Modeling the Morning Commute for Urban Networks with Cruising-for-parking: an MFD Approach

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Abstract

This study examines the morning commute equilibrium with explicit consideration of cruising-for-parking, and its adverse impacts on traffic congestion. The cruising-for-parking is modeled through a dynamic aggregated traffic model for networks or areas: the macroscopic fundamental diagram (MFD). Firstly, we formulate the morning equilibrium solution for a congested downtown network with cruising-for-parking. It is shown that the cruising-for-parking would yield smaller system or network outflow, and thus induce more severe congestion. We then develop a dynamic model of pricing for the network to reduce system travel cost including cruising time cost, moving time (the duration during which vehicles move to the destination but do not cruise for parking yet) cost and schedule delay cost. At the system optimum, the downtown network should be operating at the maximum production of the MFD, but the cruising effect is not fully eliminated. Also, it is shown that the time-dependent toll has a different shape than the classical Vickrey equilibrium fine toll. This analysis is then extended to the bi-modal commuting equilibrium with cruising-for-parking in the auto side. In this case, besides departing earlier to enjoy less cruising time, travelers can take public transportation to avoid the cruising-for-parking. Similarly, the optimal dynamic toll is introduced to reduce traffic inefficiency due to cruising-for-parking and roadway congestion, and realize the bi-modal system optimum. Finally, analytical results are illustrated and verified with numerical studies.

Keyword: morning commute, cruising-for-parking, MFD, pricing

1. Introduction

Parking is not only a headache for commuters in the morning peak, but also a challenging issue for the transport system planners, operators and regulators. The time spent on cruising or searching

¹ The leading author is a junior researcher, and wish to apply for the Best Paper by a Junior Researcher Prize.
² The full paper has been submitted to the journal of Transportation Research Part B for publication consideration.
for a vacant parking space can be up to 40% of the total travel time (Axhausen et al. 1994), due to the limitation of parking supply in the destinations such as city centers. Shoup (2006) summarized the findings of several studies done between 1927 and 2001, which shows between 8 and 74 percent of the traffic was cruising for parking, and the average time to find a curb space can be up to 14 minutes. Cruising can also influence drivers not involved in parking and create congestion even for medium level of demand conditions, as the outflow of the system (arrivals to the parking) can reach very low values. Due to its inefficiency, the phenomenon of cruising-for-parking is one of the most studied topics in the economics of parking. Understanding the effect of cruising-for-parking for congested networks can improve efficiency in the flow of vehicles and facilitate the development of more equitable strategies as trips with cruising might contribute to congestion more than trips without, e.g. trips with destinations outside the limited parking zones.

Glazer and Niskanen (1992) has modelled the congestion caused by through-traffic and by traffic destined for the area where consumers park. For the evaluation of various parking policies, Bifulco (1993) introduced the parking search times in a static stochastic traffic assignment model. Anderson and de Palma (2004) studied the parking problem under a private parking operator in a monopolistically competitive market, with an emphasis on the commuter’s time spent on searching for a vacant parking space. Arnott and Rowse (1999) developed a structural model of parking for a ring-road on which travelers’ choice of parking lot is uniformly distributed; the expected parking time, driving time and cruising distance for searching available parking spaces are derived. Arnott and Inci (2006) presented a downtown parking model that integrates traffic congestion and saturated on-street parking. There is a series of follow-up studies on the parking problem integrating traffic congestion and on-street or curbside parking (e.g., Arnott and Rowse, 2009; Arnott and Inci, 2010). These studies focusing on cruising-for-parking provide some ideas of the interaction between cruising, traffic congestion and network performance. However, they often overlook the rush hour traffic dynamics and time-varying traffic intensity. For a recent comprehensive review on the economics studies of parking, one may refer to Inci (2014).

Arnott et al. (1991) embedded the parking problem into the well-known morning commute model (Vickrey 1969), and showed that a parking fee alone can effectively increase social welfare, and that a combination of dynamic road toll and dynamic parking fee can yield the system optimum. Zhang et al. (2008) further extended the study by deriving the daily commuting pattern that combines both the morning and evening commute. More recently, attentions have been paid to how parking capacity allocations, parking fees, parking permits and parking reservations can be designed to improve efficiency for a dynamic network with one roadway bottleneck (Zhang et al. 2011; Qian et al. 2011, 2012; Fosgerau and de Palma, 2013; Yang et al. 2013; Liu et al. 2014a,b). However, in most of these studies, the cruising or searching for parking spaces is not modeled.
Qian and Rajagopal (2014) modeled how travelers make parking location choices and departure time choices to minimize their generalized travel cost, and they incorporated cruising-for-parking by using a cruising time function dependent on parking occupancy. However, their study considered cruising time as a cost at the end of trip, but ignored the impacts of cruising-for-parking on the roadway traffic congestion, as well as the interaction between cruising and moving traffic. Note that by moving, we mean that the vehicles are moving towards their destination, but have not started to find a parking space yet.

This study is the first to examine the morning commute equilibrium which explicitly incorporates not only the cruising-for-parking, but also its adverse impacts on traffic congestion and how this interactions re-shape the commuting equilibrium. Following a recent macroscopic simulation study (Geroliminis, 2015), the impact of cruising-for-parking is modeled through a recently proposed traffic model for networks or areas: the macroscopic fundamental diagram (MFD), see Daganzo and Geroliminis (2008) for empirical evidence. The MFD approach has been used to study the recurrent morning commute problem without consideration of cruising-for-parking (e.g., Geroliminis and Levinson, 2009; Arnott, 2013). By using the MFD approach, one of the advantages is that the downward-sloping part of the curve between traffic flow and density, known as hypercongestion in economic terms, can be modeled. A simplified version of the MFD model considering capacity drop facing queueing is adopted in some other studies on the morning commute problem (e.g., Fosgerau and Small, 2013; Liu et al., 2015).

Under the MFD framework, the traffic arriving at destinations or the outflow of the network depends on the traffic accumulation in the system and the average trip length of all the traffic. The existing MFD models often assume that the average trip length is constant over time and independent of destination, and ignore phenomena which may change trip length, as when vehicles are cruising for parking (increase in trip length due to route choice is analyzed in Yildirimoglu and Geroliminis, 2014). After taking into account parking, as parking vacancy goes down over time, it becomes more difficult to find a vacant parking space. It follows that the cruising distance for finding a vacant parking space will increase. This would lead to a decrease in traffic arriving at destination (find a vacant parking space) or the outflow of the downtown network. Therefore, macroscopic model that ignores this phenomenon will overestimate the outflow of the system. Furthermore, the increased travel distance due to cruising will lead to more severe congestion in the network. If we look at the traffic dynamics, given the future traffic inflow, the decreased outflow due to cruising-for-parking would in return intensify the traffic accumulation of the network in the future, and decrease the traveling speed in the system and create more severe traffic congestion.
In this study, firstly, we formulate the morning equilibrium solution for a congested downtown network with cruising-for-parking. Since the cruising-for-parking would yield smaller outflow of the system, we then develop a dynamic model of pricing for the network to reduce system travel cost including cruising time cost, moving time (the duration during which vehicles move to the destination but do not cruise for parking yet) cost, schedule delay cost. After that, we extend our analysis to the bi-modal commuting equilibrium with cruising-for-parking. In this case, besides departing earlier to enjoy less cruising time, travelers can take public transportation to avoid the cruising-for-parking. Impacts of cruising-for-parking on modal-split are examined. The optimal dynamic toll is introduced to reduce traffic inefficiency due to parking cruising and roadway congestion, and to realize the bi-modal system optimum. Due to the consideration of cruising-for-parking, the dynamic toll has quite different shape (over time) than the classical triangular toll in Vickrey’s model.

The rest of the paper is organized as follows. Section 2 presents the MFD based formulation of the traffic dynamics with cruising-for-parking. In Section 3, the morning commute equilibrium with cruising-for-parking is formulated and discussed. Section 4 introduces the optimal time-varying toll to reduce total travel cost and improve traffic efficiency. Section 5 extends our analysis to the bi-modal commuting equilibrium with cruising-for-parking. Numerical studies are presented in Section 6 to illustrate and verify the essential ideas in the paper. Finally, Section 7 concludes the paper and provides some discussions.

2. Model Formulation

2.1. The MFD representation of the traffic dynamics with cruising-for-parking

Following Geroliminis and Levinson (2009), the traffic dynamics are modeled through a recently proposed traffic model for networks or areas: the macroscopic fundamental diagram (MFD), Geroliminis and Daganzo (2008). Basically, the MFD describes the relationships among network vehicle density, network average speed of traveling traffic, and network space-mean flow. The dynamic equations for a multi-region system are described in detail in Ramezani et al. (2015).

Consider a downtown area where congestion is homogeneous distributed over space and exhibits an MFD with low scatter. Denote $n$ the accumulation (number of the vehicles in the system) of the downtown network or area. The average traveling speed of all the traffic in the area would depend on the accumulation $n$, i.e., $v = v(n)$. Let $P(n)$ be the production (vehicle kilometers traveled per unit time) of the system, where $P(n) = n \cdot v(n)$. The outflow of the system under
steady state can be approximated by $o(n) = P(n)/L$, where $L$ is the average trip length of traffic in the network. The travel time for a trip then is given as follows:

$$\tau(n) = \frac{L}{v(n)} = \frac{n}{o(n)}.$$  \hfill (1)

Note that most of the existing MFD models often assume that the average trip length $L$ is constant over time and independent of destination, and ignore phenomena which may change trip length, as for example when vehicles are cruising for parking.

If cruising-for-parking is taken into account, trip length $L$ would be composed of two parts: moving distance (vehicles move towards their destinations but do not cruise for parking spaces yet), denoted by $l_m$, and cruising or searching distance (vehicles cruise or search for vacant parking spaces), denoted by $l_s$. Thus, the trip length is $L = l_m + l_s$. In this paper, the average moving distance $l_m$ is assumed to be a constant. The cruising distance $l_s$, however, will depend on the percentage of available parking spaces, $p$, and the average distance traveled in each trial a vehicle tries to find a parking space (might be occupied or empty), $d$ (as also described in Geroliminis, 2015). On average, to find an available parking space, the distance traveled is $l_s = d/p$. The total distance traveled to complete a trip then can be written as a function of $p$, i.e.,

$$L(p) = l_m + \frac{d}{p}.$$ \hfill (2)

The percentage of available parking spaces $p = 1 - n_p/N_p$, where $n_p$ is the number of occupied parking spaces and $N_p$ is the total number of parking spaces or the parking supply in the considered network.

After taking into account the cruising-for-parking, the travel time is then a function of both accumulation $n$ and percentage of available parking spaces $p$. With Eqs.(1) and (2), it is straightforward to see that travel time would be

$$\tau(n, p) = \frac{L(p)}{v(n)}.$$ \hfill (3)

Furthermore, the output of the system would also depend on the accumulation $n$ and the percentage of vacant parking spaces $p$. From Eqs.(1) and (3), we have

$$o(n, p) = \frac{n \cdot v(n)}{L(p)}.$$ \hfill (4)
In Eq.(4), the outflow of the system depends on the accumulation through the production \( n \cdot v(n) \), and on the percentage of available parking spaces through the trip length \( L(p) \).

Before formulating the morning commute equilibrium problem, we provide the MFD used for the downtown network in the following. As shown in Figure 1, the speed \( v(n) \) is assumed to be a constant (the maximum speed) when the accumulation is less than the critical value \( n_c \), i.e., \( n \leq n_c \) and the network is not congested; and \( v(n) \) is decreasing for \( n > n_c \) where the network is congested. Note that the decreasing part of \( v(n) \) shown in Figure 1 is only illustrative. The production of the system reaches its maximum at \( n = n_c \). As long as the average trip distance remains constant, the outflow \( o(n) \) (which is more often used for presenting the MFD dynamics) would have the same shape as the production \( P(n) \), but if trip length varies then Eq.(4) should be applied. For later use, we here also define \( v^{-1}(\cdot) \) as the inverse function of \( v = v(n) \) when \( n \geq n_c \).

![Figure 1. The MFD of the downtown network](image)

2.2. Formulation of the morning commute in the rush hour

The purpose of this study is to examine the downtown parking problem in the context of dynamic user equilibrium in the morning commute. Thus, the mentioned accumulation \( n \) and percentage of vacant parking spaces \( p \) in Eqs.(1)-(4) will be time-dependent, and travel time, outflow of the system would also be time-dependent. In the following, we will set up the model for the dynamic user equilibrium problem with cruising-for-parking.
It is assumed a continuum of \( N \) commuters travelling through a network and reach their destination. They have a common desired arrival time \( t^* \) (extension to the case with differentiated desired arrival times will be briefly discussed in section 3). Let \( I(t) \) and \( A(t) \) be the cumulative departures from home and arrivals at destination at time \( t \) respectively (also the cumulative input and output of the network respectively), then the departure rate from home and arrival rate at destination are \( I'(t) = dI(t)/dt \) and \( A'(t) = dA(t)/dt \). Commuters are assumed to be aware of traffic conditions and parking vacancies after their long term experience, and they choose their departure time to minimize their individual travel cost, which is composed of travel time cost and schedule delay cost. The full trip cost of a commuter by departing from home at time \( t \) is given by
\[
\sum_{c(t,t^*)} = c_w \cdot \tau(n(t), p(t)) + c_s \cdot \left(t^* - t - \tau(n(t), p(t))\right),
\]
where \( \tau(n(t), p(t)) \) is the travel time defined by Eq.(3), \( c_w \) is the value of unit travel time, and \( c_s \) is the schedule penalty of unit time. The schedule penalty \( c_s = e \) for a unit time of early arrival, i.e., \( t^* \geq t + \tau(n(t), p(t)) \), while \( c_s = -l \) for a unit time of late arrival, i.e., \( t^* < t + \tau(n(t), p(t)) \).

Also, it is assumed that \( e < c_w < l \), which is consistent with empirical studies.

In Eq.(5), for analytical tractability, travel time of a commuter departing from home at time \( t \) is assumed to be only dependent on the accumulation at time \( t \), i.e., \( n(t) \), and the percentage of vacant parking spaces when this commuter finds a vacant parking space, i.e., \( p(t) \). Since \( I(t) \) is the cumulative departure from home at time \( t \), for the traveler departing at time \( t \), when he or she finds a parking space, the percentage of available parking spaces will be
\[
p(t) = 1 - \frac{I(t)}{N_p}.
\]
Note that Eq.(6) implies that first-in-first-out (FIFO) principle is assumed, i.e., by departing earlier, the travelers will find a parking space earlier. As mentioned, we use this percentage to estimate the travel time, as well as travel distance the traveler has to cover for finding a vacant parking space. Note that \( p(t) \) is not the percentage of available parking spaces at time \( t \). Indeed, at time \( t \), the percentage of available parking spaces is
\[
\hat{p}(t) = 1 - \frac{A(t)}{N_p},
\]
where \( A(t) \) is the cumulative arrival at time \( t \). Let \( I(t) = A(t) \), with Eqs.(6) and (7), then \( \hat{\rho}(t) = p(t) \), where \( \hat{t} \) is the departure time of commuters arriving at their parking spaces at time \( t \). Note that in Eqs.(6) and (7), without loss of generality, we consider the initial percentage at the start of morning peak is \( p_0 = 100\% \), extension to the case with \( p_0 \neq 100\% \) is straightforward.

If the percentage of vacant parking spaces \( p \) is constant during the morning peak, our study here would reduce to Geroliminis and Levinson (2009). Dynamic user equilibrium is achieved when no one can reduce travel cost by unilaterally changing his or her departure time, which will be derived and discussed in the next section.

3. Morning Commute Equilibrium with Cruising-for-parking

3.1. User Equilibrium Conditions

Similar to Geroliminis and Levinson (2009), the peak starts at time \( t_s \) when the accumulation of the system reaches the critical one \( n_c \) and the outflow (or capacity) is at its maximum, as shown in Figure 2. For traffic departing from home earlier than \( t_s \), we consider they are off-peak and not included in the considered travel demand \( N \). The last peak traffic departs at time \( t_e \) when the accumulation of the system again reaches the critical one \( n_c \), and the production reaches its maximum. However, the outflow is smaller than that at time \( t_s \) since the percentage of vacant parking spaces decreases and trip length increases, i.e., \( p(t_s) = 1 - N/N_p = p(t_e) \), with Eq.(2), it follows \( L(p(t_s)) < L(p(t_e)) \). Similar with traffic departing earlier than \( t_s \), the traffic departing from home later than \( t_e \) are regarded as non-peak traffic and not included in the considered travel demand \( N \).

However, as one can see in Figure 2, to estimate inflow of the system (departure rates from home) after \( t_s \) but before traffic departing at time \( t_s \) arrives, we have to know the outflow of the system (which are arrivals of non-peak traffic, denoted by dotted line in bottom-left of Figure 2). Here, for the non-peak traffic, we assume the percentage of available parking spaces is 100% to compute the outflow. In the other side, to estimate the outflow of the system after time \( t_e \) but before the last peak traffic arrives, we need to know the departures from home and the accumulation of the system which both depend on those non-peak traffic. Without loss of generality, we assume the
departure rate is in the pattern that the time-varying accumulation \( n(t) = n_e \) from time \( t_e \) to the time when the last peak traffic arrives (see dotted line in upper-right of Figure 2).

![Figure 2. Cumulative departure and arrival at the user equilibrium](image)

In the following, we will derive the dynamic user equilibrium conditions. As mentioned, equilibrium requires that no one can reduce its travel cost by unilaterally changing its departure time. By taking the first-order derivative of the individual travel cost given by Eq.(5) with respect to \( t \), we have

\[
\frac{\partial c(t,t^{*})}{\partial t} = (c_w - c_s) \left[ \frac{\partial \tau(n(t),p(t))}{\partial n} \frac{dn(t)}{dt} + \frac{\partial \tau(n(t),p(t))}{\partial p} \frac{dp(t)}{dt} \right] - c_s. \tag{8}
\]

Equilibrium requires that \( \frac{\partial c(t,t^{*})}{\partial t} = 0 \), indicating a traveler cannot reduce his or her travel cost by changing departure time. Then we have

\[
\frac{\partial \tau(n(t),p(t))}{\partial n} \frac{dn(t)}{dt} + \frac{\partial \tau(n(t),p(t))}{\partial p} \frac{dp(t)}{dt} = \frac{c_s}{c_w - c_s}. \tag{9}
\]

Note that in Eq.(9), the schedule penalty \( c_s \) is different for early and late arrival traffic.
For travellers departing at time $t_s$, travel time is given by $\tau_s = \tau(n_s, p_0)$ where $p_0 = 1$. The on time travellers depart at time $t_\mu$, thus $t_\mu + \tau(n_\mu, p(t_\mu)) = \tau^*$, and let $\tau_\mu = \tau(n_\mu, p(t_\mu))$. The last traveller will depart at time $t_e$ when the congestion vanishes, i.e., $n(t_e) = n_c$. As trip length increases due to cruising-for-parking, travel time of the last traveller will be longer than the travellers departing at time $t_s$, i.e., $\tau_e = \tau(n_e, 1-N/N_p) > \tau_s$, and the percentage of available park spaces at time $t_e$ is $p(t_e) = 1-N/N_p$. The estimation of the equilibrium $t_s$, $t_\mu$ and $t_e$ will be discussed later. With Eq.(9), we can derive the equilibrium travel time profile, which is given as follows

$$
\tau^*(n(t), p(t)) = \begin{cases} 
\frac{e}{c_w - e}(t-t_s) + \tau_s & \text{for } t_s \leq t < t_\mu \\
\frac{l}{c_w + l}(t-t_\mu) & \text{for } t_\mu \leq t < t_e
\end{cases}
$$

and depicted in Figure 3. As mentioned, $\tau_e > \tau_s$ holds, which is also shown in Figure 3 such that $\Delta \tau = \frac{N}{N_p-N} \cdot \frac{d}{v(n)} > 0$. This is different from that under the departure/arrival equilibrium without cruising-for-parking where $\Delta \tau = 0$ as $N_p \rightarrow \infty$. It implies that, at the equilibrium with cruising-for-parking, the last commuter experiencing a longer travel time (due to cruising) than the first commuter will experience less schedule delay cost in compensation. In an alternative way, to enjoy less cruising-for-parking, commuters are forced to encounter larger schedule delay cost. This will later be verified in numerical studies.

![Figure 3. The equilibrium travel time profile](image)

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Note that the above derivation from Eq.(5) to Eqs.(8), (9) and (10) indeed does not rely on the assumption of identical desired arrival time $t^*$. The studied case with identical $t^*$ can be extended to that with differentiated desired arrival time. In the current case with all travellers having the identical $t^*$, for given $t_s$ and $\tau_s$, the departure time of on time traveller $t_\mu$ can be explicitly determined by $\tau_\mu = e \cdot \left( t_\mu - t_s \right) / \left( c_w - e \right) + \tau_s$ and $t_\mu + \tau_\mu = t^*$, which is

$$
 t_\mu = \frac{c_w - e}{c_w} \left( t^* - \tau_s \right) - \frac{e}{c_w} t_s, 
$$

and $\tau_\mu$ can be determined accordingly. However, if $t^*$ is not identical for the population, then closed-form of $t_\mu$ is not available, and $t_\mu$ has to be determined by numerically solving the equation $t_\mu + \tau \left( n(t_\mu), p(t_\mu) \right) = t^*$.

### 3.2. Estimation of User Equilibrium

With the equilibrium travel time profile given in Eq.(10), we then can drive the equations for estimating the equilibrium time-varying accumulation, percentage of available parking spaces, and outflow of the system.

From Eq.(3), we see that $\tau \left( n(t), p(t) \right) = L \left( p(t) \right) / v \left( n(t) \right)$, then with the $\tau^* \left( n(t), p(t) \right)$ given by Eq.(10), the equilibrium $n(t)$ can be written as follows:

$$
 n(t) = \begin{cases} 
 v^{-1} \left( L \left( p(t) \right) \cdot \left[ \frac{e}{c_w - e} \left( t - t_s \right) + \tau_s \right]^{-1} \right) & \text{for } t_s \leq t < t_\mu \\
 v^{-1} \left( L \left( p(t) \right) \cdot \left[ \tau_\mu - \frac{1}{c_w + l} \left( t - t_\mu \right) \right]^{-1} \right) & \text{for } t_\mu \leq t < t_c 
\end{cases}
$$

where $v^{-1} (\cdot)$ is the inverse function of $v = v(n)$ for $n \geq n_c$. In Geroliminis and Levinson (2009), since $p(t)$ is constant (without consideration of cruising-for-parking), the $n(t)$ can be determined explicitly with Eq.(12) given the equilibrium travel time profile in Eq.(10). However, in this study, in the right hand side of Eq.(12), $p(t)$ is dependent on the cumulative departure $I(t)$, thus is related to $n(t)$ profile over the time interval $[t_s, t_c]$. Therefore, $n(t)$ and $p(t)$ should be jointly estimated as shown later.
We use Eq.(4) to estimate the outflow of the system $o(t) = o(n(t), \bar{p}(t))$. Note here $\bar{p}(t)$ is the percentage of available parking spaces at time $t$. Furthermore, the conservation of traffic requires that
\[
\frac{dn(t)}{dt} = \frac{dI(t)}{dt} - o(t) \quad \text{(13)}
\]
It is worth mentioning that $A(t)$ is the cumulative arrival of the demand $N$, for $t \in [t_s + \tau_s, t_e + \tau_e]$, we have $dA(t)/dt = o(t)$, while for $t < t_s + \tau_s$, $A(t) = 0$ since no commuter in the total demand $N$ has arrived the parking spaces yet. From the above analysis, we can see that the estimation of the Dynamic User Equilibrium solution is not trivial as those for traditional analysis of Vickrey’s bottleneck model, due to the time-varying capacity (or outflow of the system) and the consideration of cruising-for-parking. We then propose the following process for estimation of the solution:

**Estimation procedure for User Equilibrium**

**Initialize:** Set $t_s = t^* - \tau_s$.

**do (Loop 1)**

Estimate $t_s$ from initialization or $t_s = t_s - \Delta t_s$.

**do (Loop 2)**

Update the current $t$ by $t = t_s$ or $t = t + \delta t$.

**do (Loop 3)**

With the initialized or updated $I'(t)$, calculate $I(t)$.

With $I(t)$, calculate $p(t)$ with Eq.(6).

With $p(t)$, calculate $n(t)$ with Eq.(12).

**do (Loop 4)**

With initialized or updated $o(t)$, calculate $A(t)$, and further calculate $\bar{p}(t)$ with Eq.(7).

With $n(t)$, $\bar{p}(t)$, update $o(t)$ with Eq.(4).

**while** $\left| \frac{o(t)_{i+1}^{(k+1)} - o(t)^{(k)}_{i}}{o(t)^{(k)}_{i}} \right| > \text{error}$

With $n(t)$ and $o(t)$, update $I'(t)$ with Eq.(13), and update $I(t)$, $p(t)$.

**while** $\left| I'(t)_{i+1}^{(k+1)} - I'(t)_{i}^{(k)} \right| > \text{error}$

**while** $\left| \frac{n(t)_{i} - n(t)_{i-1}}{n(t)_{i}} \right| > \text{error}$

Set $t_e = t$ and $N_e = I(t_e)$.

**while** $\left| \frac{N_{max} - N_e}{N_e} \right| > \text{error}$

Note: $\text{error} = 10^{-3}$ and $\delta t = 0.1 \text{ (min)}$ are applied in this paper.
The above procedure is also depicted in Figure 4. Note that, in Loop 3, we need to check that whether \( t + \tau(n(t), p(t)) \leq t^* \) or not to use the right form for \( n(t) \). Since this can be done even if \( t^* \) is non-identical, therefore, the above process can be principally extended for the case with differentiated \( t^* \).

Figure 4. The estimation procedure for User Equilibrium
4. Optimal time-varying pricing for the downtown network

The travel delay due to roadway congestion (intense traffic because of both concentrated schedule preference and cruising-for-parking), and increased schedule delay due to competition to enjoy less cruising-for-parking are both deadweight loss of social welfare. We now introduce a time-varying (fine) toll to minimize total travel cost including travel time cost and schedule delay cost, and improve traffic efficiency. It is straightforward to show that, for a single-region system, the total travel cost will be minimized when the downtown network or system is operating at the maximum production of the MFD (of the downtown network), i.e., \( n(t) = n_c \) and \( v(t) = v(n_c) \), and \( P(t) = n_c \cdot v(n_c) \). This is consistent with those described in Daganzo (2007), Gonzales and Daganzo (2012). For multi-region cities, more complex control strategies have to be introduced (see, e.g., Haddad et al., 2013; Geroliminis et al. 2013).

Let \( T(t) \) be the toll for the commuters departing at time \( t \) or entering into the network at time \( t \), individual full trip cost including the toll can be written as follows:

\[
\tau(n(t), p(t)) + c_s \cdot (t^* - t - \tau(n(t), p(t))) + T(t).
\]  
(14)

Similar to the User Equilibrium case, we take the first-order derivative of Eq.(14) with respect to \( t \), and let it to be zero, then we have

\[
\frac{\partial \tau(n(t), p(t))}{\partial n} \cdot \frac{dn(t)}{dt} + \frac{\partial \tau(n(t), p(t))}{\partial p} \cdot \frac{dp(t)}{dt} + \frac{1}{c_w - c_s} \cdot \frac{dT(t)}{dt} = c_s.
\]  
(15)

The above is the equilibrium condition under the time-varying toll.

Suppose under the time-varying toll, the peak starts at \( t_{s,1} \), of which the estimation will be discussed later. For \( t \leq t_{s,1} \) we set \( T(t) = T_0 \). After \( t_{s,1} \), since we maintain \( n(t) = n_c \), \( dn(t)/dt = 0 \). Then after some manipulations, we have

\[
\frac{dT(t)}{dt} = c_s - (c_w - c_s) \cdot \frac{\partial \tau(n_c, p(t))}{\partial p} \cdot \frac{dp(t)}{dt}.
\]  
(16)

Given \( \tau(n(t), p(t)) = L(p(t))/v(n(t)) \), Eq.(16) can be written as

\[
\frac{dT(t)}{dt} = c_s - (c_w - c_s) \cdot \frac{1}{v(n_c)} \cdot \frac{dL(p(t))}{dp(t)} \cdot \frac{dp(t)}{dt}.
\]  
(17)

From Eq.(17), we can derive the time-varying toll to support \( n(t) = n_c \) during the peak given as follows
\[
T(t) = \begin{cases} 
  T_0 + e \cdot (t - t_{s,1}) - T_p(t) & \text{for } t_{s,1} \leq t \leq t_{\mu,1} \\
  T(t_{\mu,1}) - l \cdot (t - t_{\mu,1}) - T_p(t) & \text{for } t_{\mu,1} < t < t_{c,1}, 
\end{cases}
\] 

(18)

where \( T_p(t) \) is given as follows:

\[
T_p(t) = \begin{cases} 
  \left( c_w - e \right) \cdot \frac{L(p(t)) - L(p(t_{s,1}))}{v(n_c)} & \text{for } t_{s,1} \leq t \leq t_{\mu,1} \\
  \left( c_w + l \right) \cdot \frac{L(p(t)) - L(p(t_{\mu,1}))}{v(n_c)} & \text{for } t_{\mu,1} < t < t_{c,1}, 
\end{cases}
\] 

(19)

and \( t_{\mu,1} \) is the departure time for the on time traveller and \( t_{c,1} \) is the latest departure time. For \( t > t_{c,1} \) we can set \( T(t) = T(t_{c,1}) \).

Figure 5 shows the pattern of the optimal time-varying toll when the minimum toll is zero (negative toll or rebate is not considered at the current stage). It is worth mentioning that after taking into account the cruising-for-parking, the time-varying toll becomes non-triangular since the impact of cruising is generally non-linear over time. Furthermore, by approximating \( \tilde{p}(t) \) with \( p(t) \), i.e., \( \tilde{p}(t) = p(t) \), and noting the cumulative departure (system inflow) is parallel to the cumulative arrival (system outflow) at the system optimum, it can be shown that the time-varying toll should be concave over time as \( L(p(t)) \) is convex over time.

Furthermore, it can be proved that the first commuter would experience a higher toll than the last commuter, i.e., \( T_o > 0 \). This \( T_o \) is to avoid the additional schedule delay cost due to incentive to enjoy a lower cruising time. Last but not least, we would like to point out that the maximum toll might not always be experienced by the on time travellers (however, Figure 5 take this case as an illustrative example). This can be seen by looking at Eq.(17) which is the first-order derivative of the toll over time. The reason is that if cruising-for-parking is relatively costly, \( \frac{dT(t)}{dt} \cdot \frac{dp(t)}{dt} \) might be relatively large thus \( \frac{dT(t)}{dt} < 0 \). However, since the case we are discussing is in the system optimum, when the on time traveller arrives, the percentage of vacant parking spaces can still be relatively large, and cruising-for-parking is not such costly, we then have \( \frac{dT(t)}{dt} \geq 0 \) for \( t < t^* \).

Besides, for \( t > t^* \), it can be easily verified that \( \frac{dT(t)}{dt} < 0 \).
By utilizing the toll design as discussed in the above, choosing different $t_{s,1}$ will not affect the exact departure/arrival pattern since it is determined by $n(t) = n_c$, but translate that pattern along the time horizon. The travel time cost then would be identical under different $t_{s,1}$. To minimize total travel cost, it suffices to choose an appropriate $t_{s,1}$ to minimize schedule delay cost, and solve the following minimization problem:

$$
\min : \text{SC}(t_{s,1}) = \int_0^{\Delta t} \frac{dI(x)}{dx} \cdot c_s \cdot (t^* - (t_{s,1} + x) - \tau(x)) \, dx,
$$

where $\Delta t$ is the length of the travellers’ departure duration, $I(x)$ is the cumulative departure (inflow) at time $t_{s,1} + x$, and $\tau(x)$ is the travel time for travellers departing at time $t_{s,1} + x$. Note that as the departure/arrival pattern is exactly the same under different $t_{s,1}$, $I(x)$, $\tau(x)$ and $\Delta t$ in Eq.(20) are independent of $t_{s,1}$. Indeed, for $t_{s,1} \neq t'_{s,1}$, if we implement the discussed toll design in Eq.(18), we would have

$$
t_{s,1} = t_{s,1} + \Delta t, \quad t'_{s,1} = t'_{s,1} + \Delta t.
$$

For $t \in [0, \Delta t]$, we have $n(t_{s,1} + t) = n(t'_{s,1} + t) = n_c$, it follows

$$
I(t_{s,1} + t) = I(t'_{s,1} + t), \quad p(t_{s,1} + t) = p(t'_{s,1} + t), \quad \tilde{p}(t_{s,1} + t) = \tilde{p}(t'_{s,1} + t).
$$

Since $v(t_{s,1} + t) = v(t'_{s,1} + t) = v(n_c)$, we further have

$$
L(t_{s,1} + t) = L(t'_{s,1} + t), \quad o(t_{s,1} + t) = o(t'_{s,1} + t), \quad \tau(t_{s,1} + t) = \tau(t'_{s,1} + t).
$$
However, the toll pattern over time would be different, i.e., $T(t_{s,1} + t) \neq T'(t_{s,1} + t)$.

Taking the first order derivative of the objective function in Eq.(20) with respect to $t_{s,1}$, we have

$$
\frac{dSC(t_{s,1})}{dt_{s,1}} = \int_0^{\Delta t} d\left(\frac{d(x)}{dx} \cdot c_s \cdot (t^* - (t_{s,1} + x) - \tau(x))\right) dx = \int_0^{\Delta t} -dI(x) c_s dx .
$$

(24)

Let $\Delta t_{\mu,1} + \bar{\tau}(\Delta t_{\mu,1}) = t^* - t_{s,1}$, then $\Delta t_{\mu,1}$ corresponds to the on time traveller, and it can be verified that $d\Delta t_{\mu,1}/dt_{s,1} < 0$. Then, Eq.(24) can be rewritten as

$$
\frac{dSC(t_{s,1})}{dt_{s,1}} = -e \cdot \int_0^{\Delta t_{\mu,1}} dI(x) dx + l \cdot \int_{\Delta t_{\mu,1}}^{\Delta t} dI(x) dx .
$$

(25)

We then look at the second order derivative of the objective function in Eq.(20), which is given as

$$
\frac{d^2SC(t_{s,1})}{dt^2} = -e \cdot \left(\frac{dI(x)}{dx}\right)_{x=\Delta t_{\mu,1}} \cdot \frac{d\Delta t_{\mu,1}}{dt_{s,1}} + l \cdot \left(\frac{dI(x)}{dx}\right)_{x=\Delta t_{\mu,1}} \cdot \frac{d\Delta t_{\mu,1}}{dt_{s,1}} > 0 .
$$

(26)

Total schedule delay cost is minimized if we let Eq.(25) to be zero, i.e.,

$$
\frac{dSC(t_{s,1})}{dt_{s,1}} = 0 \Leftrightarrow \int_0^{\Delta t_{\mu,1}} \frac{dI(x)}{dx} dx = \frac{N_e - l}{N_l} \cdot e ,
$$

(27)

which says the early arrival traffic $N_e$ should be $l/e$ times as much as the late arrival traffic $N_l$. This is consistent with the case without cruising and that in Vickrey’s model.

For given $N$ and $N_p$, denote the $t_{s,1}$ under which the last traveller arrives on time by $t^1_{s,1}$, then $t^1_{s,1} = t^* - \tau_e - \Delta t$, and denote the $t_{s,1}$ under which the first traveller arrives on time by $t^2_{s,1}$, then we have $t^2_{s,1} = t^* - \tau_s$. It can be shown that Eq.(25) will be negative when $t_{s,1} = t^1_{s,1}$, and will be positive when $t_{s,1} = t^2_{s,1}$. Given Eq.(26), we see that the $t_{s,1}$ that solves Eq.(27) will be within $[t^1_{s,1}, t^2_{s,1}]$.

Based on the results above, we propose to compute the System Optimum solution as follows. Firstly, we should choose initial lower bound $t^l_{s,1} = t^1_{s,1}$ and upper bound $t^u_{s,1} = t^2_{s,1}$ for $t_{s,1}$. Then set $t_{s,1} = \frac{1}{2}(t^l_{s,1} + t^u_{s,1})$ as an initial solution. Now, we can estimate all the time-varying variables with a similar approach as the estimation for User Equilibrium starting from $t = t_{s,1}$ and ending $t = t_{e,1}$ where $I(t_{e,1}) = N$. However, when computing the $n(t)$, we should incorporate the toll $T(t)$.
given by Eq.(18) into the equilibrium condition with time-varying toll given in Eq.(15), and the resulting \( n(t) = n_c \). To check whether the current \( t_{s,1} \) is the optimal solution or not, we can compare the numbers of early and late traffic \( N_e \) and \( N_l \). If \( N_e/N_l < l/e \), it means the current \( t_{s,1} \) is too large, then we can set \( t_{s,1}^u = t_{s,1} \), and let \( t_{s,1} = \frac{1}{2} \left(t_{s,1}^u + t_{s,1}^l \right) \); If \( N_e/N_l > l/e \), it means the current \( t_{s,1} \) is too small, then we can set \( t_{s,1}^l = t_{s,1} \), and let \( t_{s,1} = \frac{1}{2} \left(t_{s,1}^l + t_{s,1}^u \right) \). The System Optimum solution is achieved as \( \left|(t_{s,1}^u - t_{s,1}^l)/t_{s,1}^u \right| < \text{error} \). We use error = 10^{-3} in this paper.

Approximate solution for \( p(t) \) under the System Optimum

The following discusses an approximate solution for \( p(t) \) under the System Optimum solution. It is mentioned that \( p(t) = 1 - \frac{\int_0^t (1-p(t)) \, dt}{N_p} \), and with Eq.(13) and the fact that at the System Optimum, \( n = n_c \) and \( dn(t)/dt = 0 \), we have

\[
\frac{dp(t)}{dt} = -\frac{1}{N_p} \cdot \frac{dL(t)}{dt} = -\frac{1}{N_p} \cdot \left(\frac{dn(t)}{dt} + o(t)\right) = -\frac{1}{N_p} \cdot o(t),
\]

(28)

With \( o(t) = o(n(t), \tilde{p}(t)) \), and let \( p(t) \) to approximate \( \tilde{p}(t) \), i.e., \( p(t) = p(\tilde{t}) = \tilde{p}(t) \), then we have

\[
\frac{dp(t)}{dt} = -\frac{1}{N_p} \cdot \frac{n_c \cdot v(n_c)}{L(p(t))}.
\]

(29)

Since \( L(p(t)) = l_m + d/p(t) \), after some manipulations of Eq.(29), we obtain

\[
l_m \cdot p(t) \cdot \frac{dp(t)}{dt} + d \cdot \frac{dp(t)}{dt} + \theta \cdot p(t) = 0.
\]

(30)

where \( \theta = n_c \cdot v_c / N_p \). We then have

\[
p(t) = \frac{d}{l_m} W \left( \frac{l_m}{d} \exp \left( \frac{k_i - \theta t}{d} \right) \right),
\]

(31)

where \( W(\cdot) \) is the inverse function of \( f(W) = W \cdot e^W \), and \( k_i \) is determined by \( p(t_{s,1}) = p_0 \). Based on Eq.(31), with information of \( d, l_m, p(t_{s,1}) = p_0, n_c, v(n_c) \) and \( N_p \), we can estimate the shape of the parking vacancy profile over time in the system optimum.

5. Bi-modal Commuting Equilibrium with Cruising-for-parking
Now we extend the previous analysis to the case with public transit as an alternative for driving. Total travel demand \( N \) is given and fixed. In the bi-modal context, travellers can either drive to their destination through the downtown network or take public transit (we assume there is no flow interaction between cars and transit). For commuters driving to the destination, denoted by \( N_a \), their full trip cost still includes schedule delay cost and travel time cost which is given by Eq.(5). For commuters taking transit, denoted by \( N_b \), trip cost \( c_b \) is assumed to be constant\(^4\). And the summation of users choosing both travel modes should be \( N_a + N_b = N \). Total number of parking spaces \( N_p \) is again assumed to be given. At the bi-modal equilibrium, \( N_a < N_p \) should hold, otherwise the time spent on cruising-for-parking would go to infinity.

Under the current bi-modal setting, the start of the peak of the auto side can be determined by equating the travel cost of two modes:

\[
\left( t^* - t_e \right) + c_w \cdot \tau_s = c_i.
\]

Therefore,

\[
t_s = t^* - \frac{c_b - c_w \cdot \tau_s}{e}.
\]  

(32)

With the \( t_s \) determined above, the User Equilibrium at the auto side can be estimated with a similar procedure as such we discussed for the single mode user equilibrium (without transit). Note that, in the current setting, \( t_s \) is known from Eq.(32) and no longer needs to be estimated. The estimation of all the time-varying variables will start at \( t = t_s \) and end at \( t = t_e \) where \( n(t_e) = n_i \). The resulting number of car users is \( N_a = I(t_e) \). Further with \( N_a + N_b = N \), we have the equilibrium mode share for transit.

The total travel cost of the bi-modal system is

\[
TC = TC_a(N_a) + c_i \cdot (N - N_a),
\]

(33)

where \( TC_a(N_a) \) is the total travel cost of all car users given equilibrium is achieved for auto mode itself, and \( c_i \cdot (N - N_a) \) is the total travel cost of transit users. In the bi-modal equilibrium, \( TC_a(N_a) = c_i \cdot N_a \), and total system travel cost would be \( TC = c_i \cdot N \). Taking the first-order derivative of Eq.(33) with respect to \( N_a \), i.e.,

\[
\frac{dTC}{dN_a} = \frac{dTCA(N_a)}{dN_a} - c_i,
\]

(34)

\(^4\) Extension to the case with an increasing transit cost function with respect to number of users is easy, while extension to consider responsive transit service, i.e., transit operator will adjust frequency and fare according to roadway capacity, as in Zhang et al. (2014), will be more challenging, and might be studied in future research.
Without consideration of time-varying toll, in the optimal modal-split, Eq.(34) should be equal to zero, which means the marginal cost of the auto mode should be equal to the marginal cost of the transit mode (note here since we consider constant transit cost, the marginal cost is equal to the constant). However, equilibrium requires the average cost of the auto mode is equal to the average cost of transit mode, i.e., \( TC_a \left( N_a \right) / N_a = c_i \). This means that without the time-varying toll, by shifting the modal split, we can improve the system by a certain amount.

\[
AC_{SO} + \text{toll}_{ave}'
\]

\[
MC_{SO}
\]

\[
N_a'
\]

\[
N_a^{re}
\]

\[
N_a^{*}
\]

**Figure 6.** The optimal modal split when the time-varying (fine) toll is introduced

Similar analysis for the case without time-varying toll can be applied to the case with optimal time-varying toll introduced. Figure 6 presents the optimal modal-split when optimal time-varying toll is introduced in the auto side. In Figure 6, \( AC_{SO} \) and \( MC_{SO} \) are the average cost and marginal cost (with respect to number of users) of the auto mode (for auto mode itself, system optimum is achieved) respectively. The bi-modal system optimum is achieved when \( MC_{SO} = c_i \). To support such a bi-modal system optimum, a well-designed \( T_o \) in the time-varying toll given by Eq.(18) should be chosen. Therefore, not only the auto side is operating at the maximum outflow, but also the modal split is optimal. A well-designed \( T_o \) indicates the average toll (total toll revenue divided by total number of drivers) levied on all drivers should be appropriate. As shown in Figure 6, a too high average toll \( \text{toll}_{ave}' \) would lead to a too small demand, i.e., \( N_a'' < N_a^{re} \); while a too low average toll \( \text{toll}_{ave}'' \) would lead to a too large demand, i.e., \( N_a'' > N_a^{re} \). It worth mentioning that we may need
to give the last commuters either a positive toll or rebate (negative toll) to support the bi-modal system optimum. This is verified in the numerical examples. Note that Figure 6 is valid for the case with identical $t^*$. If $t^*$ is not identical, travelers do not have the same travel cost and might have different preferences on the two travel modes, and the analysis of the bi-modal system would be more complicated.

6. Numerical Studies

In this section, we present some numerical examples to illustrate and verify the models and analysis in the previous sections. Table 1 summarizes the values of parameters and variables valid for all the following analysis.

<table>
<thead>
<tr>
<th>Parameters or Functions</th>
<th>Specification</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value of travel time</td>
<td>$c_w = 9.91$ (EUR$)</td>
</tr>
<tr>
<td>Early arrival penalty</td>
<td>$e = 4.66$ (EUR$)</td>
</tr>
<tr>
<td>Late arrival penalty</td>
<td>$l = 14.48$ (EUR$)</td>
</tr>
<tr>
<td>Critical accumulation</td>
<td>$n_c = 1000$ (veh)</td>
</tr>
<tr>
<td>Travelling Speed</td>
<td>$v(n) = v_0 \cdot e^{-v_n}$ (km/h) for $n \geq n_c$</td>
</tr>
<tr>
<td></td>
<td>$v(n) = v(n_c)$ (km/h) for $n &lt; n_c$</td>
</tr>
<tr>
<td>Speed function parameters</td>
<td>$v_0 = 90$ and $v_1 = 10^{-3}$</td>
</tr>
<tr>
<td>Trip distance</td>
<td>$L(p) = l_m + d/p$ for $0 &lt; p \leq 1$</td>
</tr>
<tr>
<td>Trip distance function parameters</td>
<td>$l_m = 11$(km) and $d = 2$ (km)</td>
</tr>
</tbody>
</table>

Note: $c_w$, $e$ and $l$ are from Tseng et al. (2005).

6.1. User Equilibrium

*Time-varying traffic and parking vacancy under User Equilibrium*

In the basic case, we consider $N = 6000$ and $N_p = 7000$, and $t^* = 250$ (min). Figure 7(a) presents the resulting equilibrium cumulative departure and arrival, i.e., $I(t)$ and $A(t)$, while Figure 7(b) presents the inflow (departure rate) and outflow (arrival rate). Figure 7(c) and Figure 7(d) depict
the time-varying accumulation and associated speed, and time-varying percentage of available parking spaces and associated trip distance respectively.

As shown in Figure 7(b), the system outflow decreases from 4.24 (veh/0.1min) at time $t_s = 53.2$ (min) to 2.54 (veh/0.1min) at time $t_{μ} = 145$ (min). This decrease is partly due to the increase of accumulation from 1000 (veh) to 2276 (veh) (the system becomes more congested, and speed goes down as shown in Figure 7(c)), and partly due to the decreasing percentage of vacant parking spaces from 100% to 38.64% (it becomes more difficult to find a vacant parking space, and trip length goes up as shown in Figure 7(d)). After $t_{μ} = 145$ (min), the accumulation of the system starts to decrease, this leads to increase in outflow shown in Figure 7(b). At this stage, the impact of lessening congestion (decreasing accumulation) overweighs the impact of the lengthening trip length (decreasing parking availability). However, at time $t = 227.5$ (min), the remaining available parking is about 17.84%, and the impact of decreasing parking availability is
relatively large, and overweights that of the decreasing accumulation and increasing speed. More importantly, the outflow of the system at the end of the peak can never go back to the level at the start of the peak, i.e., the maximum 3.21 (veh/0.1min) at time 227 (min) is less than the initial 4.24 (veh/0.1min) owing to the limitation of parking spaces. On the contrary, as can be seen later in Figure 9, the outflow of the system without cruising-for-parking can go back to the initial value at the end of the peak as congestion vanishes. In Figure 7(b), we also see that the inflow (departure rate from home) almost decreases from the beginning to the end. However, there is a slight increase after $t_\mu = 145$ (min), which is due to the decreasing accumulation and increasing speed (less congested network). Note that, due to cruising-for-parking and accumulation-dependent traveling speed, the inflow (departure rate from home) is time-varying, while in Vickrey’s model there are only two values. Also, as travelers have the incentive to enjoy less cruising time, there are more early arrival traffic, $N_e/N_i = 5.0$, than the case without cruising, $N_e/N_i = 2.3$, and that in Vickrey’s model, $N_e/N_i = l/e = 3.1$.

Figure 8 further shows the equilibrium travel cost, travel time cost and schedule delay cost. It is shown that the first traveler experiences more schedule delay cost than the last one, i.e., 13.43 (EUR$) > 9.83$ (EUR$), since he or she can enjoy less cruising time, as well as travel time cost, i.e., 3.88 (EUR$) < 7.48$ (EUR$).

Figure 8. Equilibrium costs based on departure time

Figures 9(a) and 9(b) depict the equilibrium inflow/outflow and the travel cost under $N = 6000$ and $N_p \to \infty$ respectively. As shown in Figure 9(a), the outflow of the system without cruising-for-parking at equilibrium can go back to the initial maximum at the end of the peak. The outflows
at the beginning and the end are 4.24 (veh/0.1min). In addition, the first and last commuter would experience the same schedule delay cost (10.3 EUR$) and travel time cost (3.9 EUR$), which is consistent with that in Geroliminis and Levinson (2009) and Vickrey’s model. The total travel cost is smaller due to lack of cruising, i.e., equilibrium individual travel cost is 14.2 (EUR$), which is less than that with cruising (17.3 (EUR$)). Also note that, without cruising, there are less early traffic, i.e., $N_e/N_l = 2.3$, which is less than 5.0.

![Figure 9. User equilibrium without cruising: inflow/outflow and travel cost](image)

As mentioned before, we estimate the travel time, outflow and the resulting user equilibrium by using the instantaneous accumulation and percentage of vacant parking spaces for analytical tractability. All the efficiency measures, such total travel cost, travel time, schedule delay, the number of early or late arrival traffic etc. are calculated according to these estimations. We compare these estimations with the exact values under the equilibrium departure/arrival pattern. These differences are generally small (ranging from ±5% to ±10%, dependent on the parking capacity). Furthermore, to have a more accurate estimation of equilibrium departure/arrival pattern (as the one shown in Figure 9(a)), we can adjust the time-varying accumulation based on the difference between the travel time under the current departure/arrival pattern and the equilibrium travel time, i.e., if the travel time is higher (lower) than the equilibrium one, we decrease (increase) the accumulation.

**System performance under varying parking capacity**

Given $N = 6000$, Figure 10(a) depicts how total travel cost, travel time cost and schedule delay cost vary with the parking capacity $N_p$, while Figure 10(b) shows how the number of early and
late arrival traffic and schedule delay cost vary with the parking capacity $N_p$, where the x-axis is the ratio of travel demand $N$ to the parking capacity $N_p$. This ratio, i.e., $N/N_p$, is indeed the parking occupancy at the end of the morning peak. In Figure 10(a), as the ratio of $N/N_p$ increases, i.e., parking supply is more limited, the total travel cost, travel time and schedule delay will increase. This is because more severe parking limitation would lead to longer trip length and smaller outflow of the system. Given schedule delay increases as parking becomes more limited, Figure 10(b) further shows that the schedule delay cost of early arrival increases more sharply than the total schedule delay cost, while the schedule delay cost of late arrival will decrease. This is because, as parking capacity decreases, travelers are more willing to depart earlier to enjoy less cruising, therefore, early arrival traffic increases with $N/N_p$ as shown in the right panel of Figure 10(b). In Figure 10(a), the case with $N/N_p=0$ corresponds to the situation with no cruising, i.e., the one depicted in Figure 9. As can be seen, the total travel cost reaches the minimum as there is no cruising effect, which is $6000 \times 14.2 = 8.52 \times 10^4$ (EUR$).
Figure 11 shows the equilibrium cumulative departure/arrival patterns under three specific $N/N_p$ (0.75, 0.60 and 0.00), where $N = 6000$. As $N/N_p$ increases from 0.00 to 0.75 (parking capacity decreases), on one hand, the departure/arrival are starting earlier, due to competition for less cruising time; on the other hand, the duration of the peak is enlarged, owing to decreased network capacity (outflow) under lower parking capacity.

![Equilibrium departure/arrival under different parking capacities](chart.png)

**Figure 11.** Equilibrium departure/arrival under different parking capacities

### 6.2. System Optimum

**Flow pattern and travel cost under System Optimum**

We still consider $N = 6000$ and $N_p = 7000$, and $t^* = 250$ (min), which is identical with those in the User Equilibrium (section 6.1). As discussed in Section 4, an appropriate $t_{s,1}$ should be chosen to minimize total social cost (without toll). Figure 12 (b) shows how the toll revenue, social cost, and travel cost including toll vary with $t_{s,1}$ when the minimum toll is zero (so there is no negative toll), while Figure 12(a) shows the first and last tolls (tolls experienced by the first or last traveler respectively) under given $t_{s,1}$. As can be seen in Figure 12(b), the social cost (without toll) is minimized when $t_{s,1} = 107.86$ (min), i.e., SO(a). In this case, the first traveler would experience a higher toll than the last traveler, i.e., 8.39 (EUR$) > 0$ (EUR$). Also, as summarized in Table 2,
the ratio of early traffic to late traffic, $N_e/N_l$, is equal to $1/e = 3.1$, which verifies our analysis in Section 4. The total cost including toll is minimized when $t_{s,l} = 81.5$ (min), i.e., SO(b). Also note that, in this case, the toll revenue is minimized, and both the first toll (experienced by first traveler) and last toll (experienced by last traveler) are zero. We can verify that toll revenue is 70050 (EUR$) at SO(a), which is 2.5 times of the toll revenue (28580 (EUR$)) when total cost (includes toll) is minimized (SO(b)). By imposing such a high level of toll for SO(a) instead of that for SO(b), the social cost can only be reduced from 62230 (EUR$) to 58890 (EUR$). This reduction is around 3.2% comparing to total travel cost under User Equilibrium (103923 (EUR$)). This means, we may set a much lower level of toll to achieve approximate efficiency of the optimal time-varying toll, which is further verified in Figure 13. In addition, in Figure 12(b), we see that a too early $t_{s,l}$ (all $t_{s,l} < 51.5$ (min)) would lead to the same toll revenue. This is because, when $t_{s,l} < 51.5$ (min), the toll pattern is exactly the same, but starts at a different $t_{s,l}$. Similar results can be observed for a too late $t_{s,l}$ (all $t_{s,l} > 223.0$ (min)).

![Figure 12. Toll revenue, social cost and total cost (the minimum toll is zero)](image)

Figures 13(a) and 13(b) show the cumulative departure and arrival, and travel costs at the SO(a), and Figures 13(c) and 13(d) show those for SO(b), while Figures 13(e) and 13(f) presents those for another case SO(c), where SO(c) is when the last toll is fixed at zero, and total cost including...
toll is minimized. Note that under SO(c), we have \( t_{s,1} = 49.5 \) (min), and the social cost is equal to that for \( t_{s,1} = 49.5 \) (min) depicted in Figure 12(b), i.e., 73890 (EUR$). However, as the minimum toll becomes negative, which is shown in Figure 13(f), thus the toll revenue and total cost including toll under SO(c) will be different from those in Figure 12(b).

As can be seen in Figures 13(a), 13(c) and 13(e), at each of the defined three system optimums, the cumulative departure and arrival are parallel to each other, thus the time-varying accumulation of the system remains at the critical level of 1000 (veh), and the speed is at its maximum of 33.11 (km/h). Also note, the cumulative departure/arrival patterns under SO(a), SO(b) and SO(c) are exactly the same while they start at different \( t_{s,1} \). This is consistent with our analytical analysis in Section 4. Furthermore, the slopes of the cumulative departure and arrival, i.e., inflow and outflow of the system, decrease over time as the percentage of available parking spaces goes down and trip length increases.

Under SO(a), as shown in Figures 13(a) and 13(b), there is zero departure from home before \( t_{s,1} = 107.86 \) (min). To support SO(a) as an equilibrium and ensure travelers will not depart earlier than \( t_{s,1} = 107.86 \) (min), we set a constant toll equal to \( T_o \) for time \( t < 107.86 \) (min). The travel cost by departing earlier than \( t_{s,1} = 107.86 \) (min) is larger than equilibrium travel cost as shown in Figure 13(b). Note that similar tolls have been designed to support SO(b) and SO(c) as an equilibrium as well, which are shown in Figures 13(d) and 13(f). Under the three defined system optimums, travel time cost increases from 3.89 (EUR$) (for the first traveler) to 7.47 (EUR$) (for the last traveler), due to the increasing trip length (as a result of decreasing parking vacancy). The time-varying toll to support the SO(a) as an equilibrium, as shown in Figure 13(b), is consistent with that depicted in Figure 5 of Section 4.2. The first traveler will experience a positive toll of 8.39 (EUR$), i.e., \( T_o \), and the last traveler encounter a zero toll. Furthermore, the individual travel cost including the toll is 21.45 (EUR$), which is higher than the equilibrium travel cost in the User Equilibrium. This means travelers are worse off although social cost decreases. To make every traveler better off, we need to decrease the average toll level (the toll revenue over the total traffic), thus some of the travelers experiencing larger cruising time would get a rebate (negative toll).

By comparing Figure 13(b) and Figure 13(d), we see that the toll under SO(a) is much larger than the toll under SO(b). This is consistent with Figure 12, i.e., the toll revenue is 70050 (EUR$) at SO(a), which is 2.5 times of the toll revenue (28580 (EUR$)) at SO(b). As mentioned, we can set the toll under SO(b) to achieve a Pareto-improving situation for all travelers, i.e., individual travel cost, 15.1 (EUR$), is smaller than that under User Equilibrium (17.3 (EUR$)), while losing social
efficiency by only 3.2% as shown in Figure 12. If we allow negative toll (rebate) and fix the last toll to be zero, the time-varying toll supporting SO(c) as an equilibrium, as shown in Figure 13(f), will minimize total cost including toll. This total cost with toll is less than the minimum social cost achieved under SO(a), i.e., 45010 (EUR$) < 58890 (EUR$), and every traveler is better off, i.e., 7.5 (EUR$) < 17.3 (EUR$). However, as can be seen in Figure 13(f), such a Pareto-improving result requires large amount of subsidies (negative tolls) to travelers.

Figure 13. Flow patterns and costs at SO(a), SO(b) and SO(c) based on departure time
Table 2 further summarizes different efficiency measures for five cases: User Equilibrium with cruising-for-parking, User Equilibrium without cruising-for-parking, SO(a) with cruising-for-parking, SO(b) with cruising-for-parking, and System Optimum without cruising-for-parking. By comparing the UE with and without cruising, we see that cruising-for-parking lead the total social cost, moving time, cruising time, and schedule delay to increase. However, the schedule delay cost of late arrival decreases, which is due to the fact that travelers are departing earlier to enjoy less cruising (there is a sharp increase in schedule delay cost of early arrival). By comparing the UE with SO(a) (both with cruising), we see huge reduction in travel cost (43.33%), moving time (55.39%), cruising time (55.76%), and schedule delay (24.97%). However, there is a small increase in schedule delay cost of late arrival. This is because, the toll avoids travelers to depart earlier (for less cruising time), and there is more late arrival traffic in the SO(a) case. Besides, we note that total cost including toll for SO(a) with cruising is larger than that under UE with cruising, i.e., $0.589 + 0.699 > 1.039$, which means all individual travelers are worse off as mentioned already. However, for the cases without cruising, total cost including toll for SO $0.482 + 0.248$ will be less than that under the UE $0.851$, which means individual are better off even if they have to pay a toll. This is consistent with Geroliminis and Levinson (2009). As discussed already, by setting a much lower level of toll, i.e., the toll for SO(b), we can achieve approximate efficiency of the optimal time-varying toll, i.e., the toll for SO(a). Indeed, the social cost increases from 0.589 to 0.622, while the total cost including toll, decreases from $0.589+0.699$ to $0.622 + 0.286$.

In Table 2, also note that, due to cruising-for-parking, there are much more early traffic, i.e., $N_e/N_i = 5.0$ for UE with cruising is much larger that $N_e/N_i = 2.3$ for UE without cruising. The toll to support SO(a) then avoids early departure of travelers and $N_e/N_i$ reduces to 3.1, which is equal to the ratio of late arrival penalty to early arrival penalty, i.e., $l/e$. This is consistent with our analytically proved results in Section 4. Besides the peak starts later under SO(a) compared to UE with cruising, the peak duration shortens, i.e., the departure duration under SO(a) (154.8) is shorter than that under UE with cruising (192.4). Furthermore, the reduction of departure duration from UE with cruising to SO(a) or SO(b) with cruising (37.6) is larger than that from UE without cruising to SO without cruising (34.3). This is because, besides the moving traffic, the cruising traffic also benefits from the reduced traffic congestion in the system.
Table 2. Various efficiency measures for three cases

<table>
<thead>
<tr>
<th></th>
<th>UE with cruising</th>
<th>UE (no cruising)</th>
<th>SO(a) with cruising</th>
<th>SO(b) with cruising</th>
<th>SO (no cruising)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Social cost (10^5 EUR$)</td>
<td>1.039</td>
<td>0.851</td>
<td>0.589</td>
<td>0.622</td>
<td>0.482</td>
</tr>
<tr>
<td>Toll revenue (10^5 EUR$)</td>
<td>0</td>
<td>0</td>
<td>0.699</td>
<td>0.286</td>
<td>0.248</td>
</tr>
<tr>
<td>Moving time (10^5 min)</td>
<td>3.167</td>
<td>3.130</td>
<td>1.413</td>
<td>1.413</td>
<td>1.413</td>
</tr>
<tr>
<td>Cruising time (10^5 min)</td>
<td>0.624</td>
<td>0</td>
<td>0.276</td>
<td>0.276</td>
<td>0</td>
</tr>
<tr>
<td>Schedule (10^4 EUR$)</td>
<td>4.132</td>
<td>3.345</td>
<td>3.100</td>
<td>3.435</td>
<td>2.490</td>
</tr>
<tr>
<td>Early arrival (10^4 EUR$)</td>
<td>3.341</td>
<td>2.150</td>
<td>2.178</td>
<td>3.187</td>
<td>1.883</td>
</tr>
<tr>
<td>Late arrival (10^4 EUR$)</td>
<td>0.791</td>
<td>1.195</td>
<td>0.922</td>
<td>0.248</td>
<td>0.607</td>
</tr>
<tr>
<td>Ratio of N_c/N_i</td>
<td>5.0</td>
<td>2.3</td>
<td>3.1</td>
<td>4.1</td>
<td>3.1</td>
</tr>
<tr>
<td>Departure duration (min)</td>
<td>192.4</td>
<td>175.5</td>
<td>154.8</td>
<td>154.8</td>
<td>141.2</td>
</tr>
</tbody>
</table>

For the System Optimum case, the differences between estimations and the exact values based on the system optimal departure/arrival pattern of measures such as total travel time are generally smaller than those for the User Equilibrium case, which are within ±5%. This is because errors from using instantaneous accumulation to calculate travel time are reduced considerably as the accumulation remains constant under the system optimum.

6.3. Bi-modal equilibrium

In the bi-modal equilibrium with cruising-for-parking, we still consider a total travel demand of N = 6000. Figure 14(a) depicts the modal-split against the parking capacity, while Figure 14(b) presents the total and average cruising times of all drivers under varying parking capacity. In Figure 14, parking capacity N_p varies from 3800 to 33800. Transit cost is c_t = 8.4 (EUR$), which is larger than the free-flow travel time cost (percentage of vacant parking spaces is 100%), which is equal to 3.9 (EUR$), as can be seen in Figures 8 and 9. Figure 14(a) shows that as parking capacity increases, more and more travelers shift from transit to auto (number of transit passengers decreases from 3480 to 3032). This is because cruising-for-parking is reduced under a larger parking capacity, which is shown in Figure 14(b), where both the total cruising time (of all drivers) and average cruising time are decreasing.
By assuming $N = 6000$ and $N_p = 7000$, Figure 15 depicts how the total social cost of the bi-modal system varies with the modal-split when the fine toll is introduced to the auto mode. As depicted in Figure 15, when the share of auto mode is 35%, the total social cost of the bi-modal system is minimized, and the marginal cost of the auto side is $14.4$ (EUR$), which is identical to the marginal cost of the transit mode, i.e., $c_t = 14.4$ (EUR$).

In the fine toll to support the optimal modal split in Figure 15 as an equilibrium, the first toll should be $T_o = 6.99$ (EUR$) while the last toll is equal to $6.54$ (EUR$) = 0$, which is shown in Figure 16. If we consider only auto mode with $N = 6000$ and $N_p = 7000$, under the optimal time-varying
toll, the first toll should be $T_o = 8.39$ (EUR$) while the last toll is zero, as also shown in Figure 16. The difference between the first and last tolls for the case with two travel modes (0.45 (EUR$)) are much smaller than that (8.39 (EUR$)) for the case with single mode. This is because, after considering public transit as an alternative for auto mode, less people are driving, and the incentive to enjoy less cruising time is less as well.

![Figure 16. The optimal tolls for auto side in the bi-modal and single mode system optimum](image)

6.4. Convergence of the estimation procedures

Figure 17(a) depicts the errors defined in the estimation procedures for User Equilibrium, i.e., $\left|\left(N_{end} - N\right)/N\right|$, and for System Optimum, i.e., $\left|\left(t_{s,1}^u - t_{s,1}^l\right)/t_{s,1}^u\right|$, against the number of iterations, while Figure 17(b) depicts how the peak start time $t_s$ evolves over iteration. For the UE and SO with cruising, $N = 6000$ and $N_p = 7000$ are applied, while for the UE and SO without cruising, $N = 6000$ and $N_p = 7 \times 10^{10}$. As can be seen, it takes more iterations for the UE with cruising (number of iteration: 27) than that without cruising (number of iteration: 14) to have the errors to be less than $10^{-3}$. As shown in Figure 17(b), for estimating UE with or without cruising, by utilizing the information of the gap $\left|N_{end} - N\right|$, and the average departure rate, i.e., $N_{end}/(t_e - t_s)$, in the first few iterations, $t_s$ is sharply reduced and becomes much closer to the final converged solution. For estimating the SO with and without cruising, the numbers of iterations are both 12 to have $\left|\left(t_{s,1}^u - t_{s,1}^l\right)/t_{s,1}^u\right| \leq 10^{-3}$. 

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7. Conclusion and discussion

In this study, firstly, we formulate the morning equilibrium solution for a congested downtown network with cruising-for-parking. As the parking vacancy goes down over time, the cruising distance and time for finding a vacant parking space goes up. We show that, at equilibrium, more traffic would arrive their destination earlier than their desired arrival time when parking is more limited. In addition, as cruising-for-parking leads to smaller outflow of the system, the morning peak becomes longer.

A dynamic model of pricing for the network is then developed to reduce system travel cost including cruising time cost, moving time (the time vehicles move to the destination but do not cruise for parking yet) cost, schedule delay cost. It is shown that under the system optimum, the network should be operating at the critical accumulation with highest traveling speed and maximum system production. However, the system outflow decreases as parking vacancy decreases over time and trip length increases. Under the optimal dynamic fine toll, the first commuter will encounter a higher toll than the last commuter. This higher toll is to avoid the schedule delay caused by competition for less cruising. Furthermore, it is proved that the total schedule delay is minimized when the ratio of early traffic to late traffic is equal to the ratio of late arrival penalty to early arrival penalty.
We then formulate the bi-modal equilibrium with cruising-for-parking in the auto side. It is shown that as total parking supply increases, more and more travelers would shift to the public transit mode. However, the average cruising time for those still driving will decrease even though more people are driving to work. The previously discussed optimal dynamic toll can be introduced into the auto side to reduce traffic inefficiency due to parking cruising and roadway congestion. As shown by our analysis, an appropriated average toll level should be chosen to minimize the total cost of the bi-modal system. Indeed, under the bi-modal system optimum, the marginal cost (social cost of introducing additional traffic into the system) of the auto side is equal to the marginal cost of the public transit side.

This study considers that the distribution of congestion over the network or region is homogeneous. If this is not the case, the MFD for the whole region might not be well defined. Recent studies (e.g., Geroliminis and Sun, 2011) have identified the spatial distribution of vehicle density as one of the important features that affect the scatter and the shape of an MFD. Based on these results, the concept of an MFD might still be applied for the heterogeneously loaded downtown network if it can be partitioned in a small number of homogeneous clusters. Recent work created clustering algorithms for heterogeneous transportation networks (e.g., Ji and Geroliminis, 2012).

Besides, the distribution of parking over the network might also be uneven, thus the cruising-for-parking phenomenon over the space are not homogeneously distributed. In this case, we may have to classify the parking spaces into groups where the parking is more evenly distributed. Then we may further examine how spatial distribution of parking and parking fees can influence travelers’ parking choices as well as traffic congestion. Also, we may take into account another type of parking, i.e., garage parking, which might have higher facility cost than on-street parking. In this case, we can examine the optimal percentage of garage parking should be supplied to minimize total system cost including travel cost and facility cost of the parking spaces.

Further research may also consider the real-time congestion/parking pricing control problem with both historical data on demand distribution over time and real-time traffic and parking information. The current study is from a long-term perspective, and relies on the recurrent behavior of travelers. However, in reality, travelers' travel choices and traffics are uncertain over time even if they are recurrent. Therefore, it is of our interest to develop a dynamic congestion/parking pricing system based on both the information of recurrent commuting behavior (e.g. distribution of travel demand over time which might be used for prediction of future traffics) and real dynamic traffic, which can maintain the transportation system running near its optimum. Extensions may also consider parking fees, parking duration and linking the morning commute with evening commute as considered in, e.g., Zhang et al. (2005), Zhang and van Wee (2011).
Acknowledgement. This research was supported by ERC Starting Grant “METAFERW: Modeling and controlling traffic congestion and propagation in large-scale urban multimodal networks”.

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