Airport Capacity and Inefficiency in Slot Allocation*

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Abstract

This paper studies the time slot allocation of flight departure when travelers have a preference for departing on peak times and the numbers of available peak time slots are constrained by airport capacities. We show that, compared to public airports, private airports may restrain their supply of peak slots strictly below their capacity levels when they serve airlines that compete to the same destinations. Such an inefficiency takes place in airports that charge low per-passenger fees and are not too busy. It does not occur in the absence of competition in destination markets.

JEL classification: R41; H21; H23.

Keywords: Slot allocation; Airport capacity; Vertical differentiation.

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1 Introduction

Over the past decades, airline traffic growth has outpaced capacities at many of the world’s major airports. Limited airport capacities are expected to become a more acute and widespread issue in the coming decades because of the expanding travel demand from emerging and developing countries.

The literature on airport capacity presents “congestion pricing” as the appropriate regulatory tool to deal with congestion. Accordingly, carriers pay a toll according to their contribution to congestion. However, despite its theoretical attractiveness, congestion pricing has seldom been adopted in the real world. By contrast, slot allocation is the usual approach to management of airport capacity. According to IATA World Scheduling Guidelines, a slot is “the permission given by a managing body for a planned operation to use airport infrastructure that is necessary to arrive or depart at a airport on a specific date and time.” Under a slot allocation system, the airport authority distributes the slots according to some allocation rule. Because of the prevalence of this system, it is important to investigate the factors underlying the slot allocation under airport capacity constraints.

A recent issue in the airport management is related to unused or misused slots. Even at congested airports, where capacity cannot satisfy the demand for slots, over 10% of the allocated slots are not eventually utilized (Zografos et al., 2013). ACI Europe provides some quantitative evidence of this problem by estimating that unused slots account for approximately Eur 20 million fewer revenues per season only for large, congested European airports (ACI Europe, 2009). Katsaros and Psaraki (2012) provide evidence of slot allocation misuse in the Greek airport system.

Moreover, there is evidence that the observed slot misuse are magnified by inefficiencies on the slot allocation mechanism per se (Zografos et al., 2012). Irrespective of the reasons, unused or misused slots indicate a poor efficiency of the allocation process itself, and they impact economically on airports, airlines, passengers and the society at large (Zografos et al., 2013). To the best of our

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1For instance, over half of Europe’s 50 largest airports have already reached or are close to their saturation points in terms of declared ground capacity (Madas and Zografos, 2008).
2The European Commission estimated that half of the world’s new traffic will come from Asia Pacific region in the next 20 years. They expect that air traffic in Europe will roughly double by 2030, and that 19 key airports will be at saturation. See MEMO/11/857.
3For an early contribution on congestion pricing see Levine (1969). Recent representative studies include Brueckner (2002, 2005). Under congestion pricing, carriers could place as many flights as they wish provided they pay the toll, thus the overall level of congestion is determined by airline decisions.
4See IATA- Worldwide slot guidelines (http://www.iata.org).
5For example, FAA (Federal Aviation Administration) capped peak hour flight movements at New York La Guardia, J.F.Kennedy, and Newark airports. As for Chicago’s O’Hare airport, FAA persuaded two major airlines United and American Airlines to reduce peak flight activities while prohibiting smaller airlines from increasing flights to fill the gap.
knowledge, this recent issue has not be analysed in the economic literature on airport policies.

The objective of this paper is to understand the airports’ choices of slot allocations in the presence of limited airport capacity, private or public management incentives and airline competition in destination markets. We examine a setting where an airport sorts the time slots according to different departure flights while airlines compete in each market to a destination city. As in Brueckner (2002), we distinguish between peak and off-peak time periods according to the travellers’ preferences for their departure time. Because travellers have preferences over the same departure time window, the airport is likely to face a capacity shortage at that time. We analyze both a public and a private airport: the public airport maximizes social welfare, while the private airport maximizes its revenue from passenger traffic. The airport is allowed to levy a uniform per-passenger fee for flight activities, this being pre-determined by administrative or regulatory bodies.

Our first finding is that private airports have incentives to allocate different type of slots to airlines that compete in same destination markets. Because peak slots occur in the passengers’ preferred time windows, the airport is able to create a quality differential by assigning competing flights to different peak and off-peak slots. As airlines choose their travel offers conditional on this quality differential, the resulting prices, passenger volumes and airport revenues depend on the slot assignments. In fact, such quality differentiation induces airlines to target different types of travellers and raises the total number of passengers in the airport while it also helps airlines to soften competition and reduce their offers. We show that the first effect dominates so that slot differentiation increases the airport

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6 We refer to a “slot” as the permission granted to a certain airline to use airport infrastructure for a planned operation at a specific time window of the day.

7 Though in practice congestion is not exclusively confined to runway congestion, and might embody other capacity dimensions such as environmental concerns, we nevertheless focus on runway congestion.

8 Our approach differs from Brueckner (2002) as follows. In Brueckner (2002)’s framework, a monopoly airport chooses the critical points on the continuum that respectively define whether to fly or not and whether to fly in peak slots or off-peak slots. Focusing on finding the optimal congestion pricing, it implicitly assumes that airport capacity is sufficient to meet peak hour demands. Unlike Brueckner (2002), our interest stems from the scarcity of peak hour slots. Thus we focus on the allocation instead of the pricing tool.

9 The analysis of a private airport seems also relevant. Although airports have long been owned by governments, there has been a significant worldwide trend towards government facilities privatization beginning from the middle of the 1980s. Following the United Kingdom, many major airports in Europe, Australia and Asia have undergone privatization or are in the process of being privatized. In principle, privatization is characterized by the transfer of ownership structure from state-owned to private enterprises.

10 The regulation of per-passenger fees is common in many countries. In EU, the Commission Regulation (EC) No.1794/2006 as well as Council Directive 96/67/EC sets out common charging scheme for air navigation and ground-handling services. In the UK, the Civil Aviation Authority regulates the charges of Heathrow and Gatwick airports to airlines since 1986. In Germany, airport charges are laid down by the main national legislative instrument German Air Traffic Act (LuftVG) (Abeyratne, 2009).
Our second finding is that private airports have incentives to keep peak slots unused when they are served by competing airlines. Airports therefore avoid fully exploiting their full peak slot capacity. As said above, private airports have incentives to create quality differentiation. They are therefore willing to allocate competing airlines on distinct peak and off-peak slots for a maximal number of destinations. When the number of peak time flights to such destinations is smaller than the number of available peak slots, airports fill only a share of the available peak slots. As a result, private airport revenues are higher if they withhold some peak slots and do not fully exploit their capacity. In this paper, we highlight the conditions under which this strategy takes place and show that it is not used by public airports. In this sense, private ownership leads to an allocative inefficiency. Such allocative inefficiency may correspond to airport managements’ practice to declare fewer slots than under their full capacities (Mac Donald, 2007). As De Wit and Burghouwt (2008, p.153) point out, “an efficient use of the slots at least requires a neutral and transparent determination of the declared capacity”. This result also explains some recent empirical evidence on allocative inefficiency (Zografos et al., 2013, Zografos et al., 2012, ACI Europe, 2009).

Things however differ if each destination is served by an airline monopoly. Monopoly airlines have indeed incentives to operate one flight to each destination. We show that they have no incentives to split their demand into different time slots and that they gather into the same airplane all the passengers departing in a same type of slot. With one flight in each destination, the airport is unable to apply the above quality differentiation strategy. As a result, private ownership does not lead to allocative inefficiency. Private and public airports choose the same slot allocation.

So far, slot allocation has drawn relatively little interest in the economic literature, with few but noteworthy contributions. As in this paper, Barbot (2004) assumes that slots for airline activities vertically differentiated products with high or low quality, but, unlike this paper, she assumes that carriers choose the number of flights they operate. Verhoef (2008) and Brueckner (2009) compare the pricing and slot policy regimes. They show that the first best congestion pricing and slot trading/auctioning generate the same amount of passenger volume and total surplus. They investigate a single congested period. Their contributions do not distinguish between peak and off-peak hours, and allow the airport to allocate slots without charges. Although this seems a plausible description of some public airports, non-profit behavior does not seem likely for a private airport. Departing from those authors, we consider that passengers have preferences for peak times and each airline
operates a single flight. We discuss the possible inefficiency when the total number of peak slots is larger than airlines’ demand.

The remainder of the paper is organized as follows. Section 2 shows some stylized facts of airline markets that motivate and support our theoretical analysis. Section 3 introduces the model. Section 4 analyzes the equilibrium in the airlines’ markets, where duopoly airlines determine their supply seats. Section 5 examines the slot allocation equilibrium, determined by the airport. Section 6.2 investigates the problem when a monopoly airline serves for each destination, while Section 7 concludes.

2 Stylized facts

To illustrate the empirical relevance of our discussion, we briefly investigate the slot allocation in the city pair markets of all the U.S. airports using the database “Airline On-Time Performance Data”, collected by the Office of Airline Information, Bureau of Transportation Statistics, U.S. Department of Transportation. We consider the flight departures during the morning (6:00 p.m. - 11:00 p.m.)\textsuperscript{11} We adopt the definition for peak load according to the website of O’Hare International Airport (ORD)\textsuperscript{12} and apply it to all airports. Also, we keep only the operating airline in each code shared destination.

In the American airport system, only four airports have adopted, over the past decades, a slot controlled system (Hawks 2015): New York’s La Guardia Airport (LGA), New York’s John Fizdgerald Kennedy airport (JFK), Newark Liberty National Airport (EWR) and Washington Regan International Airport (DCA). Therefore, along this section, we will mainly show the stylized facts from those four airports.

Tables 1 and 2 focus on the distribution of departuring flights according to airport size, city pair market structure and the distribution of flights during peak time. Here we will consider all American airports, since it is relevant to examine the market structure of city pairs, irrespective of whether an airport is slot-controlled or not. We will consider all Fridays of January 2016, so that the database

\textsuperscript{11} We focus morning flight departures because their distribution in each airport is rather well centered about the official peak time period of 8:00 to 9:00. Flight departure time distributions in mid-time and evening are less well centered on their respective official peak time periods, which introduces unnecessary variations in the data.

\textsuperscript{12} ORD defines 8-9am, 15-16pm, 17-18pm, and 19-22pm as peak hours. See Chicago O’Hare Airport Communications Guide.
includes 22,150 flight departures and 12,084 city pair connections considered. The top panel of Table 1 displays the distribution of departing flights according to airport size and city pair market structure. The rows sort airports in three groups according to their number of flight departures: less than 50 for small airports, between 50 and 125 for middle airports or more than 125 for big airports. The columns sort flights according the competition structure of their city pair market: monopoly airline, duopoly airlines, and three or more airlines. As can be observed, about three quarter of city pair service is supplied by monopoly airlines. A little bit less than a fifth is supplied by duopoly airlines. Small airports have significantly more city pairs served by monopoly airlines and big airports more city pairs served by three or more competing airlines. Middle-sized airports have a larger share of city pairs served by duopolies compared to small and big airports. When we consider the four slot-controlled airports only (LGA, JFK, EWR and DCA), we find that those airports have about 47%, 36%, and 17% of flight departures in city pair markets with one, two or more airlines.

The bottom panel of Table 1 display the share of flights departing at peak time according to airport sizes and city pair market structures. For instance, 11.52% of peak slots are used by flights operating in middle-size airports under duopoly airlines. As can be seen, peak slot occupations follow a patterns similar to the allocation of slots in the top panel. When we consider the four slot-controlled airports only, we get 47%, 38%, and 15% on average.

<table>
<thead>
<tr>
<th>U.S. Airports</th>
<th>Monopoly</th>
<th>Duopoly</th>
<th>Others</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Departing flights (%)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Small (&lt;50)</td>
<td>34.55</td>
<td>4.33</td>
<td>0.55</td>
<td>39.42</td>
</tr>
<tr>
<td>Middle (50,125)</td>
<td>28.48</td>
<td>8.93</td>
<td>2.52</td>
<td>39.93</td>
</tr>
<tr>
<td>Big (&gt;125)</td>
<td>13.29</td>
<td>4.34</td>
<td>3.01</td>
<td>20.65</td>
</tr>
<tr>
<td>Total</td>
<td>76.32</td>
<td>17.60</td>
<td>6.07</td>
<td>100.00</td>
</tr>
</tbody>
</table>

| Peak time departing flights (%) | | | | |
| Small (<50) | 26.35 | 2.77 | 1.19 | 30.32 |
| Middle (50,125) | 32.94 | 11.52 | 3.20 | 47.65 |
| Big (>125) | 12.98 | 4.55 | 4.51 | 22.03 |
| Total | 72.27 | 18.84 | 8.90 | 100.00 |

Appendix A proposes a robustness exercise on all week days of January 2016.
Table 1. Shares of departing flights: Friday mornings, January 2016, all US airports.

Table 2 shows the share of departing flights in city pairs served by only one airline. More than 75% of those include one flight only. The share of city pairs served by two flights is about four times lower while that with three flights about ten time lower. Such proportions do not change much with airport sizes. This is surprising as small airports have smaller passenger demands and give airlines less incentives to operate multiple flights in the same destination. The distribution of flights in the four slot constrained airports have similar patterns: 46%, 38% and 16%.

<table>
<thead>
<tr>
<th>U.S. Airports</th>
<th>Share of flights</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Small (&lt;50)</td>
<td>35.43</td>
<td>6.68</td>
</tr>
<tr>
<td>Middle (50,125)</td>
<td>28.07</td>
<td>6.62</td>
</tr>
<tr>
<td>Big (&gt;125)</td>
<td>13.40</td>
<td>2.86</td>
</tr>
<tr>
<td>Total</td>
<td>76.91</td>
<td>16.17</td>
</tr>
</tbody>
</table>

Table 2: Share of flights in monopoly city pairs by airport size: Friday mornings, January 2016, all US airports.

In the next empirical investigations, we are interested in examining stylized facts that can be verified in our theoretical model. Accordingly, our focus will be exclusively on the four slot-controlled airports. Since the limited number of airports considered reduces greatly the number of observations, we will examine all Fridays of all Januarys for the period 1988-2016. This allows us to exploit a sample with a number of 44,811 flight departures and 19,191 city pair connections.

Our focus is on (i) the relationship between competition and peak/off-peak slot allocation within each city pair, and (ii) city pairs served by a monopoly or duopoly. We first suggest evidence of the fact that city pairs served by a monopoly airline are more often given the same peak or off-peak slot for its flights than those served by multiple airlines. To do this, we build an index showing the airport average divergence in slot allocation between city pairs served by monopoly and duopoly airlines. In each city pair and any morning, we attribute a zero value either if all flights depart at peak or if they all depart on the off-peak slot, while we attribute a value equal to one otherwise. The index of airport average divergence in slot allocation is simply the average of those values for each

\[ \text{Index} = \frac{1}{n} \sum_{i=1}^{n} \text{value}_i \]

14 Agan, in Appendix A we consider the same exercise on all week days for of Januarys from 1988 to 2016.
Figure 1: Histograms of airport average divergence in slot allocation between city pairs offering two flights operated by monopoly or duopoly airlines. LGA, JFK, NEW and DCA Airports. Friday mornings, January from 1988 to 2016.

airport and morning. A low index suggests that many flights to the same destinations depart on the same time slot while a high index gives evidence that they are put on different slots. To control for any bias related to the number of flights, we consider the city pairs served by 2 flights only. Figure 1 shows the histograms of this index for each competition structure. The frequency distribution of monopoly city pairs is concentrated on low values of the index, suggesting small divergence in the slot allocation. Monopoly airlines are more often granted the same slot for their flights. By contrast, the frequency distribution of city pairs served by duopolies is concentrated on intermediate value of the index. This suggests that city pairs served by duopolies are less often granted the same peak slots. Their slot allocation process displays stronger divergence.

Our second empirical fact is that duopoly airlines serving a same city pair are more often granted different peak and off-peak slots in airports with tighter peak time activities. For this, we measure the peak activity tightness as the ratio of the number of peak flight departures to the number of city pairs. The higher the peak activity tightness, the more an airport uses its peak capacity to accommodate airlines. We also measure the divergence in slot allocation as the discrepancy between each airline’s distribution of peak and off-peak departures in each city pair. Towards this aim, we construct another (quadratic) divergence index that measures the (square of) difference between in each airline’s flight
distribution of peak and off-peak slot in each city pair.\footnote{This is a simpler version of the Kullback–Leibler divergence index that measures the divergence between two probability distributions and that lies between zero and infinity.} That is, $I(z) = (S_A(z) - S_B(z))^2$ with $S_A(z) \in [0, 1]$ being the share of airline $A$’s flights on the peak period within the city pair $z$ and $S_B(z) \in [0, 1]$ the share of airline $B$’s flights on the same peak period. For example, if airline $A$ has only one flight on the peak period and airline $B$ has two flights, one on the peak and the other off the peak period, those shares are respectively given by $S_A(z) = 1$ and $S_B(z) = 1/2$. The index $I(z)$ has the virtue to lie between zero and one. It takes a zero value when there is no divergence between the two airlines’ distributions over peak and off-peak slots; that is, if each airline equally shares the two slots. It takes a value of one when the divergence is maximal, which means that no airline flies in the other’s time period. The divergence indices of city pairs are averaged for each airport and date so that the airport average divergence index is given by $I = \sqrt{(1/N) \sum_{z=1}^N I(z)}$ where $N$ is the number of city pairs operated by duopoly airlines.

Figure 2 shows that the relationship between the above divergence index and peak activity tightness for each airport when we restrict the database to city pairs with duopoly airlines. By construction, there is no divergence when the activity tightness is equal to zero and one because there is no peak flight in the first case and no off-peak flight in the second case. A first glance shows that
business tightness matters in determining the level of peak slot divergence.

Our theoretical analysis below will consider the case of city pairs served by duopoly airlines and two flights. Therefore we also look at this specific case in the data. Figure 3 depicts the observed value of divergence and business activity indices computed on the subset of US city pair markets when we restrict the database to city pairs with duopoly airlines and two flights. Note that this restriction changes the value of the indices of divergence and business activity tightness in each airport. The observations are slightly and artificially "jittered" to avoid to overlapping of airport labels. The observations located on the triangle (dashed lines) indicate the maximal divergence for a given business activity: one airline has its flight departing on peak whereas the other has it off peak. The observations located at the bottom of the figure indicate the zero divergence: the two airlines have their flights allocated on the same type of departure slot. It can be observed that a significant set of airports exhibits a strong divergence in slot allocation. Also, there exists an important set of observations bunching at the point (1,1). The purpose of the next section is to give a theoretical support for this main stylized fact.
3 The model

We consider an airport offering a mass $N$ of city pair connections with heterogenous market sizes $z$ distributed according to the cumulative distribution function $G: [z, \overline{z}] \to [0, 1], 0 \leq z < \overline{z}$. The parameter $z$ being the only parameter that differentiates city pairs, each city pair is indexed by its market size $z$.

The departing time is the only quality dimension perceived by passengers. There are two travel periods $i \in \{0, 1\}$, called off-peak and peak. A peak period is the time window that consists of the most desirable travel times in a day, for instance early morning and late afternoon. The peak period may contain a collection of disjoint time intervals like 7:00-9:00 and 17:00-19:00. Off-peak periods, by contrast, are all other time intervals that are no desired as much. Each city pair market $z$ hosts a mass $z$ of vertically differentiated passengers (Gabszewicz and Thisse, 1979). Each passenger differs by her idiosyncratic taste parameter $v \in [0, 1]$, $v$ being uniformly distributed. Passengers perceive the convenience of a time slot to the value $vs_0$ and $vs_1$ where $s_0$ and $s_1$ are the intrinsic value parameter of off-peak and peak departure times ($0 < s_0 < s_1$). For simplicity we assume that each passenger flies at most once in one city pair connection and have zero reservation utility. Passengers are endowed with the utility function: $U_i(v, p) = vs_i - p$ where $i = 0$ or $1$ if they respectively take the off-peak or peak flight with fare price $p$. Under vertical differentiation, all passengers prefer peak hour at an equal price. For simplicity, destinations are assumed to be independent in the sense that they are neither substitutes nor complements; therefore the demand for one destination is irrelevant to demands for other destinations.

To operate in its city pair market, each airline pays an airport per-passenger charge $\phi$ and other variable operating costs, which we normalize to zero. For simplicity, we focus on two airline companies that operate each a single flight in various city pair markets. Passenger fees and city pair markets are assumed to be givens. For instance, in the US, passenger facility charges (PFCs) have been capped to 4.5 USD per passenger. IATA recommends "direct passenger based charges" instead of other aeronautical based charges (see IATA policy website). The above passenger fees may also

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16 Although the quality of an airline depends on many other factors, this approach allows us to concentrate on the peak capacity issue.

17 We also assume that the origin airport’s allocation decisions do not affect the flight scheduling at the destination airports. Furthermore, we focus on the case of single trip departing flights. Return trip flights can be dealt with by either an identical analysis with two runways, or simply by adding a scale factor if there is a single runway.

18 Variable costs are actually equivalent to parallel shifts in demand functions (see Appendix B). Pels and Verhoef (2004), Brueckner and Van Dender (2008), and Basso (2008) discuss non zero variable costs.

19 See Wendell H. Ford Aviation Investment and Reform Act for 21st Century, AIR-21. See
incorporate the part of departing and landing fees that varies according to the number of passengers. Moreover, it can be shown that our results hold also if the fee $\phi$ is paid by the passenger directly to the airports as it is the case in Canada and China (e.g. "explicit improvement fees"). Finally, the airport obtains significant revenues from the charges levied on shopping and parking slots. To the extent that those activities are broadly proportional to passenger flows, the associate profit per passenger can be considered as constant and equivalent to the above fixed passenger fee.

We assume that each city pair market $z$ is served by a set $\mathcal{A}(z)$ that includes two airlines (for example, $\mathcal{A}(z) = \{\text{Delta, United Airlines}\}$). Airlines compete with each other and engage in seat (quantity) competition in each destination market. In such a city pair market, airline $a \in \mathcal{A}(z)$ sets its aircraft seat capacity $q^a(z)$, taking the other airline $b$’ seat capacity $q^b(z)$, $b \in \mathcal{A}(z)$, as a given. Airline $a$’s profit in city pair $z$ is given by $\pi^a(z) = [p^a(z)q^a(z) - \phi]q^a(z)$ where $p^a(z)$ is its fare price.

The airport serves a mass $N$ of city pairs and is in charge of the slot allocation. A slot allocation is a mapping $n$ such that $n^a(z) = 1$ if the airline $a$ in city pair $z$ gets a peak slot and $n^a(z) = 0$ otherwise. The airport charges a fee per passenger $\phi$ and therefore benefits from a revenue $\sum_{a \in \mathcal{A}(z)} \phi q^a(z)$ in each city pair $z$. We study private and public airports. The private airport proposes a slot allocation that maximizes its revenues

$$\Pi = \int \sum_{a \in \mathcal{A}(z)} \phi q^a(z) \times NdG(z),$$

while the public airport implements the slot allocation that maximizes the social welfare

$$W = \int \sum_{a \in \mathcal{A}(z)} w^a(z) \times NdG(z),$$

where $w^a(z)$ is the welfare generated by the airline $a$ and the airport in the city pair market $z$. Welfare is equal to the consumer and producer surpluses generated the travel activities. To focus on the problem of peak slot capacity, we set airport operating costs to zero. The airport however faces a capacity constraint on peak slots. That is, the mass of peak time flight departures must be lower


20 Quantity competition is common in the airline economics literature. Brueckner (2009) discusses competition in "flight volume", Pels and Verhoef (2004) competition in "passengers". Brander and Zhang (1990) find empirical evidence that the rivalry between duopoly airlines is consistent with Cournot behavior.
than its capacity, say $M$. More formally, it imposes that

$$
\int_{z}^{Z} \left( \sum_{a \in A(z)} n^a(z) \right) N_d G(z) \leq M
$$

To make relevant our discussion, we impose that the airport cannot supply a slot to each flight during peak time ($M < 2N$) and the total off-peak and peak capacities are large enough to serve all flights. This assumption captures the traffic patterns of most airports.

Figure 4 shows the timing of the game. In the first stage the airport allocates peak and off-peak slots for a given fee and set of city pairs. In the second stage, the airlines operating in city pair uncooperatively choose their seat supplies based on the slot allocation. In the third stage passengers in each destination decide whether to depart during off-peak or peak time, or not to fly at all. The equilibrium concept is the sub-game perfect Nash equilibrium. The timing follows the commitment strength of each type of decision. Airports’ slot allocations are usually made for a significantly long time horizon with rather strong commitment because of the long run relationships between airports, airlines and civil aviation authorities. By contrast, the passengers’ travel choices are known to be very short run decisions while the airlines’ choice of seat capacity can be reverted with casual airline fleet restructuring. The two last stages constitute what we study below as the airlines’ markets equilibrium.

4 Airlines’ market equilibrium

In this section, we characterize the equilibrium supply of seats by the airlines (second and third stage). Towards this aim we characterize the equilibrium prices for a given aircraft seat supply by airlines operating in the off-peak and peak segments of a city pair market. We then determine the
equilibrium seat supply by the airlines in a Cournot Nash equilibrium. In this case, each airline chooses the aircraft seat capacity that maximizes its profit, taking as given by the competitor’s seat capacity and the airport’s passenger fee. In the main part of this text, we consider two airlines \( a \) and \( b \in \mathcal{A}(z) \) that flight to a same destination \( z \). Because destinations are independent for both travellers and airlines, travel demand and airline decisions can be studied separately for each market. As a result, we temporarily dispense with the reference to city pair market \( z \). Let airline \( a \) and \( b \)'s seat supply be denoted by \( q^a \) and \( q^b \). We distinguish between the cases where the airlines serve the same or different types of time slot.

### 4.1 Supplies in the same time slot

Consider first that airlines \( a \) and \( b \) supply \( q^a \) and \( q^b \) seats in flights departing in the same slot \( i \in \{0, 1\} \). Those seats are seen as homogeneous goods by passengers. In the last stage, passengers choose to travel only if \( s_i v - p \) is larger than their zero reservation utility. Given the uniform distribution of \( v \), a passenger \( v \) is willing to fly if

\[
v > v_i = \frac{p}{s_i}.
\]

(1)

The travel demand is thus given by \( x = z (1 - p/s_i) \). At the equilibrium, the market price \( p \) balances the demand and supply so that \( x = q^a + q^b \), which gives the inverse demand function

\[
p = s_i \left( 1 - \frac{q^a + q^b}{z} \right).
\]

(2)

As a result, airline \( a \)'s profits is given by

\[
\pi^a = \left[ s_i \left( 1 - \frac{q^a + q^b}{z} \right) - \phi \right] q^a,
\]

(3)

and a symmetric expression holds for airline \( b \). Profits are concave functions in seat capacities.

In the second stage, each airline chooses the number of seats that maximizes its profit, taking as given by the competitor’s seat capacity and the airport’s passenger fee. The equilibrium seat capacities solve the first order conditions so that \( q^a = q^b = z q_i \) where

\[
q_i \equiv \frac{s_i - \phi}{3s_i}.
\]

(4)
Seat capacities are symmetric across airlines and proportional to the market size \( z \). To ensure non-zero seat supply, we assume a low enough passenger fee

\[ \phi < s_0, \]  

(A0)

in the sequel. By (2) the equilibrium prices are equal to \( p^a = p^b = p_i \) where

\[ p_i = \frac{s_i + 2\phi}{3}. \]  

(5)

Since the passenger fee is a cost for airlines, prices are naturally increasing functions of \( \phi \).

### 4.2 Supplies in different time periods

Consider now that airlines \( a \) and \( b \) supply \( q^a \) and \( q^b \) seats in flights departing respectively in the off-peak and peak slot. Because of their preference for peak time departure, passengers do not see those seats as homogeneous goods. In the equilibrium of the last stage, there exist two off-peak and peak flight ticket prices \( p^a \) and \( p^b \) such that passengers with taste parameter \( v \) decide to fly off the peak time if \( vs_0 - p^a > vs_1 - p^b > 0 \) and travel at peak time if \( vs_1 - p^b > vs_0 - p^a > 0 \). Otherwise, they do not travel. The off-peak travel price must be smaller than the peak one since off-peak departure is less attractive to passengers: \( p^a < p^b \). The passenger indifferent between the peak and off-peak flights has taste

\[ v^b = \frac{p^b - p^a}{s_1 - s_0}. \]  

(6)

Likewise, a passenger is indifferent between not flying and flying has

\[ v^a = \frac{p^a}{s_0}. \]  

(7)

Given the uniform distribution of \( v \), the mass of passengers flying on the peak is then given by \( x^b = z(1 - v^b) \) while the demand for flying off peak is \( x^a = z(v^b - v^a) \). In equilibrium, market prices balance demands and supplies such that \( x^a = q^a \) and \( x^b = q^b \). Solving for those two equations, we
get the following inverse demand functions:

\[ p^a = s_0 \left[ 1 - \frac{1}{z} (q^a + q^b) \right], \tag{8} \]
\[ p^b = s_1 \left[ 1 - \frac{1}{z} \left( \frac{s_0 q^a + q^b}{s_1} \right) \right]. \]

In the aggregate, off-peak and peak travels are substitutes because the coefficients of \( q^a \) and \( q^b \) in those expressions are both negative. According to (8), airlines’ profits are expressed by

\[ \pi^a = \left[ s_0 - \frac{1}{z} s_0 (q^a + q^b) - \phi \right] q^a, \tag{9} \]
\[ \pi^b = \left[ s_1 - \frac{1}{z} (s_0 q^a + s_1 q^b) - \phi \right] q^b. \tag{10} \]

Finally, airline \( a \)'s profit is a concave function in \( q^a \) because \( p^a \) is a decreasing in it. The same applies to airline \( b \)'s profit.

In the second stage, each airline chooses the aircraft seat capacity that maximizes its profit, taking as givens by the competitor’s seat capacity and the airport’s passenger fee. The following first order conditions yield the Nash equilibrium conditions:

\[ \frac{\partial \pi^a}{\partial q^a} = -\phi + s_0 - \frac{2}{z} s_0 q^a - \frac{1}{z} s_0 q^b = 0, \]
\[ \frac{\partial \pi^b}{\partial q^b} = -\phi + s_1 - \frac{2}{z} s_1 q^b - \frac{1}{z} s_0 q^a = 0. \]

Given profits’ concavity, these conditions express a unique maximum for each firm. Solving those equations yields the equilibrium seat capacities \( q^a(z) = z q_{01} \) and \( q^b(z) = z q_{10} \) where

\[ q_{01} \equiv \frac{s_0 s_1 - \phi (2 s_1 - s_0)}{(4 s_1 - s_0) s_0} \quad \text{and} \quad q_{10} \equiv \frac{2 s_1 - s_0 - \phi}{4 s_1 - s_0}. \tag{11} \]

Seat capacities linearly increase in market size \( z \). To ensure interior solutions, we assume the condition

\[ 0 < \phi < \overline{\phi} \equiv \frac{s_0 s_1}{2 s_1 - s_0}. \tag{A1} \]
Equilibrium prices can then be computed as

\[ p^a = p_{01} = \frac{s_1(s_0 + 2\phi)}{4s_1 - s_0}, \]
\[ p^b = p_{10} = \frac{(2s_1 - s_0)s_1 + (3s_1 - s_0)\phi}{4s_1 - s_0}, \]

which are positive under the above assumption. Similarly, one readily checks that \( p_{01} < p_{10} \) and \( q_{01} < q_{10} \) so that \( p^a < p^b \) and \( q^a < q^b \). Therefore, in this model, off-peak flights supply fewer and cheaper seats than peak ones. As result, off-peak flights have lower price-cost margins and are therefore less profitable. As in the previous configuration, prices are increasing functions of the airport fee \( \phi \).

Finally, we compare the two above configurations

### 4.3 Comparison

The allocation of flights within the same or different types of slots yields different aircraft sizes. First, we get

\[ q_{10} > q_1 > q_0 > q_{01}. \]  
(12)

(see computation in Appendix C). This means that peak flights embark larger numbers of passengers than off-peak ones and that this difference is more acute when airlines are allocated to different types of time slots. Second, we have

\[ q_{01} + q_{10} > 2q_0. \]  
(13)

Accordingly, an allocation of flights to both peak and off-peak time slots generates a larger flow of passengers than an allocation of the flights to the off-peak time slot. Finally, we have

\[ q_{01} + q_{10} > 2q_1 \iff \phi < \hat{\phi} \equiv \frac{s_0s_1}{6s_1 - 2s_0}, \]  
(14)

where \( \hat{\phi} < \bar{\phi} \). At low airport passenger fee \( \phi < \hat{\phi} \), the flow of passengers is larger when the flights are differentiated in term of their departure times. The opposite holds for fees higher than \( \hat{\phi} \).

There are three forces in this setting. First, airlines have low market power when they compete for the same high valuation passengers in a peak/peak configuration. This tends to decrease prices and attract more passengers. Second, the off-peak/peak configuration allows airlines to segment
passengers in two groups and offer each one the price and departure time that it prefers. This should also increase the travel demand in the off-peak/peak configuration. Finally, because of the differentiation in departure time, each airline gets a stronger market power, which should raise prices and reduce travel demand. In property (13) the second force dominates: the introduction of a peak time departure flight permits to serve better high valuation passengers and attracts more of them. In property (14), the second force dominates only for low airport passenger fee. Indeed, when $\phi < \hat{\phi}$, travel prices are low and offering an off-peak flight at a low price permits to serve the low valuation passengers better, which increases the total passenger flow in the airport. To the contrary, when $\phi > \hat{\phi}$, prices must high to cover the fee and are unattractive to low valuation passengers. The introduction of an off-peak flight will not attract a high number of them.

We now turn to the analysis of the airport’s choice of allocation of flights to off-peak and peak slots.

5 Slot allocation

In this section we examine the slot allocation equilibrium, that entails to determine the airport behavior in the first stage of the game. In particular, we consider the slot allocation choice of private and public airports facing a limited capacity of peak slots.

5.1 Private airport

When the airport is privately owned, it maximizes its profit by allocating peak slots subject to its limited capacity. In the city pair market $z$, the number of passengers is given by the airlines’ total supply $\sum_{a \in A(z)} q^a(z)$ and their use of peak slots by $\sum_{a \in A(z)} n_a(z)$. Let us denote by $n_0(z)$, $n_{01}(z)$ and $n_1(z) \in \{0, 1\}$ the airport’s decision variables to allocate respectively an off-peak/off-peak, off-peak/peak and peak/peak configuration in the city pair market $z$. Those decisions are exclusive in the sense that only one variable takes a value equal to 1 (i.e. $n_k(z) = 1$ and $n_{k'} = n_{k''} = 0$ for $k \neq k' \neq k'' \in \{0, 1, 01\}$). We can then simplify the airport allocation problem to the following linear program:

$$\max_{n_0(z), n_{01}(z), n_1(z)} \phi \int_{\mathbb{Z}} z \left[2q_0n_0(z) + (q_{01} + q_{10})n_{01}(z) + 2q_1n_1(z)\right] NdG(z),$$
Using $n_0 = 1 - n_{01} - n_1$, we write the Lagrangian function

$$
\mathcal{L} = \int_\mathcal{Z} z \left[ 2q_0 n_{01} + (q_{01} + q_{10}) n_{01} + 2q_1 (1 - n_{01} - n_1) \right] N dG(z),
$$

where $\mu > 0$ is the Lagrange multiplier associated with the capacity constraint and where we dispense decisions variables with their argument $z$. We treat the variables $n_k$ as continuous functions on $[0, 1]$ and select only the solutions on the corner values 0 and 1. Pointwise differentiation with respect to $n_{01}$ and $n_1$ yields

$$
\mathcal{L}_{01} (z) = z (q_{01} + q_{10} - 2q_0) - \mu,
$$
$$
\mathcal{L}_1 (z) = 2z (q_1 - q_0) - 2\mu.
$$

The first expression reflects the marginal profit of shifting one flight to the peak slot in a city pair that initially hosts two airlines with off-peak flights. By (13), the marginal profit rises with the market size $z$ and falls with more stringent capacity constraint (higher $\mu$). Similarly, the second expression gives the marginal profit of shifting the two flights from off-peak to peak slots in the city pair. By (12), it also rises with larger market size. It falls with stringent capacity constraint twice as fast as the previous case because two flights are moved to the capacity constrained peak period. By (14), when the airport passenger fee is small ($\phi < \hat{\phi}$), the flow of passengers is larger in the off-peak/peak configuration than in the peak/peak configuration ($q_{01} + q_{10} > 2q_1$). As a result, the first expression is always larger than the second so that the airport has more incentives to set-up off-peak/peak configurations than peak/peak ones. It thus prefer to allocate off-peak/peak configurations in the large markets such that $\mathcal{L}_{01} (z) > 0$ and peak/peak configurations in the small market when $\mathcal{L}_{01} (z) \leq 0$.

By contrast, when the fee is large ($\phi > \hat{\phi}$), the flow of passengers is larger in the peak/peak configuration ($q_{01} + q_{10} < 2q_1$). Then, the second expression is larger than the first for large market sizes $z$. Hence, the airport has an incentive to allocate peak/peak configurations to large destination markets. Also, one can see that both expressions are negative for small enough market sizes. In this case, the airport puts the flights off the peak period. As shown in the following proposition, the
airport allocates off-peak/peak configurations for intermediate market sizes.

At this point of the paper, we distinguish between two schedules of slot allocations:

**Definition 1** A **discriminatory slot allocation schedule** grants at most one peak flight per destination market. It allocates an off-peak/peak configuration to high demand destinations \( z \in [z^*, z] \) and an off-peak/off-peak configuration to low demand ones \( z \in [z, z^*] \) where \( z^* \) solves \( G(z^*) = 1 - M/N \).

In this allocation schedule, the airport does not use its peak slots when \( M > N \) (\( z^* = z \)). In this case, all destinations are given exactly one peak slot although \( M - N \) peak slots are unused.

**Definition 2** A **balanced slot allocation schedule** with parameter \( \omega \) grants a peak/peak configuration to high demand destinations \( z \in [z^*, z] \), an off-peak/peak configuration to intermediate demand ones \( z \in [\omega z^*, z^*] \) and an off-peak/off-peak to low demand ones \( z \in [z, \omega z^*] \) where \( z^* \) solves \( G(z^* \omega^*) + G(z^*) = 2 - M/N \).

The airport balances the slot allocation according to destination market sizes. It grants two peak slots to large market destinations and none to the small ones. Destinations with intermediate market sizes receive only one peak slot. The airport allocates all available peak slots.

Let

\[
\omega^* = \frac{2q_1 - (q_{01} + q_{10})}{q_{01} + q_{10} - 2q_0} < 1,
\]

which reflects the airport’s gain from allocating a second flight to the peak period compared to its loss of moving a second flight to off-peak time. The ratio shows decreasing returns in term of passenger flows from allocating additional peak slots to a same destination.

**Proposition 1** (i) If \( \phi < \tilde{\phi} \), the private airport implements discriminatory slot allocation schedule. Otherwise, it implements a balanced slot allocation with parameter \( \omega = \omega^* \).

**Proof.** See Appendix D.

Importantly, for small passenger fees (\( \phi < \tilde{\phi} \)), the airport adopts the rule to allocate one peak flight per destination market and does not use the full airport peak capacity when \( M > N \). This reflects the above incentive to set peak/off-peak slot configurations. In this case, some peak slots are not used although they have a value to all passengers. As mentioned above, by differentiating the departure time the airport increases the number of passengers. As shown below this is the choice of a public airport.
5.2 Public airport

We here consider a public airport that maximizes the social welfare, given by the sum of passenger surplus, airlines’ and airport’s profits.

Because ticket purchases and airport fees are transfers between those agents, they cancel in the evaluation of the social welfare. Also, because we assumed zero marginal costs for airlines and airport operations, the total welfare in a city pair \( z \), \( \sum_{a \in \mathcal{A}(z)} w^a(z) \), is equal to the gross consumer surplus obtained under the equilibrium seat supplies \( q^a(z) \in \mathcal{A}(z) \). As a result, when the airport allocates the airlines in the same type of slot \( i \in \{0, 1\} \), each flight generates a welfare level equal to \( w_i(z) = \frac{1}{2} \int_{1-2q_i}^1 vs_i z \, dv \). By contrast, when it allocates them to different types of slot, the off-peak flight generates a welfare level equal to \( w_{01}(z) = \int_{1-\phi_{01}}^{1-q_{10}} vs_{0} z \, dv \) while the peak flight yields \( w_{10}(z) = \int_{1-q_{10}}^{1} vs_{1} z \, dv \). One can check that welfare levels are proportional to market sizes so that \( w_{ij}(z) = zw_{ij} \) where

\[
 w_i = \frac{(s_i + \phi)(2s_i - \phi)}{9s_i}, \tag{15}
\]

\( i \in \{0, 1\} \), and

\[
 w_{01} = \frac{s_1(s_0 - 2\phi) + s_0\phi [3s_1s_0 + \phi (2s_1 + s_0)]}{2s_0 (4s_1 - s_0)^2}, \tag{16}
\]

\[
 w_{10} = \frac{s_1(2s_1 - s_0 - \phi)(6s_1 - s_0 + \phi)}{2 (4s_1 - s_0)^2}, \tag{17}
\]

By comparing the welfare level obtained in each configuration, one can show that

\[
 2w_1 > w_{01} + w_{10} > 2w_0, \tag{18}
\]

(see details in Appendix E). The peak/peak configuration always yields higher welfare. From the welfare viewpoint, the passengers’ and airlines’ benefits from market segmentation do not outweigh the benefits of a travel service closer to passengers’ preferences for peak time and for the low prices resulting from intense competition in the peak/peak market.

The airport’s allocation problem then simplifies to the following linear program:

\[
 \max_{n_0(\cdot), n_{01}(\cdot), n_1(\cdot)} \phi \int_{\tilde{z}} \int_{\tilde{z}} [2w_0 n_0(z) + (w_{01} + w_{10}) n_{01}(z) + 2w_1 n_1(z)] N dG(z),
\]
subject to

\[ \int_{\tilde{z}}^{z} \left[ n_{01}(z) + 2n_{1}(z) \right] \cdot N dG(z) \leq M. \]

where \( n_0(z) \), \( n_{01}(z) \) and \( n_1(z) \in \{0, 1\} \) are the airport’s decision variables as defined in the previous section. The problem is solved in the same way. Pointwise differentiation of the Lagrangian function yields

\[ L_{01}(z) = z(w_{01} + w_{10} - 2w_0) - \mu, \]
\[ L_1(z) = z(2w_1 - 2w_0) - 2\mu, \]

where \( \mu \geq 0 \) is the Lagrange multiplier associated with the capacity constraint.

As in the previous section, those expressions reflect marginal increase in welfare obtained from shifting respectively one or two flights from the off-peak to the peak period. They all increase in market size \( z \) but, in contrast to the previous section, by (18), the second expression always increase faster than the first. As a result, the airport has an incentive to set up peak/peak configurations in sufficiently large destination markets. Also, because those expressions are negative for low enough \( z \), it has no incentive to choose a configuration with a peak slot for small enough destination market. For intermediate destinations market, the following proposition shows that the airport chooses to allocate a off-peak/peak configuration. Let

\[ \omega^o = \frac{2w_1 - (w_{01} + w_{10})}{(w_{01} + w_{10}) - 2w_0} < 1, \]

which reflects the welfare gain from allocating a second flight to the peak period compared to its loss of moving a second flight to off-peak time. The ratio indicates decreasing welfare gains in allocating additional peak slots to a same destination.

**Proposition 2** The public airport implements a balanced slot allocation schedule with parameter \( \omega^o \). It offers all available peak slots.

**Proof.** See Appendix F. ■

The public airport always operates at its maximum peak slot capacity. The reason is that it internalizes passengers’ loss when departure times are away from their preferences. This contrasts with the private airport that keeps some unused peak slots. This is particularly true for small fees \( \phi < \tilde{\phi} \). Indeed, when \( M < N \), the private airport implements an off-peak/peak configuration to all
city pairs whereas the public airport allocates such a configuration only to destination markets with size \( z \in [z^o, z^o] \). Hence, only a mass \( N(G(z^o) - G(z^o)) \) of city pairs gets the socially efficient slot allocation. Similarly, when \( M > N \), only the destination market market sizes \( z \in [z^o, z^o] \) get an off-peak/peak slot allocation in both public and private airports and only those with \( z \in [z^o, z^o] \) are allocated to the off-peak period in both types of airports. The rest of city pairs are misallocated from a welfare view point.

When \( \phi \geq \hat{\phi} \), both public and private airports adopt a similar structure for slot allocations, reserving two peak slots for large destination markets, one peak slot for medium sized markets, and none for small markets. The main difference lies in the values of the thresholds \((z^*, z^* \omega^*)\) and \((z^o, z^o \omega^o)\). From the identity \( G(z^\omega) + G(z) = 2 - M/N \) defining the balanced slot allocation schedule, one can deduce that\(^{21}\)

\[
\frac{d(z\omega)}{d\omega} < 0 < \frac{dz}{d\omega}
\]

So, suppose that \( \omega^o < \omega^* \).\(^{22}\) Then, we get \( z^o \omega^o > z^* \omega^* \) and \( z^o < z^* \) so that the interval \( [z^* \omega^*, z^*] \) includes \( [z^o \omega^o, z^o] \). As a result, private airports allocate too many off-peak/peak configurations than public airports.

**Proposition 3** For \( \phi < \hat{\phi} \), the private airport implements too many off-peak/peak slot city pairs from a welfare viewpoint. For \( \phi \geq \hat{\phi} \) the same conclusion applies if and only if \( \omega^o < \omega^* \).

The private airport keeps some peak slots unused because, by doing so, it expands the city pair markets by attracts more numerous low valuation passengers. Yet those passengers lose from travelling at an off-peak time. By contrast, the public airport internalizes this loss and uses all peak slots. In practice, airports may misallocate peak slots by misreporting their true handling capacity (De Wit and Burghouwt 2008).

The results of allocative inefficiency concerning private airports is consistent with the recent empirical evidence of unused or misused slots (Zografos et al., 2013, Zografos et al., 2012, Katsaros and Psaraki, 2012, ACI Europe, 2009).

\(^{21}\)We have \( \frac{dz}{d\omega} = -g(z\omega)/[g(z) + \omega g(z\omega)] < 0 \) and \( \frac{d(z\omega)}{d\omega} = \omega^{-1} g(z)/[g(z) + \omega g(z\omega)] > 0 \), where \((z, \omega)\) replaces \((z^*, \omega^*)\) or \((z^o, \omega^o)\).

\(^{22}\)The sign of \( \omega^o - \omega^* \) is ambiguous. See Appendix J.
5.3 Application to US airports

Our second stylized fact highlights a strong and significant level of divergence between the slot allocation of duopoly city pairs. The theoretical analysis shows that slot allocation can be used by airports with limited capacity as a discrimination tool in markets where several firms compete (Proposition 1 and 3). Hence the divergence can be explained by the incentive to discriminate. We now re-examine Figure 3 in the context of our model.

When city pair markets are served by two airlines and two flights, the shares of each airline’s flights on the peak period \((S_A, S_B)\) take only two values: zero or one. As a result, the divergence index \(I(z) = (S_A(z) - S_B(z))^2\) simplifies to zero if the two flights operate within the same slot and to one otherwise. More formally, we have \(I(z) = n_1(z)\) so that the square of the aggregate divergence index \(I\) is given by \(I^2 = \int_{\xi}^{\bar{\xi}} n_1(z) dG(z)\). The concept of divergence between airlines’ peak slot distribution therefore maps to the concept of slot discrimination that we have discussed above. As before, the activity tightness is defined by the ratio of peak flight departures to city pairs as \(T = \int_{\xi}^{\bar{\xi}} [n_1(z) + 2n_2(z)] dG(z)\).

We show in Appendix G that, for \(\phi < \widehat{\phi}\), the divergence index of a private airport is given by \(I^2 : [0, 1] \rightarrow [0, 1]\) such that \(I^2(T) = T\). This is because they allocate one peak flight per destination market. In airports constrained on their peak capacity, both the divergence index and activity tightness increase with peak capacity. By contrast, in unconstrained airports, the divergence index remains at its maximum level \(I = 1\) whereas the activity tightness remains at \(T = 1\) because the number of peak flight departures remain equal to the number of destinations. Thus, airports with inefficient use of peak slots are located at the point \((I^2, T) = (1, 1)\). This linear relationship between business tightness and slot divergence by the bold line and point in Figure 5.

Things are different for high airport fees \((\phi \geq \widehat{\phi})\). In this case, the divergence index of a private airport is shown to be given by \(I^2 : [0, 2] \rightarrow [0, 1]\) such that \(I^2(T) = G[z^*(T)] - G[\omega^*z^*(T)]\) where \(z^*(T)\) solves \(G(z^*\omega^*) + G(z^*) = 2 - T\). Since \(G\) is increasing it readily comes that \(z^*(T)\) is a decreasing function. The index in fact measures the share of market destinations with size \(z \in [\omega^*z^*, z^*]\). The shape of the index depends on the underlying distribution of market size \(G\). It can be shown that for low enough \(T\) so that \(\omega^*z^*(T) < \bar{z} < z^*(T)\), the index is given by the linear increasing relationship \(I^2(T) = T\) while it is given by the linear downward sloping relationship \(I^2(T) = 2 - T\) for \(T\) high enough so that \(\omega^*z^*(T) < \bar{z} < z^*(T)\). Therefore, when market size heterogeneity is very small \((\bar{z} \rightarrow \bar{z})\), the index is given by the combination of those two linear functions and the graph of the
index $I^2(T)$ is given by the triangle displayed with a dashed lines in Figure 5. It can be shown these results apply for uniform and Pareto distribution of city pair market sizes.\textsuperscript{23} The important point is that the absence of a “bunching” at the point $(I^2, T) = (1, 1)$ and the existence of a downward sloping section on the graph of the index.

The empirical prediction presented in Figure 5 is consistent with both figures 2 and 3. Furthermore, a relevant set of airports has indices that bunch to the point (1,1) (12 over 80). This suggests some inefficiency in slot allocation.\textsuperscript{24}

![Figure 5: Theoretical relationship between airport tightness and slot divergence indices.](image)

6 Discussion

Having established the factors underlying the peak slot allocation in duopoly destinations, we now discuss the case of airport with laissez-faire and the case of destinations with monopoly airlines.

6.1 Laissez-faire and slot markets

The economic literature advocates the implementation of slot allocation with auctions (Brueckner, 2009). Earlier, Grether Isaac and Plott (1979) proposed to use the competitive sealed-bid auctions for primary market. In practice, slot auctioning is a difficult task in hub airports since it requires to allocate complementary flights to various slots (combinatorial auction) (Rassenti, Smith and Bulfin,

\textsuperscript{23}The applications to uniform and Pareto distributions of city pair market sizes are available upon request.

\textsuperscript{24}See robustness checks in Appendix A.
1982). However, up to now, few airports have set up such auction mechanisms because of airlines’ resistance to a transfer of their rents and the complexity of such auction. Slot auctioning have been fully or partly implemented in a few international airports like Hong Kong, Guangzhou Baiyun .... Those experiences have got mixed success because of incumbents’ market power and market manipulations. Given its prevalence in economic analysis and recommendation, we nevertheless investigate the properties of the slot allocation through the analysis of a simple slot market mechanism.

Consider again our two airlines $a$ and $b \in A(z)$ that supply each one aircraft respectively with $q^a$ and $q^b$ seats in the destination market $z$. If they share the same type of time slot $i \in \{0, 1\}$, we know that they will set a seat capacity equal to $q^a = q^b = q_i$, where we dispense with the reference to $z$ when it does not bring confusion. They will make a total profit equal to $2\pi_i = 2\left[ \left( 1 - 2q_i \right) s_i - \phi \right] z q_i$.

Similarly, when they operate on different type of slot, we have that, say, $q^a = zq_{01}$ and $q^b = zq_{10}$, which yields the total profit $\pi_{01} + \pi_{10} = \{ s_0 \left[ 1 - \left( q_{01} + q_{10} \right) \right] - \phi \} z q_{01} + \{ s_1 \left[ 1 - \left( q_{01}s_0/s_1 + q_{10} \right) \right] - \phi \} z q_{10}$.

Comparison of individual profits yields

$$\pi_{10} > \pi_{1} > \pi_{0} > \pi_{01} \quad (19)$$

(see Appendix C). Hence, airlines prefer to be the unique recipient of a peak slot. By contrast, they make their worst profit realization when they operate the unique off-peak flight and therefore prefer to apply for peak slot. As a result, there is competition for peak slots, which pushes the airport to its peak capacity constraint.

In this section we consider a market for peak slots. It is assumed that a slot market authority (e.g. Federal Aviation Authority) sets the peak slot price $t$ and collects the demand by airlines. At the slot market equilibrium, the slot price give no incentives to airlines to change time slot so that they operate their flights in the demanded slots. As before, we assume that once slots are allocated, airlines competitively set their seat capacities and passengers choose their preferred flights. Contingent on the final slot allocation, airlines in city pair market $z \in A(z)$ obtain the profits $z\pi_0$, $z\pi_{01}$, $z\pi_{10}$ and $z\pi_1$ as defined in (19). Also, as in Brueckner’s (2009), we avoid entry issues by assuming that no airlines exit in any city pair ($z$ is sufficiently large). Moreover, to eliminate implausible equilibria, we allow airlines to re-allocate their peak slots to the most profitable flights or exchange their peak slot rights with side payments. Finally, we assume that the slot market authority has the objective to

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25 Because we assumed no capacity constraint on off-peak slots, the off-peak slot price should be zero. The implementation of a market for this type of slot is not necessary.
assign slots up to peak slot capacity fulfillment. By construction, this objective forbids inefficiencies resulting from unused peak slots but, as we now show, it does not necessarily lead to the public airport’s choice.

Airlines uncooperatively decide about their slot demand. We assume a Nash equilibrium in which they do not want to deviate from their slot demand decisions. Let us denote by $y^a \in \{0, 1\}$ the slot demanded by airline $a \in \mathcal{A}(z)$: when $y^a = 1$ the airline buys a peak slot trade at price $t$, while it takes an off-peak slot at zero price when $y^a = 0$. The game between airlines $a$ and $b \in \mathcal{A}(z)$ is given by the following normal form:

$$
\begin{array}{c|c|c}
\pi^a, \pi^b & y^b & \pi^a, y^b \\
\hline
0 & (z\pi_0, z\pi_0) & (z\pi_{01}, z\pi_{10} - t) \\
1 & (z\pi_{10} - t, z\pi_{01}) & (z\pi_1 - t, z\pi_1 - t) \\
\end{array}
$$

Table 3: Normal form game for two airlines.

This game has four types of equilibria according to the destination demand size $z$. For large $z$, both airlines demand peak slots while they both prefer off-peak slots for low $z$. For intermediate $z$, the equilibrium differs according to the threshold $\omega^e$ which reflects the airline’s additional profit from being the first mover to peak time period compared to being the second one. When this ratio is larger than one, it indicates a first mover advantage. Then, if $\omega^e > 1$, only one airline demands a peak slot while the other does not. By contrast, when $\omega^e \leq 1$, both airlines make the same decision, either for two peak or for two off-peak slots.

In the slot market equilibrium, the slot market authority fills slots up to capacity. In appendix H we show that, if $\omega^e > 1$, the authority implements a balanced slot allocation with parameter $\omega^e$ and sets a slot market price equal to $t = z^* (\pi_{10} - \pi_0)$ (where $z^*$ is defined in Definition 2). This is because there exist destinations that use only one peak slot at the equilibrium. Conversely, if $\omega^e \leq 1$, every destination uses the same type of slot. The slot market equilibrium can only lead to the following allocation schedule:
Definition 3 A pooled slot allocation schedule assigns the same type of slot to the flights on the same destination: a peak/peak configuration for $z \in [z^*, \bar{z}]$ and an off-peak/off-peak for $z \in [\underline{z}, z^*)$ where $G(z^*) = 1 - M/(2N)$.

Comparing those results to the ones for private and public airport we can state the following proposition:

Proposition 4 Suppose $\omega^c > 1$. Then, in equilibrium, the slot market results in a balanced slot allocation schedule with parameter $\omega^c$. Otherwise, if $\omega^c < 1$, it yields a pooled slot allocation schedule.

Proof. See Appendix H. ■

The important conclusion from this Proposition is that the slot allocation in a slot market equilibrium can be different from the one chosen by a public and private airport. The first reason is that the willingness to pay for peak slots are driven by profits rather than welfare or passenger flows. It is also a function of individual airline’s gain rather than city pair’s gains in the public and private airports. Indeed, $\omega^c$ reflects the airline individual profits rather than city pair profit, welfare or passenger flow. The second reason lies in the peak slot competition between airlines and the equilibrium in each city pair. In a slot market with a same price for each type of slot, airlines end up with the same decision on their slot purchases. This is why the slot market equilibrium is biased toward same slot structure when $\omega^c < 1$.

Finally, it must be noted that the above slot markets has a rather simple structure and appears more equitable from the airlines’ point of view than a full fledge auction mechanism. The slot markets applies the same slot price to all airlines and also implies lower rent extraction. However, the above discussion does not explain how the slot market authority determines equilibrium slot prices. This question is left for further research.

6.2 Monopoly

Having examined competition between duopolists in each destination market, we shall now investigate the case in which each destination market is served by a single airline that acts as a monopolist operating two flights. As with the baseline model, we analyze the second stage in each possible configuration separately, while the analysis of the third stage remains unchanged.
Consider the case where a monopoly airline $m$ is offered several slots of type $i \in \{0, 1\}$ for city pair with market size $z$ ($A(z) = \{m\}$). Since passengers perceive the same utility for the flights and the airline operates under constant returns to aircraft size, the airline should be indifferent to any aircraft size. In reality, increasing returns to aircraft sizes are non-negligible and entice airlines to operate only one aircraft in the time slots of type $i$. As a result, we state that monopolist airline operates only one flight on this slot type. Its profit is given by:

$$\pi^m(z) = \left[ \left( 1 - \frac{q^m(z)}{z} \right) s_i - \phi \right] q^m(z),$$

where the number of seats is $q^m(z)$, and $s_i$ is the passenger’s intrinsic value parameter of departure times. Under the above concave profit function and ($A0$), the equilibrium travel price and number of seats are given by $p^m(z) = p^m_i$ and $q^m(z) = zq^m_i$ where

$$p^m_i = \frac{s_i + \phi}{2} \text{ and } q^m_i = \frac{s_i - \phi}{2s_i}. \quad (20)$$

The aircraft capacity is larger at peak time since $q^m_1 > q^m_0$.

The monopoly airline can be offered a off-peak and peak slot. This offer is nevertheless not valuable for the monopoly airline as the airline prefers to not to use the off-peak slot. This result is well-known in the theory of ‘menu pricing’ or ‘versioning’ that studies firms’ incentives to offer a menu of versions of a same good with different quality levels and prices (see Belleflamme and Peitz, 2010, Ch. 9.2.2.). Salant (1989) shows that monopoly firms have no incentives to offer such a menu of goods when production cost is unrelated to product quality, which is the case here because airlines’ costs are the same for all time slots. A menu allows the firm to accommodate the high and low willingness to pay and therefore leads to an expansion of the destination market, as in the case of duopoly airlines. However, the monopoly faces a cannibalization effect as the introduction of a peak flight reduces the demand for the off-peak one. As in the above mentioned literature, the negative effect of cannibalization here outweighs the benefit of market expansion. Hence, given the uniform per-passenger fee, the monopoly airline prefers to put all seats in the peak flight. In practice, it assigns a bigger airplane flying on that destination in peak time rather than operating two smaller airplanes on different types of slot. Note that this result does not hinge on fixed costs.

---

26 This configuration is mainly made for the sake of comparison. It is certainly the case in configurations where there are two (morning and evening) peak slots per day. The case where the airline merges the two flights is left for future research.
and economies of scales, as usually argued in transport literature. It adds on this argument. This result is consistent with the empirical fact presented in Table 2, according to which more than 75% of monopoly destinations adopt one flight only. The adoption of more than one flight may be explained with other factors such that as preference for frequency and hub connectivity.

The airport choice is formulated in the same way as in the previous section. A private airport allocates the slots according to the passenger flows $q^m(z)$. Its allocation problem is

$$\max_{m(\cdot)} \phi \int z [q^m_0 (1 - m(z)) + q^m_1 m(z)] N dG(z),$$

subject to peak capacity constraint $\int z m(z) N dG(z) \leq M$ where $m(\cdot) \in \{0, 1\}$ is the airport’s decision variable to allocate a peak slot in the city pair market $z$. The same allocation problem applies for the public airport, replacing the passenger flows $q^m_0$ and $q^m_1$ by welfare levels $w^m_0 = \int v0^1 v s_0 dv$ and $w^m_1 = \int v1^1 v s_1 dv$ ($w^m_0 < w^m_1$). The slot allocation is simpler than in the duopoly structures and summarized in the following proposition:

**Proposition 5** Suppose all destination markets are served by monopoly airlines. Then, airlines operate one flight per destination and the (public or private) airport fills the peak slots starting from the largest airline destination markets, until capacity is reached.

**Proof.** See Appendix I. □

Very intuitively, the airport gives a peak slot to large destination flights because the latter yields larger passenger flows. The important point is that city pairs served by monopoly airlines are given the same type of slots. By contrast, those served by duopolies receive different slot configurations. This result is consistent with our empirical fact in Section 2, which shows that monopoly airlines more often operate flights in the same type of slots.

7 **Concluding remarks**

This paper studies the allocation of departing slots in airports with limited capacity. Because slot management is the standard tool to manage airport capacity, it is important to investigate the impact of the capacity constraint on one of the main factors underlying the use of time slots, namely the preference for peak time travel.
We found that private airports have incentives to grant their departure slots so that airlines competing in the same destination market fly on different types of slots. Depending on the fees that they impose on passengers, private airports may also keep some peak slots unused in the destination markets with competing airlines. This entails that airports may not exploit their full peak slot capacity, leading to an allocative inefficiency. By contrast, airlines operating alone in a city pair market prefer to operate one flight to each destination. Airports have no incentives in using different time slots to segment those markets. This entails one flight in each destination, so that the airport cannot differentiate quality as with city pairs served by duopolies. Finally, in each airport, authorities may design and implement a market for peak slots so that no peak slot becomes unused. However the outcome of such a market does not achieve the welfare optimal slot allocation. Our results help to explain some empirical regularities in the U.S. airline markets. These results also provide a theoretical explanation to the recent phenomenon of allocative inefficiency in several airport systems.

References


Appendix A: Robustness check

In our empirical illustration above, we have gathered flights’ departure data from U.S. airport on the Fridays of January. To ensure that this day is representative of the airline traffic, we compare it with the same data on all weekdays of the same month. Weekdays are chosen to exclude irregular influx for air travel that happens on weekends. Table 4 is similar to Table 1, with all airports for the year 2016, but with all weekdays data. The total number of flights and city pairs considered now are 96,273 and 52,033, respectively. A quick comparison shows that the distribution among departing flights and peak time departures is qualitatively similar.

<table>
<thead>
<tr>
<th>U.S. Airports</th>
<th>Monopoly</th>
<th>Duopoly</th>
<th>Others</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Departing flights (%)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Small (&lt;50)</td>
<td>33.29</td>
<td>4.16</td>
<td>0.55</td>
<td>37.99</td>
</tr>
<tr>
<td>Middle (50,125)</td>
<td>30.15</td>
<td>9.06</td>
<td>2.63</td>
<td>41.83</td>
</tr>
<tr>
<td>Big (&gt;125)</td>
<td>12.80</td>
<td>4.36</td>
<td>3.02</td>
<td>20.17</td>
</tr>
<tr>
<td>Total</td>
<td>76.24</td>
<td>17.57</td>
<td>6.19</td>
<td>100.00</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Peak time departuring flights (%)</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Small (&lt;50)</td>
<td>24.35</td>
<td>2.50</td>
<td>1.18</td>
<td>28.02</td>
</tr>
<tr>
<td>Middle (50,125)</td>
<td>35.32</td>
<td>11.55</td>
<td>3.50</td>
<td>50.37</td>
</tr>
<tr>
<td>Big (&gt;125)</td>
<td>12.49</td>
<td>4.59</td>
<td>4.53</td>
<td>21.60</td>
</tr>
<tr>
<td>Total</td>
<td>72.16</td>
<td>18.63</td>
<td>9.20</td>
<td>100.00</td>
</tr>
</tbody>
</table>

Table 4. Shares of departing flights: Week day mornings, January 2016, all US airports.

We then turn on the relationship between competition and peak/off-peak slot allocation within each city pair. As in Section 2 we focus on the four slot controlled airports for the period 1988-2016, so that the number of flight departures is 222,963, while the number of city pairs is 95,522. We exploit again the index taking value 0 either if all flights depart at peak or if they all depart on the off-peak slot, while it takes 1 otherwise, and we consider the city pairs served by 2 flights only. Figure 6 shows the histograms of this index for each competition structure. As for the case in the main text, the distribution of monopoly city pairs is concentrated on low values of the index, suggesting small divergence in the slot allocation. Monopoly airlines are more often granted the same slot for their flights. The distribution of city pairs served by duopolies is concentrated on intermediate value.
Figure 6: Histograms of slot divergence in city pairs offering two flights operated monopoly or duopoly airlines. Morning time of all weekdays, of all Januaries in the period 1988-2016. Airports: LGA, JFK, NEW and DCA.

Figure 7: Airport average divergence and business tightness indices. City pairs with two airlines. LGA, JFK, NEW and DCA Airports. Week day mornings, January from 1988 to 2016.

of the index. As in the main text, this suggests that city pairs served by duopolies are less often granted a slot in the same time period.

We replicate the analysis performed in Section 2 for city pairs served by duopolies for all weekdays. The comparison of Figure 7 with Figure 2 shows that observations are qualitatively similar.

Finally, Figure 8 replicates Figure 3 for all weekdays. Figure 8 is consistent with 3.
Figure 8: Divergence index $I^2$ and activity tightness indices. City pairs with two airlines and two flights. LGA, JFK, NEW and DCA Airports. Week day mornings, January from 1988 to 2016.

**Appendix B: Airline operating costs**

We here discuss the case where airlines have non-zero operating costs, say $c > 0$. For conciseness we focus on the case of a private airport. The analysis can be developed in a similar vein as before, where the airline marginal cost is now $c + \phi$ rather than $\phi$ only. Naturally, in both configurations the volume of passengers is larger without operating cost. The conclusion drawn from the comparison of off-peak and peak configurations also applies here. It follows that, with positive operating costs airport profit is also smaller in each configuration. The condition required to guarantee positive passenger volumes in equilibrium is

$$0 < \phi < \phi' \equiv \frac{s_1s_0}{2s_1 - s_0} - c,$$

while the threshold determining the preference between peak/peak and peak/off-peak is

$$\hat{\phi}' \equiv \frac{s_1s_0}{6s_1 - 2s_0} - c.$$

Therefore, the above proposition now reads with $\phi'$ and $\hat{\phi}'$ substituting for $\bar{\phi}$ and $\hat{\phi}$. If the cost $c$ is small enough so that $\hat{\phi}' > 0$, the proposition presents the same configurations and the same issue of allocative inefficiency. The configuration peak/off-peak induces more passenger volume than configuration peak/peak so that the airport does not distribute all available peak slots and inefficiency arises. However, if $c$ is large enough so that $\hat{\phi}'$ becomes negative, all available peak slots
are distributed and allocative inefficiency never arises. Similar considerations hold for the public airport.

**Appendix C: Equilibrium seat capacities and profits**

To compare individual seat capacities, we compute

\[
q_{10} - q_1 = \frac{(s_1 - s_0)(2s_1 + \phi)}{3s_1(4s_1 - s_0)} > 0 \quad \text{and} \quad q_0 - q_{01} = \frac{(s_1 - s_0)(s_0 + 2\phi)}{3s_0(4s_1 - s_0)} > 0
\]

\[
q_1 - q_0 = \frac{(s_1 - s_0)\phi}{3s_1s_0} > 0 \quad \text{and} \quad q_{01} + q_{10} - 2q_0 = \frac{(s_1 - s_0)(s_0 + 2\phi)}{3s_0(4s_1 - s_0)} > 0.
\]

where the inequalities obtain because \(s_1 > s_0\). Also, we have

\[
q_{01} + q_{10} - 2q_1 = \frac{(s_1 - s_0)\left[s_0(s_1 + 2\phi) - 6s_1\phi\right]}{3s_1s_0(4s_1 - s_0)} > 0 \iff \phi < \bar{\phi} \equiv \frac{s_0s_1}{6s_1 - 2s_0}
\]

The equilibrium profits can be computed as

\[
\pi_i = \frac{1}{9} \frac{(\phi - s_1)^2}{s_i}, \quad \pi_{10} = s_1 \left(\frac{2s_1 - \phi - s_0}{4s_1 - s_0}\right)^2 \quad \text{and} \quad \pi_{01} = \frac{1}{s_0} \left[\frac{s_0s_1 - \phi (2s_1 - s_0)}{4s_1 - s_0}\right]^2
\]

To compare individual airline profits, we compute

\[
\pi_{10} - \pi_1 = \frac{(s_1 - s_0)\left[(7s_1 - s_0)\phi - 2s_1(5s_1 - 2s_0)\right]}{9s_1(4s_1 - s_0)^2} > 0,
\]

\[
\pi_1 - \pi_0 = \frac{(s_1 - s_0)\left(s_1s_0 - \phi^2\right)}{9s_1s_0} > 0,
\]

\[
\pi_0 - \pi_{01} = \frac{z(s_1 - s_0)(s_0 + 2\phi)\left[(s_0(7s_1 - s_0) - 2(5s_1 - 2s_0)\phi)\right]}{9s_0(4s_1 - s_0)^2} > 0,
\]

where the inequalities obtain by (A1). Also, we compute

\[
\pi_{10} - \pi_0 = -\frac{(s_1 - s_0)\left[\phi^2(16s_1 - s_0) + \phi(2s_0^2 + 4s_0s_1) + s_0(-36s_1^2 + 16s_0s_1 - s_0^2)\right]}{9s_0(4s_1 - s_0)^2} > 0,
\]

The inequality obtains because the convex quadratic function of \(\phi\) in the square brackets is negative for \(\phi \in [0, \bar{\phi}]\). Indeed, the latter has one positive root, is increasing for \(\phi > 0\) and is negative at both
\( \phi = 0 \) and \( \phi = \bar{\phi} \). Hence the profit ranking is given by

\[
\pi_{10} > \pi_1 > \pi_0 > \pi_{01}.
\]

**Appendix D: Proof of Proposition 1**

We show that for \( \phi < \bar{\phi} \), the private airport’s optimal allocation \((n_{01}, n_1, n_0)\) is given by \((1, 0, 0)\) if \( M > N \); otherwise, it is given by \((1, 0, 0)\) for \( z > z_{01} \) and \((0, 0, 1)\) for \( z \leq z_{01} \) where \( G(z_{01}) = 1 - M/N \).

For \( \phi \geq \bar{\phi} \), the optimal slot allocation is given by \((n_{01}, n_1, n_0) = (0, 1, 0)\) for \( z > z^* \), \((1, 0, 0)\) for \( \omega^* z^* < z < z^* \) and \((0, 0, 1)\) for \( z < \omega^* z^* \) where \( \omega^* < 1 \) and \( z^* \) solves \( G(\omega^*) + G(z) = 2 - \frac{M}{N} \).

Indeed, we have

\[
\mathcal{L}_{01}(z) > 0 \iff z \geq z_{01} \equiv \frac{\mu}{q_{01} + q_{10} - 2q_0},
\]
\[
\mathcal{L}_1(z) > 0 \iff z \geq z_1 \equiv \frac{\mu}{q_1 - q_0}.
\]

On the one hand, suppose \( \phi < \bar{\phi} \) so that \( q_{01} + q_{10} > 2q_1 \). Then, \( \mathcal{L}_{01}(z) > \mathcal{L}_1(z) \) for all \( z > 0 \) while \( \mathcal{L}_{01}(z) \geq 0 \) iff \( z \geq z_{01} \). The solution is \((n_{01}, n_0, n_1) = (0, 1, 0)\) for \( z > z_{01} \) and \((1, 0, 0)\) for \( z \leq z_{01} \). The binding constraint requires \( \int_{z}^{z_{01}} n_0 N dG(z) = N \int_{z_{01}}^{z} dG(z) = N(1 - G(z_{01})) = M \). That is, we get the value of \( z_{01} \) that solves \( G(z_{01}) = 1 - M/N \).

On the other hand, suppose \( \phi \geq \bar{\phi} \) so that \( 2q_1 \geq q_{01} + q_{10} \). Then \( \mathcal{L}_{01}(z) > \mathcal{L}_1(z) \) iff \( z < z^* \) where

\[
z^* = \frac{\mu}{2q_1 - (q_{01} + q_{10})}.
\]

One can check that \( z_{01} \equiv z^* \omega^* \) where

\[
\omega^* = \frac{2q_1 - (q_{01} + q_{10})}{q_{01} + q_{10} - 2q_0} < 1.
\]

One can also check that \( z_{01} < z_1 < z^* \). As a result, \( \mathcal{L}_1(z) > \max(0, \mathcal{L}_{01}(z)) \) for \( z > z^* \), \( \mathcal{L}_{01}(z) > \max(0, \mathcal{L}_1(z)) \) for \( z_{01} < z < z^* \) and \( 0 > \mathcal{L}_1(z) > \mathcal{L}_{01}(z) \) for \( z < z_{01} \). The solution is \((n_{0}, n_{01}, n_1) = (0, 0, 1)\) for \( z > z^* \), \((0, 1, 0)\) for \( z_{01} < z < z^* \) and \((1, 0, 0)\) for \( z < z_{01} \). The constraint is binding if \( \int_{z}^{z_{01}} (n_{01} + 2n_1) N dG(z) = M \), or after simplification, if \( N(2 - G(z_{01}) - G(z^*)) = M \). Therefore the optimal allocation is given by the solution \( z^* \) of \( G(z^*) + G(z) = 2 - M/N \).
Appendix E: Public airport

Note first that interior solutions obtains for \( 0 < 1 - 2q_1 < 1 \) and \( 0 < 1 - q_{01} - q_{10} < 1 - q_{10} < 1 \). This implies that \( 0 < w_i < \frac{1}{2}s_i \int_0^1 vdv = \frac{1}{4}s_i, \) \( 0 < w_{01} + w_{10} = \int_{1-q_{01}-q_{10}}^1 vs_0dv < \frac{1}{4}s_0, \) \( 0 < w_{10} = \int_{1-q_{10}}^1 vs_1dv < \frac{1}{4}s_1 \) and finally \( 0 < w_{01} = \int_{1-q_{01}-q_{10}}^1 vs_0^2dv \). It can be shown from (15), (16) and (17) that the last inequality is satisfied for \( \phi < \bar{\phi} \) and the rest for \( \phi < s_0/2 \).

We further need to show that \( 2w_1 > w_{01} + w_{10} > 2w_0 \). We compute

\[
w_1 - w_0 = \frac{(s_1 - s_0)(\phi^2 + 2s_1s_0)}{9s_1s_0} > 0,
\]

\[
2w_1 - w_{10} - w_{01} = K \left[s_1^2s_0^2 (20s_1 + s_0) + 4s_1s_0\phi (2s_1 + s_0) + \phi^2 (36s_1^2 - 19s_1s_0 + 4s_0^2)\right] > 0,
\]

\[
w_{10} + w_{01} - 2w_0 = \frac{K}{9s_0^2} \left[108s_1s_0 + s_0 (8s_0^2 - 4s_0\phi - 13\phi^2) + s_1 (28\phi^2 - 65s_0^2 - 8s_0\phi)\right] > 0
\]

where \( K = (s_1 - s_0) / [18s_0(4s_1 - s_0)^2] > 0 \). The positive sign in the second and third expressions results from \( \phi < \phi \) and \( \phi < s_0/2 \).

Appendix F: Proof of Proposition 2

We prove that the airport allocates in peak/peak configuration the destination markets \( z \in [z^*o, z^*] \) where \( z^o \) solves \( G(z^o\omega) + G(z^o) = 2 - \frac{M}{\alpha} \). It gives peak/off-peak configuration to the destination markets \( z \in [z^o\omega, z^o] \) and an off-peak/off-peak configuration to those with \( z \in [z^o, z^o\omega] \). In any case, the airport uses all available peak slots.

Pointwise differentiation yields

\[
\mathcal{L}_{01} = z (w_{01} + w_{10} - 2w_0) - \mu,
\]

\[
\mathcal{L}_1 = z (2w_1 - 2w_0) - 2\mu
\]

where \( \mu \geq 0 \) is the Lagrange multiplier associated with the capacity constraint. By (18) the second expression rises faster in \( z \) than the first one. Also, we have \( \mathcal{L}_{01} \geq 0 \) if and only if \( z \geq z_{01}, \mathcal{L}_1 \geq 0 \) if and only if \( z \geq z_1 \) and \( \mathcal{L}_1 \geq \mathcal{L}_{01} \) if and only if \( z \geq z^o \) where

\[
z_{01} \equiv \frac{\mu}{(w_{01} + w_{10}) - w_0}, \quad z_1 \equiv \frac{2\mu}{2(w_1 - w_0)} \quad \text{and} \quad z^o \equiv \frac{\mu}{w_1 - (w_{01} + w_{10})}.
\]
By (18), we have \(0 < z_{01} < z_1 < z^o\) when the constraint is binding, \(\mu > 0\). As a result, when the constraint is binding, we get that \(L_1 > \max\{L_{01}, 0\}\) so that the optimal allocation is equal to \((n_0, n_{01}, n_1) = (0, 0, 1)\) for any \(z > z^o\). Similarly, we have that \(L_{01} > \max\{L_1, 0\}\) and the optimal allocation is equal to \((0, 1, 0)\) for any \(z_{01} < z < z^o\). Finally, we get that \(0 > \max\{L_{01}, L_1\}\) and the optimal allocation is given by \((1, 0, 0)\) for \(z < z_{01}\). The binding constraint implies that \(\int_{\frac{z}{z_0}T}^T (n_{01} + 2n_1) \ N dG(z) = N \int_{T_{01}}^T dG(z) + N \int_{z_{01}}^T dG(z) = N [2 - G(z_{01}) - G(z^o)] = M\). One finally can check that \(z_{01} = z^o \omega^o\) where \(\omega^o = \frac{2w_1 - (w_{01} + w_{10})}{(w_{01} + w_{10}) - 2w_0} < 1\).

This yields \(z^o\) as the solution of the following equation: \(G(z^o \omega^o) + G(z^o) = 2 - M/N\).

Finally, suppose that the constraint is not binding \((\mu = 0)\). Then, by (18) \(L_1 > L_{01}\) for any \(z\). Therefore, the optimal allocation is \((n_0, n_{01}, n_1) = (0, 0, 1)\). This case occurs if \(\int_{\frac{z}{z_0}T}^T (n_{01} + 2n_1) \ N dG(z) = 2N < M\), which we have assumed away in this paper.

### Appendix G: Divergence and activity tightness indices

For low airport fee \(\phi < \hat{\phi}\), we have that \(T = 1\) if \(M > N\) and \(T = M/N\) otherwise. In each destination \(z\), the divergence index is equal to \(T(z) = 1\) if \(M > N\) and \(T(z) = 1\) if \(z > z^*\) otherwise. The aggregate index is equal to \(T^2 = 1\) if \(M/N > 1\) and \(T^2 = 1 - G(z_{01}) = M/N\) otherwise. Therefore, we can write \(T^2 = T\) for \(T \in [0, 1]\).

For \(\phi \geq \hat{\phi}\) and \(M < 2N\), we have \(T^2 = G(z^*) - G(\omega^* z^*)\) and \(T = M/N < 2\). Writing \(z^*\) as the function \(z^*(N/M)\), we get \(T^2(T) = G(z^*(T)) - G(\omega^* z^*(T))\), \(T \in [0, 2]\). Note that, when \(\omega^* z^*(T) < \bar{z} < z^*(T)\), the index simplifies to \(T^2(T) = G(z^*(T))\) and \(T = \int_{\bar{z}}^T [n_1(z) + 2n_2(z)] dG(z)\) simplifies to \(G(z^*(T)) + 2 [1 - G(z^*(T))]\) or equivalently \(2 - G(z^*(T))\). Therefore, \(T^2(T) = 2 - T\). By contrast, for \(\omega^* z^*(T) < \bar{z} < z^*(T)\), we get \(T^2(T) = 1 - G(\omega^* z^*(T))\) and \(T = \int_{\bar{z}}^T n_1(z) dG(z) = 1 - G(\omega^* z^*(T))\), which gives \(T^2(T) = T\).

### Appendix H: Slot market equilibrium

We first determine the Nash equilibrium in the game between the two airlines in each destination market \(z\) for a given slot price \(t\) and then determine the slot price \(t\) that is compatible with an
equilibrium. Let us define \( z_0 = t / (\pi_{10} - \pi_0) \) and \( z_1 = t / (\pi_1 - \pi_{01}) \), which are positive by (19). We get the threshold

\[
\omega^e \equiv \frac{z_1}{z_0} = \frac{\pi_{10} - \pi_0}{\pi_1 - \pi_{01}}
\]

Suppose first that \( z_0 < z_1 \) so that \( \omega^e > 1 \). Then, it can readily be checked that, the slot game of Table 2 yields a Nash equilibrium in which both airlines ask a peak slot when the city pair \( z \) has a high passenger demand \( z \geq z_1 \) and ask no peak slots at all for low passenger demands \( z \leq z_0 \). For intermediate passenger demands \( z \in (z_0, z_1) \), only one airlines asks a peak slot while the other does not (there are actually two equilibria). In a slot market equilibrium, the slot market authority fills slots up to capacity. To achieves this it should be that \( M = 2N(1 - G(z_1)) + N(G(z_1) - G(z_0)) \) and \( t = z_0(\pi_{10} - \pi_0) \). The former condition yields

\[
G(z_0) + G(z_0\omega^e) = 2 - M/N
\]

This corresponds to the condition for a balanced slot allocation schedule with parameter threshold \( \omega^e \). However, since \( \omega^e \) is generically different from \( \omega^* \) and \( \omega^o \), it implies a different slot allocation.

Suppose finally that \( z_0 \geq z_1 \) so that \( \omega^e \leq 1 \). Then, for a given slot price \( t \), the Nash equilibrium of the above game implies that both airlines ask a peak slot when the city pair \( z \) has a high passenger demand \( z \geq z_0 \) and ask no peak slots at all for low passenger demands \( z \leq z_1 \). For demands \( z \in (z_1, z_0) \), two equilibria arise where both airlines jointly ask either two peak slots or none. The important thing is that equilibrium slot choices are always the same within each city pair. In those equilibria, nothing imposes that city pairs \( z \in (z_1, z_0) \) be allocated the peak slots as a function of their demand sizes. However, any airline with multiple destinations have incentives to re-allocate its peak slots to its destinations with larger demand \( z \). Similarly, an airline operating in a higher demand destination has incentives to make side payments to obtain the peak slot of another airline operating in a lower demand destination. To become robust to re-allocation and side payments, the slot market equilibrium must grant two peak slots to the larger demand sizes amongst destinations \( z \in (z_1, z_0) \). In this case, there exists a destination size \( \hat{z} \in [z_1, z_0] \) such that airlines with \( z \geq \hat{z} \) purchase a peak slot allowance whereas others do not. Since the most profitable destinations are granted peak slots, no airline wants to exchange its slot.

\(^{27}\)To be more precise, destinations \( z > z_0 \) have slot allocation equilibrium at \( (y^a, y^b) = (1, 1) \). Destinations \( z \) with \( z_0 > z > \hat{t} / (\pi_{10} - \pi_0) \) have equilibrium at \( (y^a, y^b) = (1, 1) \) and \( (0, 0) \) but the authority selects \( (1, 1) \). Those \( z \) with \( \hat{t} / (\pi_{10} - \pi_0) > z > z_1 \) have equilibrium at \( (y^a, y^b) = (1, 1) \) and \( (0, 0) \) but the authority selects \( (0, 0) \). Given this.
the price must be set such that peak slot demand meets peak capacity. Full peak capacity is reached for \( \hat{z} \) that satisfies \( M = 2N (1 - G(\hat{z})) \). Hence, any \( z_1 \) and \( z_0 \) such that \( z_1 \leq \hat{z} \leq z_0 \) are consistent with a slot market equilibrium. Since \( z_0 = z_1/\omega_e = t/ (\pi_{10} - \pi_0) \), slot equilibrium price are given by \( t \in [\hat{t}, \hat{t}/\omega_e] \) where \( \hat{t} \) solves \( G(\hat{t}/(\pi_{10} - \pi_0)) = 1 - M/(2N) \).

### Appendix I: Monopoly

The Lagrangian writes as

\[
\mathcal{L}(z) = \int \{ \phi z [q_0^m (1 - m(z)) + q_1^m m(z)] + \mu \left[ \frac{M}{N} - m(z) \right] \} N dG(z),
\]

Taking \( m(z) \) as a continuous function, pointwise differentiation w.r.t. \( m(z) \) yields

\[
\mathcal{L}_1(z) = \phi z (q_1^m - q_0^m) - \mu
\]

Hence, \( m(z) = 1 \) if \( \mathcal{L}_1(z) > 0 \iff z > \mu / [\phi (q_1^m - q_0^m)] \). Otherwise, \( m(z) = 0 \). When the binding constraint yields \( \mu = [\phi (q_1^m - q_0^m)] G^{-1}(1 - M/N) \). Hence, the solution is \( m(z) = 1 \) if \( G(z) > 1 - M/N \) and \( m(z) = 0 \) otherwise.

### Appendix J: comparison of \( \omega^* \) and \( \omega^0 \)

We set \( \theta = s_1/s_0 > 1 \) and \( \tau = s_0/\phi > 2 \), since \( \phi < s_0/2 \). Moreover, we are interested in the comparison for \( \phi \geq \hat{\phi} \), where

\[
\hat{\phi} = \frac{s_0 s_1}{6s_1 - 2s_0} = \frac{\theta \tau}{2 (3\theta - 1)} \phi,
\]

from which follows that \( \phi \geq \hat{\phi} \iff \tau \leq 2 (3 - 1/\theta) \). Hence, \( \phi \geq \hat{\phi} \) if \( \tau \leq 4 \). We examining the sign numerically in the space \((\theta, \tau)\) in Figure 9. We focus on the area where \( \theta > 1 \) and \( \tau \in \left( 2, \frac{2(3\theta - 1)}{\theta} \right) \) (the grey and blue areas). It emerges that \( \omega^0 > \omega^* \) whenever the gap between the benefit of off-peak time flight and per passenger fees, \( \tau \), is sufficiently high (the grey area). Conversely, for low \( \tau \) and a sufficiently high \( \theta \) (the blue area), the relationship is \( \omega^0 - \omega^* \) is negative.

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selection, no airline wants to trade its slot for cash.
Supplementary material

Divergence indices under uniform and Pareto distributions

In what follows, we study the divergence index for uniform and Pareto market size distributions. When market size is uniformly distributed, it is shown that the triangular shape in Figure 5 also applies if the market size heterogeneity is not too strong: \( z/\tau > \omega^* \). Otherwise, when \( z/\tau > \omega^* \), the triangular shape is truncated at its upper corner.

Suppose that \( G \) is uniform on the interval \([\underline{z}, \overline{z}]\) so that \( G(z) = (z-\overline{z})/(\overline{z}-\underline{z}) \) and \( g(z) = 1/(\overline{z}-\underline{z}) \). Suppose that \( \underline{z}/\omega^* < \overline{z} \) so that those functions have five pieces. The divergence index with uniform distribution is summarized in the following lemma.

Lemma 4 Suppose a uniform distribution of city pair market sizes on the interval \([\underline{z}, \overline{z}]\). If \( \underline{z}/\overline{z} > \omega^* \), the divergence index is given by

\[
[I(T)]^2 = \begin{cases} 
T & \text{if} \quad 0 \leq T < 1 \\
2 - T & \text{if} \quad 1 < T \leq 2
\end{cases}
\]

Otherwise, it is equal to

\[
[I(T)]^2 = \begin{cases} 
T & \text{if} \quad 0 \leq T < \frac{\pi(1-\omega^*)}{\overline{z}-\underline{z}} \\
(1-\omega^*) \left(2 - T + \frac{2\omega^*}{\overline{z}-\underline{z}}\right) & \text{if} \quad \frac{\pi(1-\omega^*)}{\overline{z}-\underline{z}} \leq T < \frac{2\pi - \frac{2\omega^*}{\overline{z}-\underline{z}}}{\overline{z}-\underline{z}} \\
2 - T & \text{if} \quad \frac{2\pi - \frac{2\omega^*}{\overline{z}-\underline{z}}}{\overline{z}-\underline{z}} \leq T < 2
\end{cases}
\]
Proof. On the one hand, suppose $z/\bar{z} > \omega^*$. Let us define $H(z^*) = G(\omega^* z^*) + G(z^*)$ so that

$$H(z^*) = \begin{cases} 
0 & \text{if } z^* \leq \bar{z} \\
(z^* - \bar{z}) / (\bar{z} - \bar{z}) & \text{if } \bar{z} < z^* \leq \bar{z} \\
1 & \text{if } \bar{z} < z^* \leq \bar{z} / \omega^* \\
1 + (\omega^* z^* - \bar{z}) / (\bar{z} - \bar{z}) & \text{if } \bar{z} / \omega^* < z^* \leq \bar{z} / \omega^* \\
2 & \text{if } z^* \geq \bar{z} / \omega^* 
\end{cases}$$

So, the identity $H(z^*) = \frac{2N-M}{N}$ is equivalent to $H(z^*) = 2 - T$ where $T = M/N$, which yields

$$\begin{cases} 
T = 2 & \text{if } z^* \leq \bar{z} \\
z^* = \bar{z} + (2 - T) (\bar{z} - \bar{z}) & \text{if } \bar{z} < z^* \leq \bar{z} \iff 0 < 2 - T \leq 1 \\
T = 1 & \text{if } \bar{z} < z^* \leq \bar{z} / \omega^* \\
z^* = \frac{(2-T)(\bar{z}-\bar{z})-(\bar{z}-\bar{z})}{\omega^*} & \text{if } \bar{z} / \omega^* < z^* \leq \bar{z} / \omega^* \iff 1 < 2 - T \leq 2 \\
T = 0 & \text{if } z^* \geq \bar{z} / \omega^* 
\end{cases}$$

Then we compute $G(z^*) - G(\omega^* z^*)$ as

$$G(z^*) - G(\omega^* z^*) = \begin{cases} 
0 & \text{if } z^* \leq \bar{z} \\
G(z^*) = (z^* - \bar{z}) / (\bar{z} - \bar{z}) & \text{if } \bar{z} < z^* \leq \bar{z} \\
G(z^*) - G(\omega^* z^*) = 1 & \text{if } \bar{z} < z^* \leq \bar{z} / \omega^* \\
1 - G(\omega^* z^*) = 1 - (\omega^* z^* - \bar{z}) / (\bar{z} - \bar{z}) & \text{if } \bar{z} / \omega^* < z^* \leq \bar{z} / \omega^* \\
0 & \text{if } z^* \geq \bar{z} / \omega^* 
\end{cases}$$

Finally

$$[\mathcal{I}(T)]^2 = G(z^*(T)) - G(\omega^* z^*(T)) = \begin{cases} 
0 & \text{if } z^* \leq \bar{z} \\
2 - T & \text{if } \bar{z} < z^* \leq \bar{z} \\
1 & \text{if } \bar{z} < z^* \leq \bar{z} / \omega^* \\
T & \text{if } \bar{z} / \omega^* < z^* \leq \bar{z} / \omega^* \\
0 & \text{if } z^* \geq \bar{z} / \omega^* 
\end{cases}$$
Then

\[
[I(T)]^2 = \begin{cases} 
0 & \text{if } z^* \leq \bar{z} \\
2-T & \text{if } 2 > T \geq 1 \\
1 & \text{if } \bar{z} < z^* \leq \bar{z}/\omega^* \\
T & \text{if } 1 > T \geq 0 \\
0 & \text{if } z^* \geq \bar{z}/\omega^* 
\end{cases}
\]

On the other hand, suppose \( \bar{z}/\bar{z} < \omega^* \). Let us define \( H(z^*) = G(\omega^* z^*) + G(z^*) \) so that

\[
H(z^*) = \begin{cases} 
0 & \text{if } z^* \leq \bar{z} \\
(z^* - \bar{z}) / (\bar{z} - \bar{z}) & \text{if } \bar{z} < z^* \leq \bar{z}/\omega^* \\
(z^* (1 + \omega^*) - 2 \bar{z}) / (\bar{z} - \bar{z}) & \text{if } \bar{z}/\omega^* < z^* \leq \bar{z} \\
1 + (\omega^* z^* - \bar{z}) / (\bar{z} - \bar{z}) & \text{if } \bar{z} < z^* \leq \bar{z}/\omega^* \\
2 & \text{if } z^* \geq \bar{z}/\omega^* 
\end{cases}
\]

So, the identity \( H(z^*) = \frac{2N-M}{N} \) is equivalent to \( H(z^*) = 2 - T \) where \( T = M/N \), which yields

\[
\begin{align*}
T = 2 & \quad \text{if } z^* \leq \bar{z} \\
z^* = \bar{z} + (2 - T)(\bar{z} - \bar{z}) & \quad \text{if } \bar{z} < z^* \leq \bar{z}/\omega^* \\
\frac{2\bar{z} + (2-T)(\bar{z} - \bar{z})}{1+\omega^*} & \quad \text{if } \bar{z}/\omega^* < z^* \leq \bar{z} \\
\frac{(2-T)(\bar{z} - \bar{z}) - (\bar{z} - 2\bar{z})}{\omega^*} & \quad \text{if } \bar{z} < z^* \leq \bar{z}/\omega^* \\
T = 0 & \quad \text{if } z^* \geq \bar{z}/\omega^* 
\end{align*}
\]

Then we compute \( G(z^*) - G(\omega^* z^*) \) as

\[
G(z^*) - G(\omega^* z^*) = \begin{cases} 
0 & \text{if } z^* \leq \bar{z} \\
G(z^*) = (z^* - \bar{z}) / (\bar{z} - \bar{z}) & \text{if } \bar{z} < z^* \leq \bar{z}/\omega^* \\
G(z^*) - G(\omega^* z^*) = (1 - \omega^*) z^* / (\bar{z} - \bar{z}) & \text{if } \bar{z}/\omega^* < z^* \leq \bar{z} \\
1 - G(\omega^* z^*) = 1 - (\omega^* z^* - \bar{z}) / (\bar{z} - \bar{z}) & \text{if } \bar{z} < z^* \leq \bar{z}/\omega^* \\
0 & \text{if } z^* \geq \bar{z}/\omega^* 
\end{cases}
\]
Finally

\[ [I(T)]^2 = G(z^*(T)) - G(\omega^* z^*(T)) = \begin{cases} 
2 - T & \text{if } 2 > T \geq \frac{2\pi - (1+\omega^*) \varpi}{\varpi - \Delta} \\
(1-\omega^*) \left( 2 - T + \frac{2\varpi}{\varpi - \Delta} \right) & \text{if } \frac{2\pi - (1+\omega^*) \varpi}{\varpi - \Delta} > T \geq \frac{\pi (1-\omega^*)}{\varpi - \Delta} \\
T & \text{if } \frac{\pi (1-\omega^*)}{\varpi - \Delta} > T \geq 0 
\end{cases} \]

When the market sizes are drawn from a Pareto distribution with a tail for high markets \((G : (\tilde{z}, \infty) \rightarrow [0, 1])\) such that \(G(z) = 1 - (\tilde{z}/z)^{\kappa}\) with \(\varpi > 0\) and \(\kappa > 0\), the left hand ridge of the dashed triangle gets lower slope whereas the right hand side ridge remain unchanged. Suppose a Pareto distribution \(G : (\tilde{z}, \infty) \rightarrow [0, 1]\) such that \(G(z) = 1 - (\tilde{z}/z)^{\kappa}\) with \(\varpi > 0\) and \(\kappa > 0\). The following lemma shows the divergence index with Pareto distribution.

**Lemma 5** Suppose a Pareto distribution of city pair market sizes on the interval \((\tilde{z}, \infty)\). The divergence index is given by

\[ [I(T)]^2 = G(z^*(T)) - G(\omega^* z^*(T)) = \begin{cases} 
\frac{1-(\omega^*)^{\kappa}}{1+(\omega^*)^{\kappa}} T & \text{if } T < 1 + (\omega^*)^{\kappa} \\
2 - T & \text{if } 1 + (\omega^*)^{\kappa} \leq T < 2 
\end{cases} \]

**Proof.** Let us define \(H(z^*) = G(\omega^* z^*) + G(z^*)\) so that

\[ H(z^*) = \begin{cases} 
0 & \text{if } z^* \leq \tilde{z} \\
1 - (\tilde{z}/z^*)^{\kappa} & \text{if } \tilde{z} < z^* \leq \tilde{z}/\omega^* \\
2 - (1 + (\omega^*)^{-\kappa}) (\tilde{z}/z^*)^{\kappa} & \text{if } \tilde{z}/\omega^* < z^* 
\end{cases} \]

So, the identity \(H(z^*) = \frac{2N-M}{N}\) is equivalent to \(H(z^*) = 2 - T\) where \(T = M/N\), which yields

\[ \begin{cases} 
T = 2 & \text{if } z^* \leq \tilde{z} \\
\frac{\tilde{z}}{(T-1)^{\frac{1}{\kappa}}} & \text{if } 2 > T \geq 1 + (\omega^*)^{\kappa} \\
\left( \frac{1+(\omega^*)^{-\kappa}}{T} \right)^{1/k} \tilde{z} & \text{if } T < 1 + (\omega^*)^{\kappa} 
\end{cases} \]
Also,

\[ G(z^*) - G(\omega^* z^*) = \begin{cases} 
0 & \text{if } z^* \leq \bar{z} \\
1 - (\bar{z} / z^*)^\kappa & \text{if } \bar{z} < z^* \leq \bar{z} / \omega^* \\
((\omega^*)^{-\kappa} - 1) (\bar{z} / z^*)^\kappa & \text{if } \bar{z} / \omega^* < z^* 
\end{cases} \]

Therefore

\[ \left[ \mathcal{I}(T) \right]^2 = G(z^*(T)) - G(\omega^* z^*(T)) = \begin{cases} 
0 & \text{if } z^* \leq \bar{z} \\
2 - T & \text{if } 2 > T \geq 1 + (\omega^*)^\kappa \\
\frac{1 - (\omega^*)^\kappa}{1 + (\omega^*)^\kappa} T & \text{if } T < 1 + (\omega^*)^\kappa 
\end{cases} \]

To sum up, the stronger heterogeneity squashes the triangle shape of the index down but does not eliminate its non-monotonic property.