bar{p} \) tends to be less sensitive to \( n \).

In the elastic cases, we predict a break at the upper switching points, i.e. \( \bar{n}^u = \{7, 10, 12\} \) for \( u = 5 \) and \( \bar{n}^u = \{10, 12, 12\} \) for \( u = 10 \) when \( \bar{Q} = \{80, 85, 90\} \), but the results show a lower switching point as the elasticity increases. As we will see below, this is consistent with a break at the lower switching point. Simulations are similar across rows, which indicates that the impact of \( u \) on \( \bar{p} \) is, at best, modest.

Proposition 4 would predict that equilibrium prices are increasing in \( \bar{Q} \) and we find support for that claim in the simulations. First, a higher \( \bar{Q} \) decreases the upper switching point and region (i) expands and (ii) contracts. Second \( \bar{Q} \) increases \( p_m^* \). This would also be consistent with the empirical evidence (REFERENCIA).

Overall, the demand results are quite, but not perfectly, consistent with theory. First, although monopoly prices are close to \( \Psi \) and the relationship between \( n \) and \( \bar{p} \) is decreasing, post-threshold
prices are far from $c$. Second, inelastic simulations fit the break predictions better than those with elasticity. Third, smaller excess capacity ($K - Q$) results in higher prices. Fourth, results do not vary too much within our elasticity ranges, which are comparable to those in the literature. This is due to some RL mechanics that we discuss in detail below.

**Supply specifications** Figure 4 reports the bidding assumption results. The $m = 1, 2, 3$ simulations are in columns, and stepwise and SF bidding in rows. We use RL with uniform initial distribution, $Q = 80$ and $u = 0$ in all cases.

<<Figure 4: The influence of supply parameters on prices>>

As predicted, the relationship between $n$ and $\bar{p}$ monotonously decreases both for stepwise and SF assumptions, and its shape changes around $n = 5$. In the stepwise case, prices remain above the competitive levels after the threshold. $\bar{p}$'s sensitivity to $n$ decreases with $m$ so that deviations from theory grow with it. For example, when $m = 1$, post-threshold prices are around 40, when $m = 2$ around 75 and when $m = 3$ around 100. This is consistent with more difficult agent coordination, which complicates convergence to a theoretically sound steady state.

Although the price variability increases, SF yields a better fit. This is probably because its higher “expressiveness” overcomes the difficulties to coordinate in the theoretical prediction. The standard deviation grows in $m$, especially around $n$. Under SF, bids above the equilibrium price may be reinforced because they also obtain substantial profits. It is therefore more difficult for agents to tell good from bad bids and dispersion grows. In comparison, stepwise bids are either on- or out-of-the-money. Hence it is easier to identify the good bids and the simulations become crisper.

Overall, prices are less sensitive to $n$ under stepwise bidding and SF is better at capturing the extreme monopoly and competitive predictions. Still, SF’s dispersions increase substantially around the pivotal breaks.

**Behavioural specifications** Figure 5 reports $\bar{p}$ for the different behavioural specifications with $m = 1$, stepwise bidding, $Q = 80$ and $u = 0$). RL is on top, BR in the middle panel and FP in the bottom. The columns include results for the four initial assumptions in the following order: agents believe their opponents will bid the maximum price, randomly, the minimum and the medium point.

<<Figure 5: The influence of behavioral assumptions on prices>>

The panels consistently show that the relationships between $n$ and $\bar{p}$ is decreasing. All confidence intervals are narrow. In reinforcement learning, monopoly prices are close to $\Psi$, decrease until the theoretical switching point and approach $c$ after it, but the break is visually not too striking. The best response monopoly prices are far from $\Psi$ at about 140, but there is a clear breaking point for $n = 5$, after which they exactly converge to $c$. The fictitious play specification departs substantially from the predictions: Monoply prices are around 140, decrease slowly in $n$ and stay patently above competitive levels. Overall, RL matches the theory best but is not as reactive to market conditions as BR. FP performs worst. These results are to our knowledge the first on the robustness of ACE techniques to behavioural choices in electricity markets.
We now trace back the results to the algorithms’ features. Under BR, prices are competitive when no agent is pivotal. Agents choose actions below those of their opponents in \(t - 1\), so that \(p\) can only stay constant or decrease. The resulting unravelling yields \(p = c\). When all agents are pivotal, equilibrium forces tend to increase \(\bar{p}\) through coordination so that agents choose high actions with some probability. Simultaneously, unravelling prevents \(\bar{p}\) from staying very high. On balance, \(\bar{p}\) never reach \(\Psi\), not even in monopoly.

There is no in-built unravelling under RL. Strategies that represent "higher than best prices" are reinforced because they allow the agent to sell some capacity at a high price. Thus, \(\bar{p}\) is higher than under BR for all \(n\).

FP weights heavily initial periods where outcomes are quasi-random. Hence, agents assign propensities to inadequate actions, adaptation slows down and there is strong path dependence. There are two countervailing forces at work. On the one hand, the higher \(n\) the more likely it is that there will be unravelling as non-pivotal firms best-respond to lower prices. On the other hand, initial random prices are more likely to be high, so that reductions start from a higher base. \(\bar{p}_{n=1}^{FP} \neq 200\) because of unravelling. \(\bar{p}_{n=12}^{FP}\) is far from zero owing to high initial prices. On balance, the \(n\) to \(p\) relationship decreases slowly and stays far from the theory extremes.

Further, comparing across columns, the simulations outcomes depend on pre-game beliefs. Initial conditions do not have an impact under BR due to the algorithm’s lack of memory. Experimentation under RL eliminates most initial effects. However, FP keeps that noise. For example, when all agents belief the others will bid the maximum price, the best response is to bid randomly, so that the initial beliefs will not be fulfilled.

Competitive pre-game beliefs render by far the most homogeneous prices across behavioral assumptions. Why? Because they are self-fulfilling. Everyone’s best response is to bid \(c\) when they each belief that the others will bid \(c\). BR and FP lock themselves up in that value. Experimentation in RL is not powerful enough to depart from it.

One final point. The theoretical soundness of RL with competitive initial beliefs is strikingly good and the best of the twelve. Monopoly prices are only slightly below \(\Psi\), and decrease clearly in \(n\) for \(n < 5\), so that \(\bar{p}_{n=1}^{RL} > 100\). For \(n > 5\), \(\bar{p}_{n=1}^{RL} = c\).

### 5.3 Non-linear relationships and pivotal switching points

In this section, we formalise the previous visual inspection with threshold regressions. Specifically, we carry out tests of whether the data features switching points in the predicted locations. To that purpose, we estimate a piecewise linear model between \(n\) and \(\bar{p}\) for each demand, supply and behavioural combination. The models are uniquely specified by a dummy variable associated with the threshold value \(n_v\),

\[
P_i = \beta_0 + \beta_1 D_i + \beta_2 \alpha_i + \beta_3 D_i n_i + u_i, \text{ where } D_i = 0 \text{ if } n_i < n_v, D_i = 1 \text{ when } n_i \geq n_v. \tag{5}
\]

The pre- and post- breaking points regression estimates are specified by

\[
E(P_i | D_i = 0, n_i) = \beta_0 + \beta_2 n_i \text{ and } E(P_i | D_i = 1, n_i) = (\beta_0 + \beta_1) + (\beta_2 + \beta_3) n_i.
\]

We test the null hypothesis of linearity against the alternative of structural breaks at the pivotal switching points \(n_v\). Evidence supporting the existence of a breaking point can come either from
significant intercept or slope change coefficients, i.e. $\beta_1$ and $\beta_3$ different from zero.

Table 2 reports the results. On the left-hand side, we provide a summary of the parameters under each specification. Those characterising the simulation batches are in boxes.

<<Table 2: Hypotheses’ tests of breaking point regression estimates>>

The right-hand side reports regression estimates. The coefficients correspond to eq. 5. In all cases, at least either $\beta_1$ or $\beta_3$ is significant at standard levels. There are three non-significant coefficients (one $\beta_1$, two $\beta_3$) but the other coefficient is always significative. The tests provide preliminary support for the hypotheses. One could therefore conclude that all simulations are consistent with the pivotal breaking point’s theory.

However, simple hypothesis tests might not be enough if there are significant breaking points in several locations. This could explain why the tests identify breaks also in cases in which they are not visually identifiable (e.g. under fictitious play). One way to discriminate between breaking points is to single out those producing best-fitting models.

### 5.4 Identification of best-fitting breaking points

We create a procedure to select the “optimal” data breaking point and establish whether it is statistically equivalent to the theoretical one. We start by providing an optimal breaking point definition.

**Definition 7** The optimal breaking point $\check{n}$ satisfies $F(\check{n}) \geq F(n)$ for any threshold $n$, where $F(n)$ denotes the F-statistic obtained from a piecewise linear regression with threshold $n$.

We approximate the distribution of the structural breaks through a subsampling procedure. For each assumption, we extract 99 random subsamples of 240 observations stratified for $n$ (i.e. twenty observations for each $n$). Then, we use Definition 7 to obtain the subsamples’ optimal breaking points.

Figure 6 reports the comparison between theoretical ($\check{n}$) and the simulated breaking points. The upper panel (6a) provides results with RL, $m = 1$ and stepwise bidding. For $u = 0$, they statistically match the theory ($p < .05$): $\check{n}_{Q=80} = 5$, $\check{n}_{Q=85} = 6$ and $\check{n}_{Q=90} = 10$. Under $u > 0$, we use the upper ($\Psi$ - pink) and lower ($c$ - blue) definitions to calculate $\check{n}$. Recall that $Q(\Psi) < Q(c)$, $\check{n}_\Psi > \check{n}_c$. The simulations match $\check{n}_\Psi$, but $\check{n}_c = 12$ is far from the prediction. This is consistent with high RL prices under standard initial conditions.

<<Figure 6: Comparison of simulated and theoretical switching points>>

The middle panel (6b) reports breaking points under stepwise bidding and SF, and $m = \{1, 3, 5\}$. RL, $Q = 80$ and $u = 0$ are used in all cases. The prediction is unique ($\check{n} = 5$) and all combinations match it well ($p < .05$).

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16 One of the main complications in this type of models arises in the case when there are an unknown number of change points. However, the problem is easier when (like in our case) there is prior theory suggesting the number of breaking points.
The lower panel (6c) includes the results under BR, RL and FP with stepwise $m = 1$ bidding, $Q = 80$ and $u = 0$. BR and RL match the theory well, but not FP with a much higher break location.

To summarise, the pivotal breaking point prediction is one of the simplest and most widely accepted in electricity markets. RL and BR specifications identify it well, but not FP. Further, its optimal break estimates do not coincide with the simple hypotheses tests and subsampling appears as a good way to discriminate between several possible breaking points.

5.5 Latent intensity of competition

Agents prioritise some actions over others. In a simulation environment, we can inspect the probability distributions from which they choose their bids. Thus, it is possible to study how the parameters influence the agents’ “competitive attitude” and not only market outcomes.

Figure 7 depicts some end-of-simulation individual latent probability distributions. It reports probabilities under $n = \{1, 12\}$ and the three behavioural assumptions, averaged across simulation runs for $u = 0$ and step bidding. Actions are identified on the horizontal axis, with numbers ranging from 1 for the more competitive to 50 for the highest possible bid. Cumulative probabilities appear on the vertical axis. The convergence definition implies that the probabilities’ concentration is largely invariant beyond the convergence point, so that these distributions are a good approximation to the agents’ long-term strategies. Agents will bid mixed strategies unless the probabilities of playing all but one action are zero. Probabilities concentrated on higher actions result in less competition and vice-versa.

The Figure offers a number of general insights linking market structure, trading behaviour and prices. Distributions for $n = 1$ stochastically dominate those for $n = 12$, so that monopolist bids are less competitive than those of competitive agents. For a given $n$, there are substantial differences across behavioural algorithms. When $n = 12$, trading priors are extremely competitive under BR and RL. FP induces less competition. In contrast, when $n = 1$, BR and FP lead to the same probability distribution, which is more competitive than for RL.

Overall, the Figure provides solid support for the mechanisms underpinning the influence of each behavioural model on $\bar{p}$. It also shows that RL is more sensitive to $n$ than the others, but also that it leads to less competitive outcomes than BR. As expected, $n$ does not have a strong influence on FP distributions.

6 Discussion and conclusions

Firms and regulators alike have started to use agent-based computational economics (ACE) to study the properties of many markets. However, the literature has advanced little in creating a set of standards, and this paper is an attempt to advance in that direction. We study the properties of different ACE techniques and how they compare to each other and to theory. As a particular case, we focus on a general wholesale electricity trading model. The theoretical claim is so well-established in the literature that any model should be able to replicate it.
However, the bidding, initial conditions, and, particularly, behavioural comparisons call into question an important part of the extant ACE literature on electricity markets. Best-response and reinforcement learning perform quite well, but not fictitious play. Learning is more difficult with multi-step bids. Flat and upward slopping supply functions yield similar results, and also several plausible elasticity assumptions. Competitive pre-game beliefs render the best match to theory. Simulations perform best when they combine reinforcement learning, competitive initial beliefs and single-bin bids.

Let us be clear in one aspect. Just as much as Cournot, Bertrand, Stackelberg or Supply Function Equilibria yield different results, it might be appropriate for ACE researchers to use different algorithms. This paper underlines the need to understand, justify, and even sometimes exploit, what each modelling choice entails.

In that context, the paper has several implications. First, some preceding work makes choices that are not consistent with economic theory in our simple setting. There might be an important venue for research in checking the robustness of their findings to alternative assumptions. Second, future modelling efforts should also incorporate more systematic robustness tests. Third, we have not included all possible assumptions. For example, we have left out variable marginal costs and behavioural rules like genetic algorithms and Q-learning that are also prominent, and we have not compared the implications of different convergence definitions. Further, our strongest result relates to fictitious play. Its main difference with best-response is mainly one of memory, so that how much memory to retain, and how it should decay, are intriguing, and still unresolved, algorithmic questions. Fourth, this paper focuses on electricity markets. We should continue doing similar exercises in other settings, as in Fagiolo et al. (2007), Leombruni et al. (2007), Marks (2007) and Midgley et al. (2007). Finally, the question of which models best fit real data is complementary to our research and deserves future attention both in the behavioural laboratory (e.g. Duffy, 2006) and empirically if ACE is to be more widely used in practice.
Appendix A: Supply functions

We show here how one can construct the market supply functions. For simplicity we assume that marginal costs are zero \((c = 0)\) and firms can only use one bin \((m = 1)\). Take first the case of stepwise bidding. We proceed as follows. First, we order the “bids” from lowest to highest. Abusing of the notation, we denote them here as \(b_1, b_2, ... b_n\). The horizontal sum of the individual supply functions \(S(q)\) can be defined as

\[
S(q) = b_i \text{ for } \kappa_{i-1} < q < \kappa_i \quad (i = 1, ..., n) \text{ where } \kappa_i \equiv i k_n \text{ and } \kappa_0 \equiv 0.
\]

Market clearing occurs at where this equation and the demand intersect. The market price \(\hat{p}\) is the given by \(\hat{p} = b_j\) where \(j\) is the unique index which satisfies \(\kappa_{j-1} < Q(b_j) < \kappa_j\).

Second, take the case of supply function bidding. Again, order the “bids” \((b(s))\) from lowest to highest and abusing the notation, denote them as \(b'_1, b'_2, ... b'_n\). As shown in the text, the linear supply schedule for each firm \(i\) is given by \(\min(b'_{0i} q, \Psi)\), which is an increasing linear function from \((0, 0)\) until \((k_n, b'_i)\) capped at \(\Psi\). Denoting \(b'_0 = 0\), the horizontal sum of the individual supply functions \(S'(q)\) can be defined as

\[
S'(q) = \min\left\{ \frac{(b'_i - b'_{i-1}) q + b'_{i-1} \kappa'_i - b'_i \kappa'_{i-1}}{\kappa'_i - \kappa'_{i-1}}, \Psi \right\} \text{ for } \kappa'_{i-1} < q < \kappa'_i \quad (i = 1, ..., n),
\]

where

\[
\kappa'_i \equiv \left( i - 1 + \sum_{j=i}^{n} \frac{b'_i}{b'_j} \right) k_n \text{ and } \kappa'_0 \equiv 0.
\]

Market clearing occurs at where this equation and the demand intersect. The market price \(\hat{p}\) can be found by solving the following implicit equation

\[
\hat{p} = \frac{(b'_i - b'_{i-1}) Q(\hat{p}) + b'_{i-1} \kappa'_i - b'_i \kappa'_{i-1}}{\kappa'_i - \kappa'_{i-1}}.
\]
Appendix B: Market clearing

Here we find \( Q(p^*) \):

\[
Q(p^*) = \bar{Q} - \frac{u}{(\Psi - v)}(p^* - v) \text{ and } \\
Q(p^*) = \bar{Q} - \frac{u}{(\Psi - v)} \left( \frac{Q + \frac{u(v+c)}{(\Psi-v)} - (n-1)\frac{K}{n}}{2} \right) + \frac{u}{(\Psi - v)} v \\
Q(p^*) = \bar{Q} - \frac{Q + \frac{u(v+c)}{(\Psi-v)} - (n-1)\frac{K}{n}}{2} + \frac{u}{(\Psi - v)} v \\
Q(p^*) = \bar{Q} - \frac{Q + \frac{u(v+c)}{(\Psi-v)} + (n-1)\frac{K}{n} + \frac{2uv}{(\Psi-v)}}{2} \\
Q(p^*) = \bar{Q} + \frac{\frac{uv}{(\Psi-v)} - \frac{uc}{(\Psi-v)} + (n-1)\frac{K}{n}}{2} \\
Q(p^*) = \bar{Q} + \frac{\frac{u(v-c)}{(\Psi-v)} + (n-1)\frac{K}{n}}{2}
\]

Appendix C: Proofs

Proof of Proposition 4

For part (a) note that \( \hat{n} = K / (K - Q(0)) \) is the unique solution to \((n-1)k_n = Q(0)\).

For part (b), suppose first that firms can only use one bin and they bid using stepwise bidding functions. For part (i), suppose that all but firm \( i \) are bidding equal to the marginal costs and firm \( i \) is bidding at \( p_m^* \). Firm \( i \) has no incentives to deviate since it is setting the monopoly price given the actions of the others. The other firms do not have an incentive to deviate either (1) at a price equal to the monopoly price because they would sell less at the same price; or (2) at a price below the monopoly because they would earn exactly the same. If \( p_m^* < \Psi \), the remaining firms could set a price above the monopoly price. The optimal deviation should then be at a price marginally above \( p_m^* \), but the deviator would sell less at the corresponding price. For part (ii), it is not profitable to unilaterally deviate because the remaining firms would serve the demand.

The equilibria is the same if firms can use multiple step bids per firm. Given the uniform pricing, the price-setting firm(s) cannot do better if the other firms submit the lowest bid for all their bins. The equilibria in the case of supply function bidding are essentially the same. Firms can again submit (almost) flat bids at the marginal cost level and almost flat bids at the price cap level.
Proof of Corollary 5

For part (a), substituting,

\[ p_m^* = \arg \max_{0 \leq p \leq \Psi} (p - c) \left[ \hat{Q} - \frac{u}{(\Psi - v)} (p - v) - (n - 1) \frac{K}{n} \right]. \]

Clearly, \( p_m^* \) is never binding from below. The Kuhn-Tucker conditions,

\[ \hat{Q} - \frac{u}{(\Psi - v)} (p - v) - (n - 1) \frac{K}{n} - (p - c) \frac{u}{(\Psi - v)} - \lambda = 0 \quad \text{and} \quad \lambda(\Psi - p) = 0 \]

and the second derivative is negative (and therefore a maximum). If \( \lambda = 0 \), then

\[ p^* = \left( \frac{\Psi - v}{2u} \right) \left( \hat{Q} + \frac{u(v + c)}{(\Psi - v)} - (n - 1) \frac{K}{n} \right), \]

whereas if \( \lambda > 0 \) then \( p^* = \Psi \). Deriving, in the interior case,

\[ \frac{\partial p^*}{\partial n} = -\frac{K}{n^2} \frac{1}{2u(\Psi - v)} < 0. \]

For part (b), evaluating the demand at \( p^* \), we have

\[ Q(p^*) = \frac{1}{2} \left( \hat{Q} - \frac{u(c - v)}{(\Psi - v)} + \frac{(n - 1)K}{n} \right) = \frac{Q(c) + (n - 1)k_n}{2}, \]

and deriving with respect to \( n \)

\[ \frac{\partial Q(p_m^*)}{\partial n} = \frac{K}{2n^2} = \frac{k_n}{2n}. \]

Now, we have that the derivative of the relative quantities sold (and therefore profits) of a price-setting with respect to a non-price setting firm are

\[ \frac{\partial [Q(p_m^*)/k_n - (n - 1)]}{\partial n} = \frac{\partial Q(p_m^*)}{\partial n} - \frac{\partial k_n}{\partial n} Q(p_m^*) - 1 \]

and given that \( \frac{\partial k_n}{\partial n} = -k_n/n \), we have

\[ \frac{\partial [Q(p_m^*)/k_n - (n - 1)]}{\partial n} = \frac{Q(c) + nk_n}{2nk_n} - 1 = \frac{Q(c) - nk_n}{2nk_n} < 0 \]

since \( Q(p) \leq K = nk_n \) for all \( p \), and in particular that \( Q(c) \leq nk_n \).
References


<table>
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<tr>
<th>No.</th>
<th>Paper</th>
<th>Journal</th>
<th>Research question</th>
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**Table 1. Published papers**

The Table includes an alphabetical list of electricity agent-based modeling papers published as journal articles, with the year of publication and abbreviated journal title. In addition, the Table briefly summarizes the research issue in each paper together with their supply bidding, demand representation and behavioral algorithm assumptions. Full citations appear in the references list.
### Demand Estimates

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### Behavioural choices Estimates

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**Table 2: Hypotheses’ tests of breaking point regression estimates**

The parameters considered in each simulations batch are marked with boxes. The parameters outside those boxes stay constant. The estimates correspond to the threshold equation specified in eq. 5. Beta1 and Beta3 estimate the post-breaking point changes in intercept and slope. They support the existence of a breaking point when they are statistically significant: **significant at the 0.05 level; ***significant at the 0.01 level.
Figure 1: Examples of stepwise bidding (left panel) and supply functions (right panel)
In each panel, we include two individual bids (green and blue) and their horizontal aggregation (black) and three potential demands with no, low and high elasticity (purple, magenta, red, respectively)
Figure 2: Initial cumulative distribution of probabilities
Beliefs: opponents bid maximum, price, randomly, minimum price and center of the distribution
Competition cases: n = 1 and n = 12
**Figure 3:** The influence of demand parameters on prices
(mean +/- two standard deviations)
Demand cases $Q = \{80, 85, 90\}$ in columns.
Elasticity cases $u = \{0, 5, 10\}$ in rows.
Figure 4: The influence of supply parameters on prices  
(mean +/- two standard deviations)  
Number of bins \( m = \{1, 3, 5\} \) in columns  
Upper row: stepwise; Lower row: SF.
Figure 5: The influence of behavioral assumptions on prices  (mean +/- two standard deviations)
Rows: Behavioral algorithms (best-response, reinforcement learning and fictitious play)
Columns: Initial conditions (left to right: rivals bid maximum price, random, minimum price and center of the distribution)
Figure 6: Comparison of theoretical and simulated switching points
(Simulations' mean +/- two standard deviations)
Upper panel: demand alternatives; middle panel: supply alternatives
Lower panel: behavioral representation alternatives (BR best response, RL reinforcement learning, FP fictitious play; PB pre-game beliefs)
Figure 7: Latent intensity of competition
End of simulation cumulative distribution of probabilities
BR best response, RL reinforcement learning, FP fictitious play;
n = 1 monopoly; n = 12 competition