“Bidding the Project” vs. “Bidding the Envelope” in Public Sector Infrastructure Procurements*

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I. Introduction

Governments around the world are looking at innovative ways to procure large-scale public projects such as roads, bridges, hospitals, schools and prisons. The widely-recognized “infrastructure deficit” experienced in many countries, with both developed and developing economies, helps explain this interest. A recent report by the World Economic Forum suggested that an investment of the equivalent of US$2 trillion would need to be made each year for the next twenty years to bring the world’s infrastructure to proper levels.¹

Among the innovations being considered by governments eager to address these challenges with the minimum drain on the public purse, for example, are public-private partnerships (PPPs). PPPs vary in structure from project to project but a fairly general definition describes a PPP as: “an agreement between the government and one or more private partners (which may include the operators and the financiers) according to which the private partners deliver the service in such a manner that the service delivery objectives of the government are aligned with the profit objectives of the private partners.”² Essentially, PPPs involve a greater use of the private sector to deliver public services than is traditional, for example by having the private partner provide financing and operating the new project for some period once completed. The hope is to marshal the alleged efficiency and innovativeness of the private sector, driven by the forces of competition between rival potential private partners, to deliver public services at a lower cost to taxpayers and/or users.³ There is now a large and growing literature in the economics, policy and public management fields on the use of PPPs to deliver public services.⁴

³ See, e.g., de Bettignies and Ross (2004) for a discussion of some of the key economic issues associated with the choice between PPPs and more traditional procurement methods.
⁴ Much of this work focuses on the alleged advantages and disadvantages of the PPP model and on case studies of individual projects. A good textbook treatment of PPPs is that by Yescombe (2007). A recent collection of papers on various aspects of PPPs and the experience of different countries is Hodge et al. (2011). The chapter by de Bettignies and Ross (2011) reviews many of the most significant theoretical analyses of PPPs by economists.
The broader public procurement literature, beyond that on PPPs, has explored many questions related to public sector procurement such as the design of optimal procurement auctions, scoring rules for multidimensional projects, second-sourcing, contract design for complex projects, collusion in bidding and transparency issues. Our goal here is to contribute to this literature by focusing on a question that has received relatively little attention; rather than focusing, as others have, on optimal procurement mechanisms we study certain efficiency properties of two basic mechanisms which, while not being optimal in any formal sense, are actually employed in practice.

In the standard representation of a large-scale public procurement the government defines the project it would like delivered. It may leave a lot of discretion to bidders about how that project is to be delivered, but what we will call the “quantity” of services or “size of the project” to be delivered is precisely defined before bidding begins. Potential private partners will then bid competitively to provide that quantity at the lowest possible price to the government with the winner being the party with the lowest (quality controlled) price. We refer to this kind of procurement as “bidding the project” (or BTP). Competitive bidding will then lead to the provision of the defined quantity/project at a price close to the private sector partner’s costs. The optimality of the mechanism then turns on the degree to which the government correctly specified the project before it asked for bids. If the government is uncertain about either the benefits of the project and/or the costs of delivery, it may not specify the optimal project size – determined by balancing the marginal benefits and costs of larger and small projects– before asking for bids. The final project, while delivered at close to cost, may not then be of the optimal size, resulting in some deadweight loss through this procurement.

A second method for procuring this project would involve the government determining how much money it was prepared to spend on the project (the “envelope”) and then letting bidders compete through the quantity or size of the project they will provide for that amount of money. We refer to this as “bidding the envelope” (BTE). This approach has been used, for example by the Province of British Columbia, Canada in its PPP procurement of large-scale improvements to the Sea-to-Sky Highway linking Vancouver with the mountain resort community of Whistler. A variant of this approach, widely applied, involves governments specifying a project quantity but

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5 See, for example, the collection of essays in Piga and Thai (2007).
6 On the Partnerships BC website, see: www.partnershipsbc.ca/files-4/project-seatosky.php. To be precise, the original call did specify baseline requirements that bidders must satisfy, but then let them offer up further improvements beyond that. The winning bidder provided many additional benefits beyond the baseline and within the envelope provided (e.g. more kilometers of passing lanes, better lighting and signage, and improved highway maintenance etc).
then also a maximum amount they will pay (an “affordability ceiling”). If their specified project is not feasible given this affordability limit, the bidding becomes essentially a BTE competition to give the government a project as large as possible with that envelope.

If a total welfare maximizing government has full information and bidding is competitive these two bidding mechanisms would clearly produce identical results. The government would select as the project size the quantity that maximized social welfare. Knowing the costs of the bidding firms it would be able to predict the costs of delivering that project. It could then either specify the quantity and pay those costs or offer that amount as an envelope and the winning bidder in a competitive contest would commit to provide that same quantity.

As we show below, this equivalence and efficiency result breaks down when there is uncertainty about costs and/or benefits. In these cases, the government will likely incorrectly (ex post) set the quantity or envelope. For example, after setting a project size based on expected benefits and costs, a government finding through bidding that firms actually had lower costs than expected would prefer a larger project. Under the BTE mechanism the firms will indeed bid greater quantities than previously expected; however, under the BTP mechanism the project size will not change.

The purpose of this paper, then, is to explore the conditions under which each of these mechanisms will be superior to the other in terms of minimizing the deadweight losses associated with inefficient project sizes ex post. We will see that the relative advantages of the mechanisms will depend on a number of factors including the general level of benefits derived from the project and the shape of the marginal benefit curve. While we will discuss these bidding mechanisms in the context of public-private partnerships – because of the attention such large-scale procurements have been receiving in recent years – it should be clear that our analysis is more broadly applicable to government procurement generally. It should also be clear that when we talk about the project “size” or “quantity” we could alternatively be talking about “quality” as long as, in this case, the quantity is fixed and quality is a measurable and contractible output.

The next section reviews the related literature including that on the regulation of prices versus quantities and scoring auctions. Section III then presents an overview of the model with the key results presented in Section IV. Section V offers our conclusions.

II. Related literature: Prices vs. Quantities and Scoring Auctions
The ideas here are clearly related to the pioneering work on the uses of prices versus quantity controls as regulatory mechanisms. In Weitzman’s (1974) classic contribution, he asked whether it was better to control the behavior of a regulated private firm by setting the price it receives for its output and letting it choose profit-maximizing quantities, or by directly setting the quantity to be produced by the firm. As is true here, these mechanisms will trivially produce identical results when the regulator has full information. However, when there is uncertainty about the benefits and/or the costs of output, introduced much as we have here, the mechanisms are not equivalent and the superiority of one over the other will depend on the shapes of the benefit and cost functions.

Laffont (1977) clarified and extended Weitzman’s results, distinguishing between “genuine randomness” – random elements of costs and benefit functions unknown to all players (regulator/planner, producers and consumers) – and random elements that, while unknown to the regulator/planner, are known to the consumer (in the case of benefits) and producer (in the case of costs). This second type of randomness contributes to the information gap that really drives the differences between mechanisms. In a similar way, we show below that genuine randomness in project benefits will not affect the relative merits of the two procurement mechanisms we study. 7

Despite these similarities, there are significant differences between the present paper and this prior literature. For one, while our BTP mechanism is very much like choosing quantities in this earlier literature, our BTE mechanism is not the same as choosing prices. This becomes most apparent when unit (“marginal”) costs are constant: a firm responding to a fixed price per unit would either supply zero output (if the price was below its unit costs) or an infinite quantity (if the price was above).

The most important difference here, however, derives from the fact that we are exploring a procurement model in which competitive bidding modifies firms’ behavior in an important way. In fact, it is largely the competitive bidding that regulates firms in our model and, without it, neither of our mechanisms would produce satisfactory results.

Our focus here on two very simple mechanisms – both in current use and each one-dimensional – also sets this paper apart from the literature on scoring auctions. That literature,

7 Laffont (1977, p. 180) does recognize that if the different parties have different expectations about genuine randomness, the mechanisms will not be equivalent. This would be true in our model as well.
for example Che (1993), and Asker and Cantillon (2008 and 2010), considers procurements in which the government invites prospective suppliers to quote on multiple dimensions of a project including price and possibly numerous aspects of project quality. In contrast, we consider mechanisms in which prospective suppliers need quote just one number — either a price or a quantity — so there is no need for the procuring authority to weight multiple attributes. And, importantly, one of these mechanisms — unlike virtually all of those discussed in the scoring literature — requires the government to specify an envelope and does not ask the bidders to quote a price as part of their bids. In other words, in a scoring auction the Government does not know what the final cost of the auction will be; in a BTE mechanism it does.

In its focus on the efficiency properties of practical procurement mechanisms, the papers that comes closest to the present paper are those by Engel et al. (1997, 2001). In their articles the authors study the choice between two variants of what we are calling BTP procurements for new highways. In both cases, the project is specified by the government. In one mechanism bidders are asked to bid based on tolls to be charged or (if tolls are regulated) by franchise length, with the lowest tolls or shortest franchise lengths generally winning. In their second mechanism — newly proposed in this article but since then put into practice — bidders bid on the net present value of toll revenues they will accept, recognizing that their franchise will terminate when that amount has been collected. In contrast, in this paper we compare the more common version of the BTP mechanism with a different mechanism in which the scope of the project is not defined by the public body — as assumed by Engel et al. — but rather is itself the dimension over which bidders compete.

III. Model

In our model, the government wishes to procure some assets, and possibly also services delivered using those assets. Collectively the assets and any associated services are referred to here as the “project”. We assume that the government has decided to procure the project using a PPP model and focus on how it should structure the competitive bidding by prospective

8 Of course, it may be the case in the BTE mechanism that there are multiple dimensions of “quantity” that the government cares about, in which case it will have to create some scoring mechanism to determine which of a set of different bids provides the greatest aggregate quantity for the purposes of winning the competition.

9 This second mechanism can more efficiently allocate demand risk in their model in which bidders are risk averse. Beginning in 1998, this mechanism has been employed four times in Chile. The largest case was the tendering, in 1998, of the Santiago-Valparaiso route.
suppliers. A contract between the public and private parties will specify two key elements: the amount of services to be provided \((q)\) and the price to be paid for those services by the public partner to the private partner. For some projects it will make sense to interpret \(q\) as the quantity of services provided, such as the number of patients treated by a PPP hospital. In other cases \(q\) may be better thought of as a measure of quality, for example the safety (measured in reduced vehicular accident costs) of a PPP road. It is assumed that this contract is fully enforceable on both parties.

The government will choose between two alternative bidding mechanisms, described below, for procuring this project. In making this choice the government seeks to maximize total social surplus – the difference between the benefits of the project and the costs of delivering it. There are many equally qualified bidders such that we will approximate the result of any bidding as leading to complete profit dissipation for the winning bidder. As a result, the private partner in the PPP will not be sharing in the surplus generated and the choice can then also be interpreted as the one that delivers the greatest “value for money” for the government.\(^\text{10}\)

In a world in which the public partner had full information about the benefits of the project and the costs of delivery, it would be a straightforward matter for it to specify the optimal project “size”, \(q^*\), and then conduct bidding to find a supplier to provide that project at minimum cost. The optimal size would clearly come from a simple balancing of the marginal benefits of increases in \(q\) with the marginal costs of those increases.

However, the key to our model is that the public partner will not have full information about costs and/or benefits. Uncertainty about benefits of a project of size \(q\) is represented by the function \(B(q; \theta)\) where \(\theta\) is random variable capturing the imprecision with which the government assesses the public’s need for the services. This uncertainty is shared – \(\theta\) is also unknown to all potential private partners. Conventionally, we assume that the first and second derivatives of the benefit function with respect to \(q\) obey: \(B_q > 0\) and \(B_{qq} < 0\). We also assume that higher levels of \(\theta\) are associated, for a given project size, with higher levels of benefits, \(B_\theta > 0\), as well as higher levels of marginal benefits, \(B_{q\theta} > 0\). The common inability

\(^{10}\) Value for money is the criterion often discussed in the professional (government and private sector) literatures on PPPs. While a universally acknowledged definition of value for money is not available, it essentially involves the government procuring the assets/services it wants at the lowest total cost to taxpayers and users (if users pay). It differs from total social surplus in that it is not indifferent to transfers – for example higher profits to providers may not reduce total social surplus (if they do not affect quantities) but they do reduce the value for money obtained by the government (if they arise from higher prices). See, e.g., Boardman and Vining (2011). Of course, if such transfers are costly – due to taxation-induced deadweight losses – they can have implications for total social surplus as well.
of governments to properly assess the value of large public projects is widely recognized -- governments have been notoriously poor at predicting the demand for transportation projects, for example.\textsuperscript{11}

Similarly, the government will not have precise information about the private sector’s costs of delivering the public services. We represent these imperfectly known costs with the function $T$C $= C(\eta)q$. Notice that this functional form assumes that marginal costs, $C(\eta)$, are constant. We assume that higher levels of $\eta$ correspond to higher levels of marginal costs: i.e. $C_\eta > 0$. While the public partner knows the shape of the total cost function, it does not know the value of the random component, $\eta$. The private firms do know their own costs when they bid, however, so this is not general uncertainty of the sort we saw with respect to the benefits function, it is a case of asymmetric information.

Though it is entirely possible that potential private partners may have different costs, we assume that there are enough private bidders with the lowest costs to drive bid prices down to the level of these minimum costs. For ease of exposition then, and without any further loss of generality, we will assume that all firms have the same minimal costs, which will be unknown to the government at the time it is awarding the contract. Bidding firms are assumed to know their own costs which, combined with the enforceability of the contracts, implies that the firms will never bid prices below their costs. In this way, bidding firms reveal their realization of $\eta$.

Again, in choosing between the mechanisms, the government will look to maximize expected social welfare where social welfare is given by $W = B(q; \theta) - C(\eta)q$.\textsuperscript{12}

In the first mechanism, which we term the “bid the project” (BTP), the government specifies the project, which here means it indicates what level of $q$ it wants provided, $q^p$, and bidders compete by offering to provide the amount $q^p$ at the lowest possible price to the government. With competitive bidding, this will mean that the project is delivered for a price equal to $C(\eta)q^p$ where $C(\eta)$ is the lowest marginal cost among all private providers.

In the second mechanism, “bid the envelope” (BTE), the government determines the total amount it is prepared to pay for the project (the “envelope”) and bidders compete by offering to provide the greatest quantity (or quality) for that total price. If we denote this envelope as

\textsuperscript{11} See, e.g. Flyvbjerg et al (2005a, 2005b, and 2010).

\textsuperscript{12} To the extent that there are implementation costs associated with these mechanisms, we assume that they are not different between the two mechanisms and that they are largely fixed costs that will not affect the optimal size of a project.
$T^e$, we can see that competitive bidding would then result in a quantity defined by: \( q^e = T^e/C(\eta) \).

In the first best, when the government knows that, for example, $\theta = \theta^*$ and $\eta = \eta^*$, the social welfare maximizing project will be known to come from setting the marginal benefits of $q$ equal to the marginal cost of $q$, giving $q^*$:

\[
B_q(q^*; \theta^*) = C(\eta^*)
\] (1)

With full information, the government can clearly achieve this first best using either mechanism. Under BTP, specifying $q^*$ will call forth bids offering to deliver the project at cost, $C(\eta^*)q^*$. Under BTE, if the envelope size is set to $T^* = C(\eta^*)q^*$, the winning bidder will offer to supply $q^*$ at that price.

With asymmetric information, however, the two mechanisms will in general perform differently. Consider the case of known benefits but unknown costs, and suppose first that the Government believes that the constant marginal cost will be $C_0$. It would then, under the BTP mechanism, choose a quantity that equates marginal benefits with $C_0$. If, however, actual costs come out lower than $C_0$, the actual first-best quantity will be higher, though under the BTP mechanism, the same quantity is provided. The fact that costs are lower does mean that the government will pay less than it expected for its project, but there will be a deadweight loss associated with the inefficient choice of quantity. This deadweight loss is illustrated in Figure 1a. Here $B_q$ represents the known (in this case) marginal benefits curve, $C_0$ the anticipated marginal costs curve and $C_1$ the realized, lower, marginal costs curve. Under a BTP mechanism the government will specify a project $q_0$ and the winning bidder will provide it at a price of $C_0q_0$. The true first-best quantity will be $q_1$, however, and the resulting deadweight loss associated with the lower output is illustrated as area $DWL_a$ in Figure 1a.

Under a BTE mechanism, there will be output adjustments in response to costs that are above or below expected levels. This is illustrated for this case in Figure 1b. If the government specifies an envelope based on the expected costs of a project $q_0$ it chooses $T_0 = C_0/q_0$. Given that costs are lower than expected in this case, competitive bidders will be able to provide a larger quantity – indeed, competitive bidding leads to a quantity given by

\[
q_2 = T_0/C_1 = \left( \frac{C_0}{C_1} \right) q_0
\]
While, with lower costs some increase in quantity is socially efficient, there is no guarantee that the quantity produced by bidding in the BTE mechanism will not overshoot the true first best quantity ($q_1$). Figure 1b illustrates a case where such overshooting has taken place, generating a deadweight loss represented by area $DWL_b$.

These two figures illustrate the key tradeoffs studied in this paper – essentially the BTP mechanism is inflexible with respect to quantity in the face of changing costs while the BTE adapts imperfectly to changing costs. The next section presents the full model and our formal results, examining the trade-offs illustrated here.

IV. Results

In this section we provide the formal analysis of these two mechanisms and demonstrate conditions under which either might be preferred to the other in terms of the government’s objectives. Again, project benefits and costs are given by $B(q, \theta)$ and $C(\eta)q$ respectively with $\theta$ and $\eta$ both randomly and independently distributed, each with mean 0.

To carry the analysis further at certain points, below we will add further structure to the problem in which the random components enter in a linear additive fashion and the marginal benefit curve is itself linear. Specifically:
Therefore in this special case θ and η shift, respectively, the marginal benefit and marginal cost functions in a parallel fashion. Note that, once we have assumed that the error terms are additive, irrespective of whether the non-random part is linear or not, assuming that the random variables θ and η have zero mean comes at no further loss of generality.

With full information (about θ and η) the government can either: (i) pick a quantity q∗ to satisfy the first-order condition (1), and firms will bid a total price of C(η∗)q∗ to deliver the project; or (ii) it can offer an envelope of T∗ = C(η∗)q∗ and firms would offer to provide up to q∗ for that envelope. Either way the first-best is achieved. With imperfect information though, the government has to choose a procurement mechanism.

### 4.1 Bidding the Project (BTP)

Knowing only the distributions of θ and η but not their realized values, a government choosing the BTP mechanism will specify a project size q^P to maximize expected social surplus given by E_{θ,η}[B(q^P; θ) − C(η)q^P], yielding first-order conditions:

\[ E_{θ,η}[B(q^P; θ)] = E[C(η)] \]

Notice directly that if either random component, θ and η, were to enter their respective marginal benefits or marginal costs functions in an additively separable way they will not affect the (second-best) project size since E(θ) = E(η) = 0. This means that the prescribed project size under BTP would be the one that corresponds to average (expected) demand and cost conditions. Moreover, expected social welfare will not depend on the distributions of θ and η either:

\[
E[SW^P] = E\left[ \int_0^{q^P} B_q(q; θ)dq − C(η)q^P \right] = E\left[ \int_0^{q^P} (g(q) + θ)dq − (c + η)q^P \right]
\]

\[
E[SW^P] = E\left[ \int_0^{q^P} g(q)dq + θq^P − (c + η)q^P \right] = \int_0^{q^P} g(q)dq − cq^P
\]

where, \( g(q) \) is the non-random part of the marginal benefit function.
Further, in the special linear case we have that \( g(q) = a - bq \) and, therefore, the selected project size—which comes from equating expected marginal benefits and expected marginal costs—is:

\[
q^p = \frac{a - c}{b}
\]

and expected social welfare under this mechanism will be given by:

\[
E[SW^p] = \frac{(a - c)^2}{2b}
\]

Moreover, the expected costs of a project using the BTP mechanism, which we denote by \( T^p \), will be given by

\[
T^p = E[(c + \eta) \cdot q^p] = c \cdot q^p = \frac{c(a - c)}{b}
\]

Note that all three values, \( q^p, E[SW^p], T^p \) correspond to the ones that would appear should average conditions prevail with certainty.

### 4.2 Bidding the Envelope (BTE)

A government choosing the BTE mechanism will select an envelope \( T^e \) knowing that the project size will then be determined—because of competition—as the maximum quantity that the private partner can deliver for a total cost of \( T^e \), a quantity given by \( q = T^e / C(\eta) \).

The government’s problem is to maximize expected social welfare with respect to the envelope:

\[
\text{Max } T^e \ E[SW^e] = E \left[ B \left( \frac{T^e}{C(\eta)} ; \theta \right) - T^e \right]
\]

yielding first-order conditions:

\[
E \left[ B_q \left( \frac{T^e}{C(\eta)} ; \theta \right) \cdot \frac{1}{C(\eta)} \right] = 1
\]
Notice that if marginal benefits are linearly additive in \( \theta \), this random component will again drop out of the first-order conditions and therefore the randomness of the benefits function will not determine the choice of envelope size. However, in this case even if the random element of costs, \( \eta \), enters marginal costs in a linear additive way, its distribution will still affect the optimal envelope size through its effect on the quantities achieved under the envelope.

Thus, in the linear special case we can solve for the optimal envelope size from first-order conditions:

\[
E \left[ \frac{1}{c + \eta} \left( a - \frac{b T^E}{c + \eta} + \theta \right) \right] = 1
\]

Yielding

\[
E \left[ \frac{a + \theta}{c + \eta} \right] - 1 = b T^e E \left[ \frac{1}{(c + \eta)^2} \right]
\]

Which, given the independence of \( \theta \) and \( \eta \), and the fact that \( E[\theta] = 0 \), we can write as:

\[
E \left[ \frac{a}{c + \eta} \right] - 1 = b T^e E \left[ \frac{1}{(c + \eta)^2} \right]
\]

Solving this for the optimal envelope size:

\[
T^e = \frac{a E \left[ \frac{1}{c + \eta} \right] - 1}{b E \left[ \left( \frac{1}{c + \eta} \right)^2 \right]}
\]

Jensen’s Inequality indicates that \( E[1/(c + \eta)] > 1/(c + E[\eta]) = 1/c \). And using the fact that \( a > c \), it follows that \( a E[1/(c + \eta)] > a/c > 1 \). This ensures that that \( T^e > 0 \).

For the BTP case, the prescribed quantity \( q^p \) was the same that would be chosen if the Government knew, with certainty, that average (expected) conditions for cost and demand prevailed. This is not the case for the envelope, though. The prescribed envelope size is different than \( T = \frac{(a-c)c}{b} = T^p \), which is the optimal value for average conditions. As we show below, when we compare the two mechanisms, the prescribed envelope under BTE will be sometimes above and sometimes below this figure. One way to picture this is to think of the simple case where \( a = 2c \) and \( b = 1 \); in that case, the envelope for average conditions is simply given by \( c^2 \). But it is easy to see that, in every other possible outcome, the optimal ex-
post envelope is smaller, given by \((c - \eta)(c + \eta) = c^2 - \eta^2\). Thus choosing an envelope equal to the one optimal for average conditions would ensure overshooting the quantity ex-post almost always; that is, the optimal ex-ante envelope should be smaller. In a similar fashion, and assuming different values for \(a\) and \(b\), one can construct simple examples in which the envelope for average conditions is much smaller than most of the envelopes for other conditions; for example, when considering large \(a\) and \(b\) such cases will arise. Finally, since the optimal ex-ante envelope is not equal to the one for average conditions, one could conjecture that the ex-ante average envelope could be optimal, but this is not true either. The ex-ante average envelope is given by

\[
E \left[ \frac{(a + \theta - (c + \eta))(c + \eta)}{b} \right] = \frac{ac - E[(c + \eta)^2]}{b}
\]

which is different than \(T^e\).

Expected social welfare under BTE can then be calculated straightforwardly:

\[
E[SW^e] = \frac{\left( aE \left[ \frac{1}{c + \eta} \right] - 1 \right)^2}{2bE \left[ \left( \frac{1}{c + \eta} \right)^2 \right]}
\]

which clearly will depend on the distribution of \(\eta\), though not on the distribution of \(\theta\), as explained.

### 4.3 Comparing the two mechanisms

We are now in a position to compare the two mechanisms. We do this by looking at the expected levels of welfare each mechanism reach, but also by looking at the expected expenditure and expected project size under each mechanism. In all cases, we use the results from the special linear case. We push the analytics as far as we can but this will prove insufficient at times; in order to analyze the effects of the distribution of \(\eta\) on the comparison, or the actual magnitude of differences we will need to resort to numerical techniques. For this, we use a family of distributions for \(\eta\) given by:

\[
h(x, \alpha, d) = \chi([-d, d]) \times \begin{cases} 
\sqrt{\alpha} - \alpha d + \alpha |x|, & \alpha > 0 \\
\frac{1}{2d}, & \alpha = 0 \\
\sqrt{-\alpha} + \alpha |x|, & \alpha < 0
\end{cases}
\]
where $\chi([-d,d])$ is the indicator (or characteristic) function of the interval $[-d,d]$; outside the interval the value of $h$ is zero. This family of distributions, always symmetrical around the zero mean, allow us to place the weight on the borders or the center of the interval $[-d,d]$, depending on the value that $\alpha$ takes. Figure 2 shows three examples for a value of $d = 1$.

**Figure 2: Family of distributions**

![Figure 2: Family of distributions](image)

**Efficiency Comparison**

In order to compare the efficiency of the two mechanisms we construct the ratio of expected social welfare and call this $ESW_{e/p}$:

$$ESW_{e/p} = \frac{E[SW_e]}{E[SW_p]} = \frac{\left[\alpha E\left(\frac{1}{c+\eta}\right) - 1\right]^2}{(\alpha - c)^2 E\left[\left(\frac{1}{c+\eta}\right)^2\right]}$$

Therefore, when $ESW_{e/p} > 1$, the BTE mechanism delivers greater expected social welfare, while the BTP mechanism dominates when $ESW_{e/p} < 1$.

Notice, first, that this ratio does not depend on the slope of the marginal benefits function, $b$ or, again, on the random component of the benefits function, $\theta$. It will depend on the vertical intercept of the marginal benefits curve $a$ (an indicator of the size and importance of the project) as well as the non-random and random components of the cost function.

We first establish that no mechanism will dominate the other under all conditions, i.e., that $ESW_{e/p}$ can be either greater than or less than one for some parameter values; we proceed in two steps, mapped out in the Appendix. First we show that $ESW_{e/p}$ is negatively related to the
marginal benefits intercept term, \(a\). We then demonstrate that there is a value of \(a\), which we label \(a'\), at which \(ESW^{e/p} = 1\) and which also satisfies the condition \(a' > c\) (so that the project is worth undertaking at some scale). Taken together we see that these results imply that for \(a > a'\) the BTP mechanism will generate greater expected welfare, while for \(a < a'\), the BTE mechanism will be superior.

**Lemma 1:** The expected social welfare from the BTE mechanism, relative to that from the BTP mechanism will be negatively related to the level of \(a\), i.e. \(dESW^{e/p} / da < 0\).

*Proof:* See appendix.

**Lemma 2:** There exists a value of \(a, a'\), such that: (i) \(ESW^{e/p}\) evaluated at \(a'\) will equal one; and (ii) \(a' > c\).

*Proof:* See appendix

This then gives us our first key comparative finding.

**Proposition 1:** For some values of the parameters \(a\) and \(c\), the BTP mechanism will generate greater expected social welfare and for other values the BTE mechanism will generate greater expected social welfare.

*Proof:* Follows directly from discussion above applying Lemmas 1 and 2.

Proposition 1, when combined with Lemma 1, tells us that for a given level of costs, \(c\), the BTP mechanism will generate larger social welfare than the BTE mechanism for values of \(a\) beyond some critical level, and that the BTP relative advantage grows as \(a\) increases further above that critical level. But, of course, it also confirms that the BTE mechanism will generate larger social surplus for levels of \(a\) below that critical value and the further below that level \(a\) is, the greater will be the BTE mechanism’s advantage.

A natural question that may arise is how the comparison between the two mechanisms changes as \(c\) increases. Perhaps a first natural reaction would be to think that a decrease in \(a\) or in increase in \(c\) should have the same effect as they both shrink the market; i.e. it would be expectable that the ratio of expected social welfare increases with \(c\). However, \(c\), as opposed to \(a\), appears inside the expectation operator which makes things more complex as the distribution of \(\eta\) now plays a role. In fact, not even making the simple assumption that \(\eta\) is
uniformly distributed helps: for a uniform distribution over \([-d; d]\), where \(|d| < c\), straightforward calculations lead to

\[ ESW_{e/p} = \frac{(c^2 - d^2)(a \text{ ArcTanh}(\frac{d}{c}) - d)^2}{d^2(a - c)^2} \]

whose derivative with respect to \(c\) we have been unable to sign analytically. A particular case that can be solved, though is for a uniform distribution for \(c\) where the support grows with \(c\), specifically let the support = \(b \cdot c\), where \(b \in (0,1)\). In that case, the ratio of expected welfare reduces to

\[ ESW_{e/p} = \frac{(1 - b^2)(a \text{ ArcTanh}(b) - b \cdot c)^2}{b^2(a - c)^2} \]

This leads to

\[ \frac{\partial ESW_{e/p}}{\partial c} = \frac{2a(1 - b^2)}{b^2(a - c)^3} (\text{ArcTanh}(b) - b)(a \text{ ArcTanh}(b) - b \cdot c) \]

which is positive since \(b < 1\), \(\text{ArcTanh}(b) - b > 0\) and \(a \text{ ArcTanh}(b) - b \cdot c > 0\) since \(a > c\). Therefore, in this case, increases in \(c\) favour the BTE mechanism relative to the BTP mechanism.

One way to simultaneously see the behavior of \(ESW_{e/p}\) as \(c\) grows and as the distribution changes is to graph it using different values for \(\alpha\) and \(d\) in the family of distributions described above. We choose to do this for a given value of \(a = 1,000\) and for continuous values of \(c\) and \(d\) under three values of \(\alpha\), which controls for the amount of weight that the distribution has on the center or at the boundaries (see Figure 2). The result is shown in Figure 3 below:
The graphs above are plotted only where $c - d > 0$ and $c + d < \alpha$ so that costs never end up being negative or above the maximum willingness to pay. The graphs show, for these cases, essentially, three things. First, in the largest portion of the feasible parameter space, BTE beats BTP as shown in the second panel of Figure 3; this panel, together with the other panels, also show that differences between mechanisms may be above 10% in large sections of the parameter space. One can also see that it is only when marginal costs are very uncertain, given in the graphs by the combination of small $c$ and large $d$, that the BTP mechanism wins. This occurs because in that case is when the overshooting of quantity adjustments under BTE can be large. The second aspect that is clear is that as more weight is put on the boundaries of the support of the distribution, i.e. as $\alpha$ increases, the difference in favour of the BTE mechanism increases. This is reasonable: with a distribution that places more weight on the borders, the probability of “getting it right” by playing the expectation, which is what the BTP mechanism does, disappear. For example, Figure 2 shows that for the case when $d = 1$ and $\alpha = 6$, the expected value of $c$ never occurs. Therefore, the output adjustment that the BTE mechanism induces becomes particularly valuable. As $\alpha$ decreases, uncertainty becomes less important.
and, therefore the mechanisms tend to become more similar. This is very clear seeing that for a uniform distribution ($\alpha = 0$) differences are still sizeable but smaller, while for the case $\alpha = -2$, where most of the probability weight is in the center, the ratio becomes almost indistinguishable from 1 everywhere. Finally, the graphs show that at least for symmetric distributions, as $c$ increases ceteris paribus, BTE improves its performance over BTP (although the rate changes of course).

Project cost comparison

Governments choosing between these mechanisms may also care about which will result in more costly projects in expectation. The expected costs of projects using the BTP and BTE mechanisms were made explicit above. We define the relative expected costs of the two mechanisms to be $T^{e/p}$, which will then be given by:

$$T^{e/p} = \frac{T^e}{T^p} = \frac{a\eta \left[ \frac{1}{c + \eta} \right] - 1}{(a - c)\eta \left[ \left( \frac{1}{c + \eta} \right)^2 \right]}$$

With this definition, whenever $T^{e/p}$ is larger than one, it means that the BTE mechanism leads to higher expected costs for the project and vice versa. Also, given that $T^p$ is the optimal value for average conditions, a value larger than one for $T^{e/p}$ means that the BTE mechanism prescribes an envelope that is larger than the one for average conditions.

A similar approach to that applied for social welfare allows us to establish the following two lemmas:

**Lemma 3:** as the value of $a$ increases, the expenditures under the BTE mechanism fall relative to those under the BTP mechanism (i.e. $dT^{e/p}/da < 0$).

**Proof:** See appendix.

**Lemma 4:** there is a value of $a$, called here $a''$ such that $a'' > c$ and at which $T^{e/p} = 1$.

**Proof:** See appendix.

Together these results imply Proposition 2.
Proposition 2: The expected costs of projects under BTE mechanism can be higher or lower than the expected costs under the BTP mechanism.

Proof: It follows from the discussion above (and the associated proofs in the appendix) that $T^{e/p}$ will be greater than one – implying that the expected costs of a BTE project are higher than those for a BTP project – for $a < a''$ and that $T^{e/p}$ will be less than one – implying lower BTE project costs for $a > a''$.

We can now ask when the BTE mechanism is more costly –in expectation– than the BTP mechanism and by how much? In order to assess this, we again use the family of distributions for $\eta$ in order to graph $T^{e/p}$. Figure 4 shows the result for $a = 1,000$. The overall picture shows that, in expectation, the BTE mechanism is cheaper in the largest portion of the parameter space and is never more than 10% more expensive than the BTP mechanism.\(^\text{13}\)

Importantly, since the area in which the BTE is more efficient (in a social welfare sense) than the BTP contains the area in which the BTE mechanism is more expensive, we can conclude that, at times the efficiency will come from cheaper projects and other times from more expensive projects. When the latter happens, it has to be true that the extra (expected) cost is worthwhile: the mechanisms bids on efficient adaptation of size. As we show now, whenever the BTE mechanism is more expensive, it delivers a larger project.

\(^\text{13}\) The reader may wonder, following the discussion in the previous section, how the optimal envelope compares to the average envelope. Using the family of distributions it can be shown that no value will be always larger than the other; it all depends on the parameter values chosen.
Figure 4: Expected costs comparison

Project size comparison

Governments will of course care also about the size of the project they will end up buying. For example, they might not want to fall short of textbooks for schools, or meals for children. In that sense, the BTP mechanism gives them an amount that is certain, but the BTE mechanism does not. Which mechanism generates a larger project in expectation and under what conditions? Recall that in the BTP mechanism the quantity is certain and given by

\[ q^p = \frac{a - c}{b} \]

The BTE mechanism quantity is not known in advance, we can only calculate an expectation:
Therefore, the ratio of (expected) project size is given by

\[ q^e = E[T^e/(c + \eta)] = T^e E[1/(c + \eta)] = \frac{aE\left(\frac{1}{c + \eta}\right) - 1}{bE\left[\left(\frac{1}{c + \eta}\right)^2\right]} E\left(\frac{1}{c + \eta}\right) \]

such that, whenever \(q^e/p\) is larger than one, the BTE mechanism delivers a larger project size than the BTP mechanism.

Following the same procedure as before, we can show that \(q^e/p\) may be larger or smaller than one, depending on parameter values. But, again, the most important question is when and by how much the BTE mechanism project size is larger. Reorganizing terms it is easy to see that:

\[ q^e/p = \frac{aE\left(\frac{1}{c + \eta}\right) - 1}{(a - c)E\left[\left(\frac{1}{c + \eta}\right)^2\right]} E\left(\frac{1}{c + \eta}\right) \]

Therefore, \(q^e/p\) can be written as a constant times \(T^e/p\), where that constant is strictly larger than one since \(c E\left(\frac{1}{c + \eta}\right) > 1\), implying that \(T^e/p < q^e/p\). It follows that the graphs we obtain for the expected project size (shown in the Appendix) look, essentially, like the ones we showed before for \(T^e/p\) (Figure 4). More importantly, though, it means that, at times, the BTE mechanism may be less expensive in expectation while delivering more output in expectation than the BTP mechanism; this because there will be parameter values that induce \(T^e/p < 1 < q^e/p\).

Also, the parameter space where the BTE delivers more output is completely contained by the parameter space where the BTE is more efficient. This means that whenever the BTE delivers more output in expectation, it will be more efficient. But it can also happen that it delivers less output in expectation and still be more efficient; this is a reflection of the fact that the quantity adaptation of the BTE mechanism can go both ways.

\[14\] Because the proof is rather similar to the analogous proofs for \(E\Theta^e/p\) and \(T^e/p\) we omit it here and in the appendix. It is available upon request from the authors.
The following figure summarizes the areas in which each situation can occur:

**Figure 5: BTE dominance (illustrated for the case \( a = 1,000 \) \( \alpha = 0 \))**

In expectation, in area A the BTP mechanism is better, social welfare wise, than the BTE mechanism. In areas B, C, and D the BTE mechanism dominates but, in area B it does so by providing both smaller project size and project cost, in area C it provides larger size but smaller cost and in area D it provides both larger size and cost.

**V. Extensions**

**5.1 Costly public funds**

Recognizing that large scale procurements, including many PPPs, involve large commitments of public funds and that raising those funds from taxpayers comes at a social cost – the deadweight loss of taxation – we can ask how changes in marginal cost of raising tax revenues to pay for the procurement will affect the relative merits of the two mechanisms. To examine
this let $\lambda > 1$ represent the cost to the economy of withdrawing a dollar by taxation, something known as the *marginal cost of public funds (MCPF)*.\(^\text{15}\) Then social welfare is given by:

$$SW = B(q, \theta) - \lambda T + T - (c + \eta)q$$

for a project leading to an output of $q$ and a government payment to the provider of $T$. It is straightforward then to show that under the BTP mechanism the government will specify a project quantity:

$$q^p = \frac{a - \lambda c}{b}$$

and expected social welfare under the BTP mechanism will be:

$$E[SW^p] = \frac{(a - \lambda c)^2}{2b}$$

We note that a MCPF larger than 1 diminishes both the project size and expected welfare. For the BTE mechanism we can show that the new envelope will be given by:

$$T^e = \frac{aE \left[ \frac{1}{c + \eta} \right] - \lambda}{bE \left[ \frac{(a - \lambda c)}{b} \left( \frac{1}{c + \eta} \right)^2 \right]}$$

And the expected social welfare under the BTE mechanism will be:

$$E(SW^e) = \frac{\left[ aE \left( \frac{1}{c + \eta} \right) - \lambda \right]^2}{2bE \left[ \left( \frac{1}{c + \eta} \right)^2 \right]}$$

We note that a MCPF larger than 1 diminishes both the size of the envelope and expected welfare.

\(^{15}\) See Auerbach and Hines Jr (2002) for a theoretical discussion. The actual value of $\lambda$ has been calculated empirically, by a number of authors. Its value depends on the years, the tax being considered, and the country among other things. For example Harrison, Rutherford and Tarr (2002) find a mcpf for Chile that is between 1.08 and 1.18 depending on the tax considered. Ballard, Shoven and Whalley (1985) estimate a range of 1.17 to 1.33 for the U.S., while Auriol and Warlkers (2012) find an average mcpf for 38 African countries of 1.2.
Using these revised measures of expected social welfare, we can now express $ESW^{e/p} = E[SW^e]/E[SW^p]$ as a function including the shadow price of public funds, $\lambda$:

$$ESW^{e/p}(\lambda) = \frac{[aE(\eta) - \lambda]^2}{[a - \lambda c]^2E(\eta^2)}$$

From here we are then able to establish how increases in the deadweight loss from taxation affect the relative merits of the two mechanisms.

**Proposition 3**: Increases in the marginal cost of taxation will increase the expected social welfare of the BTE mechanism relative to that for the BTP mechanism, that is $dESW^{e/p}/d\lambda > 0$.

**Proof**: see appendix

We first note that this property does rely on the linearity assumptions but not on any specific distribution for $\eta$. In this case both mechanisms have now to consider the the MCPF is larger than one and that, therefore, transfers are more costly than before. But exact control over transfers is exactly what the BTE mechanism does while the BTP only delivers the transfer value ex-post, and therefore, cannot do as well.

**5.2 Different shapes of benefit functions**

In the linear special case developed here, the slope of the marginal benefits curve, while affecting the expected social welfare under either mechanism, does not affect their ratio. This is due to the fact that, in this model, changing the slope of the marginal benefit curve without changing the intercept term amounts to a rotation of the marginal benefit curve around the intercept term $a$. So the optimal scale of the project changes along with the sensitivity of marginal benefits to project size.\(^{16}\) Alternatively, if we consider two different linear marginal benefits curves that intersect the expected marginal cost at the same point, the flatter curve will also correspond to a smaller intercept term. By Proposition 2 this will favor the BTE mechanism, which takes advantage of the greater sensitivity of the optimal project size with respect to changes in costs when the marginal benefit curve is flatter.

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\(^{16}\) This implies that the elasticity of project size with respect to marginal benefits will be constant with respect to changes in the slope for a constant level of marginal benefits.
Referring back to Figure 1a, it is easy to see that the flatter the marginal benefit curve, the greater the increase in optimal project size \((q_1 - q_0)\) when costs are lower than expected. When the optimal project size is so different from that under average values, the BTE mechanism will dominate the fixed-size project under the BTP mechanism.

However, if the marginal benefit curve took the shape of a 0-1 demand curve (or even a kinked curve that was vertical below the kink but negatively sloped above), there would be no value to larger projects and smaller projects would lose great value, with the result that only a project of a very specific size would be efficient. The BTP mechanism will secure that project at a minimum cost, preserving the full-information solution.

This type of case is illustrated in Figure 6 below. In this case, for considerable variation in marginal costs, the same project \((q_0)\) is socially desirable. The BTP mechanism will deliver \(q_0\) for a price that just covers its actual costs. There will be no deadweight loss in this case (as long as cost marginal costs remain below the kink). However, the BTE mechanism will lead to extra, unvalued output when costs are lower than expected (the case illustrated in Figure 6 where the chosen output level is \(q_2\)) and a loss of highly valued output when costs are higher than expected.

![Figure 6](image-url)
VI. Discussion and Conclusions

In this paper we have described two mechanisms for the procurement of large public projects, such as those commonly provided using public-private partnerships. The more common BTP mechanism involves the government specifying the size of the project it wishes delivered and competitive bidders compete to provide that project by offering bids close to cost levels. In the second mechanism, BTE, the government specifies a budget or envelope that represents the total amount it is willing to spend on the project and bidders compete by offering greater quantities until the quantity they provide results in costs equal to the size of the envelope.

While the mechanisms will be equally effective—and will generate ex post optimal projects—with no uncertainty about benefits or costs, this is no longer true when costs and/or benefits are uncertain at the time the project is put out to bids. Clearly, with either kind of uncertainty, the project put out to bids may not be of the optimal size ex post. In addition, uncertainty on the part of government about costs will lead to differences in the efficiency of the two mechanisms. In contrast, uncertainty about benefits, if shared by all parties, does not affect the relative efficiency of the mechanisms.

Our first key result is that either mechanism can dominate the other in terms of the social welfare generated, depending on the values taken by various parameters of the model. We have shown that which mechanism dominates the other will depend on a number of factors. With respect to the shapes of the benefit and cost curves, the key determinant is the degree to which efficiency requires significant changes in project size when costs change. The BTE mechanism does allow project size to adjust in the initial direction of ex post optimality, though there is no guarantee that it will not overshoot. The BTP mechanism allows for no change in project size.

We have also seen that characteristics of the distribution of the random component of the cost functions are also very important. Reviewing the implications of different distributions among a family of symmetric distributions revealed that when weight is placed on the borders of the support, the BTE mechanism, with its capacity for adjustment, does better than the BTP mechanism which essentially selects an average that almost never happens. When weight is put towards the center of the support, uncertainty diminishes making both mechanisms to look more similar.

What may be surprising about these results to many is the fact that the unconventional BTE mechanism dominates its BTP counterpart in a large fraction of the parameter space we
explore, suggesting that its superiority does not only arise with extreme or unlikely parameter configurations. We have also seen that the performance difference between the BTE and BTE mechanisms can be substantial.

Further work on this model could elaborate on the conditions under which one mechanism dominates the other, and we offer a few conjectures here. First, our model above assumes that there is no correlation between the random elements of the benefit and cost functions. Given the analysis above, the introduction of non-zero correlation would have reasonably predictable qualitative effects. If benefits and costs are negatively correlated (e.g. higher benefits tend to occur with lower costs) the quantity adjustments coming from the BTE mechanism will be even more valuable at least initially – so this favours BTE procurement, other things equal. On the other hand, if benefits and costs positively correlated the quantity adjustments from the BTE mechanism will be less valuable – and this favours the BTP mechanism, other things equal.

Second, the implications of non-constant (with respect to quantity) marginal costs are also quite intuitive. If marginal costs are rising with quantity, the output adjustments that come from the BTE mechanism are less valuable. To be specific, the optimal adjustments to changing rising or falling marginal costs are smaller with a rising marginal cost curve and the deadweight loss associated with any given inefficiency in output is less as well. If marginal costs are falling (but more slowly than marginal benefits) the desired output adjustments with changed costs are larger and any changes are more valuable, suggesting a greater relative advantage of the BTE mechanism.

It is easy to see the implications should the uncertainty with respect to costs be with fixed rather than marginal costs. If it is only the firms’ fixed costs that are uncertain the government, (and for simplicity assume there is no uncertainty about benefits) changes in costs from expected levels will have no implications for optimal project size. In such a case the BTP mechanism will continue to provide the optimal project size. The BTE mechanism, however, will adjust project size away from the optimal with fixed costs that are higher or lower than expected.

Our analysis here has focused on purely economic trade-offs, but there is no denying that political considerations often weigh heavily in decision-making about large infrastructure projects and this could be an interesting avenue for future work. Some political considerations

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17 This assumes that the fixed costs do not rise so high as to make the project inefficient at any size. Part of the fixed costs could also represent the opportunity cost for firms associated with not pursuing other projects (if they do not have the capacity to take any number of projects).
suggest themselves immediately. Governments that want certainty about their expenditures on a project might be attracted to the BTE mechanism whereas those that want more certainty about the scale of the project they are getting would get that from the BTP mechanism. It may also be the case that no government will be absolutely pure in its application of either mechanism, for example if bids under a BTP mechanism diverge dramatically from what the government expected, it may adjust the scale of the project and re-tender. In a similar way, if the bids received under a BTE envelope are for quantities far from what the government anticipated and wanted, it may go back to the drawing board. Of course, rational bidders anticipating these possibilities will adjust their bids in ways that would need to be analyzed as well.

Further work could also explore hybrid versions of these mechanisms that might try to make better use of the private information held by firms. For example, consider an altered BTP mechanism in which each bidder can quote a price for the stated project but also offer the government project enhancements (at stated prices) that it has determined would pass the government’s benefit/cost test.\textsuperscript{18}

Taken together, our results suggest, to us at least, that governments may wish to take a closer look at BTE mechanisms for public procurement. Of course there may be other challenges associated with the BTE approach that we have not considered here – for example, if “output” or “project size” is not such a simple one-dimensional variable as modelled here, some sort of scoring rule will be needed to compare the size of competing bids that differ on multiple dimensions. However, we think the results here are strong enough in support of the BTE mechanism that it merits serious consideration under the right conditions.

References


\textsuperscript{18} Or it could suggest to the government that some of what was requested (by the government) was too costly for the benefit derived and recommend the project be scaled back in some respect. Early stages of complex PPP bidding can have characteristics like this, with information flowing between the government and bidders about what various aspects of projects will cost and whether they are all worth it.


APPENDIX
Proofs

We define $\psi = \frac{1}{c + \eta}$. We already establish in the text that

$$E(\psi) > \frac{1}{c}$$

(a.1)

And we will also use

$$cE(\psi^2) - E(\psi) > 0$$

(a.2)

Which holds since

$$cE(\psi^2) - E(\psi) = c(\text{Var}(\psi) + E(\psi)^2) - E(\psi) = c\text{Var}(\psi) + E(\psi) \left[ E(\psi) \cdot c - 1 \right] > 0$$

We now move on to prove lemmas and propositions.

**Lemma 1:** The expected social welfare from the BTE mechanism, relative to that from the BTP mechanism will be negatively related to the level of $a$, i.e. $\partial ESW^{e/p}/\partial a < 0$.

**Proof:** $ESW^{e/p} = \frac{[aE(\psi)-1]^2}{[a-c]^2E(\psi^2)}$. Differentiating with respect to $a$:

$$\frac{\partial ESW^{e/p}}{\partial a} = \frac{2(aE(\psi)-1)E(\psi)(a-c)^2E(\psi^2)-2(a-c)E(\psi^2)(aE(\psi)-1)^2}{([a-c]^2E(\psi^2))^2}$$

$\frac{\partial ESW^{e/p}}{\partial a} = \frac{2(a-c)E(\psi^2)(aE(\psi)-1)^2}{([a-c]^2E(\psi^2))^2} \left[ E(\psi)(a-c) - (aE(\psi) - 1) \right]$

Where the first term in parentheses is positive since $aE(\psi) - 1 > cE(\psi) - 1 > 0$, by (a.1). The second term is negative directly by (a.1) as well. It thus follows that

$$\frac{\partial ESW^{e/p}}{\partial a} < 0$$

**Lemma 2:** There exists a value of $a$, $a'$, such that: (i) $ESW^{e/p}$ evaluated at $a'$ will equal one; and (ii) $a' > c$.

**Proof:** $ESW^{e/p}(a) = \frac{[aE(\psi)-1]^2}{[a-c]^2E(\psi^2)}$ is a continuous and differentiable function of $a$ over $(c, +\infty)$. Lemma 1 established that $\frac{\partial ESW^{e/p}}{\partial a} < 0$. It is also easy to see that:

$$\lim_{a\to c} ESW^{e/p} = +\infty$$
since the numerator is finite and positive and the denominator goes to 0. Also, applying L’Hôpital’s rule twice, we get that
\[ \lim_{a \to +\infty} ESW^{e/p} = \frac{E(\psi)}{E(\psi^2)} < c, \]
following (a.2). It follows from the intermediate value theorem that there exists \( a' > c \) such that \( ESW^{e/p}(a') = 1 \).

**Proposition 1:** For some values of the parameters \( a \) and \( c \), the BTP mechanism will generate greater expected social welfare and for other values the BTE mechanism will generate greater expected social welfare.

**Proof:** none required, see text.

**Lemma 3:** as the value of \( a \) increases, the expenditures under the BTE mechanism fall relative to those under the BTP mechanism (i.e. \( \partial T^{e/p} / \partial a < 0 \)).

**Proof:** As obtained in the paper, \( T^{e/p} = \frac{T^e}{T^p} = \frac{aE[\psi]-1}{(a-c)E[\psi^2]} \)
\[
\frac{\partial T^{e/p}}{\partial a} = \frac{E(\psi)(a-c)cE(\psi^2) - cE(\psi^2)(aE(\psi) - 1)}{[(a-c)cE(\psi^2)]^2} \\
= \frac{cE(\psi^2)}{[(a-c)cE(\psi^2)]^2} \left[ E(\psi)(a-c) - aE(\psi) + 1 \right] \\
= \left( \frac{cE(\psi^2)}{[(a-c)cE(\psi^2)]^2} \right) (1 - cE(\psi))
\]
Where the first term is positive and the second negative by (a.2), leaving us with:
\[
(\partial T^{e/p})/\partial a < 0
\]

**Lemma 4:** there is a value of \( a \), called here \( a'' \) such that \( a'' > c \) and at which \( T^{e/p} = 1 \).

**Proof:** We show that the \( a'' \) that makes \( T^{e/p} = 1 \) is such that \( a'' > c \) and therefore that \( T^{e/p} \) will be above 1 for \( c < a < a'' \) and it will be below 1 for \( a > a'' \).

Consider \( a'' \) such that \( T^{e/p} = 1 \)
\[
a'' E(\psi) - 1 = (a'' - c) \cdot c \cdot E(\psi^2) \\
\Rightarrow \quad c^2 E(\psi^2) - 1 = a''(cE(\psi^2) - E(\psi)) \\
\Rightarrow \quad a'' = \frac{c^2 E(\psi^2) - 1}{c^2 E(\psi^2) - E(\psi)} = c \frac{E(\psi^2) - 1/c}{cE(\psi^2) - E(\psi)}
\]
Next, (a.2) ensures that that denominator is positive. And (a.1) ensures that
\[
cE(\psi^2) - 1/c \geq cE(\psi^2) - E(\psi)
\]
Therefore, \( a'' > c \)

**Proposition 2:** The expected costs of projects under BTE mechanism can be higher or lower than the expected costs under the BTP mechanism.

**Proof:** none required – see text.

**Proposition 3:** Increases in the marginal cost of taxation will increase the expected social welfare of the BTE mechanism relative to that for the BTP mechanism, that is \( dESW^{e/p} / d\lambda > 0 \).

**Proof:** We showed in the text that

\[
ESW^{e/p}(\lambda) = \frac{[aE(\psi) - \lambda]^2}{[a - \lambda c]^2 E(\psi^2)}
\]

And differentiating:

\[
\frac{\partial ESW^{e/p}}{\partial \lambda} = \frac{2(aE(\psi) - \lambda)(-1)(a - \lambda c)^2 E(\psi^2) - 2(a - \lambda c)(-c) E(\psi^2)(aE(\psi) - \lambda)^2}{[(a - \lambda c)^2 E(\psi^2)]^2}
\]

\[
\frac{\partial ESW^{e/p}}{\partial \lambda} = \frac{2(a - \lambda c)E(\psi^2)(aE(\psi) - \lambda)[c(aE(\psi) - \lambda) - a + \lambda c]}{[(a - \lambda c)^2 E(\psi^2)]^2}
\]

\[
\frac{\partial ESW^{e/p}}{\partial \lambda} = \left( \frac{2(a - \lambda c)E(\psi^2)(aE(\psi) - \lambda)}{[(a - \lambda c)^2 E(\psi^2)]^2} \right) \left[ aE(\psi) - \lambda c - a + \lambda c \right] = \frac{(2(a - \lambda c)E(\psi^2)(aE(\psi) - \lambda))}{[(a - \lambda c)^2 E(\psi^2)]^2} \left[ a \left( cE(\psi) - 1 \right) \right]
\]

Where the first term in parentheses is positive by virtue of the fact that (as above)

\[
aE(\psi) - \lambda > a \cdot \frac{1}{c} - \lambda > 0
\]

and the second is positive by Jensen’s inequality. Therefore we have:

\[
dESW^{e/p} / d\lambda > 0
\]
Figure A.1: Expected project size comparison

α = 0

α = 6

α = -2