Optimal transit lines structures on a general parametric city: the role of heuristics

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Abstract
Using a synthetic parametric description of a city, the merits of selected heuristics to obtain transit lines structures are analyzed using predetermined optimal strategic designs as a basis for comparison. We show that a) direct-type lines dominate for a large range of city types with the help of heuristics that extend the longer routes collecting more passengers, and b) city-specific strategic designs can be helpful to start a search for detailed design.

1. Introduction
Designing an optimal transit lines structure, i.e. the spatial layout of a public transport system, is an NP-Hard problem (Quak 2003, Schoebell & Scholl 2008) that becomes unmanageable for any real urban system and its associated street network. To overcome this issue, there are two ways to approach this aspect of transit design. One is based on heuristics, i.e. algorithms with an intuitive basis, which are conceived to deal with large networks (graphs) but do not guarantee optimality. Other approach is based on the representation of the urban system through a simple or regular network in order to work out exact solutions over it, restricting a priori the family of feasible line structures. In this second approach we can distinguish two kinds of networks: some are composed just by a few nodes and arcs and represent a specific aspect to be analyzed, while other emphasize regularity by using ring-radial or grid-type representations of a city. Obviously, this approach depends on the ability of the simplified network to capture the essential elements of the city.

Heuristic approaches have been developed along many years. Some examples are Dubois et al (1979), Ceder and Wilson (1986), Pattnaik et al (1998), Van nes et al (1998), Goossens et al (2001), Borndörfer et al (2005), Schöbel and Scholl (2008), and Cenek (2010). These articles vary in their methodology (search procedures, evaluation function, etc.) but they share one important characteristic: the possibility of working over any graph. On the other hand, articles that assume a small network to solve transit design problems can be traced back to Mohring (1972, 1976) and Jansson (1980, 1984) on a single line, later expanded by Kocur and Hendrickson (1982) and Chang and Schonfeld (1991), who work over grids but with a single destination. Jara-Díaz and Gschwender (2003), Jara-Díaz et al (2014) and Gschwender et al (2016) worked over

1 For reviews and synthesis of heuristics see Guihaire and Hao (2008) and Kepaptsoglou and Karlaftis (2009).
cross-shaped and Y-shaped networks to examine specific spatial properties. There exist some graphs of intermediate complexity where the implicit models are grids or monocentric cities (e.g. Byrne, 1975; Daganzo, 2010; Tirachini et al 2010).

Until recently there was no bridge between these two approaches. In Fielbaum et al (2016a) a parametric description of a city was proposed and used to describe different actual cities. The model was shown to be a better representation of cities than usual simple or regular network models; it was based on the idea of peripheral zones - each with a subcenter - and a center (CBD). Trips originate both at the periphery and the subcenters, ending at the subcenters or the CBD. The network is radial plus a ring that connects all subcenters. Geometric characteristics and flow pattern are parametrically described such that the model can represent monocentric, polycentric or dispersed patterns over different urban shapes. On this parametric model we have found the optimal transit lines structures by identifying first the possible strategic systems (exclusive, direct, hub-and-spoke or feeder-trunk) and then optimizing fleet and vehicle sizes, showing the conditions (value of the parameters) under which each system could become optimal (Fielbaum et al, 2016b).

In this paper we will address the question of how effective the heuristic approach is when used to find strategic structures. We will do this by applying selected heuristics to the parametric city in order to compare the resulting structures with those previously obtained from the exact approach. As we will see, some heuristics can help finding structures that improve over the strategic designs for some combinations of parameters.

In the following section we make a brief description of the parametric urban pattern described in Fielbaum et al (2016a) and summarize the strategic structures found by Fielbaum et al (2016b) that dominate on this city for different values of the parameters. In section 3 we describe the heuristics that we are going to use and we explain how we applied them to this urban scheme, showing the structures that they generate. In section 4 we first optimize (on fleets and vehicle sizes) the structures obtained with the heuristics and then we compare them with the strategic ones. We deepen the analysis in section 5 comparing the point of view of users and operators. Finally in section 6 we present the conclusions.

2. Parametric urban scheme and optimal structures

2.1 Parametric description of the urban area.

Let us summarize the parametric representation of an urban area proposed, justified and applied in Fielbaum et al (2016a,b). It is a city model based on previous studies,
flexible enough to represent many of the phenomena described in the literature about modern cities and, at the same time, simple enough to admit a precise analysis of public transport lines. Through its parameters, the city model can represent different degrees of monocentricity or polycentricity and its road structure is hierarchical, as observed in most of the cities. It is useful for our purpose because it has a recognizable structure that allows the design of (alternative) strategic line structures.

The city has a CBD and $n$ zones, each one containing a subcenter and a periphery. There are arcs linking the CBD with each subcenter, each subcenter with the periphery of its zone, and neighbouring subcenters. The distance between a subcenter and the CBD (measured in the time needed by a bus) is $T_0$, and the distance between a subcenter and its periphery is $gT_0$. The city presents radial symmetry, so the distance between consecutive subcenters is known as well. The CBD is only an attractor and the peripheries are only generators of trips; subcenters generate and attract trips. The city and the demand structure are shown in Figure 1, where $Y$ is the total patronage and $\alpha$ is the proportion of trips that start from the peripheries, out of which a proportion $\alpha$ goes to the CBD, $\beta$ to the own subcenter and $\gamma$ to the other subcenters such that $\alpha + \beta + \gamma = 1$. A proportion $b = 1 - \alpha$ of trips starts at the subcenters and go to the CBD and to other subcenters in proportions $\bar{\alpha} = \frac{\alpha}{1-\beta}$ and $\bar{\gamma} = \frac{\gamma}{1-\beta}$ respectively. There are only four flow related independent parameters: $Y, \alpha, \alpha$ and $\beta$.

Figure 1. Parametric representation of a city (symmetric version) and its demand structure.

The parameters $\alpha, \beta$ and $\gamma$ represent the degree of monocentricity, polycentricity or dispersion of the city, respectively: most trips to the CBD for $\alpha \to 1$, most trips to own subcenter for $\beta \to 1$, and most trips distributed towards other subcenters for $\gamma \to 1$. Along with $Y$ they will be varied during the analysis to explore how they affect the results.
2.2 The strategic line structures

In Fielbaum et al (2016b) we analyzed four strategic structures represented in Figure 2:

- Direct lines structure (DIR): there are lines connecting each Origin-Destination (O-D) pair, including short lines for specific pairs; nobody needs to transfer.
- Exclusive lines structure (EXC): there is one line for each pair O-D without intermediate stops.
- Hub & spoke structure (HS): lines connect the zones with the CBD, where a transfer can be made to other zones. A circular line serves the subcenters ring, so shorter trips are sometimes feasible.
- Feeder trunk (FT): feeder lines connect each periphery with its subcenter. A direct lines structure serves the subsystem composed by the subcenters and the CBD.

Figure 2. Graphic representation of the strategic lines structures
2.3 Main results

The best strategic lines structures were obtained in Fielbaum et al (2016b) by finding the cost function (minimum value of the resources consumed) for each structure, optimizing frequencies and bus sizes. Operators incur in a fixed cost for each bus plus a cost that is proportional to its capacity ($C_0$). Users perceive three types of costs: waiting time, in-vehicle time and a fixed cost for each transfer ($C_u$). The value of the resources consumed is then:

$$VRC = C_o + C_u = \sum_{l \in L} B_l (c_0 + c_1 K_l) + Y (P_{tr} \bar{t}_{tr} + p_w \bar{t}_{w}) + p_R R$$  \hspace{1cm} (1)

$B_l$ is the total fleet of line $l$ and $K_l$ the capacity of its vehicles. $\bar{t}_{tr}$ and $\bar{t}_{w}$ are the average in-vehicle and waiting times respectively, and $R$ the total amount of transfers in the system. $c_0, c_1, P_{tr}, p_w$ and $p_R$ are exogenous price related parameters.

 Capacities, travel times and waiting times can be written as a function of the frequencies for each line on each structure, such that the frequency vector becomes the main decision variable that can be optimized as explained in Fielbaum et al (2016b) in order to find the minimum $VRC$. Passenger assignment sometimes needs iterations, because of more than one possible route. The overall minimum yields the best line structure for different combinations of $\alpha, \beta, \gamma$ and $Y$, which we present in Figure 3 (from Fielbaum et al, 2016b), using colours for each structure.

In Figure 3a we analyze the effect of $\alpha$ (between 0 and 1) and $Y$ (between 800 and 480,000, logarithmic scale), keeping $\beta = \gamma$, i.e peripheric passengers whose destinations are subcenters split in half between the own subcenter and the foreign subcenters. In figure 3b patronage is fixed at $Y = 24,000$ in order to analyze the effect of $\alpha$ and $\beta$. The rest of the parameters are shown in Appendix A. In both figures each color represents one of the structures presented in Figure 2, such that the conditions (parameters values) under which each one dominates emerges clearly. To facilitate the interpretation, the monocentric ($M$), polycentric ($P$) and dispersed ($D$) cases are shown when possible.
Figure 3. Optimal strategic structure for different combination of the parameters

Figure 3a shows that \( Y \) has a very clear role: as patronage increases the structures with no transfers (DIR and EXC) dominate. HS and FT are convenient up to a moderate demand range, with FT advantageous only for very low proportion of trips to the CBD. For a mid-range patronage (Figure 3b), DIR dominates for most cases where \( \alpha + \beta \) is larger than \( Y \) (from mono to polycentric cities), except when \( \alpha \) is small. When \( Y \) is larger, EXC becomes the best. As evident, every structure can become optimal under certain conditions.

DIR does not work well for dispersed cities (no ability to collect trips) but is optimal when most of the trips are radial; routes are not always the shortest ones, inducing the largest in-vehicle times. EXC becomes best only for large patronage; requires a large fleet of relatively small buses inducing large waiting times but the smallest in-vehicle times (no intermediate stops). HS collects trips and induces large frequencies, reducing in-vehicle times; it dominates for low levels of patronage using a small fleet of large vehicles. Finally, the virtues of FT (little idle capacity) show up only if the city is dispersed.

3. Description and application of the heuristics

Out of the many heuristics available in the literature we selected four for this analysis: Dubois et al (1979), Ceder and Wilson (1986), Borndörfer et al (2005) and Cenek (2010). The selection is based on diversity - date of publication and methodological approach - and feasibility, i.e. applicable to our scheme. When required, the fixed parameters (city and cost related) are set using the same values applied to obtain the results for the strategic structures presented in the previous section. The specific details of the procedures to apply each heuristic to our city model are given in Appendix B.
3.1 Dubois et al (1979) - DBL.

This heuristic follows the principle that “Lines must be rather straight and at the same time a sufficient number of passengers must be picked up” (p. 801). Let us present the pseudo-code description of the DBL algorithm in order to provide an idea on how heuristics work in general.

1. Define the set $M$ that contains all the minimum-length routes that can not be extended preserving this condition.
2. Build the set $M^*$: For each route $L \in M$, denote $x$ its initial node and $y$ its final node, and for each $z$ not in $L$, define $L_z$ the route that starts in $x$, goes to $z$ and then to $y$ always following minimum-length paths. If the length of $L_z$ exceeds the length of $L$ by a fraction smaller than an exogenously fixed tolerance $\sigma$, then add $L_z$ to $M^*$. If for a certain route $L$ there is more than one $z$ that verifies the previous condition, add the path with the smallest length.
3. Define $\tilde{M}$ as the union of $M^*$ and the paths in $M$ connecting points that are not connected by paths in $M^*$. The routes in this set are the candidates.
4. Arrange the routes in $\tilde{M}$ according to some rule.
5. Define $M'$ as an empty set. Add the paths in $\tilde{M}$ to $M'$ until the city is fully connected (admitting transfers).
6. Calculate the portion of trips that need one or more transfers. If this portion is “small”, go to step (8). Otherwise, go to (7).
7. Add the route in $\tilde{M}$ that minimizes the number of required transfers. Go to (6).
8. Stop if the difference between the average travel time in the system and the average time in a system where all the trips go through the minimum-length path is “small”. Otherwise, go to (9).
9. Add the route in $\tilde{M}$ that decreases the most the average travel time. Go to (8).

Some of the elements in the pseudo-code are subject to interpretation, e.g. the meaning of “small” in steps (6) and (8), so when needed some necessary assumptions have to be made explicitly. In essence, this heuristic begins by identifying the set of shortest direct routes that connect the most distant O-D pairs covering all of them. Then small route deviations are admitted (through parameter $\sigma$ in step 2) in order to collect more passengers, thus reducing the number of routes and waiting times (same patronage on fewer lines), but increasing in-vehicle times. Finally some routes are selected such that the number of transfers and total travel time are within reasonable limits. It is worth noting that although $\sigma$ was conceived as a fixed tolerance in the original heuristic, we explored different values up to 0.5 (i.e. fifty percent longer than the min distance trips) that yielded different lines structures. In this process, for some $\sigma$, the parameter $\gamma$ (fraction of trips going to foreign subcenters) plays a relevant role as it affects directly the potential number of transfers (step 6 above). Besides $\gamma$, the original heuristic is
practically insensitive to the O-D structure, i.e. $\alpha, \beta, Y$ and $a$ do not affect the resulting lines structure. In fact, for $\sigma > 0.328$ ($\approx \frac{1}{3}$) the demand pattern plays no role at all. As shown in Figure 4, this yields three mostly direct route structures\(^2\). By construction, this heuristic allows the emergence of longer routes as is the case in structures I and III. Figure 5 shows the structure that presents the minimum $VRC$ for different combinations of $\alpha, \beta, \gamma$ and $Y$.

![Figure 4. Lines structures obtained with the DBL heuristic.](image)

DBL-I. $\sigma < 1/3, \gamma > 10\%$. DBL-II. $\sigma < 1/3, \gamma < 10\%$. DBL-III. $\sigma > 1/3$, all $\gamma$.

Structure III - with a larger allowed deviation from the shortest routes - dominates in most cases, because it is efficient collecting passengers and reducing waiting times; note that Figure 5b shows that this happens for $0.1 < \gamma < 0.5$ and Figure 5a suggests that this extends to $\gamma = 1$. This advantage weakens when patronage is large because

\(^2\)In all cases transfers might occur, but generally by a small fraction of the total patronage. In our application with eight zones structure II imposes transfers towards four foreign subcenters out of nine destinations, but involving less than 5% of the total flow. Although structure III imposes transfers only towards two foreign subcenters, the corresponding flow could be much larger in a dispersed city (large $\gamma$); for example, if $\gamma$ was 70% the flow requiring transfers could reach up to 20%.
large frequencies diminish the relevance of waiting times, favouring structures I and II with small or no deviations from the shortest-path. In highly monocentric cities structure II is optimal because it is based on trips to the CBD.

3.2 Ceder & Wilson (1986) - CW

This heuristic considers two steps. Routes are built starting from the possible terminals previously defined, passing through unconnected new nodes but not exceeding by more than $\sigma$ the length of the shortest route that connects the same nodes (similar to Dubois et al, 1979). The procedure is repeated until no new routes are possible from the same terminal. Then the optimal set of lines is found with their respective frequencies, similar to the optimization process explained in Section 2.3. When this is applied to our parametric city, routes start from a periphery (predefined as terminals) and necessarily goes to the own subcenter. Then it is possible to go to neighbor subcenters or to the CBD. We will start exploring the routes through the CBD such that, depending on the value of $\sigma$, we will reach $H = 3$, 5 or 7 subcenters. Again, we will treat $\sigma$ as an endogenous parameter with bound $\sigma \leq 0.5$ as in DBL; doing so, $H = 3$ if $\sigma < 0.2517 (\approx \frac{1}{4})$ and $H = 5$ otherwise. Once the routes through the CBD have been explored, the rest of the routes tour the subcenter ring until the whole graph is covered. Afterwards it is impossible to extend any route without reaching a previously built route that started from the same terminal. The two structures obtained are shown in Figure 6; they are optimized as in the previous heuristic, which yields the results shown in Figure 7 only in the $(\alpha, Y)$ space for $\beta = \gamma$, as in the $(\alpha, \beta)$ space with $Y = 24,000$ structure CW-I dominates everywhere.

Figure 6. Lines structures obtained with the CW heuristic.
The CW heuristic shares many features with DBL; it generates structures based on direct trips, and it is insensitive to the O-D matrix. The demand pattern is relevant for the second step though. To understand the results, notice that for small $\sigma$ there is predominance of shortest trips, while a bigger $\sigma$ is able to pick up more passengers with the same line. This last advantage is useful when patronage is small because passengers perceive higher frequencies with a smaller fleet, but when $Y$ increases the advantage vanishes. Coherently, structure II is more competitive when the destinations are more evenly distributed across the city.

3.3 Borndörfer et al (2005) - BOR
This heuristic starts with all possible lines; then an approximated optimization process is solved minimizing total costs and imposing that all passengers in an O-D pair are assigned to a potential route, not necessarily optimal from the viewpoint of each passenger. For our purpose, the structure is composed by those routes whose frequency is not null after that optimization process. The result is shown in Figure 8.
Although this heuristic yields also a direct lines structure, it is quite different from the previous ones. As in CW, the O-D matrix does not affect the result. Later on, for a fair comparison with the other structures, we will calculate the frequencies according to our method.

### 3.4 Cenek (2010)

This heuristic starts defining a subset of nodes as the centers of the city, which follows the same idea used to build our city model where the natural choice is to use the CBD and the subcenters. It is a purely algorithmic heuristic, building routes that pass through the arcs that would be mostly used if all passengers had shortest routes, prioritizing connections between centers. This makes this heuristic sensitive to the O-D structure when applied over this city pattern, such that the results depend on whether $a$ is larger than $a(\alpha + \frac{3}{7} \gamma) + b(\bar{\alpha} + \frac{3}{7} \bar{\gamma})$. The two possible structures are represented in Figure 9.

![Figure 9. Lines structures obtained with the Cenek heuristic.](image)

Cenek-I. $a > a(\alpha + \frac{3}{7} \gamma) + b(\bar{\alpha} + \frac{3}{7} \bar{\gamma})$

Cenek-II. $a < a(\alpha + \frac{3}{7} \gamma) + b(\bar{\alpha} + \frac{3}{7} \bar{\gamma})$

The first structure makes all peripheric passengers that go to the CBD choose between a transfer at the own subcenter and a deviation to a neighbouring subcenter, which seems disadvantageous for the users without diminishing operators’ costs. The second case is very similar to the original HS structure presented in Section 2. Grossly speaking, structure I emerges when $a \to 1$ (i.e. most trips start at the peripheries) and structure II emerges when $\alpha \to 1$ (i.e. the city is monocentric). Note that each structure emerges in specific zones of the $\alpha, \beta, \alpha$ space (i.e. they do not compete).

### 4. Results and analysis

#### 4.1 Main results

Using the approach described in section 2.3 here we present the dominant structures, i.e. those that minimize $VRC$, in the two spaces previously used: the $(\alpha, \beta)$ space with
To facilitate comparison, in both figures we present also the results obtained by Fielbaum et al (2016b) using the basic strategic structures shown in Figure 2, which happen to be inferior to the structures generated applying DBL, CW or BOR (shown in Figures 4, 6 and 8 respectively) in different zones of the city description, with some exceptions for very low and very high patronage. In general, however, the structures obtained with heuristics are (improved) extended versions of the DIR structure, so the gross result is that heuristics reinforce the strength of DIR by improving its performance by means of non-trivial modifications, such that they dominate nearly everywhere; out of these, a few transfers might occur only in the DBL structure.

![Figure 10. Optimal structures, $Y = 24,000$](image1)

Let us examine these results further. The original strategic structures dominate only for very small or very large patronage. EXC is optimal when patronage is very high, which is coherent with previous results (Fielbaum et al, 2016b; Gschwender et al, 2016), as its main problem (high waiting times) diminishes importance when frequencies have to be high. An interesting remark here is that the heuristics are not able to produce exclusive services, as they do not allow a route to skip stops. HS is optimal for small cities with an important CBD but not fully monocentric; its structure - that exploits the relevance of the CBD - is useful in these cases. FT is never optimal; the zones where it was optimal in
the base case are now dominated by CW-I, which is a direct-type structure but that has a line very similar to the feeder lines in FT.

The heuristics generate DIR-like structures by admitting deviations from the shortest possible lines in order to collect passengers along the routes, constraining the increase in travel times and sometimes the number of transfers, which is exactly the declared objective of Dubois et al (1979), as quoted above. For example, DIR has one line for each foreign zone reached through the CBD (what we called the H set in Fielbaum et al, 2016b), while CW covers all zones in H with only two lines from each periphery; by reducing the lines a good combination of a smaller fleet size with high frequencies is achieved. This same idea lies behind DBL-III (quite similar in shape to CW-I). By doing so, the resulting structures based on DBL and CW improve on the strategic structures generally by less than 20% of total cost. For highly monocentric cities and intermediate demand levels a 50% improvement can be achieved.

Figure 10 shows that as polycentrism grows (larger $\beta$) the CW heuristic works better; when $\beta < \gamma$, DBL structure is optimal, in a zone where previously DIR or EXC dominated; and when $\beta > \gamma$ CW is generally better, replacing DIR and FT. From Figure 11, if $Y \in [5,000; 100,000]$ then DBL is always optimal if $\alpha>0.3$, where previously mostly DIR won in Figure 11a; moreover, it also dominates for most of the demand range if $\alpha>0.8$, beating mainly HS and DIR. For a wide $Y$ range and $\beta = \gamma$, increasing monocentrism makes DBL more effective but CW dominates for $\alpha < 0.25$. Cenek is never optimal. Borndörfer is optimal only in the extreme scenario of a very large monocentric city (when $\beta = \gamma$).

Conceptually these results can be summarized by saying that the more dispersed the city the better is the DBL structure (usually DBL-III), while polycentrism is better served with the CW-I structure, as shown in Figure 12 where we detail which type of $\sigma$ related structure (shown in Figures 4 and 6) is the winner when either DBL or CW are optimal. The intuition behind this is that the dominant structures are nearly identical but, as explained above, DBL has a line that specifically serves various external subcenters instead of many lines which is quite good for collecting passengers. This is convenient when there is a high degree of dispersion; otherwise large fleets would be required or large waiting times would be obtained. Note that this is achieved because, in spite of the similarities, the DBL heuristic extends a path as much as possible while CW searches all possible forward movements from a node.
The flexibility of DBL and CW to obtain different line structures seems to be a relevant advantage to adapt well to different kinds of cities, but this arises because we treated the exogenously fixed tolerance ($\sigma$) parametrically, implicitly searching for a good $\sigma$. Figure 12 shows that up to $\alpha = 0.8$ when DBL dominates it is almost always with structure III. In the case of CW structure I is the dominant one but structure II is better for low patronage.

Let us recall that, as shown in Figure 3, the analysis involving only the four basic strategic structures indicated that each one dominates for different urban schemes. Although the analysis including the heuristics shows that the direct-type structures increase their range of dominance to all non-extreme cases, it is relevant to point out that both CW and DBL generate DIR-type because they are conceived as such. One might wonder whether heuristics based on EXC, HS or FT-type structures could be optimal in the zones where they originally dominated.

In summary, it is interesting to realize that, depending on the demand pattern, the solution may vary widely. Actually, five structures out of eight are optimal under certain conditions. This means that to find a good structure for a specific city, it is necessary to start observing the global conditions to determine first the type of strategic structure to be implemented. In our example, two original strategic designs - HS and EXC - dominate for very low or very high patronage respectively, and the selected heuristics are not able to generate superior schemes because they are conceived with the idea of improved direct lines. In these cases heuristics built upon the HS or EXC concepts could do a better job. For a specific city it seems convenient to use a heuristic search after a careful analysis of the advantages and disadvantages of what we have called strategic designs, such that the detailed design could be constructed with the help of a heuristic adequately conceived for that structure.
4.2. Analysis of operators’ and users’ costs

Which are the elements that make each line structure win under different urban structures represented by the parameters combinations? To explore this, we will look at the two main components of the VCR - i.e. users’ and operators’ costs - behind the different dominant structures. Note that we will not be looking for the optimal structures under the specific interest of users or operators, i.e. we will not optimize operators’ or users’ costs by their own; we simply want to capture how each overall dominant structure is perceived by each type of actor by using the optimal frequencies and capacities obtained to evaluate separately $C_D$ and $C_U$.

In Figure 13 we show the variation of $C_O$ (solid lines) and $C_U$ (dotted lines) for each structure as total patronage increases for $\alpha = 0.5$, keeping $\beta = \gamma$; the minimum total cost curve is also presented (top solid line), showing the colour of the structure that dominates as $\gamma$ grows. The structure generated after Cenek’ heuristic was suppressed as it systematically showed the largest costs for both operators and users in the whole range analyzed. Only in this figure is not represented in a logarithmic scale in order to emphasize linearity$^3$. $\alpha = 0.5$ was chosen because - as shown in Figure 12 - five structures become optimal as $\gamma$ increases.

![Figure 13. Operators, Users and Total Costs as functions of $\gamma$; $\alpha = 0.5$, $\beta = \gamma$.](image)

Figure 13 shows that - given the exogenous parameters used in our study - users costs weigh more (0 to 10,000) than operators’ (0 to 3,000) and both tend to increase with

$^3$ This is why EXC looks dominant in a larger space.
patronage for all structures\(^4\). DBL presents, by far, the lowest operator costs, but is relatively costly in terms of users’ costs. Recall that this structure is DIR-type but with a stronger tendency to collect passengers. This characteristic is good for operators, but users need to tolerate longer trips. So when DBL wins, it does because of operators’ costs.

On the other hand, leaving DBL aside momentarily, Figure 13 shows that operators’ costs are quite similar across structures while users’ costs present a larger variance, which explains a) why FT - the structure with the largests users’ costs - never wins, and b) the advantages of EXC and CW, the two structures that present the lowest users cost. In the case of EXC, the low travel times compensate its large waiting times. As explained earlier, the case of CW is similar to DBL but prioritizing shorter trips. Notice that DIR is also very convenient from this point of view.

The analysis of costs as patronage increases yields two very interesting by-products. First, note that the min total cost in Figure 13 nearly coincides with the users costs for the FT structure, which clearly shows that the inferiority of FT is due to the bad service to the users. Second, the min cost curve is quite linear with an intercept nearly at the origin, evolving from HS to CWII, DBLIII, CWI and to EXC (as shown in Figure 12). This means that the optimal design including lines structure besides fleet and vehicle sizes would induce an optimal monetary fare - equal to total marginal cost minus users’ average cost (Jara-Diaz, 2007) - that is quite close to average operators cost, i.e. covering costs (no subsidy needed); however, this should be taken with care as some important aspects in transport optimal pricing are missing, as congestion and mode choice. Nevertheless, this certainly reinforces the relation between optimal design and optimal pricing that we have emphasized earlier (Jara-Diaz and Gschwender, 2009).

In Figure 14 we present operators (solid lines) and users costs (dotted lines) for \(\beta \in (0,0.5)\) and \(Y = 24000\) (Cenek omitted as well). This figure permits a closer look at the role of the centers structure, while simultaneously allowing for a better view of costs for a medium-low value of \(Y\), something difficult to analyze from Figure 13. Here we clearly see that increasing polycentricity (\(\beta\)) while diminishing dispersion (\(\gamma\)) makes costs decrease steadily because trips are shorter. DBL, for example, presents really low users costs when \(\beta\) approaches 0.5 (and \(\gamma\) approaches 0) because the longer trips become irrelevant. Note that this provides a different view - from the perspective of public transport - to the intuitive idea of a healthy urban growth by means of subcenters.

\(^4\) CU for HS is the only non-monotonic function. This is caused by the fact that its circular line (see Figure 1) does not exist (frequency is nil) for the smallest values of \(Y\), so when this line emerges it becomes quite convenient for many passengers.
4.3 Role of the transfer penalty

Now we examine the role played by the transfer penalty $p_R$, which has been shown in previous analyses to be relevant in determining the optimal line structure (Fielbaum et al., 2016b; Gschwender et al., 2016). Although we have chosen an intermediate value within the wide range reported in the literature (e.g. Currie, 2005) it is worth looking at the results that would be obtained if we assume an extreme case in which the only cost associated to a transfer is the additional waiting time, i.e. $p_R = 0$.

The results are shown in Figures 15, where the most interesting novelty is the re-emergence of FT for low levels of monocentrism, particularly for intermediate values of $\beta$ and a wide intermediate range of patronage. The other structure that relies on transfers, HS, significantly increases its dominance area. Interestingly, DBL now dominates for small values of $\gamma$ (i.e. those close to the hypotenuse in Figure 15a). Note that the best DBL structure in the area $\gamma < 0.1$ is DBL II, different from Figure 5a (with $p_R = 24 p_{tr}$), which now dominates over CW in that area. This happens because DBL II presents mandatory transfers which are now less penalized. This confirms that $p_R$ is a key parameter indeed; given the large variability reported in the literature further research on this is badly required (see Raveau et al., 2014 for some results).
5. Synthesis, conclusions and further research.

In this paper we have presented a novel analysis of the goodness of several heuristics to build transit line structures (networks), using predetermined strategic structures as a basis for comparison under different urban conditions. The urban scheme, justified and explained in Fielbaum et al (2016a), is a simple but flexible representation of cities, with parameters that allow us to recreate different types of cities according to their size, population and centers structure.

We considered the four generic strategic structures defined and studied in Fielbaum et al (2016b), and the eight structures resulting from the application of four slightly modified heuristics available in the literature. For each structure we calculated the optimal frequencies and bus sizes of all lines, i.e. those that minimize the total value of the resources consumed, parametrically dependent on the type of city represented by patronage and the weights assigned to the CBD and subcenters. The comparison among the generated designs was then done for different combinations of the representative urban parameters that define the O-D structure of the city, obtaining best structures for different types of cities. The rest of the parameters (summarized in Appendix A) was chosen using a particular reference (Santiago, Chile).

Some structures obtained with the heuristics performed better than the four generic structures in most of the cases, showing that heuristics are a valuable tool for the detailed design, even in small networks. Particularly, the DBL (Dubois et al, 1979) and CW (Ceder and Wilson, 1986) heuristics were dominant in most of the urban scenarios analyzed. Both DBL and CW yield direct (DIR) type structures, i.e. a family of routes that allow (almost) every trip to be completed without transferring. Our original generic DIR structure was built using shortest path lines. DBL and CW improve over this, by
allowing deviations to collect more passengers and thus reducing the number of lines. In both heuristics the deviation of the lines is constrained by a tolerance $\sigma$ which is fixed a priori. Although the structures obtained with the original heuristics are mostly insensitive to the demand structure, we introduced a slight but effective change, namely we searched for the best value of $\sigma$ for each urban scenario, obtaining different lines structures with the same heuristic. By doing so, the heuristics became much more sensitive to the demand structure and thus more competitive over a wider range of cities.

The generic structures Hub and Spoke (HS) and Exclusive (EXC) resulted to be dominant when the total number of passengers was very small or very large, respectively. This result, which was also obtained in previous analyses (Gschwender et al. 2016, Fielbaum et al, 2016b and Jara-Díaz et al. 2017), remains valid now with the heuristics, although HS and EXC win only in extreme cases. Similarly, the HS structure remains optimal for monocentric cities.

The analysis with heuristics confirms the important role of the pure transfer cost as it affects strongly which structure is better under different conditions. Disregarding the pure transfer penalty (the perception of a transfer beyond additional waiting and walking times) evidently favors those structures conceived on the basis of transfers (HS, FT), which are generally inferior otherwise; to be precise, FT emerges as an alternative only for a nil transfer penalty. This is an important conclusion as it makes the valuation of such penalty a relevant line of research with strong strategic implications. Such value may be influenced by local conditions, like the weather, the habits, the quality of the bus stops and others.

The analyses of users and operators costs show some additional interesting aspects that are worth noting. One is the relatively low level of operators’ costs for the DBL structure regarding the rest, which is the basic reason why DBL dominates in some cases. Another aspect is the large variance of users’ costs across structures as patronage increases; leaving DBL aside, this is not the case for operators’ costs, whose variance is relatively low; this helps understanding the inferiority of FT and the advantages of EXC and CW in some cases.

An important lesson from our analysis can be summarized as a question: is it possible to create a generic heuristic able to produce good line structures for every city? Cities are indeed complex and multi-dimensional, such that important phenomena might well be left out when designing a specific heuristic. As seen here and elsewhere, optimal generic structures do depend on total patronage and degrees of monocentrism and polycentrism, among others, i.e there are relations between some specific urban
phenomena and some strategies that present good response to them. This is the case of - for example - direct and exclusive structures when transfers are especially unpleasant; a hub and spoke strategy (with the hub in the CBD) for monocentric cities; (potential) feeder-trunk strategies for polycentric cities; and exclusive lines for cities with many passengers. This is why it seems better to start defining global strategic structures (as DIR, EXC, HS or FT) in order to identify and prioritize some qualitative characteristics of the city that appears to be particularly important when creating a best line structure. We believe that future research should include the design of new heuristics that create structures based on HS or FT. For example, in HS the selection of the hub node (or nodes, if FT) might play a relevant role in the search for a good structure. Something similar happens with the selection of used and skipped stops and express services within the framework of an EXC based heuristic. In the first case the so-called p-hub problem from graph theory could be of use with p=1 for HS or p>1 for FT (see, for example, Ernst and Krishnamoorthy, 1996). In the second case, introducing the possibility of dead heading or short turnings could be rewarding (e.g. Cortés et al, 2011).

The development of new heuristic approaches based on a strategic view of lines structures is the most evident next step in this area of research. However, even within the logic of the direct-based heuristics presented and used here, there is a need to take into account congestion potentially caused by larger fleets and interaction with other modes. From the viewpoint of the underlying urban structure, in Fielbaum et al (2016a) different types of asymmetries were introduced to show that most city types could be represented with the parameters introduced here; finding good lines structures over asymmetric cities is yet another line of research. Regarding the demand structure, a comparison with results in other periods of the day, particularly with the afternoon peak and off-peak, is also necessary, in order to see a) which structures would be optimal with - for example - an inverted O-D matrix, and b) to study the convenience of adapting the system to allow different line structures in a single city for different times of the day. Finally, the relation between these analyses and optimal pricing should be further examined.

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References


APPENDIX A. Parameters used in the applications.

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<th>$c_1$</th>
<th>$T_0$</th>
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<th>$t$</th>
<th>$p_{tr}$</th>
<th>$p_w$</th>
<th>$p_r$</th>
<th>$\alpha$</th>
<th>$n$</th>
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<td>1.48</td>
<td>4.44</td>
<td>0.592</td>
<td>0.8</td>
<td>8</td>
</tr>
</tbody>
</table>

[$US/min$]  

APPENDIX B. Some details on the application of each heuristic to the city model.

Dubois et al (1979)

The set $M$ is composed by all the routes connecting two peripheries following an optimal path (crossing the CBD or through the subcenters ring, depending how far are the respective zones). First we assume $\sigma < 1/3$. We check all the possible routes $L_x$; we verify that $M^*$ happens to be composed only by the routes that go from one periphery to a periphery that is 3 zones away through the subcenters ring (instead of crossing the CBD). So the candidate set $\tilde{M}$ is composed by paths going through the subcenters ring that reach peripheries that are 1, 2 or 3 zones away, and paths that go the opposite subcenter (i.e., the one that is 4 zones away) crossing the CBD. To arrange $\tilde{M}$ (step 4), the authors propose criteria such as the most used or the least costly routes. We adapt the last one, arranging the set from the shortest to the largest route. In step (6), if we only added the shortest routes (i.e., the routes that go the neighbor zones) we would connect almost the whole graph: only the CBD would remain unconnected. For this not to happen we add the routes that go from each periphery to its opposite crossing the
CBD. The graph is now connected, but all the trips that go to two or more zones away (with the exception of the opposite zone) require a transfer, so the number of transfers is at least $5 (a\gamma + b\overline{\gamma})(n - 4)Y$. If $\gamma < 0.1$, we skip step (7); otherwise we add the route that go from one periphery to the periphery that is three zones away and we eliminate all the transfers. As the travel time cannot be reduced adding more routes, the final line structure has been reached.

Afterwards, we solve for $\sigma > 1/3$. Doing so, the line that goes from a periphery to the opposite subcenter presents a small variation: after going to the CBD, it goes to one neighbor of the opposite subcenter and then finishes in the opposite subcenter. Notice that in this case the number of transfers will always be smaller than $\frac{n-5}{n} (a\gamma + b\overline{\gamma})$, and when $n = 8$ this is smaller than $\frac{2}{\gamma}$; therefore, for $\sigma > 1/3$ a single structure is obtained. It is worth adding that in the case $\sigma < 1/3$ with $\gamma > 0.1$ passengers whose destination is located three zones away from their origin could go either through the subcenters ring, without making transfers, or take a shorter trip changing buses in the CBD; as their choice depends on the frequencies finding the optimal frequency requires iterations. We first assign all these passengers with no transfers and calculate the resulting optimal frequencies; then we verify whether the choice with no transfers is in fact the min user cost. If not, we re-calculate considering transfers. As expected the results depend on the parameters.

**Ceder & Wilson (1986)**

Nothing needs to be added in this case, as all the important information is included in the main text.

**Borndörfer et al (2005)**

The optimization problem presents the following built-in characteristics:

- The assignment is determined assuming that the passengers will choose the socially optimal alternative (min total cost).
- Transfers do not add any cost.
- The operator costs depend on the fleet (assuming fixed capacities) and on a fixed cost per route$^6$. The user costs are represented only by the travel time.

Finding and solving for all possible lines is a complex problem by itself. In our scheme, however, the simplicity of the city model allows us to solve that problem easily. To identify the structure emerged from this heuristic, we solve the optimization problem according to its own procedure and objective function treating vehicle sizes

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$^5$ The routes that minimize the number of transfers are those in which the user goes to the CBD first and take the second bus there. Other routes require more transfers.

$^6$ We assumed these route fixed costs to be zero to be coherent with the rest of the study.
parametrically (they are assumed to be fixed in the heuristic), finding that capacities did not change the line structure that emerges as a result including all routes with positive frequency.

Cenek (2010)
Some route notation is needed. A route will be denoted by its initial node (whose zone will be always denoted by \( i \)) and the final node (denoted by \( j \) if different from \( i \)). Note that if the destination is the own subcenter or the CBD, there is only one possible route. If the destination is a foreign subcenter, when the route goes through the CBD it is marked with an \( H \); if it goes across the subcenters ring is marked with an \( F \). For example \( LP_iCBD \) is a line that starts in a periphery, stops in the subcenter from the same zone and finish its tour in the CBD. In this case there is no need to specify \( H \) or \( F \) because there is only one route. \( LSC_iSC_jH \) is a route that starts in a subcenter and goes to another, but stopping previously in the CBD.

This heuristic assigns an initial weight to each arc equal to the number of passengers that would use that arc if they all choose the shortest route to go from their origin to their destination. With this initial setup, the algorithm starts creating and incorporating new routes using the following procedure:

1. Select the arc with the largest weight from those that finish in a center (if that weight is 0, finish). Define the node \( u \) as the other extreme of that arc.
2. Extend the line. Select the arc with the highest weight from those that are incident in \( u \). Update \( u \) as the other extreme of that arc. If \( u \) is a center or it does not have more incident arcs, go to (3). Otherwise, repeat (2).
3. The line is complete and added to the set of lines. Calculate \( w \) as the minimum weight of the arcs present in this line. Subtract \( w \) from the weight of all the arcs in the line. Go to (1).

This algorithm allows short deviations of the type \( x \rightarrow y \rightarrow x \) but in our case this never happens.

To apply this heuristic to our scheme, let us notice that the weight of each arc depends only on the types of nodes that it links, e.g. \( w \) of \( P_i - SC_i \) is \( a \) or \( w \) of \( SC_i - CBD \) is

\[
\alpha \left( \frac{3}{7} y \right) + b \left( \tilde{\alpha} + \frac{3}{7} \tilde{y} \right). 
\]

Depending on the values of the parameters \( a, \alpha \) and \( \beta \), the largest weight of an arc will be either \( P_i - SC_i \) or \( SC_i - CBD \).

**First case:** \( a > a(\alpha + \frac{3}{7} y) + b(\tilde{\alpha} + \frac{3}{7} \tilde{y}) \)
The first type of lines is $LP_i - SC_i$; the second type starts with arc $SC_i - CBD$ and is then extended to $CBD - SC_j$, so the line is $LSC_iSC_jH$ (we assume that the final subcenter is the opposite to the initial subcenter); finally, the last type starts again with arc $SC_i - CBD$ but is then extended to $SC_i - SC_{i+1}$; this line is denoted $LSC_iCBDb$. As a result, when finding optimal frequencies the line $LSC_iCBDb$ will always present a null frequency. This means that the structure is a mixture between the feeder-trunk structure (because all the passengers from the periphery take the feeder bus to their subcenter) and the hub and spoke structure (because the CBD will be a hub where almost all the passengers that go to a foreign subcenter will transfer).

**Second case**: $a < a(a + \frac{3}{7} \gamma) + b(\bar{a} + \frac{3}{7} \bar{\gamma})$

In this case the first type of lines starts with arc $SC_i - CBD$, and it extends to $P_i - SC_i$, so the line is $LP_iCBD$; the second type starts again with arc $SC_i - CBD$ and it extends to $CBD - SC_j$, so the line is $LSC_iSC_jH$ (as in the first case, we assume that the final subcenter is the opposite to the initial subcenter); finally, the last type of line starts with arc $SC_i - SC_{i+1}$, it extends to $SC_{i+1} - SC_{i+2}$ and so on, so finally the line is the circular line presented in some previous structures.