Modeling and Managing the Parking Sharing Problem

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Abstract

This study models the parking sharing problem in urban cities, where private parking owners can share their vacant spaces to parking users via an e-platform, and then examines the platform operator's pricing strategies for revenue-maximization or social-cost-minimization. The model takes into account the spatial dimension of parking that both public curbside spaces and private ones potentially available for sharing are distributed along the travel corridor between the city center and the residential area. On the supply side, private parking owners can rent or "sell the right-of-use" of their spaces to the platform based on the rent they can receive and the inconvenience they would suffer by sharing. On the demand side, travelers make their parking choices of space type (curbside or shared) and location (distance from the city center) under given parking capacities and prices. The resulting parking choice equilibrium is formulated as a minimization problem and the underlying properties of the equilibrium are identified and discussed. Based on the supply-demand equilibrium, the pricing strategies (rent paid to space owners and price charged on space users) of the platform operator are investigated. Analytical and numerical examples are presented to illustrate the models and results, and also to provide further insights.

Keywords: Parking sharing, parking location, parking choice equilibrium, two-sided market, e-platform operator.

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1 Introduction

To find a suitable parking space in downtown areas (or mixed areas with both residential and business functionalities) is often a headache for travelers in most large cities. Finding a parking spot generally constitutes an appreciable fraction of the individual travel time. The desperate scramble for parking can also add to the problems of chronic congestion and choking pollution (Arnott and Inci, 2006; Van Ommeren et al., 2012; Liu and Geroliminis, 2016). As pointed out in Shoup (2006), 30% of traffic congestion in road networks is caused when people are cruising for vacant parking spaces, and about 8.1 min is spent in finding a parking space (based on the data of a number of cases). Ayala et al. (2011) also found out that every year in Chicago, there is 63 million miles for vehicles to travel in order to find a vacant space to park, which generates 48,000 tons of carbon dioxide.

However, in the context of growing car ownership and shrinking usable land in mega-cities such as London, Hong Kong, Beijing, it is infeasible to solve the parking problem simply by continuing the construction of new parking facilities. Instead, many studies have proposed and evaluated parking pricing, parking reservation, parking permit systems or mechanisms to effectively manage parking supply and traffic congestion (Arnott et al., 1991, 2015; Zhang et al., 2011; Qian et al., 2012; Yang et al., 2013; Liu et al., 2014; Inci and Lindsey, 2015; Chen et al., 2015, 2016). A latest review of economic analysis and modeling of parking is given by Inci (2015). It is also noteworthy that there has been a number of recent studies that look at parking spot allocation and pricing using game theory or incorporate parking searching process as part of the traffic equilibrium problem in a network context (Boyles et al., 2015; He et al., 2015; Zheng and Geroliminis, 2016) under different scales of network modeling. Besides, there is a branch of studies looking into park-and-ride facilities in a multi-modal system (Wang et al., 2004; Liu et al., 2009).

Peer-to-peer markets, collectively known as the sharing economy, have emerged as alternative suppliers of goods and services traditionally provided by long-established industries (e.g., Airbnb, Uber, DiDi Dache). Shared parking emerges as a new notion of making more efficient use of parking facilities. It uses existing gaps intended for parking cars when the owner is not using it. Availability of parking gaps for others stems from the fact that most parking spaces are only used part time by a driver or owner who lives in one location and works in other, and the utilization and availability patterns follow predictable daily, weekly and annual cycles. With the revolution of information and communications technology and especially the latest rise of the mobile internet, private parking sharing can be enabled through an "e-parking platform" to help the supply to match the demand. In fact, some e-parking platforms already appear in the smartphone era (or shared parking apps), which are shaking up the parking industries and reshaping our daily lives (see, e.g., Air-Parking, a Wechat account based parking Apps, https://airparking.cn/indexEN.html). The emergence of e-parking apps not only help alleviate the aforementioned severe shortage of parking spaces, but also provide the owners an easy way to make money from their idle parking spaces.

Realizing the above new business opportunity and model particularly in an area with mixed commercial and residential land use developments, parking management companies intend to temporarily repurchase some private parking spots and sell them to public users during certain time of a day. Guo et al. (2016) was the first to develop a simulation-optimization based decision method to determine the repurchase strategy. Shao et al. (2016) then considered advanced booking and allocation of shared parking spots and propose a binary integer linear programming model to optimize the allocation of parking requests to specific parking spots so as to maximize the parking spot utilization or accommodate as many requests as possible under parking space and time constraints. Recently, Xu et al. (2016) addressed the private parking spot sharing problem during regular working hours in a big city by using the market design theory so that money flow is allowed in the matching mechanisms. While these studies are insightful in showing that parking sharing holds great potential to alleviate the parking limitation issues and cruising for parking, the literature fails to offer a systematical modeling framework for the parking sharing problem with a parking sharing platform.

This study aims to fill the mentioned gaps, and further models the parking sharing problem in urban cities with a hybrid supply of curbside parking and shared parking, and examines the pricing strategies of the parking sharing platform operator to maximize revenue or minimize social cost. The spatial dimension of parking has also been considered in this paper, i.e., both free curbside parking spaces and shared spaces are distributed along a corridor with increasing distance to the city center. Particularly, three parties are involved in the sharing problem, i.e., the travelers who need a parking space; the residents or other parking lot owners who can supply a private parking space; and the operator who manages the parking sharing platform. We found that, given the parking capacities and prices, the parking choice problem (parking type and location) of travelers with an e-platform can be formulated as a minimization problem. Several properties of this choice equilibrium have then been discussed. Furthermore, the pricing strategies of the e-platform operator to maximize profit or reduce social cost have been examined. In summary, this study improves and enhances our understanding of the parking sharing, pricing, and management, and also enriches the literature on the two-sided markets, where ride sharing/sourcing has attracted much more attention, and also on the sharing concept in transportation (Wang et al., 2016;

Zha et al., 2016; Chen et al., 2017; Liu et al., 2017; Zhang et al., 2017).

The rest of the paper is organized as follows. Section 2 describes the parking sharing problem and presents the basic formulations. Section 3 formulates and further discusses the properties of the parking choice equilibrium of travelers under given public curbside parking and private shared parking supplies. Section 4 further examines the pricing strategies of the parking platform operator to achieve different objectives. Section 5 numerically illustrates the models and analysis. Finally, Section 6 concludes the paper.

2 Problem Description and Basic Formulation

In this section, we start with a description of the rush-hour commuting problem with an e-platform for parking sharing; and then describe behaviors of the three parties that are involved: road users who need a parking space; private parking sharers who hold the potential to offer parking spaces; and the operator of the e-platform for parking sharing.

We consider a linear traffic corridor depicted in Figure 1, where each location is indexed by its distance x to the city center ($x = 0$), or we can call it "location x" hereafter. Every day there is a total number of N users traveling from home location (D) to the city center. All travelers drive and need a parking space around the city center, i.e., $x \in [0, d]$ along the corridor. The linear corridor framework allows the consideration of the spatial distribution of parking. Particularly, we consider that free curbside parking spaces are distributed along the corridor and for location $x \in [0, d]$ the parking density is $m_f(x)$. We assume that $m_f(x) > 0$.[1] Besides the free but limited curbside parking spaces, which often result in cruising-for-parking, we consider that there are residents (or other land operators or owners) living at $x \in [0, d]$ with a density of $M(x)$, who are the private parking owners and also the potential private parking suppliers or sharers. If appropriate rents are paid to these parking owners, they can offer their private spaces (which is vacant as e.g., they may work elsewhere and leave their parking spots empty during daytime). To summarize, there are two types of parking available: free curbside spaces and shared spaces (it is feasible while tedious to incorporate other options such as garage parking and alternative public transport modes).

[1]It is straightforward to incorporate the case with $m_f(x) = 0$. Indeed, under this situation the location x can be simply excluded when considering curbside parking. Under this setting, $m_f(x)$ can always be put in the denominator for calculating the parking occupancy rate in Eq.(2).

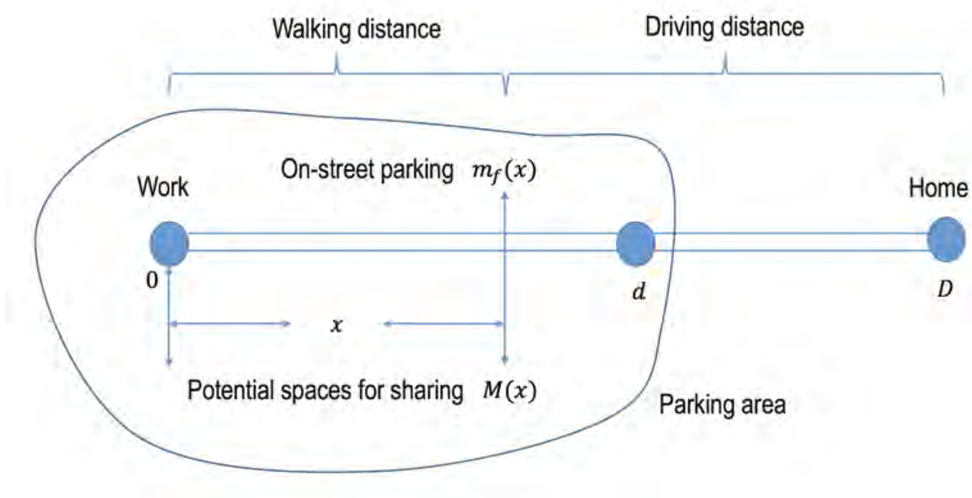

Figure 1: The linear corridor network with continuously distributed parking

2.1 Travelers

Based on the above setting, the travelers who need a parking space have two options: (1) park at the free curbside parking, but need to search for vacant spaces; and (2) park at the shared but reserved/guaranteed parking space, but need to pay a fee for the shared space. Under both options, travelers still have a location choice to make.

Under the first option, travelers can drive to location x, search for a parking space there and then walk to the center (final destination). His or her individual travel cost would be

$$c_f(x) = \alpha_d \cdot \left(\frac{D - x}{v} + t_c(x) \right) + \alpha_w \cdot \frac{x}{w}. \tag{1}$$

where $D - x$ is the driving distance (without the searching for parking process) and x is the walking distance as shown in Figure 1; $t_c(x)$ is the cruising-for-parking time to find a vacant space at location x; v is the driving speed and w is the walking speed. Moreover, α_d and α_w are the value of driving time and value of walking time respectively. It is assumed that $v > w$ and $\alpha_d < \alpha_w$, which means that travelers prefer to driving (i.e., $\alpha_d \cdot v^{-1} < \alpha_w \cdot w^{-1}$). Fees for free public curbside space are assumed to be zero, which can be easily extended to the case with a non-zero fee (however, indeed only the pricing difference between curbside parking and shared parking matters, therefore, we will just keep the pricing in the shared parking side, as will be discussed in Eq.(5)).

Now we turn to formulate the cruising time for parking. For the free curbside parking at location x, average occupancy rate for travelers is

$$q(x) = \frac{n_f(x)}{m_f(x)}, \tag{2}$$

where $n_f(x)$ is the number of drivers choosing to parking at the curbside parking spaces at location x, and $n_f(x) \leq m_f(x)$. The searching time at location x can be estimated as a function of parking occupancy rate $q(x)$, which is

$$t_c(x) = \kappa\left(q(x)\right), \tag{3}$$

where $\kappa(\cdot)$ is the searching time function, which is assumed to be strictly increasing and convex. Note that $q(x) \leq 1$, and when $q(x) \to 1$, we should have $\kappa \to +\infty$ (Anderson and De Palma, 2004; Qian and Rajagopal, 2015; Liu and Geroliminis, 2016; Arnott and Williams, 2017; Leclercq et al., 2017) .

Then the cost of choosing to park at a free curbside parking space at location x can be determined as a function of the numbers of users choosing this option.

$$c_f(x) = C_f(x, n_f(x)) = \alpha_d \cdot \left(\frac{D-x}{v} + \kappa\left(\frac{n_f(x)}{m_f(x)}\right)\right) + \alpha_w \cdot \frac{x}{w} \tag{4}$$

Under the second option (shared parking), travelers can drive to location x, park at the shared (and reserved) parking space without searching, but with a price. Travel cost for him or her then includes driving time, walking time and parking fee for the shared parking space, which is

$$c_s(x) = \alpha_d \cdot \frac{D-x}{v} + \alpha_w \cdot \frac{x}{w} + p(x). \tag{5}$$

where $p(x)$ is the fee for a shared space at location x. The users pay this price to the e-platform operator for the shared parking space (note that the platform operator has to pay a rent to the parking owner or we may call it "repurchase" as discussed in Guo et al. (2016), which will be discussed later).

Travelers will have to make choices on both parking types (free or shared) and parking locations to minimize their travel cost. Denote the numbers of travelers choosing the two options by N_f, N_s, then we have $N_f + N_s = N$. Furthermore, let the users choosing free curbside space at location x be $n_f(x)$, and those choosing shared space at location x be $n_s(x)$. We have $\int_0^d n_f(x)dx = N_f$ and $\int_0^d n_s(x)dx = N_s$. The total travel cost of all users can then be written as

$$TC = \int_0^d [c_f(x) \cdot n_f(x) + c_s(x) \cdot n_s(x)]\, dx. \tag{6}$$

It is evident that $n_f(x) \leq m_f(x)$ and $n_s(x) \leq M(x)$ should hold. However, the constraint $n_s(x) \leq M(x)$ might not be binding since the realized number of shared parking at x denoted by $m_s(x)$ can be smaller than total potential number of shared spaces $M(x)$, i.e., some of the parking owners might not share their spaces, and we will have $n_s(x) \leq m_s(x) < M(x)$. We will discuss this in the following for the parking sharers (also owners).

2.2 Parking owners (or sharers)

At location x, there are $M(x)$ residents that hold the potential to share a space, which is also shown in Figure 1. To share their spaces to others, they will encounter an inconvenience cost of δ, which can be visible or invisible to others. However, the parking owners will know it themselves. We assume that δ is distributed over $[0, \delta_x]$ with a probability density of $f_x(\delta)$ and cumulative density of $F_x(\delta)$ for the parking owners at location x. Note that δ_x can approach infinity, which means that some private parking space owners might never want to share their spaces. For analytical purpose, we consider that $F_x(\delta)$ is strictly increasing and differentiable over $\delta \in [0, \delta_x)$.

If a fee/rent (from the e-platform operator) $r(x)$ is paid to the potential parking sharers, those with a $\delta \leq r(x)$ will share or "sell the right-of-use" of their parking spaces to the platform operator. This is the so-called "repurchase" strategy from the platform operator's point of view. The total number of realized shared parking space under given rent $r(x)$ is then

$$m_s(x) = F_x(r(x)) \cdot M(x). \tag{7}$$

It is obvious that $m_s(x) \leq M(x)$. Moreover, $n_s(x) \leq m_s(x)$ should hold, i.e., the total number of drivers using the shared parking spaces at location x should be less than or equal to the total number of available ones.

The total net benefit of all parking owners (sharers) in the linear corridor can then be determined, which is equal to the difference between the total parking rent and the total inconvenience caused, i.e.,

$$R_s(\boldsymbol{r}) = \int_0^d \left[m_s(x) \cdot r(x) - \int_0^{r(x)} \delta \cdot f_x(\delta) \cdot M(x) \cdot d\delta \right] dx \tag{8}$$

where $m_s(x)$, as given in Eq.(7), will depend on $r(x)$. It is obvious that the net benefit of parking owners R_s increases with $r(x)$ or at least does not decrease.

2.3 Parking sharing platform operator

For the sharing platform operator, as mentioned earlier, it charges the travelers (parking users) a fee of $p(x)$ for a shared parking at location x, and compensates a private parking sharer at location x a fee of $r(x)$ or we call it "rent". The interactions between the operator and the users or owners are briefly depicted in Figure 2. As can been noticed, the pricing strategies of the platform operator, i.e., the combinations of $p(x)$ and $r(x)$ would significantly affect the demand and supply for the parking sharing platform, as well as its performance. This highlights the importance to appropriately explore the pricing strategies of the platform operator.

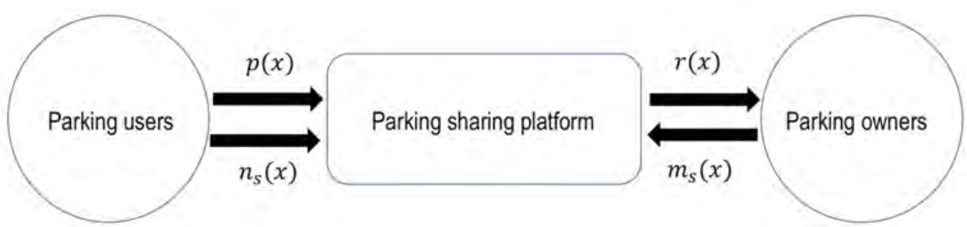

Figure 2: The parking sharing platform and its interactions with parking users and owners

For the parking sharing platform operator, it can adjust its pricing strategy, i.e., $p(x)$ and $r(x)$ for all x to maximize its profit, which is

$$R_p(\boldsymbol{p}, \boldsymbol{r}) = \int_0^d [n_s(x) \cdot p(x) - m_s(x) \cdot r(x)] \, dx \tag{9}$$

subject to the road users' parking choice equilibrium and the parking sharers' sharing equilibrium, i.e., $n_s(x)$ will depend on $p(x)$; and $m_s(x)$ will depend on $r(x)$; and furthermore $n_s(x)$ also depends on $m_s(x)$ over $x \in [0, d]$. Note that $\boldsymbol{p}$ and $\boldsymbol{r}$ are the vectors for the parking fee and rent. While here we only illustrate the case with a profit-seeking transit operator, later we will also discuss a public operator to reduce total social cost and then compare their performances. Given the pricing strategy of the platform operator, the parking choice equilibrium of travelers will be formulated and further discussed in Section 3.

3 Parking Choice Equilibrium

In this section, we firstly present the formulations for the parking choice equilibrium, and then discuss several properties of the choice equilibrium. In the end we will derive the discrete form for the equilibrium formulation and discuss the associated algorithm for solving the choice equilibrium problem.

3.1 Formulation and Properties

This section formulates the parking choice problem of users given the pricing strategy $(\boldsymbol{p}, \boldsymbol{r})$ of the platform operator. We begin with listing the following assumption to guarantee that there exists at least one feasible parking choice equilibrium solution.

Assumption 1 (Sufficient Parking)**.** *It is assumed that* $\int_0^d m_f(x)dx > N$.

Assumption 1 means that we always have sufficient parking supply for the N travelers even if there is no shared parking, which is to guarantee that the parking choice problem always has a feasible solution. This is reasonable since when d can be sufficiently large, there will always be available curbside parking spaces to travelers. Alternatively, in the long-run, the number of travelers that drive to work will be bounded by the total parking supply as travelers can shift to other options (e.g., public transit, shared-ride) when parking is costly (either costly searching time or costly fees for reserved and shared spaces).

The parking choice equilibrium can then be obtained by solving the following optimization problem:

$$\min : Z(\boldsymbol{n_f}, \boldsymbol{n_s}) = \int_0^d \left[\int_0^{n_f(x)} C_f(x,n)dn + c_s(x) \cdot n_s(x) \right] dx \tag{10}$$

subject to

$$\int_0^d n_f(x)dx + \int_0^d n_s(x)dx = N \tag{11a}$$

$$n_i(x) \leq m_i(x), \forall x, \forall i \in \{f, s\} \tag{11b}$$

$$n_i(x) \geq 0, \forall x, \forall i \in \{f, s\} \tag{11c}$$

where $\boldsymbol{n_f}$ and $\boldsymbol{n_s}$ denote $\{n_f(x), x \in [0, d]\}$ and $\{n_s(x), x \in [0, d]\}$, respectively. The optimization problem defined by Eqs.(10)-(11) follows the well-known Beckmann's formulation (Beckmann et al., 1956), but with parking capacity constraints, which is similar to those traffic assignment problem with road capacity constraints (e.g., Yang and Bell, 1997; Tong and Wong, 2000; Nie et al., 2004; Ryu et al., 2014). The feasible parking flow set defined by the conditions in Eq.(11) can be written as

$$\Omega \equiv \{n_i(x), i \in \{f, s\}, x \in [0, d] \mid \text{Constraints in Eq.(11)}\} \tag{12}$$

It is evident that Ω is non-empty, compact and convex. As the feasible flow set is non-empty and compact, and the objective function is continuous, the existence of solutions is always guaranteed as per Weierstrass' theorem (Bazaraa et al., 1993). Furthermore, we denote the set of flows $(\boldsymbol{n_f}, \boldsymbol{n_s})$ solving the minimization problem in Eq.(10) by Ω^*, thus $\Omega^* \subseteq \Omega$.

In summary, the above problem is an equilibrium parking traffic assignment problem with parking capacity constraint. With this in mind, the necessary and sufficient conditions of its optimality can be stated explicitly according to Karush-Kuhn-Tucker (KKT) conditions (similar conditions can be found in, e.g., Hearn, 1980). Let u, $\eta(x)$, and $\lambda(x)$ denote the optimal values of multipliers associated with parking demands and parking capacities respectively, the KKT conditions are:

$$n_f(x) \cdot (c_f(x) + w(x) - u) = 0, \forall x \in [0, d] \tag{13a}$$

$$c_f(x) + \eta(x) - u \geq 0, \forall x \in [0, d] \tag{13b}$$

$$n_f(x) \geq 0, \forall x \in [0, d] \tag{13c}$$

$$\eta(x) \cdot (m_f(x) - n_f(x)) = 0, \forall x \in [0, d] \tag{13d}$$

$$m_f(x) - n_f(x) \geq 0, \forall x \in [0, d] \tag{13e}$$

$$\eta(x) \geq 0, \forall x \in [0, d] \tag{13f}$$

$$n_s(x) \cdot (c_s(x) + \lambda(x) - u) = 0, \forall x \in [0, d] \tag{13g}$$

$$c_s(x) + \lambda(x) - u \geq 0, \forall x \in [0, d] \tag{13h}$$

$$n_s(x) \geq 0, \forall x \in [0, d] \tag{13i}$$

$$\lambda(x) \cdot (m_s(x) - n_s(x)) = 0, \forall x \in [0, d] \tag{13j}$$

$$m_s(x) - n_s(x) \geq 0, \forall x \in [0, d] \tag{13k}$$

$$\lambda(x) \geq 0, \forall x \in [0, d] \tag{13l}$$

$$\int_0^d n_f(x)dx + \int_0^d n_s(x)dx = N \tag{13m}$$

As mentioned earlier, if $n_f(x) \to m_f(x)$, then $q(x) \to 1$, and thus searching time for parking $\kappa \to +\infty$ and $c_f(x) \to +\infty$. Also, as we have Assumption 1 to guarantee feasibility of the parking choice problem, the parking capacity constraints for curbside spaces will never be binding. Therefore, the Lagrangian multipliers $\eta(x) = 0$. The first six equations in

Eq.(13) can be simplified as

$$n_f(x) \cdot (c_f(x) - u) = 0, \forall x \in [0, d] \tag{14a}$$

$$c_f(x) - u \geq 0, \forall x \in [0, d] \tag{14b}$$

$$n_f(x) \geq 0, \forall x \in [0, d] \tag{14c}$$

$$m_f(x) - n_f(x) > 0, \forall x \in [0, d] \tag{14d}$$

Differently, for the shared parking side, the multipliers $\lambda(x)$ might be positive. If we define a generalized cost for those using shared parking as follows:

$$c_s^g(x) = c_s(x) + \lambda(x) \tag{15}$$

The corresponding conditions in Eq.(13) can be simplified as

$$n_s(x) \cdot (c_s^g(x) - u) = 0, \forall x \in [0, d] \tag{16a}$$

$$c_s^g(x) - u \geq 0, \forall x \in [0, d] \tag{16b}$$

The physical meanings of the generalized costs for shared parking is as follows. $c_s^g(x)$ includes the direct and observable cost $c_s(x)$ associated with the shared parking choice at location x, while $\lambda(x)$ reflect the additional indirect cost experienced by the users to ensure this chosen option (e.g., book this shared space in advance in the e-platform). While this is not the focus of the paper, in future research we will examine the reservation schemes for parking sharing (Shao et al., 2016). Conditions in Eq.(13) lead to the following:

$$\lambda(x) > 0 \Rightarrow m_s(x) - n_s(x) = 0, \forall x \in [0, d] \tag{17a}$$

$$\lambda(x) = 0 \Rightarrow m_s(x) - n_s(x) \leq 0, \forall x \in [0, d] \tag{17b}$$

With the generalized cost formulation for shared-parking options $c_s^g(x)$, Wardrop's first principle is then satisfied.

We now turn to the uniqueness issue and other properties of the solutions $(\boldsymbol{n_f}, \boldsymbol{n_s}) \in \Omega^*$ for the passenger choice equilibrium. It is obvious that the objective function in Eq.(10) is convex, but not strictly convex as $c_s(x)$ is constant for potentially different $n_s(x)$. For example, if at the parking choice equilibrium, for locations $x_1 \neq x_2$, we have $c_s(x_1)$=$c_s(x_2)$, and $0 < n_s(x_1) < m_s(x_1)$ and $0 < n_s(x_2) < m_s(x_2)$. Then every combination of $(n_s(x_1), n_s(x_2))$ satisfying $0 < n_s(x_1) < m_s(x_1)$, $0 < n_s(x_2) < m_s(x_2)$ and $n_s(x_1) + n_s(x_2) = \textit{constant}$ could also be a part of an equilibrium solution ($n_f(x)$ and $n_s(x)$ for other locations remain unchanged). We further explore the uniqueness/non-uniqueness of choice equilibrium in the following.

Proposition 3.1. *For any* $(\boldsymbol{n_f}, \boldsymbol{n_s}) \in \Omega^*$, $N_f = \int_0^d n_f(x)dx$ *and* $N_s = \int_0^d n_s(x)dx$ *is unique.*

Proof. Suppose the pair of (N_f, N_s) is not unique for the problem in Eq.(10), and there are (at least) two pairs (N_f', N_s') and (N_f'', N_s''). Without loss of generality, let $N_f' > N_f''$ and thus $N_s' < N_s''$.

Since $N_f' > N_f''$, at least for some locations y, we have $n_f'(y) > n_f''(y) \geq 0$. As $c_f(y)$ is a strictly increasing function of $n_f(y)$, then $u' = c_f'(y) > c_f''(y) \geq u''$ (and therefore $u' > u''$), where u' and u'' are the minimum travel cost in Eq.(13) and Eq.(14) for the two different cases.

Since $N_s' < N_s''$, at least for some locations y, we have $0 \leq n_s'(y) < n_s''(y) \leq m_s(y)$. Based on Eqs.(13)-(14) and the derived result $u' > u''$ in the above, we then have $c_s'(y) + \lambda'(y) \geq u' > u'' = c_s''(y) + \lambda''(y)$. Note that $c_s'(y) = c_s''(y) = c_s(y)$, therefore, we have $\lambda'(y) > \lambda''(y)$. As $n_s'(y) < n_s''(y) \leq m_s(y)$ indicates that $\lambda'(y) = 0$, we then have $0 = \lambda'(y) > \lambda''(y) \geq 0$, which is contradictory. This completes the proof. $\square$

Proposition 3.2. *For any* $(\boldsymbol{n_f}, \boldsymbol{n_s}) \in \Omega^*$, $\boldsymbol{n_f}$ *is unique.*

Proof. Suppose $\boldsymbol{n_f}$ is not unique for the problem in Eq.(10), we then have at least two solutions $\boldsymbol{n_f'} \neq \boldsymbol{n_f''}$. For some locations y, we have $n_f'(y) \neq n_f''(y)$. Without loss of generality, let $n_f'(y) > n_f''(y) \geq 0$. Based on Proposition 3.1, we then must have $0 \leq n_f'(y') < n_f''(y')$ for some locations y'.

Since $n_f'(y) > n_f''(y) \geq 0$, based on Eqs.(13)-(14), we have $u' = c_f'(y) > c_f''(y) \geq u''$, where u' and u'' again are the minimum travel cost in Eq.(13) for the two different cases. Similarly, based on $0 \leq n_f'(y') < n_f''(y')$, we have $u' \leq c_f'(y') < c_f''(y') = u''$. These two results conflict with each other. This completes the proof. $\square$

Propositions 3.1 and 3.2 together say that even though the parking choice equilibrium might not be unique, the split between free and shared parking usages is unique, and furthermore, the parking flow pattern in the free curbside parking side is unique at the choice equilibrium.

For $(\boldsymbol{n_f}, \boldsymbol{n_s}) \in \Omega^*$, if $N_f = 0$, then $\boldsymbol{n_f} = 0$. We now focus on the more general case with $N_f > 0$. For location x with $n_f(x) > 0$, we let $x \in X_+^f$; otherwise, we let $x \in X_0^f$. For $x \in X_+^f$, $u = \alpha_d \cdot \left(\frac{D-x}{v} + \kappa\left(\frac{n_f(x)}{m_f(x)}\right)\right) + \alpha_w \cdot \frac{x}{w}$, where u is the minimum travel cost for free curbside space users. Therefore, the equilibrium free curbside parking flow can be determined as

$$n_f(x) = \begin{cases} \kappa^{-1}\left(\frac{u-\left[\alpha_d\cdot\frac{D-x}{v}+\alpha_w\cdot\frac{x}{w}\right]}{\alpha_d}\right)\cdot m_f(x) & x < \frac{u-\left[\alpha_d\cdot\frac{D}{v}+\kappa(0)\right]}{\alpha_w\cdot w^{-1}-\alpha_d\cdot v^{-1}} \\ 0 & x \geq \frac{u-\left[\alpha_d\cdot\frac{D}{v}+\kappa(0)\right]}{\alpha_w\cdot w^{-1}-\alpha_d\cdot v^{-1}} \end{cases}, \tag{18}$$

where $\kappa^{-1}(\cdot)$ is the inverse function of searching time for curbside parking.

Proposition 3.3. *If $x_1 \in X_+^f$, for every $x \leq x_1$ we must have $x \in X_+^f$.*

Based on Eq.(18), Proposition 3.3 is evident. This is because, the walking is more costly than driving, i.e., $\alpha_d \cdot \frac{D-x}{v} + \alpha_w \cdot \frac{x}{w}$ is increasing with respect to x. This also suggests that the parking occupancy rate is non-increasing over x. Furthermore, if $m_f(x)$ is constant over x , then $n_f(x)$ decreases with respect to x when $x < \frac{u-\left[\alpha_d\cdot\frac{D}{v}+\kappa(0)\right]}{\alpha_w\cdot w^{-1}-\alpha_d\cdot v^{-1}}$.

Now we further discuss some properties (including uniqueness/non-uniqueness issues) of $\boldsymbol{n_s}$ at the solution $(\boldsymbol{n_f}, \boldsymbol{n_s}) \in \Omega^*$ to the problem in Eq.(10). Similarly, let X_+^s denote the set of locations with positive demand, i.e., for any location x with $n_s(x) > 0$, we have $x \in X_+^s$. For other locations x with $n_s(x) = 0$, we let $x \in X_0^s$. Furthermore, when $X_+^s \neq \emptyset$, we define $x_{cri} \in X_+^s$ as the location yielding the largest travel cost for road users, i.e., $c_s(x_{cri}) \geq c_s(x)$ for any $x \in X_+^s$.

Proposition 3.4. *Suppose $\boldsymbol{n_s} \neq 0$ where $(\boldsymbol{n_f}, \boldsymbol{n_s}) \in \Omega^*$, i.e., $X_+^s \neq \emptyset$, there must exist the location x_{cri} where $0 < n_s(x_{cri}) \leq m_s(x_{cri})$ satisfying the following: (i) for all locations with $c_s(x) < c_s(x_{cri})$ we have $n_s(x) = m_s(x)$; and (ii) for all locations with $c_s(x) > c_s(x_{cri})$ we have $n_s(x) = 0$; and (iii) for all locations with $c_s(x) = c_s(x_{cri})$ we have $0 \leq n_s(x) \leq m_s(x)$.*

Proof. When $X_+^s \neq \emptyset$, it is evident that we can construct x_{cri} as described in the above. Note that x_{cri} might not be unique, as shared-parking at different locations might lead to identical travel cost. For x_{cri}, we have $0 < n_s(x_{cri}) \leq m_s(x_{cri})$. It means that $c_s(x_{cri}) + \lambda(x_{cri}) = u$ where u is the minimum generalized travel cost at equilibrium.

For a location x with $c_s(x) < c_s(x_{cri})$, we have $c_s(x)+\lambda(x) \geq u$. Therefore, $c_s(x)+\lambda(x) \geq c_s(x_{cri}) + \lambda(x_{cri})$, and furthermore $\lambda(x) > \lambda(x_{cri}) \geq 0$. The KKT conditions suggest that $m_s(x) - n_s(x) = 0$ must hold. This proves Part (i) of the proposition.

For a location x with $c_s(x) > c_s(x_{cri})$, x must not belong to X_+^s (this is how we define x_{cri}), thus $x \in X_0$ and $n_s(x) = 0$. This completes the proof of Part (ii) of the proposition.

Part (iii) of the proposition is evident. This completes the proof. □

Assumption 2. *For $x_1 \neq x_2$, we have $c_s(x_1) \neq c_s(x_2)$.*

Proposition 3.5. *Under Assumption 2, $\boldsymbol{n_s}$ is unique where $(\boldsymbol{n_f}, \boldsymbol{n_s}) \in \Omega^*$.*

Proof. As per Propositions 3.1 and 3.2, the equilibrium N_f, N_s, and $\boldsymbol{n_s}$ are unique. If $N_f = 0$, then $n_s(x) = 0$ for all x and $\boldsymbol{n_s}$ is unique. We now turn to the more general case where $N_s > 0$.

Suppose there are two different solutions $\boldsymbol{n_s'} \neq \boldsymbol{n_s''}$ and there are some locations $x \in X_d$ where $n_s'(x) \neq n_s''(x)$. Based on Proposition 3.4, we can find x'_{cri} and x''_{cri} for the two solutions respectively. Under Assumption 2, x'_{cri} and x''_{cri} must be unique for both cases. If $x'_{cri} = x''_{cri}$, then $\boldsymbol{n_s'} = \boldsymbol{n_s''}$ as per Proposition 3.4. Then we must have $x'_{cri} \neq x''_{cri}$, and $c_s(x'_{cri}) \neq c_s(x''_{cri})$ as per Assumption 2. Without loss of generality, let $c_s(x'_{cri}) < c_s(x''_{cri})$. Then for all locations $x \in (X_+^s)'$ we must have $x \in (X_+^s)''$, i.e., $n_s'(x) > 0 \Rightarrow n_s''(x) > 0$ and $n_s''(x) \geq n_s'(x)$ based on Proposition 3.4; and for some locations $y \notin (X_+^s)'$ we have $y \in (X_+^s)''$, i.e., $n_s'(y) = 0$ but $n_s''(y) > 0$. We thus have $(X_+^s)' \subset (X_+^s)''$, and furthermore, $(X_+^s)'' - (X_+^s)' = X_d - \{x'_{cri}\}$ if $n_s(x'_{cri}) = m_s(x'_{cri})$ and $(X_+^s)'' - (X_+^s)' = X_d$ if $n_s(x'_{cri}) \neq m_s(x'_{cri})$. With the above, we then have $N_s' = \int_{x \in (X_+^s)'} n_s'(x)dx \leq \int_{x \in (X_+^s)'} n_s''(x)dx < \int_{x \in (X_+^s)'} n_s''(x)dx + \int_{x \in ((X_+^s)'' - (X_+^s)')} n_s''(x)dx = N_s''$, which contradicts to Proposition 3.1. This completes the proof. $\square$

Assumption 3. *For $x_1 \neq x_2$ but $c_s(x_1)$=$c_s(x_2)$, the travelers prefer to the shared parking space at location x_1 if $x_1 < x_2$.*

Proposition 3.6. *Under Assumption 3, $\boldsymbol{n_s}$ is unique where $(\boldsymbol{n_f}, \boldsymbol{n_s}) \in \Omega^*$.*

Proof. The proof is similar to those for Proposition 3.5. $\square$

Later in this paper, we adopt Assumption 3 and consider that travelers tend to park closer to the final destination (when shared-parking at different locations yield the same total individual travel cost). In this case, solving problem in Eq.(10) may not give the exact parking choice equilibrium solution. The parking flows for shared-parking spaces $\boldsymbol{n_s}$ should be further arranged to respect Assumption 3. We discuss the computation of parking choice equilibrium solutions in the next subsection.

3.2 Solving Approach

To find the choice equilibrium solution, firstly we discretize the continuous model formulation in the last subsection into a discrete one. Specifically, the parking corridor $[0, d]$ is discretized into K space intervals of length $\Delta x = \frac{d}{K}$. The distance of the $i-th$ interval to the city center is $x_i = i \cdot \Delta x$. The number of free and shared parking spaces at location i is then $M_f(i) = \int_{x_{(i-1)}}^{x_i} m_f(x)dx$ and $M_s(i) = \int_{x_{(i-1)}}^{x_i} m_s(x)dx$, respectively. Similarly, the number of potential shared parking spaces for location i is then $M'(i) = \int_{x_{(i-1)}}^{x_i} M(x)dx$. The distribution of inconvenience δ of potential parking sharers at interval i is an aggregation of the distributions for travelers at locations $[x_{i-1}, x_i]$, of which the probability and cumulative density functions are denoted by f_i and F_i respectively. Without adding additional notation burden, we replace x in the continuous formulation with i. Therefore, travel costs are $c_f(i)$ and $c_s(i)$, searching time is $t_c(i)$, parking flows are $n_f(i)$ and $n_s(i)$.

The objective in Eq.(10) can be written in the discrete form as follows:

$$\min : Z'(\boldsymbol{n_f}, \boldsymbol{n_s}) = \sum_{i=1}^{K} \left[\int_0^{n_f(i)} c_f(i) \cdot dn \right] + \sum_{i=1}^{K} c_s(i) \cdot n_s(i) \tag{19}$$

subject to

$$\sum_{i=1}^{K} n_t(i) = N_t, \forall t \in \{f, s\} \tag{20a}$$

$$N_s + N_f = N \tag{20b}$$

$$n_t(i) \leq M_t(i), \forall i = 1, 2, 3, ..., K, \forall t \in \{f, s\} \tag{20c}$$

$$n_t(x) \geq 0, \forall i = 1, 2, 3, ..., K, \forall t \in \{f, s\} \tag{20d}$$

where $c_f(i)$ is a function of $n_f(i)$, and we still use the notation $\boldsymbol{n_f}$ and $\boldsymbol{n_s}$ to represent the vectors for parking fee and rent (in the discrete formulation).

One may notice that $Z'(\boldsymbol{n_f}, \boldsymbol{n_s}) = Z'_f(\boldsymbol{n_f}) + Z'_s(\boldsymbol{n_s})$, where $Z'_f(\boldsymbol{n_f}) = \sum_{i=1}^{K} \left[\int_0^{n_f(i)} c_f(i) \cdot dn \right]$ and $Z'_s(\boldsymbol{n_s}) = \sum_{i=1}^{K} c_s(i) \cdot n_s(i)$. If N_s and N_f are given, the flow conservation constraint $N_s + N_f = N$ in Eq.(20) also becomes redundant. To minimize $Z'(\boldsymbol{n_f}, \boldsymbol{n_s})$ under given N_s and N_f, it is equivalent to solve the following problems for $\boldsymbol{n_f}$ and $\boldsymbol{n_s}$ separately as constraints for $n_f(i)$ and $n_s(i)$ are also independent, i.e., for $t \in \{f, s\}$

$$\min : Z'_t(\boldsymbol{n_t}) \tag{21}$$

subject to

$$\sum_{i=1}^{K} n_t(i) = N_t \tag{22a}$$

$$0 \leq n_t(i) \leq M_t(i), \forall i = 1, 2, 3, ..., K, \forall t \in \{f, s\} \tag{22b}$$

Our general idea to solve the choice equilibrium with capacity constraint is to utilize the special structure of the problem defined in Eq.(19), i.e., to solve the sub-problems defined by Eq.(21) for given N_s and N_f, and then utilize information from the solution to adjust N_s

and N_f towards the equilibrium values. Once N_s and N_f are adjusted to the equilibrium values, then solving the above sub-problems would yield the choice equilibrium flow.

Firstly, we discuss how to minimize Z'_f (sub-problem I). Following the continuous formulation, we have $t_c(i) \to +\infty$ if $n_f(i) \to m_f(i)$, and $\sum_{i=1}^{K} m_f(i) > N$. Therefore, at the choice equilibrium we should have $n_f(i) < m_f(i)$, i.e., $1 - \frac{n_f(i)}{m_f(i)} \geq \epsilon > 0$ if ϵ is appropriately chosen. To avoid considering the free curbside parking constraints, we can expand the definition of $t_c(i)$ as follows:

$$t_c(i) = \begin{cases} \kappa\left(q(i)\right) & q(i) \in [0, 1-\epsilon] \\ \kappa\left(1-\epsilon\right) + \kappa'\left(1-\epsilon\right) \cdot \left(q(i) - (1-\epsilon)\right) & q(i) \in (1-\epsilon, +\infty) \end{cases}, \tag{23}$$

We then can utilize the expanded definition of $t_c(i)$ to minimize Z'_f where we do not need to consider the constraints $n_f(i) \leq M_f(i)$ any more. Therefore, convex combination based algorithms such as the well-known Frank-Wolf algorithm or its variants can be directly adopted (Nguyen, 1974; Fukushima, 1984). Note that while we expand the feasible domain to cover $n_f(i) > M_f(i)$, the solution to this problem will satisfy $n_f(i) < M_f(i)$ given our construction of the expanded $t_c(i)$, which is also the solution to the original problem (see, e.g., Nie et al., 2004).

Now, we turn to discuss how to minimize Z'_s (sub-problem II). It is evident that the problem in Eq.(21) for $t = s$ is a linear programming problem with linear constraints, which can be readily solved by the simplex method. However, as mentioned earlier, for $i \neq i'$, as $c_s(i) = c_s(i')$ might occur, solving the above mentioned linear programming problem may not give a solution that satisfies Assumption 3. We now discuss a further step to respect this objective. As $c_s(i)$ is predetermined, we can identify the locations i with identical cost c_s. Suppose there are J cost levels c_s^j where $j = 1, 2, ..., J$ with at least two locations having the same cost. For cost c_s^j, suppose location $k \in K_j$ will lead to this cost level for travelers.

After minimizing $Z'_j(\boldsymbol{n_s})$, we obtain a flow pattern $\boldsymbol{n_s^0}$. However, to respect Assumption 3, we can further minimize the following:

$$\min : Y_j = \sum_{k \in K_j} n_f(k) \cdot x_k \tag{24}$$

subject to

$$\sum_{k=1}^{k_j} n_f(k) = \sum_{k=1}^{k_j} n_f^0(k) \tag{25a}$$

$$0 \leq n_f(k) \leq M_f(k), \forall i = 1, 2, 3, ..., K \tag{25b}$$

If the unique N_f^* and N_s^* at the parking choice equilibrium are given to us, then it is obvious that solving the above problems in Eq.(21) and Eq.(24) will yield the choice equilibrium solution. The problem now becomes how to determine N_f^* and N_s^*. Before moving further, we define two values u_f and u_s, where u_f is the minimum travel cost for users using free curbside space given N_f, and u_s is the maximum cost of users using shared-parking. We list several evident results. Firstly, u_f increases with N_f. Secondly, u_s is non-decreasing over N_s. Moreover, we can define $N_s^{max} = \min\{N, \sum_{k=1}^{K} M_s(k)\}$, and thus $N_f^{min} = N - \min\{N, \sum_{k=1}^{K} M_s(k)\}$; $N_s^{min} = 0$ and thus $N_f^{max} = N$. Therefore, $N_f^{min} \leq N_f \leq N_f^{max}$ and $N_s^{min} \leq N_s \leq N_s^{max}$. At the interior equilibrium, we should have $u_f^* = u_s^*$ and $N_f^{min} < N_f^* < N_f^{max}$ and $N_s^{min} < N_s^* < N_s^{max}$; and at the boundary equilibrium, we should have $u_f^* \geq u_s^*$ if $N_f^* = N_f^{min}$ and $N_s^* = N_s^{max}$; and at the other boundary equilibrium, we should have $u_f^* \leq u_s^*$ if $N_f^* = N_f^{max} = N$ and $N_s^* = N_s^{min} = 0$. These can be derived readily with the discrete version of the KKT conditions in Eqs.(13)-(14). According to these results, we can obtain the parking choice equilibrium by the following bi-section based procedure, where u_f and u_s under given N_f and N_s give information on how to adjust N_f and N_s towards the equilibrium values N_f^* and N_s^*.

Table 1: The procedure to solve the parking choice equilibrium

Step 0: Set an initial upper bound $N_f^u = N_f^{max}$ and lower bound $N_f^l = N_f^{min}$.
Step 1: Let $N_f = \frac{N_f^u + N_f^l}{2}$ and $N_s = N - N_f$.
Step 2-1: Given N_f from Step 1, use Frank-Wolfe algorithm (convex combination) to solve the problem in Eq.(21) where $t = f$, and determine u_f.
Step 2-2: Given N_s from Step 1, use Simplex-method to solve the problem in Eq.(21) where $t = s$, and determine u_s.
Step 3: If $\frac{N_f^u - N_f^l}{N} < \epsilon$, go to Step 4; otherwise update N_f^u and N_f^l as follows: If $u_f \geq u_s$, let $N_f^u = N_f$ and $N_f^l = N_f^l$; otherwise let $N_f^u = N_f^u$ and $N_f^l = N_f$.
Step 4: Given $n_s(k)$ for $k = 1, 2, ..., K$ from Steps 1-3, use Simplex-method to solve the problem(s) in Eq.(24).

4 Revenue Maximization and System Optimum

In this section, we firstly discuss some general results for a profit-maximizing platform operator, and for a social cost minimizing operator and the system optimum parking flow pattern when all the three parties (parking users, parking owners, and parking operators) are considered. We then move to an analytical example with a further simplified network setting to gain further insights.

4.1 General Results

We start with describing the parking sharing e-platform's problem. For an e-platform operator for parking sharing, the problem is to maximize the net benefit.

$$\max : R_p(\boldsymbol{p}, \boldsymbol{r}) \tag{26}$$

subject to the parking choice equilibrium studied in Section 3, and the parking owners' supplying equilibrium discussed in Section 2.2, where $R_p(\boldsymbol{p}, \boldsymbol{r})$ is defined in Eq.(9). Note that we only need to consider that $\boldsymbol{p} \geq 0$ and $\boldsymbol{r} \geq 0$ for a profit-maximizing operator. Firstly, $r(x) < 0$ would simply yield zero parking owners to share their parking spaces (this is the case without parking sharing, and the platform operator has zero profit). Secondly, by letting $p(x) < 0$, the operator must be better off when increasing $p(x)$ to zero. Suppose $(\boldsymbol{p}^*, \boldsymbol{r}^*)$ is the solution to the problem in Eq.(26), the following conditions hold.

Proposition 4.1. *At the user parking choice equilibrium under $(\boldsymbol{p}^*, \boldsymbol{r}^*)$, we must have $n_s(x) = m_s(x)$ for every location $x \in [0, d]$.*

Proposition 4.1 means that the shared parking spaces should be fully utilized under a profit-maximizing sharing platform operator. This can be readily verified. Firstly, if $m_s(x) = 0$, then we must have $n_s(x) = m_s(x) = 0$. For $m_s(x) > 0$, if $n_s(x) < m_s(x)$, then the rent $r^*(x)$ must not be optimal. This is evident as we can reduce the rent by a small amount Δr where $[F_x(r^*(x)) - F_x(r^*(x) - \Delta r)] \cdot M(x) \leq m_s(x) - n_s(x)$. Doing so, the operator will save a rent in the amount of $r^*(x) \cdot m_s(x) - (r^*(x) - \Delta r) \cdot (m_s(x) - [F_x(r^*(x)) - F_x(r^*(x) - \Delta r)] \cdot M(x))$ (reduced rent per shared parking space, and reduced number of rented shared parking). However, as $m_s(x) - [F_x(r^*(x)) - F_x(r^*(x) - \Delta r)] \cdot M(x) \leq n_s(x)$, the operator will receive the same parking fees from the travelers. Therefore, the original $r^*(x)$ must not be optimal.

Proposition 4.2. *At $(\boldsymbol{p}^*, \boldsymbol{r}^*)$, we must have $p^*(x) = \alpha_d \cdot \kappa\left(\frac{n_f(x)}{m_f(x)}\right)$ for any location x with $n_s(x) > 0$, i.e., $x \in X_+^s$.*

Proposition 4.2 means that a profit-maximizing sharing platform operator would increase the price of parking spaces to the level thus travelers will be indifferent between shared and free curbside parking options at the same location. This is explained as follows. Firstly $p^*(x) > \alpha_d \cdot t_c(x)$ cannot occur if $n_f(x) > 0$. This is because if $p^*(x) > \alpha_d \cdot t_c(x)$ then $c_s(x) > c_f(x)$, and no users will choose shared parking spaces, i.e., $n_s(x) = 0$, and the operator will earn zero amount parking fees while it has to pay rents to the parking owners. However, by reducing $p^*(x)$ until $n_s(x) > 0$, the operator can earn a positive amount. Secondly, if $p^*(x) < \alpha_d \cdot t_c(x)$, the platform operator can increase the price to $\alpha_d \cdot t_c(x)$ where no users will shift his or her choice, but additional profit of $[\alpha_d \cdot t_c(x) - p^*(x)] \cdot n_s(x)$ can be gained.

Proposition 4.3. *At $(\boldsymbol{p}^*, \boldsymbol{r}^*)$, we must have $r^*(x) = F_x^{-1}\left(\frac{n_s(x)}{M(x)}\right)$ for any location x with $n_s(x) > 0$, i.e., $x \in X_+^s$.*

Proposition 4.3 simply means the optimal rent paid to parking sharers has a determined relationship with the number of parking users at the profit-maximizing equilibrium. Based on Eq.(7), we know $m_s(x) = F_x(r^*(x)) \cdot M(x)$. Further with Proposition 4.1, we have $n_s(x) = F_x(r^*(x)) \cdot M(x)$. Therefore, $r^*(x) = F_x^{-1}\left(\frac{n_s(x)}{M(x)}\right)$ should hold.

The above results together indicate that if the target parking flow pattern under $(\boldsymbol{p}^*, \boldsymbol{r}^*)$ is known to us, we can calculate the optimal pricing and rent for shared parking based on Proposition 4.2 and Proposition 4.3. Note that when $n_s(x)$ should be zero, we can simply set $r^*(x) = 0 = F_x^{-1}(0)$, and parking price can be set still as $p^*(x) = \alpha_d \cdot \kappa\left(\frac{n_f(x)}{m_f(x)}\right)$ without affecting the parking flow patterns. It is worth mentioning that $\frac{r^*(x)}{p^*(x)} = \frac{F_x^{-1}\left(\frac{n_s(x)}{M(x)}\right)}{\alpha_d \cdot \kappa\left(\frac{n_f(x)}{m_f(x)}\right)}$. As $F_x^{-1}(\cdot)$ and $\kappa(\cdot)$ are generally nonlinear functions, and $M(x)$ and $m_f(x)$ may also vary in different ways over x, the ratio $\frac{r^*(x)}{p^*(x)}$ generally will not be a constant. Therefore, it is generally a sub-optimal solution for the operator to fix $\frac{r(x)}{p(x)}$.

We now will further discuss the social cost minimizing operator, and the social optimum parking choice and parking sharing supply pattern. The total social cost of our interest is determined as follows:

$$TSC = TC - R_s - R_p, \tag{27}$$

where TC is defined in Eq.(6), and R_s is defined in Eq.(8), and R_p is defined in Eq.(9). We then have

$$TSC = \int_0^d \left[c_f(x) \cdot n_f(x) + (c_s(x) - p(x)) \cdot n_s(x) + \int_0^{r(x)} \delta \cdot f_x(\delta) \cdot M(x) \cdot d\delta\right] dx. \tag{28}$$

One can see from the above that the total social cost contains two parts: the cost of travelers without the parking fees, and the inconvenience cost of the parking sharers who have supplied a private space.

We now examine the optimal $(\boldsymbol{n_f}, \boldsymbol{n_s})$ and the corresponding $\boldsymbol{m_f}$ to achieve the minimum TSC. Similar to Proposition 4.1, in the social optimum situation, we should have $m_s(x) = n_s(x)$. This is explained as follows. Firstly, we must have $m_s(x) \geq n_s(x)$. Secondly, if

$m_s(x) > n_s(x)$, by reducing $m_s(x)$ to $n_s(x)$, while travelers' parking choice does not change as well as their costs, the parking owners' side can save the inconvenience cost for a total number of $m_s(x) - n_s(x)$ parking owners. This saving amount is equal to $\int_{F_x^{-1}\left(\frac{n_s(x)}{M(x)}\right)}^{F_x^{-1}\left(\frac{m_q(x)}{M(x)}\right)} \delta \cdot f_x(\delta) \cdot d\delta$.

Since $m_s(x) = n_s(x)$, $r(x)$ in Eq.(28) can be replaced by $r(x) = F_x^{-1}\left(\frac{n_s(x)}{M(x)}\right)$, and the total social cost can be written as a function of only $n_f(x)$ and $n_s(x)$ as follows:

$$TSC' = \int_0^d \left[c_f(x) \cdot n_f(x) + (c_s(x) - p(x)) \cdot n_s(x) + \int_0^{F_x^{-1}\left(\frac{n_s(x)}{M(x)}\right)} \delta \cdot f_x(\delta) \cdot M(x) \cdot d\delta \right] dx. \tag{29}$$

Note that we still have $\int_0^d n_f(x)dx + \int_0^d n_s(x)dx = N$. We can then define the following Lagrangian:

$$L(\boldsymbol{n_f}, \boldsymbol{n_s}, \nu) = TSC' + \nu \cdot (N - \int_0^d n_f(x)dx - \int_0^d n_s(x)dx), \tag{30}$$

where we further have

$$\frac{\partial L}{\partial n_f(x)} = c_f(x) + \frac{dc_f(x)}{dn_f(x)} - \nu. \tag{31}$$

$$\frac{\partial L}{\partial n_s(x)} = c_s(x) - p(x) + F_x^{-1}\left(\frac{n_s(x)}{M(x)}\right) - \nu. \tag{32}$$

$$\frac{\partial L}{\partial \nu} = N - \int_0^d n_f(x)dx - \int_0^d n_s(x)dx. \tag{33}$$

It can be readily verified that L is a convex function of $\boldsymbol{n_f}$ and $\boldsymbol{n_s}$. Suppose an interior optimal solution, we then have $\frac{\partial L}{\partial n_f(x)} = 0$ and $\frac{\partial L}{\partial n_s(x)}$=0, and $\nu > 0$. We then immediately have the following results for an interior system optimum flow solution $(\boldsymbol{n_f^*}, \boldsymbol{n_s^*})$ to the problem in Eq.(29).

Proposition 4.4. *At $(\boldsymbol{n_f^*}, \boldsymbol{n_s^*})$, we have $\alpha_d \cdot \left(t_c(x) + \frac{dt_c(x)}{dn_f(x)}\right) = F_x^{-1}\left(\frac{n_s(x)}{M(x)}\right)$ for any location $x \in X_+^s$, where $t_c(x) = \kappa\left(\frac{n_f(x)}{m_f(x)}\right)$.*

Proposition 4.5. *At $(\boldsymbol{n_f^*}, \boldsymbol{n_s^*})$, we have $t_c(x) + \frac{dt_c(x)}{dn_f(x)} = t_c(y) + \frac{dt_c(y)}{dn_f(y)}$ for locations $x \neq y$.*

Proposition 4.6. *At $(\boldsymbol{n_f^*}, \boldsymbol{n_s^*})$, we have $F_x^{-1}\left(\frac{n_s(x)}{M(x)}\right) = F_y^{-1}\left(\frac{n_s(y)}{M(y)}\right)$ for locations $x \neq y$.*

Propositions 4.4, 4.5, and 4.6 together state that, at the system optimum parking flow pattern, (1) the marginal cost of an additional driver to cruise for free parking at location x, i.e., $\alpha_d \cdot \left[t_c(x) + \frac{dt_c(x)}{dn_f(x)}\right]$ should be equal to the marginal cost of an additional driver to use shared parking, which is the inconvenience cost of the additional parking owner who supplies this shared space, i.e., $F_x^{-1}\left(\frac{n_s(x)}{M(x)}\right)$; (2) the marginal costs of an additional driver to cruise for free parking at locations $x \neq y$ should be identical; and (3) the marginal costs of an additional driver to use shared parking at locations $x \neq y$, which are the inconvenience costs of an additional parking owner who supplies this shared space at locations $x \neq y$ should be identical. These results while appear similar to those for system optimum in traffic equilibrium (Yang, 1999), they are indeed different. This is because, for the sharing problem, social cost includes those of parking users and sharers in the two-sided markets (i.e., both the demand and supply sides).

To support the above interior system optimum $(\boldsymbol{n_f^*}, \boldsymbol{n_s^*})$ as an equilibrium solution, the parking pricing $p(x)$ and parking rents $r(x)$ should have the following relationships.

Based on Proposition 4.4, at the system optimum to minimize TSC', we should have $\alpha_d \cdot \left(t_c(x) + \frac{dt_c(x)}{dn_f(x)}\right) = F_x^{-1}\left(\frac{n_s(x)}{M(x)}\right)$. At equilibrium, as $c_f(x) = c_s(x)$, we have, $p(x) = \alpha_d \cdot t_c(x)$. To support the condition $\alpha_d \cdot \left(t_c(x) + \frac{dt_c(x)}{dn_f(x)}\right) = F_x^{-1}\left(\frac{n_s(x)}{M(x)}\right)$ to hold at the equilibrium, we then should have $p(x) + \alpha_d \cdot \frac{dt_c(x)}{dn_f(x)} = F_x^{-1}\left(\frac{n_s(x)}{M(x)}\right)$, and thus $p(x) = F_x^{-1}\left(\frac{n_s(x)}{M(x)}\right) - \alpha_d \cdot \frac{dt_c(x)}{dn_f(x)}$. As mentioned earlier, under the system optimum shared parking supply we should have $r(x) = F_x^{-1}\left(\frac{n_s(x)}{M(x)}\right)$, so the following condition holds: $p(x) = r(x) - \alpha_d \cdot \frac{dt_c(x)}{dn_f(x)}$. In this case, since $\frac{dt_c(x)}{dn_f(x)} > 0$, the system optimum shared parking price could be negative if $r(x) < \alpha_d \cdot \frac{dt_c(x)}{dn_f(x)}$, and $p(x) < r(x)$ always holds, so the profits of the sharing platform operator is always negative.

Proposition 4.7. *To support the interior social optimum $(\boldsymbol{n_f^*}, \boldsymbol{n_s^*})$ as an equilibrium, the parking sharing platform operator will have negative net benefit.*

4.2 An Analytical Example

We now further present an analytical example to gain more insights. For tractability, we adopt the following simplifications: (i) only one location $x = 0$ is considered, and thus the location index is omitted; (ii) inconvenience of parking sharers is uniformly distributed over $[0, \delta_0]$, and δ_0 is sufficiently large and rent r is always less than δ_0; (iii) we consider an interior equilibrium thus both parking options are used by some users and yield the same travel cost for users.

Based on the above, we are now ready to state the following results. Firstly, given parking price p, at an interior equilibrium, $p = \alpha_d \cdot \kappa\left(\frac{n_f}{m_f}\right)$ should hold and thus $n_f = \kappa^{-1}\left(\frac{p}{\alpha_d}\right) \cdot m_f$ and $n_s = N - \kappa^{-1}\left(\frac{p}{\alpha_d}\right) \cdot m_f$. Secondly, given parking rent r, the number of sharers is equal to $m_s = \frac{r}{\delta_0} \cdot M$, which should be greater than or equal to n_s. To maximize profit or achieve social optimum, we should have $n_s = m_s$, therefore, $N - \kappa^{-1}\left(\frac{p}{\alpha_d}\right) \cdot m_f = \frac{r}{\delta_0} \cdot M$, and thus

$r = [N - \kappa^{-1}\left(\frac{p}{\alpha_d}\right) \cdot m_f] \cdot \frac{\delta_0}{M}$, or alternatively, $p = \alpha_d \cdot \kappa\left(\frac{N - \frac{r}{\delta_0} \cdot M}{m_f}\right)$. The net profit of the platform operator can then be written as a function of r

$$R'_p(r) = \frac{r}{\delta_0} \cdot M \cdot \left[\alpha_d \cdot \kappa\left(\frac{N - \frac{r}{\delta_0} \cdot M}{m_f}\right) - r\right]. \tag{34}$$

The total net benefit of all parking sharers can be written as

$$R'_s(r) = \frac{r}{\delta_0} \cdot M \cdot \frac{r}{2}. \tag{35}$$

The total user cost is

$$TC' = N \cdot \alpha_d \cdot \frac{D}{v} + N \cdot \alpha_d \cdot \kappa\left(\frac{N - \frac{r}{\delta_0} \cdot M}{m_f}\right). \tag{36}$$

We firstly look at the situation with revenue-maximizing platform operator. The first-order derivative of Eq.(34) with respect to r is

$$\frac{dR'_p(r)}{dr} = \frac{1}{\delta_0} \cdot M \cdot (\alpha_d \cdot \kappa - r) + \frac{r}{\delta_0} \cdot M \cdot \left(\alpha_d \cdot \kappa' \cdot \left(-\frac{1}{\delta_0} \cdot \frac{M}{m_f}\right) - 1\right), \tag{37}$$

where $\kappa = \kappa\left(\frac{N - \frac{r}{\delta_0} \cdot M}{m_f}\right)$ and $\kappa' = \frac{d\kappa\left(\frac{N - \frac{r}{\delta_0} \cdot M}{m_f}\right)}{d\left(\frac{N - \frac{r}{\delta_0} \cdot M}{m_f}\right)}$. At the equilibrium, a marginal change in r will lead to a marginal change in the corresponding shared parking price $p = \alpha_d \cdot \kappa\left(\frac{N - \frac{r}{\delta_0} \cdot M}{m_f}\right)$ and in the number of shared parking users $n_s = m_s = \frac{r}{\delta_0} \cdot M$. In the right-hand side of Eq.(37), the first term is the marginal revenue change due to the marginal change in the number of shared-parking users, and the second term is the change due to the marginal change in the shared parking price and rent, which is negative.

By letting $\frac{dR'_p(r)}{dr} = 0$, we immediately have

$$r = \frac{\alpha_d \cdot \kappa}{\alpha_d \cdot \kappa' \cdot \frac{1}{\delta_0} \cdot \frac{M}{m_f} + 2}, \tag{38}$$

Note that in Eq.(38) κ and κ' both depend on r, and $p = \alpha_d \cdot \kappa\left(\frac{N - \frac{r}{\delta_0} \cdot M}{m_f}\right)$. Therefore, we have the following result for the operator's optimal pricing strategy.

Proposition 4.8. *Under the optimal pricing strategy (p, r) to maximize R'_p given in Eq.(34), we should have*

$$\left(\frac{p}{r}\right)^* = \alpha_d \cdot \kappa' \cdot \frac{1}{\delta_0} \cdot \frac{M}{m_f} + 2 > 2. \tag{39}$$

We now explore the case where we try to minimize the total social cost, i.e., $TC' - R'_p(r) + R'_s(r)$. It can be obtained from Eq.(34), Eq.(35), and Eq.(36) that $r = \frac{\alpha_d \cdot \kappa + N \cdot \alpha_d \cdot \kappa' \cdot \frac{1}{m_f}}{\alpha_d \cdot \kappa' \cdot \frac{1}{\delta_0} \cdot \frac{M}{m_f} + 1}$.

Proposition 4.9. *Under the optimal pricing strategy (p, r) to maximize $TC' - R'_p - R'_s$, where R'_p is given in Eq.(34), and R'_s is given in Eq.(35), and TC' is given in Eq.(36), we have*

$$\left(\frac{p}{r}\right)^{**} < 1. \tag{40}$$

The above result is not trivial, we thus present more details. To minimize total social cost, we should have $r = \frac{\alpha_d \cdot \kappa + N \cdot \alpha_d \cdot \kappa' \cdot \frac{1}{m_f}}{\alpha_d \cdot \kappa' \cdot \frac{1}{\delta_0} \cdot \frac{M}{m_f} + 1}$, or equivalently, $r \cdot \left(\alpha_d \cdot \kappa' \cdot \frac{1}{\delta_0} \cdot \frac{M}{m_f}\right) + r = \alpha_d \cdot \kappa + N \cdot \alpha_d \cdot \kappa' \cdot \frac{1}{m_f}$. Note that $r \cdot \left(\alpha_d \cdot \kappa' \cdot \frac{1}{\delta_0} \cdot \frac{M}{m_f}\right) < N \cdot \alpha_d \cdot \kappa' \cdot \frac{1}{m_f} \leftarrow r \cdot \frac{1}{\delta_0} \cdot M < N \leftarrow r \cdot \frac{1}{\delta_0} \cdot M = n_s < N$ since an interior equilibrium is assumed. Therefore, $r > \alpha_d \cdot \kappa$. Besides, as it is socially preferable for some travelers to use the shared parking spaces, therefore, the price p should be set in a way where $c_f \geq c_s$, which indicates that $\alpha_d \cdot \kappa \geq p$. It follows that $r > \alpha_d \cdot \kappa \geq p$, and Eq.(40) holds. It is also obvious that

$$\left(\frac{p}{r}\right)^{*} > \left(\frac{p}{r}\right)^{**}. \tag{41}$$

This is expected that a profit-maximizing platform operator will set a higher price-rent ratio than a social cost minimizing operator, which will also be verified in the numerical analysis in Section 5.

We now compare the case when a positive number of travelers choose shared parking and the case without shared parking. We particularly would like to highlight the potential for a "win-win-win" situation for the three parties involved in the parking sharing problem, which has also been highlighted in the numerical experiments in Section 5.

At the interior equilibrium with parking sharing, the number of free curbside parking users is $n_f > 0$, and the number of shared parking users is $n_s > 0$. For the case without shared parking, all the N travelers use free curbside parking, denote the travel cost of users as c'_f, which is equal to $\alpha_d \cdot \frac{D}{v} + \alpha_d \cdot \kappa \left(\frac{N}{m_f}\right)$.

For the travelers, after parking sharing is introduced, equilibrium travel cost changes from $c'_f = \alpha_d \cdot \frac{D}{v} + \alpha_d \cdot \kappa \left(\frac{N}{m_f}\right)$ to $c_f = \alpha_d \cdot \frac{D}{v} + \alpha_d \cdot \kappa \left(\frac{n_f}{m_f}\right)$. It is evident $c_f - c'_f = \alpha_d \cdot \kappa \left(\frac{n_f}{m_f}\right) - \alpha_d \cdot \kappa \left(\frac{N}{m_f}\right) < 0$, which means that the travelers are better off.

For the M parking owners, we arrange them in an increasing order based on the inconvenience cost (the first owner has zero inconvenience and the M-th owner has the maximum inconvenience δ_0). After parking sharing is introduced, the y-th parking owners has a net benefit of $max\{r - \delta_0 \cdot \frac{y}{M}, 0\} \geq 0$, this means that some parking owners who have shared their parking are better off (the first n_s owners) and others are at least not worse off.

For the platform operator, it is evident from Proposition 4.9 that if the objective is to minimize social cost, the platform should be subsidized. However, if the platform is profit-seeking, by setting $p \geq r$, the platform operator will earn non-negative revenue (e.g., Proposition 4.8 describes a profit-maximizing situation) and is better off or at least not worse off.

5 Numerical Studies

This section presents some numerical experiments to illustrate the proposed model. Particularly, with a linear traffic corridor, we firstly illustrate the effectiveness of the developed solving algorithm. We then examine the platform revenue, owner net benefit, total social cost, and total user cost under different platform pricing strategies for both the linear traffic corridor case and the simplified case without location choice.

We start this subsection with summarizing the basic common numerical setting in Table 2.

Table 2: Summary of basic numerical settings

Parameters or Functions	Specification
Total demand	$N = 4000$
Driving and walking speeds	$v = 25(km/hr)$; $w = 5(km/hr)$;
Value of driving and walking times	$\alpha_d = 25(GBP/hr)$; $\alpha_w = 30(GBP/hr)$;

Moreover, we adopt a cruising time function as follows: $\kappa(q) = \kappa_0 + \kappa_1 \cdot (\kappa_2 + q)^{\kappa_3}$, where $\kappa_0 = 0.5(min)$, $\kappa_1 = 2(min)$, $\kappa_2 = 1$, and κ_3 is a piece-wise linear and increasing function of p. The following Figure 3 displays the cruising time against parking occupancy rate. As can be seen, when parking occupancy goes beyond 80%, and cruising time starts to grow more sharply.

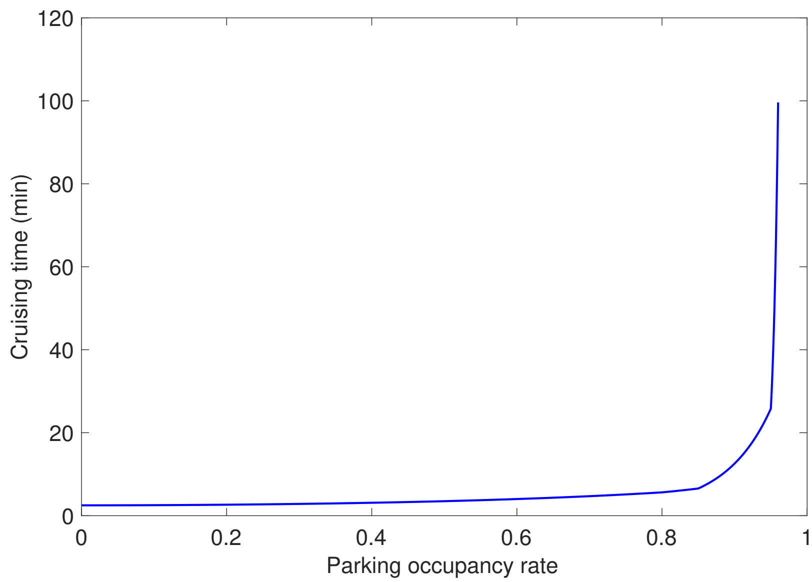

Figure 3: The cruising time function: time vs. occupancy rate

5.1 Linear Traffic Corridor

For the linear traffic corridor case, we let $d = 2(km)$ and $D = 5(km)$, and discretize d into $K = 20$ space intervals with a length of $\Delta x = 0.1(m)$. Furthermore, for the i-th interval, total number of free curbside parking spaces is $m_f(i) = 400$, and the total number of potential shared parking is $M(i) = 400$, Inconvenience cost distribution $\delta_i \sim \mathcal{U}[0, 8](GBP)$ for all $i \in \{1, 2, ..., K\}$.

To show the convergence of the bi-section based algorithms discussed in Section 3.2, we define the following term to measure the discrepancy for the convex combination method (mentioned in Table 1).

$$e_1 = \sum_i \frac{n_f(i)}{N_f} \cdot \left| c_f(i) - \min_j \{c_f(j)\} \right|. \tag{42}$$

where $i \in \{1, 2, ..., K\}$. Furthermore, we define the following gap for the bi-section based method to adjust the split between curbside parking and shared parking usage, i.e.,

$$e_2 = \frac{\left| N_f^u - N_l^u \right|}{N}. \tag{43}$$

where N_f^u and N_f^l are the bounds mentioned in Table 1. If $e_1 \to 0$, it means that the total curbside parking flow N_f is equilibrated over space x, i.e., for the locations with positive flow, travel cost approaches the minimum cost. If $e_2 \to 0$, it means that overall choice split between curbside and shared parking also converges to the equilibrium value.

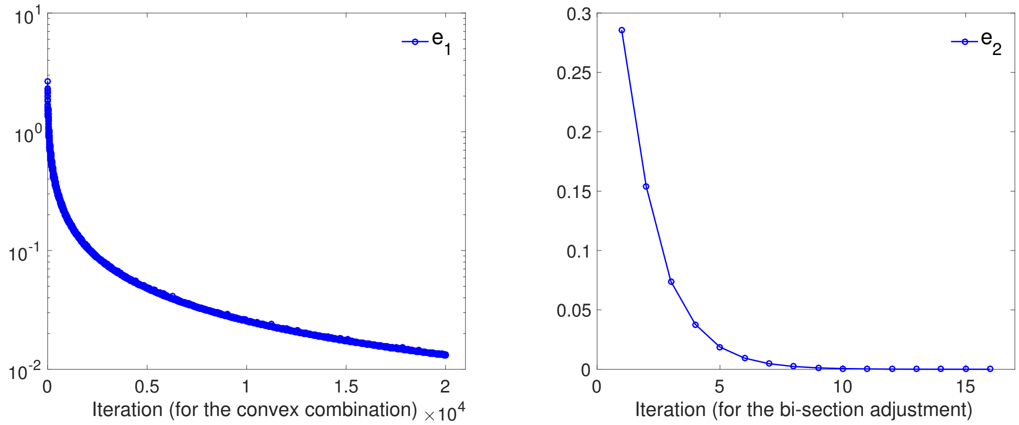

Figure 4: The evolution of the two terms e_1 and e_2 over iterations

Figure 4 shows the evolution of the two terms e_1 and e_2 defined in the above. Moreover, for illustration purpose we only show the evolution of e_1 under a given N_f. As can be seen, these terms approach zero over iterations and the solution converges. Note that for the results in Figure 4 the parking prices and rents are set as those depicted in Figure 6 for the Revenue Maximization (RM) case.

We have defined three cases for comparison purpose, namely, Revenue Maximization (RM) case (to minimize operator revenue R_p) as mentioned in the above, and System Optimum (SO) case (to minimize total social cost TSC), and the Original User Equilibrium (OUE) without parking sharing ($m_s(x) = 0$ for all x).

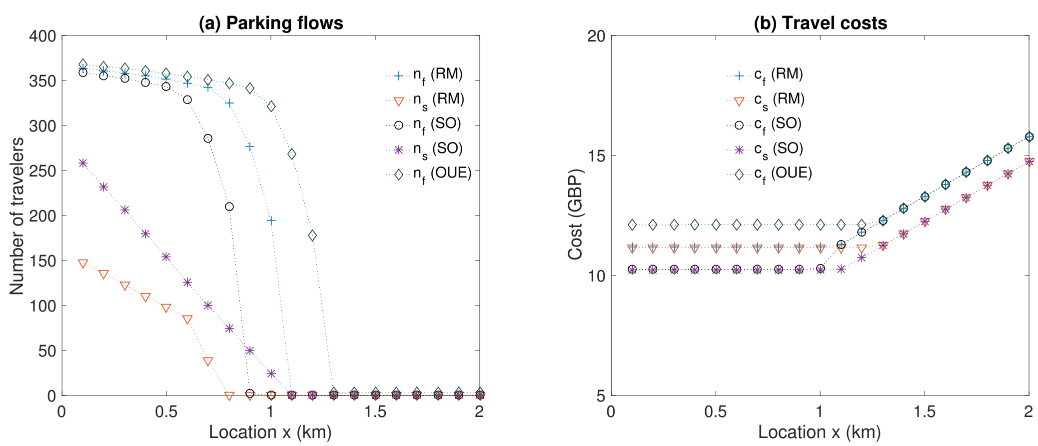

Figure 5: Comparison of three cases: RM, SO, and OUE

Figure 5 displays the (curbside and shared) parking flows over the space x and the travel costs (of curbside and shared parking option) against the parking location under the RM, SO, and OUE cases. It is evident that without parking sharing (OUE case), all the travelers have to use curbside parking, and the curve n_f for OUE is above those for the other two cases. Moreover, due to costly cruising for parking, the equilibrium travel cost for travelers is also higher, as shown in Figure 5b. Now we compare the other two cases: RM and SO. As can be seen in Figure 5a, there are more shared parking supplies under the SO case (n_f for SO is over that for RM). This is because, it is socially preferable to have sufficient shared parking supply and encourage travelers to user these shared spaces thus cruising for parking in the curbside parking side would not be too severe. And to encourage travelers to use shared parking, the travel cost c_s in the SO case is lower than the RM case.

Following the above analysis, Figure 6 further displays the prices and rents under the RM and SO cases. As can be seen, to have encourage more travelers to use shared parking (SO vs. RM), the price charged on users $p(x)$ lower in the SO case, while the rent $r(x)$ is higher in the SO case to provide more capacities. Moreover, it is evident in Figure 6 that in the SO case, the operator will have a negative benefit (thus requires subsidy from government) as the curve for p (SO) is under the curve for r (SO) slightly. In contrast, in the RM case, the operator will have a positive benefit as the prices are higher than the rents (refer to Figure 6).

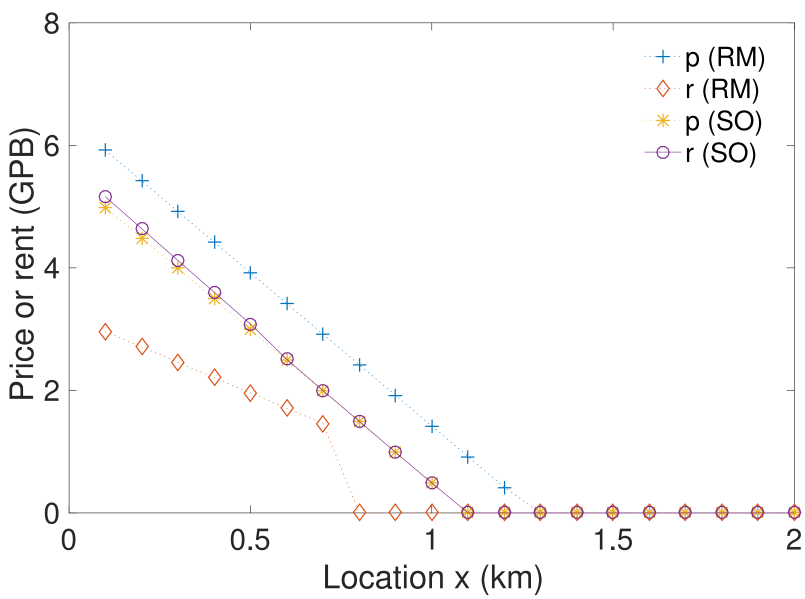

Figure 6: Price and rent under the two cases: RM and SO

Table 3 summarizes four efficiency measures (namely total operator revenue R_p, total net benefit of parking sharers R_s, total social cost TSC, and total user cost TC) for the three cases (RM, SO, and OUE). It further verifies our arguments in the above that in the SO case, the platform operator should be subsidized (negative profit of -0.14×10^3). However, while the RM case would yield a positive profit for the platform operator, the social cost is increased by $(4.21 - 3.86) \times 10^4$, and total user cost is increased by $(4.47 - 4.10) \times 10^4$ when compared to the SO case. This is mainly due to that there are less shared parking users and more curbside parking users in the RM case than the SO case, as described in Figure 6, and there are more cruising-for-parking. It is also noteworthy that in the SO case, the parking owners or sharers as a whole are also better off when compared with the RM case ($2.53 > 0.98$). Besides, if we compare the RM and SO cases with the OUE case without parking sharing, we would see the potential to reduce both total social cost and total user cost through parking sharing. Moreover, if the platform operator also gain positive net benefit (e.g., RM case), it will be a win-win-win situation when compared with the OUE.

Table 3: Comparison of three cases: RM, SO, and OUE (monetary unit: GBP)

Cases	$R_p(10^3)$	$R_s(10^3)$	$TSC(10^4)$	$TC(10^4)$
Revenue Maximization (RM)	1.71	0.98	4.21	4.47
System Optimum (SO)	-0.14	2.53	3.86	4.10
Original User Equilibrium (OUE)	0.00	0.00	4.85	4.85

5.2 Simplified case without location choice

To gain further insights and test our analysis for the analytical example in Section 4.2, we also numerically analyze the simplified parking sharing problem without location choice (there is no x any more, and all the parking is located at location $x = 0$).

We again let $D = 5(km)$. Furthermore, total number of free parking spaces $m_f = 4500$ and total number of potential shared parking is $M = 4500$. Inconvenience cost $\delta \sim \mathcal{U}[0, 3]$(GBP). We then can plot the contours of total operator revenue, total sharers' net benefit, total social cost, and the total user cost in the domain of pricing strategy (p, r), which are shown in Figure 7.

As can be seen in Figure 7a, the operator revenue R_s reaches the maximum at ($p = 1.9, r = 0.7$) with $\frac{p}{r} = 2.7 > 2$, while the minimum social cost TSC reaches the minimum at ($p = 0.5, r = 1.7$) with $\frac{p}{r} = 0.3 < 1$ in Figure 7c. These results are consistent with our analytical analysis in Section 4.2. Furthermore, 7b displays the total net benefit of the parking owners or sharers, where it is expected that its value increases with the rent paid to the parking owners. 7d shows the total user cost, which reaches the minimum when shared parking supply is large (rent should be high), and shared parking is cheap (price should be low).

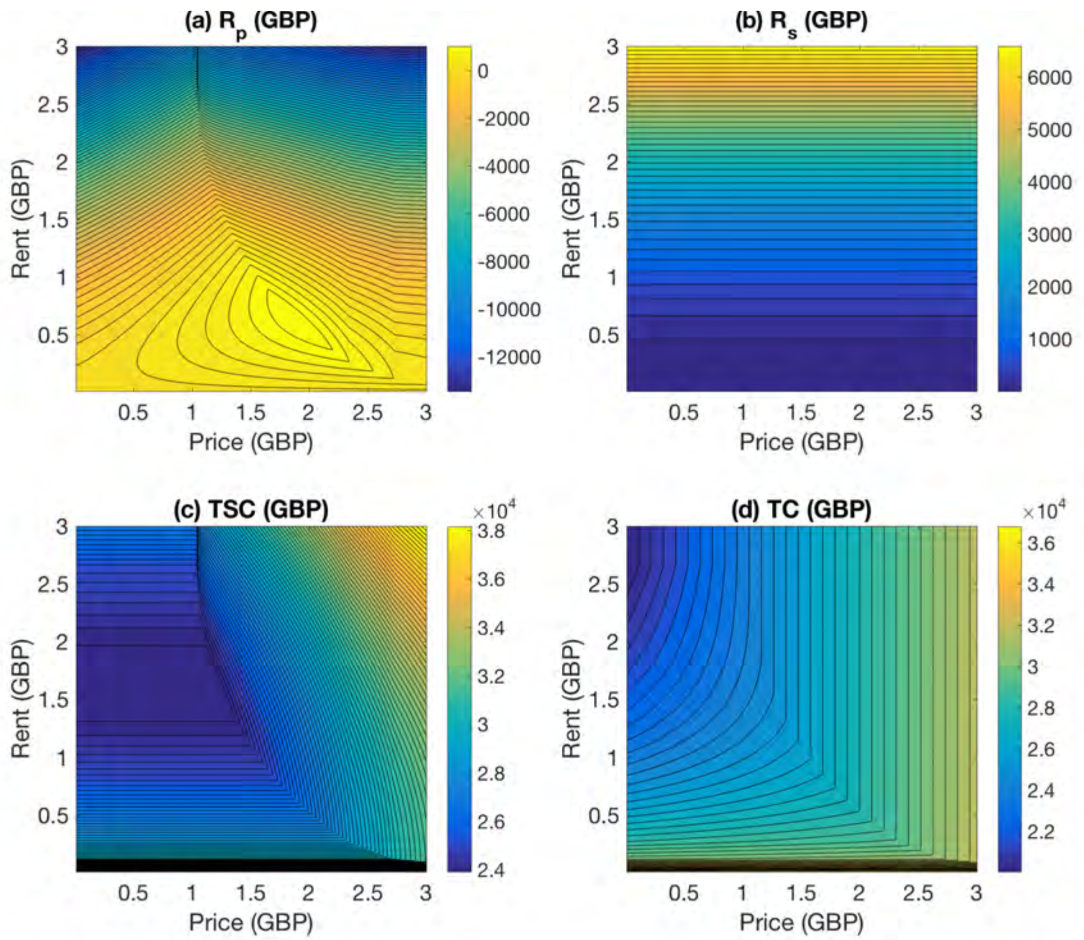

Figure 7: Revenue and Cost Contours in the domain of (p, r)

6 Conclusion

This paper formulates the commuting problem in urban cities with a hybrid supply of curb-side parking and shared parking, and examines the pricing strategies of the parking sharing platform operator. Particularly, three parties have been modeled in the two-sided market for parking sharing, i.e., the travelers who need a parking space; the residents or other parking lot owners who can supply a private parking space; and the operator who manages the parking sharing platform. The interactions among the three parties jointly determine the user parking choice equilibrium and sharer (owner) parking supply equilibrium.

We find that the parking choice equilibrium can be formulated as a minimization problem, and that its solution might not be unique unless we assume the weakly preference over parking closer to city center (refer to Assumption 3). However, without this assumption, even if the equilibrium parking flow at the shared parking side might not be unique, the parking flow at the curbside parking side, and the overall split between curbside and shared parking usage can be uniquely determined. A discretization scheme and a bi-section based approach are then proposed to the solve the parking choice equilibrium.

We then further analyze the pricing strategies of the parking sharing platform operator under profit-seeking and social-cost-minimizing objectives. Particularly, we find that in both cases, shared parking supply should be fully utilized. Moreover, for the profit-seeking operator, the shared parking price should be equal to the cruising-time-cost for a curbside space at the same location. A social-cost-minimizing operator, however, should be compensated so that a lower price/rent ratio than the profit-seeking case can be set, which in turn encourages shared parking usage and reduces cruising for parking.

This study improves and enhances our understanding of the parking sharing, pricing, and management. Beyond this, the study also further enriches the literature on the two-sided markets, where ride sharing/sourcing has attracted much more attention (Wang et al., 2016; Zha et al., 2016; Chen et al., 2017). Given the fast growth of the sharing economy worldwide, this paper opens up an avenue for the shared parking sector and delivers insightful information to both parking business stakeholders and policy makers.

This paper can be extended in several ways. Firstly, this study focuses on a linear corridor to replicate the situation where parking is distributed with an increasing distance to the city center. In future research, a general parking network such as those in Boyles et al. (2015) can be adopted, where a more detailed parking distribution over space can be modeled. Secondly, the current study does not consider other alternatives for travelers. Future research will integrate the parking sharing problem into the multi-modal system, especially when the transit service is responsive to roadway conditions (Zhang et al., 2014, 2016). Thirdly, the current study assumes a dominating parking sharing platform operator. This can be extended to the cases with multiple operators where competition exists among different operators. Fourthly, the current study assumes steady-state or static parking flow analysis, i.e., the time dimension of parking is not considered. Future research will explore the parking sharing and parking pricing problem in a time-dependent context such as those in Zhang et al. (2008).

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