Vicrey’s bottleneck model with a cooperative sub-network

Raphaël Lamotte\textsuperscript{a} André de Palma\textsuperscript{b} Nikolas Geroliminis\textsuperscript{a}

\textsuperscript{a}School of Architecture, Civil and Environmental Engineering, Urban Transport Systems Laboratory, Ecole Polytechnique Fédérale de Lausanne (EPFL), GC C2 389, Station 18, 1015 Lausanne, Switzerland

\textsuperscript{b}Département Économie et Gestion, Ecole Normale Supérieure de Cachan, 61, avenue du Président Wilson, 94230 Cachan, France

Introduction

The aggregate performances of a road network result from the choices of a great number of atomic drivers, all competing to minimize their own individual cost. Unlike for other multi-agent games, this competition is not benefiting anyone. In fact, Vickrey [1] showed that it represents a sizeable part of the costs due to congestion (exactly half these costs if all users are identical - see e.g. [2]). On the basis of this observation, Vickrey and the following literature have proposed time-dependent tolls that can ensure a cooperative behavior, thus converting the time wasted in congested traffic into real money (that can be redistributed), without affecting the arrival times at work. Despite the extraordinary expected benefits of such a scheme, several reasons have hindered the wide-scale adoption of cooperation-ensuring measures. On the one hand, tolling is in general very poorly accepted or understood by the population, especially if the tolling infrastructure are themselves expensive. On the other hand, the question of how the money collected should be redistributed remains the subject of debate, with fears that it would disappear into a bureaucratic black hole. Consequently, only very few major cities (e.g. Singapour and Stockholm) have taken such a step.

Separate sub-networks might be the solution to the first issue. In fact, cooperation is more acceptable if it is enforced only on some part of the network and remains a choice, as demonstrated by the rapid development of High Occupancy Toll (HOT) lanes on American highways. In addition, Fosgerau [3] showed that if properly designed, subnetworks can be used to make all users
better-off. Nevertheless, the very high percentage of HOT lanes applying a flat or dynamic rate instead of a deterministic time-of-day rate [4] indicates that HOT lanes have traditionally been considered more as traffic control measures than as tools to shift departure times. Besides, HOT lanes do not remove the concerns about toll redistributions. The scheme proposed in this work is also based on the spatial segregation of cooperative and non-cooperative users but it relies on the self-registration of users to a service that allocates departure times. To be fair, compliance to such a service is given an inconvenience cost that represents a natural aversion to cooperation, innovation and/or automation. While the inconvenience is obvious here, we believe that tolls should also be associated to such a cost since they push users to shift their departure time.

This scheme is especially interesting since it might arise naturally as the trends toward autonomous vehicles and towards car-sharing converge. On the first hand, while autonomous vehicles have the potential to greatly improve the safety and the efficiency of the road network, their development is made difficult by the interactions with traditional vehicles. Thus, the development of autonomous vehicles is likely to lead to the emergence of separated road networks, at least as a first step toward full automation. On the other hand, car-sharing is currently expanding at a rapid rate and is at the cutting edge of technology, as evidenced by the numerous services offering electric cars to their customers (to mention just a few: Autolib’ in Paris, ElectricEasy in Switzerland, Zipcar in the United States). This shift from a paradigm of individual vehicle owners to a paradigm in which transportation is seen as a service provided by a central operator means that traffic would change from an ill-organized first-come-first-served equilibrium to a problem of logistics in which the operator (either public or private) would internalize the cost of congestion.

In this work, the only heterogeneity considered among users is their cost of compliance defined above, in case they choose to be cooperative. This is in agreement with an important part of the literature which considered users to be all identical to allow for elegant results or generalizations (e.g. [2]). To ensure fairness on the long term, departure times are assumed to be allocated to the cooperative users differently from day to day (in a real implementation, personal preferences should of course be considered).

Several versions of this problem are addressed in this paper. In the first part, it is assumed that the demand and capacity splits between the two sub-networks are decision variables and we look for their socially optimal values. These optimal values are shown to exist, to be unique, and numerical values are given for specific distributions of the cost of compliance. With distributions of the cost of compliance that were deemed reasonable, it was found that such a scheme could reduce the social cost by approximately 12%.
Then, different user equilibria were considered, depending on the existence of constant tolls and on the objective of the operators setting the tolls. With no toll, the cost of anarchy was found to be relatively small, reducing the gains in terms of social cost by just a few percents. In addition, this loss can easily be avoided by applying a small toll on the cooperative service. However, if a private operator is given the freedom to set the toll on one of the two sub-networks, these gains in terms of social cost decrease with the capacity the private operator is given to manage and the social cost can become much worse than in the original "atomic only" situation. A Stackelberg equilibrium was then considered in which the government would first set a toll on the other sub-network, knowing how the private would react. This solution allows the social optimum to be obtained again but at the cost of considerable tolls, which were deemed unacceptable and were one of the motivation of this work. Finally, such a scheme could be efficient from a social point of view only if both sub-networks are free or managed by the government to maximize social welfare.

1 Social optimum

The social cost can be expressed as a function of the demand and capacity splits by:

\[
SC = \begin{cases} 
N\kappa \int_0^{\hat{c}_c} xf(x)dx + \delta \frac{N_{na}^2}{2gS_{na}} + \delta \frac{(N-N_{na})^2}{S-S_{na}} & \text{if } (N_{na}, S_{na}) \in [0, N] \times ]0, S[ \\
N\kappa \int_0^1 xf(x)dx + \delta \frac{N^2}{2S} & \text{if } (N_{na}, S_{na}) = (N, S) \\
\delta \frac{N^2}{S} & \text{if } (N_{na}, S_{na}) = (0, 0)
\end{cases}
\]

With \(N_{na}(\hat{c}_c) = N \int_0^{\hat{c}_c} f(x)dx\), where \(N_{na}\) and \(S_{na}\) are the non-atomic (or cooperative) demand and road capacity while \(N\) and \(S\) are the total demand and capacity. \(g\) is a constant such that \(g \geq 1\), accounting for a better use of the physical space in the case cooperative vehicles are high-performance autonomous vehicles. Using the standard notations \(\delta = \frac{\beta\gamma}{(\beta+\gamma)}\), where \(\beta\) and \(\gamma\) are the values of earliness and lateness. Finally, \(f\) is the probability density function of the normalized cost of compliance, \(\kappa\) is the scale and \(\hat{c}_c\) is the critical normalized cost of compliance, such that individuals are cooperative if and only if their normalized cost of compliance is smaller than \(\hat{c}_c\).
By differentiating this social cost with respect to \( \hat{c}_c \), one can the following proposition.

**Proposition 1** For a given capacity split, if the pdf of the cost of compliance has a support \([0, \kappa]\) or \([0, +\infty]\), then there is a unique demand split that minimizes the social cost. If \( S_{na} \in ]0, S[ \), then the solution is also interior and it is specified by the implicit equation Eq. 2.

\[
\kappa \hat{c}_c + \frac{\delta}{gS_{na}(S-S_{na})}N_{na}(\hat{c}_c) - \frac{2N}{(S-S_{na})} = 0
\] (2)

In addition, an explicit expression of the critical cost of compliance can be obtained by assuming a specific type of distribution.

Then, if both the demand and capacity splits can be varied, the socially optimal pair is found by first expressing the social cost as a function of the demand split only (assuming that the capacity is optimally allocated) and then by differentiating this function of one variable.

**Proposition 2** There is a unique pair of demand and capacity splits that minimizes the social cost.

In addition, one can show that if the solution is interior (which is always the case if the cost of compliance has a support \([0, +\infty]\)), then it is given by the implicit equation

\[
\kappa \hat{c}_c + \frac{\delta}{gS}((\sqrt{2g} - 2g)N + (1 - \sqrt{2g})^2 N_{na}(\hat{c}_c)) = 0.
\] (3)

Again, an explicit expression can be obtained by assuming a specific distribution for the cost of compliance. In order to provide some idea of the scale of the expected gains, the demand splits and the different components of the social cost were plotted in Figure 1 with three different distributions for the normalized cost of compliance (uniform, log-normal and exponential - parameters were chosen such that their average is \( \frac{1}{2} \), and with \( \sigma = 1 \) for the log-normal distribution) and for \( g = 1 \). These results were obtained for a peak-hour duration of 1h and with all costs normalized with respect to the cost of one time unit of travel time. Even though there is no known value for the cost of compliance, an educated guess would be that its average is in the order of 0.5 (half an hour of travel time, which is equivalent to \( \kappa = 1 \)). For such a value of \( \kappa \), a rough estimate for the reduction of the social cost would be approximately 12% (but it could be much more for smaller values of \( \kappa \)).

With higher values of \( g \), the social cost can be even further reduced, although this gain is more related to technical progress than to shifts of departure times.
Fig. 1. Comparison of the (a) demand split and of the different components of the social cost (b, c and d) under Social Optimum for a uniform, an exponential and a log-normal distribution of the cost of compliance and in the "atomic only" case

2 User Equilibria

The user equilibrium corresponds to a situation where no user can reduce his individual cost by changing his decision. Thus, if there is no toll:

$$\kappa \hat{c_c} + c_{na} = c_a$$  \hspace{1cm} (4)

This case can be simply addressed with the following proposition:

Proposition 3 If the two sub-networks exist \((S_{na} \in ]0, S[)\), the User Equilibrium demand split for any given \(\kappa^*\) and \(S_{na} = S_{na}^*\) is the Social Optimum demand split for \(S_{na} = S_{na}^*\) and \(\kappa = 2\kappa^*\).

Thus, all the work that was done for the social optimum remains valid after replacing the value of \(\kappa\). In particular, there is also existence and uniqueness of a solution for a given capacity split. Another consequence of Proposition 3 is that the user equilibrium can be made socially optimal simply by adding a toll \(\tau = \frac{\kappa}{2} \hat{c_c}\) on the atomic sub-network.

While the analysis of the impact of the capacity split on the social cost was relatively easy under the assumption of social optimum, it is much more tedious under User Equilibrium since there is no closed-form relationship between the equilibrium demand and the capacity split (except if the toll is assumed to be socially optimal of course). Thus, the analyses of the price of anarchy, of the equilibrium with a private operator and of Stackelberg equilibria were done by assuming a given distribution (uniform) for the cost of compliance.
With such an assumption, the price of anarchy was found to be relatively small (for instance for \( \kappa = 1 \) and with the numerical values used earlier, the minimum social cost is 374 under Social Optimum and 378 under User Equilibrium, compared to 400 in the "all-atomic" case).

Nevertheless, it was found that whenever a private operator imposes a toll on one of the two sub-networks (without any toll on the other sub-network), the profit-maximizing strategy leads to a dramatic increase of the social cost, which increases with the proportion of the capacity that is privately managed.

Finally, in a Stackelberg equilibrium in which the government is leader and sets his toll first, it is always possible for the government to set a toll such that the social cost is minimized, knowing that the private will react by maximizing its profits. However, such an equilibrium leads to extremely high tolls and to exactly the type of situation that was deemed unacceptable by the public and that motivated this work. Thus, the cooperative scheme proposed here has the potential to be acceptable by the public and to reduce the social cost only if there are no tolls or if these tolls obey to a welfare-maximizing strategy rather than to a profit-maximizing strategy.

References


