

Economics of a reservation system for morning commute

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Abstract

Reservation or booking is not a new concept, which indeed has been widely used both in transportation industries such as rail, airline, and liner shipping, and some other industries such as parking, hotel and entertainment. This paper conducts an economic analysis of the highway reservation system which will allocate the highway space to potential users during different time intervals. We show that, since the system manager can reject reservation request beyond highway capacity, traffic congestion can be relieved. Also, efficiency of the reservation system has been evaluated. The factors affecting the performance of the reservation system is discussed, and efficiency loss due to these factors are examined.

Keywords: reservation, bottleneck model, dynamic traffic equilibrium, efficiency

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1. Introduction

Reservation or booking is not a new concept, which indeed has been widely used both in transportation industries such as rail, airline, and liner shipping, and some other industries such as parking, hotel and entertainment. Based on a stated-preference survey, Akahane and Kuwahara (1996) quantitatively evaluates the trip reservation systems that adjust departure times of travelers on motorways in a similar way to train seat reservations. Wong (1997) then conceptually described the booking system for highway use to control traffic and improve road performance.

Koolstra (1999) studied the potential benefits of slot reservation on highways by analyzing the difference between user equilibrium departure times and system optimal departure times. de Feijter et al. (2004) have proposed trip booking as a method for improving the travel time reliability and increasing the effective usage of road capacity. The demand management concept known as the Highway Space Inventory Control System was presented in Teodorovic and Edara (2005) and Edara and Teodorovic (2008). Zhao et al. (2010) introduced a Downtown Space Reservation System to facilitate the mitigation of traffic congestion in a cordon-based downtown area. More recently, Yang et al. (2013), Liu et al. (2014a, 2014b) proposed the parking reservation system for managing both parking competition and traffic congestion.

While all these papers have discussed the advantages and issues related to trip reservation systems, none of them have developed a comprehensive economic analysis of the highway reservation system which will allocate the highway space to potential users during different time intervals and accept or reject user travel requests based on the highway capacity and the number of reservations made. Under the discussed highway reservation system for morning commute, each commuter have to make a reservation before they use the highway, free access is no longer available. In this situation, on one hand, the system manager can reject reservation request beyond highway capacity thus traffic congestion can be relieved; on the other hand, the system manager would know the real travel demand over time through the reservation system, and efficiency loss due to demand uncertainty can be reduced. It thus of our interest to study how the reservation system can help to decrease commuters’ total travel cost and improve system performance. The factors affecting the performance of the reservation system will be discussed, and efficiencies are evaluated. (e.g., reservations can only be differentiated in a limited way, user heterogeneity in values of travel time and schedule penalties, tactical
waiting behavior of commuters)

In this paper, bottleneck model (Vickrey, 1969) is adopted to capture the congestion dynamics in the morning commute. Smith (1984) and Daganzo (1985) established the existence and uniqueness of the time-dependent equilibrium distribution of arrivals at a single bottleneck respectively. Thanks to its tractability, the bottleneck model can generate insights concerning traffic congestion, e.g., congestion pricing and tradable mobility credits (Arnott et al., 1990; Laih, 1994; Nie and Yin, 2013; Tian et al., 2013; Xiao et al., 2012; Xiao et al., 2013a; Yang and Meng, 1998), Pareto-improving strategies (Daganzo and Garcia, 2000; Xiao and Zhang, 2013). Besides, it has been extended to consider demand elasticity (Arnott et al., 1993; Yang and Huang, 1997), stochastic capacity and demand (Arnott et al., 1999; Lindsey, 2009; Xiao et al., 2013b), heterogeneous commuters (Arnott et al., 1994; Doan et al., 2011; Lindsey, 2004; Liu and Nie, 2011; Newell, 1987; van den Berg and Verhoef, 2011; van den Berg, 2014), integration of both morning and evening peak hour commute (Zhang et al., 2005; Gonzales and Daganzo, 2013; Daganzo, 2013), travel time reliability (Yin et al., 2004; Li et al., 2012), congestion derivatives (Yao et al., 2010 and 2012).

The remainder is organized as follows. Section 2 introduces the basic model formulation of the morning commute. In section 3, the reservation system for highway use in the morning commute is introduced, the traffic equilibrium under such a system and efficiency of the system in reducing travel cost have been discussed. Section 4 discusses the efficiency loss due to tactical waiting of commuters and user heterogeneity respectively. Lastly, Section 5 concludes the paper.

2. Basic Model

Consider a continuum of $N$ commuters travel from homes to workplaces through a highway every morning. Commuters have a common preferred arrival time $t^*$ at the destination. Travel cost, including travel time cost and schedule delay cost, departing at time $t$ is given by

$$c(t) = \alpha \cdot T(t) + \beta \cdot \max\{0, t^* - t - T(t)\} + \gamma \cdot \max\{0, t + T(t) - t^*\}$$

(1)

where $T(t)$ is the travel time at departure time $t$, $\alpha$ is the value of unit travel time, and $\beta$ and $\gamma$ are the schedule penalty for a unit time of early arrival and late arrival respectively. Without loss of generality, we assume $T(t)$ only contains queuing time at
the highway bottleneck whose service capacity is $s$. Under these standard setting in the literature, the morning commuting equilibrium can be depicted by Figure 1, where the blue line and red line represents the cumulative departure from home and arrival at the destination respectively. Note that in Figure 1, $r_1 = \frac{n}{\alpha - \beta} s$ and $r_2 = \frac{n}{\alpha + \gamma} s$ denote the departure rates from home for commuters arriving at destination earlier and later than $t^*$ respectively. At equilibrium, total travel cost of all commuters is

$$TC_o = \delta \frac{N^2}{s},$$

(2)

where $\delta = \frac{\beta \gamma}{\beta + \gamma}$.

![Figure 1. No policy commuting equilibrium](image)

3. The Reservation System for Highway Use

In this section, we will first introduce the reservation system and then discuss the commuting equilibrium under such a system and examine efficiency of the system in reducing total travel cost.

3.1. Reservation system

Since the capacity of the highway bottleneck is limited, a commuter needs to make a reservation in order to pass the highway bottleneck. Further, when a commuter makes a reservation, he or she need to specify whether he will pass the bottleneck earlier than or later than a certain time point (the system manager can set a series of choices in advance). Suppose there is an electronic device located at the highway bottleneck which checks whether the passing vehicle has made a reservation or not. We consider a commuter
passing the highway bottleneck would encounter a very high penalty fee once he or she has not made a reservation. Therefore, all commuters will make reservations before they pass the highway bottleneck.

Under a reservation system, the system manager can set a series of time points, i.e., \( \{t_1, t_2, \ldots, t_k, t^*, t_{k+1}, \ldots, t_n\} \) where \( t_x < t_y \) for \( x < y \) and \( t_k < t^* < t_{k+1} \). For all commuters, they can make reservations and specify they will pass the highway bottleneck before \( t_i \) where \( i \leq k \) or just before \( t^* \); or they can make reservations and specify they will pass the highway bottleneck after \( t_i \) where \( i \geq k+1 \) or just after \( t^* \). Generally, commuters would prefer to reservations with more flexibility, i.e., for early arrival, a larger \( t_i \); and for late arrival, a smaller \( t_i \). To take advantage of his or her reservation, an early arrival commuter would pass the highway bottleneck before \( t_i \), and a late arrival commuter would pass the highway bottleneck after \( t_i \).

3.2. Efficiency of the reservation system

Now we want to look at the optimal design of the reservation system and the corresponding commuting equilibrium. Without loss of generality, we consider the whole morning peak is divided into \( n \) time intervals equally, i.e., \( t_{i+1} - t_i = t^* - t_k = t_{k+1} - t^* \) where \( i = 1, 2, \ldots, k - 1, k + 1, \ldots, n \). Under the socially preferable design: firstly, the number of reservations accepted for a given time interval should be just equal to the total capacity of the highway during this time interval, i.e., \( t_{i+1} - t_i = t^* - t_k = t_{k+1} - t^* = \frac{n}{m} \); secondly, the total schedule delay cost should be minimized, this requires \( \frac{k}{n} \) to be optimally determined thus Eq.(3) is minimized for given \( n \). It is expected that \( \frac{k}{n} \to \frac{\gamma}{\beta + \gamma} \) for relatively large \( n \). Besides, for commuters with reservation allowing them to pass the highway bottleneck after \( t_i \), we assume all of they will depart just at \( t_i \) and join a queue, their arrival order will be decided randomly. This mass departure assumption is firstly used in Arnott et al. (1990). Under the above setting, the departure time equilibrium can be depicted by Figure 2.
Figure 2. Commuting equilibrium under the $n$-step reservation scheme

Total travel cost under the commuting equilibrium depicted in Figure 2 can be expressed as follow:

$$TC_n = \frac{k(k+1)}{2n^2} \beta \frac{N^2}{s} + \frac{n-k}{2n^2} (\alpha + \gamma) \frac{N^2}{s} + \frac{(n-k)^2}{2n^2} \gamma \frac{N^2}{s}.$$  \hspace{0.5cm} (3)

By looking at the first-order partial derivative of Eq.(3) with respect to $\frac{k}{n}$, we obtain the optimal $\frac{k}{n}$ that minimizes Eq.(3), i.e., $\frac{\gamma}{\beta+\gamma} + \frac{\alpha+\beta-1}{\beta+\gamma} \frac{1}{2n}$. Note that, as $n \rightarrow \infty$, we have $\frac{k}{n} \rightarrow \frac{\gamma}{\beta+\gamma}$, and $TC_n \rightarrow \frac{1}{2} \frac{\beta \gamma N^2}{s}$. Compared to the original departure/arrival equilibrium with no policy introduced, the reservation system would yield 50% reduction in total travel cost. In this case, the queue at the highway is eliminated completely and queuing delay cost is reduced to zero, and schedule delay is minimized, which is half of total travel cost in the original departure/arrival equilibrium, $TC_o = \frac{\beta \gamma N^2}{s}$.

Since in reality, it is certainly unlikely to set a $n \rightarrow \infty$, we now derive the inefficiency due to a limited $n$. We begin with defining the efficiency loss due to the limited $n$ as follows:

$$\theta_n = \frac{\min \{TC_n\} - TC_\infty}{TC_o - TC_\infty},$$  \hspace{0.5cm} (4)

where $TC_n - TC_\infty$ is the loss of travel cost reduction of a $n$-step reservation system, and $TC_o - TC_\infty$ is maximum of total travel cost reduction. We then have the following proposition. Note that $\min \{TC_n\}$ is determined by Eq.(3) under $\frac{k}{n} = \frac{\gamma}{\beta+\gamma} + \frac{\alpha+\beta-1}{\beta+\gamma} \frac{1}{2n}$.

**Proposition 1.** The efficiency loss defined in Eq.(4) satisfying
\[
\theta_n \leq \frac{\alpha + 2\gamma}{\gamma} \frac{1}{n}.
\]

(5)

**Proof.** For given \( n \), if we set \( k = \frac{k}{\beta + \gamma} n \), then

\[
\frac{TC_n - TC_{\infty}}{TC_{\infty} - TC_n} = \frac{\alpha + 2\gamma}{\gamma} \frac{1}{n}.
\]

Since \( \min \{ TC_n \} \leq TC_n \), we then have Eq. (5) holds. \( \square \)

Proposition 1 implies that a reservation system of ten steps, i.e., \( n = 10 \), would yield an efficiency loss less than 30% (note that \( \alpha < \gamma \)). If we consider \( \frac{\alpha}{\gamma} \approx 2 \), then the efficiency loss would be within 25%.

Note that for given number of step, \( n \), the socially optimal \( k = \frac{k}{\beta + \gamma} \) relies on commuters’ value of time and schedule preference. However, the central manager of the reservation system does not need to know this exact information. This is because, the central manager can open a sufficiently large time window for commuters to make reservations, and then commuters will choose to make reservations close to their desired arrival time \( t^* \) and total schedule delay cost will be minimized accordingly. This means one can get rid of commuters’ value of time information and schedule preference when looking for the optimal design of the reservation system. In this sense, the reservation system is superior to the first best time-varying pricing. Besides, the performance of the reservation system with limited number of steps \( n \) is robust to variation of the population’s value of time. This can be seen in Eq. (5), where \( \frac{\alpha + 2\gamma}{\gamma} \) will always be less than 3 and \( \theta_n \leq \frac{3}{n} \).

4. **Factors Influencing System Performance**

We have bounded the inefficiency due to a limited \( n \). In this section, we will further consider two factors influencing the performance of the reservation system, and examine whether the inefficiency due to these factors can be bounded.

4.1. **Tactical waiting of commuters**
In the previous section, we assume the late arrival commuters with reservation would depart immediately after they are allowed to pass the highway bottleneck (mass departure). However, some commuters may depart earlier than the time point they are allowed and wait at the highway bottleneck (tactical waiting) thus to enjoy a lower travel cost. This would lead the reservation system to lose efficiency. Without loss of generality, we assume the commuters are divided into \( n \) groups equally, therefore, for \( 1 \leq x < k \), we have \( t_x - t_{x-1} = t^* - t_k = \frac{n}{m} \), and for \( k + 1 \leq x < n \), we have \( t_{x+1} - t_x = t_{k+1} - t^* = \frac{n}{m} \) where \( r_s = \frac{n}{\alpha + \gamma} \). Under this setting, the commuting equilibrium can be depicted by Figure 3.

**Figure 3.** Commuting equilibrium under the \( n \)-step reservation scheme (tactical waiting)

Total travel cost under the commuting equilibrium depicted in Figure 3 can be expressed as follow:

\[
TC_n'' = \frac{k(k+1)}{2n^2} \beta \frac{N^2}{s} + \frac{(n-k+1)(n-k)}{2n^2} \gamma \frac{N^2}{r_s^2}.
\]  

Similarly, to minimize Eq.(6), we found that \( \frac{k}{n} = \frac{(\alpha + \gamma) n}{\alpha + (\alpha + \gamma) n} - \frac{1}{2n} \). To see the efficiency loss due to tactical waiting, we look at \( \Delta TC_1 = TC_n'' - TC_n \) which can be expressed as

\[
\Delta TC_1 = \frac{n-k}{2n^2} \left( \frac{\gamma}{\alpha} - 1 \right) (\alpha + \gamma) + \frac{(n-k)\gamma}{\alpha} \frac{N^2}{s}.
\]  

Note that for comparison purpose, we look at the cost difference \( \Delta TC_1 \) for the same \( k \) and \( n \). Suppose \( k = \lambda n \), when \( n \to \infty \), \( \Delta TC_1 \to \frac{1}{2} (1-\lambda) \frac{\gamma^2}{\alpha} \frac{N^2}{s} \). In this case, if we set \( \lambda = \frac{r_s}{\alpha + \gamma} \) which minimizes \( TC_n \) for \( n \to \infty \), \( \Delta TC_1 \to \frac{1}{2} \frac{\gamma^2}{\alpha} \frac{N^2}{s} \). This indicates even we can set \( n \to \infty \), commutters’ tactical waiting would yield an efficiency loss. Unless \( \lambda \to 1 \), i.e., \( k = n \), \( \Delta TC_1 > 0 \) holds. This is because the efficiency loss comes from late arrival
commuters’ tactical waiting behavior. Once \( \lambda \to 1 \) or late arrival is not allowed, i.e., \( \gamma \to \infty \), there would be no tactical waiting (since there is no late arrival commuters), and one can verify that \( \Delta TC_i \to 0 \).

### 4.2. User heterogeneity

When commuters’ value of time (VOT) varies but \( \beta \) and \( \gamma \) are identical, then the user heterogeneity would not influence the performance of the reservation system since who reserve to pass the bottleneck close to \( t^* \) makes no difference to the system. However, as reported in empirical studies (e.g., Tseng and Verhoef, 2008), not only commuters’ value of time, but also schedule delay penalty can vary. In this case, under the socially preferable commuting equilibrium, commuters with higher \( \beta \) and \( \gamma \) should obtain reservations with more flexibility, i.e., their departure and arrival should be more close to \( t^* \). However, without other instrument, the reservation system cannot deal with user heterogeneity and would lose efficiency. The following discussion will focus on the inefficiency due to user heterogeneity for two types of user heterogeneity, under both commuters’ schedule preferences vary.

#### 4.2.1 Proportional heterogeneity

Assume commuters’ VOT, \( \alpha \), continuously increases from \( \underline{\alpha} \) to \( \bar{\alpha} \) and follows the following cumulative distribution function:

\[
F(x) = \Pr\{\alpha \leq x\}. \tag{8}
\]

The corresponding probability density function is denoted by \( f(\alpha) \), which is positive for \( \alpha \in [\underline{\alpha}, \bar{\alpha}] \) and is zero otherwise. For ease of presentation, let \( \bar{F}(x) = 1 - F(x) \). For a specific commuter, his or her early and late arrival penalty is proportional to VOT, i.e., \( \beta = \rho_1 \alpha \) and \( \gamma = \rho_2 \alpha \), where \( \rho_1 < 1 < \rho_2 \) and \( \rho_1 \) and \( \rho_2 \) are identical for all commuters. This is similar to that considered in Vickrey (1973), Xiao et al. (2011), Xiao and Zhang (2013), Xiao et al. (2013a). Note that the two ratios measure the willingness of a commuter to reduce schedule delays by increasing travel times, which determine the arrival order of commuters. Under other types of heterogeneity, these two ratios may vary, and there is a separation of users over time.
When no policy is introduced, the departure/arrival equilibrium will be identical to that in the homogeneous case, which is depicted in Figure 1. For a commuter with VOT equal to $\alpha$, travel cost would be

$$C_o(\alpha) = \rho\alpha \frac{N}{s},$$

where $\rho$ is given by

$$\rho = \frac{\rho_1\rho_2}{\rho_1 + \rho_2}.$$  \hfill (10)

Total travel cost at this no policy equilibrium is

$$TC_o = \int_\alpha^\pi \rho\alpha \frac{N}{s} f(\alpha) Nd\alpha.$$  \hfill (11)

Under the system optimal equilibrium, queuing should be eliminated and schedule delay cost should be minimized. Therefore, commuters with larger $\alpha$ should arrive closer to the desired arrival time, $t^*$. For a commuter with VOT equal to $\alpha$, travel cost is

$$C_{so}(\alpha) = \rho\alpha \frac{N}{s} \bar{F}(\alpha).$$  \hfill (12)

Total travel cost at this system optimal equilibrium is

$$TC_{so} = \int_\alpha^\pi \rho\alpha \frac{N}{s} \bar{F}(\alpha) f(\alpha) Nd\alpha.$$  \hfill (13)

Unfortunately, the reservation system or scheme with $n \to \infty$ proposed here cannot differentiate commuters according to their VOTs. We consider two cases: (i) the average case, i.e., for each specific reservation, all commuters have the same probability to obtain it; (ii) the worst case, i.e., commuters with higher VOT obtain the reservation with less flexibility and arrives farther from $t^*$. For case (i), from a long term perspective, the total travel cost is an expectation, which is given by

$$TC_{av} = \int_\alpha^\pi \rho\alpha \frac{1}{2} \frac{N}{s} f(\alpha) Nd\alpha.$$ \hfill (14)

In case (ii), for a specific commuter with VOT equal to $\alpha$, travel cost is

$$C_w(\alpha) = \rho\alpha \frac{N}{s} F(\alpha).$$ \hfill (15)

Further, total travel cost is

$$TC_w = \int_\alpha^\pi \rho\alpha \frac{N}{s} F(\alpha) f(\alpha) Nd\alpha.$$ \hfill (16)

Now we are ready to present the following proposition.

**Proposition 2.** For all the cases, the total travel costs satisfies
\[ TC_{so} < TC_{av} < TC_w < TC_o. \]  

(17)

**Proof.** See Appendix A. □

The above proposition indicates that the proposed reservation system or scheme can reduce total travel cost while it cannot differentiate commuters with different VOTs, i.e., even the worst case would lead to a travel cost reduction compared to that when no policy is introduced. Further, the performance in the average case lies between the system optimal case and the worst case. Besides, we note that, in the average case, i.e., case (i), total travel cost is half of that under the no policy equilibrium, i.e., \( TC_{av} = \frac{1}{2} TC_o \). This indicates the reservation system would yield an efficiency of 50% in an average sense.

**Proposition 3.** Total travel costs in the system optimum and the worst case satisfy

\[ \frac{TC_w}{TC_{so}} \leq \frac{\bar{\alpha}}{\alpha}. \]

(18)

**Proof.** See Appendix B. □

Although the reservation system, without other instrument, would loss efficiency due to user heterogeneity, i.e., \( TC_w > TC_{so} \), however, this efficiency is upper bounded. If \( \frac{\alpha}{\bar{\alpha}} \to 1 \), the efficiency loss approaches zero, and \( TC_w \to TC_{so} \). If \( \frac{\alpha}{\bar{\alpha}} = 2 \), \( TC_w \leq 2TC_{so} \) holds. Since \( TC_w + TC_{so} = TC_o \), from Proposition 3, one can verify that \( TC_w \leq \frac{\alpha}{\bar{\alpha} + \bar{\alpha}} TC_o \). This indicates even in the worst case, the reservation system can at least reduce a certain percentage of cost, i.e., \( \frac{\alpha}{\bar{\alpha} + \bar{\alpha}} \). If \( \frac{\alpha}{\bar{\alpha}} \to 1 \), this percentage approaches 50%. If \( \frac{\alpha}{\bar{\alpha}} = 2 \), the percentage is 33%.

To further look at the efficiency loss due to user heterogeneity, we consider \( \alpha \) follows a uniform distribution, i.e.,

\[ f(\alpha) = \begin{cases} \frac{1}{\bar{\alpha} - \alpha} & \alpha \in [\alpha, \bar{\alpha}] \\ 0 & \text{otherwise} \end{cases}. \]

(19)

Then we have the following propositions regarding the efficiency of the reservation system.

**Proposition 4.** When \( \alpha \sim U[\alpha, \bar{\alpha}] \), we have
\[ \frac{1}{3} < \frac{TC_{so}}{TC_o} < \frac{1}{2}; \frac{TC_{av}}{TC_o} = \frac{1}{2}; \frac{1}{2} < \frac{TC_{wc}}{TC_o} < \frac{2}{3}; \frac{1}{3} < \frac{TC_{sc}}{TC_{so}} < 2. \]  

(20)

**Proof.** Note that \( TC_{av} = \frac{1}{2} TC_o \) always holds. When \( \alpha \sim U[\underline{\alpha}, \bar{\alpha}] \), from Eqs.(11), (13) and (16), it can be shown that

\[ \frac{TC_{so}}{TC_o} = \frac{1}{3} \left( 1 + \frac{\alpha}{\bar{\alpha} + \underline{\alpha}} \right); \quad \frac{TC_{wc}}{TC_o} = \frac{1}{3} \left( 1 + \frac{\bar{\alpha}}{\bar{\alpha} + \underline{\alpha}} \right); \quad \frac{TC_{sc}}{TC_{so}} = \frac{2\bar{\alpha} + \alpha}{\alpha + 2\bar{\alpha}}. \]  

(21)

With Eq.(21), one can verify that

\[ \frac{\overline{\alpha}}{\underline{\alpha}} \uparrow \Rightarrow \frac{TC_{so}}{TC_o} \downarrow; \quad \frac{TC_{wc}}{TC_o} \uparrow; \quad \frac{TC_{sc}}{TC_{so}} \uparrow. \]  

(22)

Since \( 1 < \frac{\underline{\alpha}}{\bar{\alpha}} < \infty \), with Eq.(22), one can verify Eq.(20) holds.  \( \Box \)

Proposition 4 provides a meaningful measure of efficiency loss when VOT is uniformly distributed. In this case, the efficiency loss would solely depends on the ratio of \( \frac{\underline{\alpha}}{\bar{\alpha}} \), as shown in Eq.(21). Consistent with Proposition 2, when \( \frac{\underline{\alpha}}{\bar{\alpha}} \to 1 \), since commuters are almost homogeneous, the efficiency loss approaches zero, i.e., \( \frac{TC}{TC_{so}} \to 1 \). However, when \( \frac{\underline{\alpha}}{\bar{\alpha}} \to \infty \), unlike Proposition 3, the efficiency loss can still be bounded, i.e., \( \frac{TC}{TC_{so}} \to 2 \), and even in the worst case, travel cost can be reduced by 33%, i.e., \( \frac{TC}{TC_{so}} \to \frac{1}{3} \).

4.2.2 Non-proportional heterogeneity

We now turn to consider another type of user heterogeneity, i.e., commuters may have different \( \rho_1 \) and \( \rho_2 \). To focus on heterogeneity in \( \rho_1 \) and \( \rho_2 \), we consider all commuters’ VOT, \( \alpha \), are identical. Further, we assume that \( \rho_2 = \eta \rho_1 \) is also the same for all commuters. Commuters’ \( \rho_1 \) continuously increases from \( \underline{\rho}_1 \) to \( \overline{\rho}_1 \) and follows the following cumulative distribution function:

\[ F_1(x) = \Pr \{ \rho_1 \leq x \}. \]  

(23)

The corresponding probability density function is denoted by \( f_1(\rho_1) \), which is positive for \( \underline{\rho}_1 \leq \rho_1 \leq \overline{\rho}_1 \) and is zero otherwise. Note that, even \( \rho_1 \) and \( \rho_2 \) can vary, it is assumed \( \rho_1 < 1 < \rho_2 \), then we see that, \( \overline{\rho}_1 < 1 \) and \( \overline{\rho}_2 = \eta \rho_1 > 1 \). Arranging commuters in a decreasing order of \( \rho_1 \) and thus \( \rho_1(m) \) specifies the \( m \)-th commuter’s \( \rho_1 \), which can be
given as follows:

\[ \rho_1(m) = F^{-1}_1\left(1 - \frac{m}{N}\right). \]  

(24)

This type of user heterogeneity is similar to that considered in van den Berg and Verhoef (2011) and Tian et al. (2013).

Under this setting, at the no policy equilibrium, commuters’ arrival is ordered in relation to their \( \rho_1 \), i.e., commuters with lower \( \rho_1 \) will arrive closer to the desired arrival time \( t^* \). This is because a lower \( \rho_1 \) means the commuter is less willing to reduce schedule delay by increasing travel times, or is more willing to reduce travel times by increasing schedule delay. The commuter with \( \rho_1 \) would experience a travel cost as follows:

\[ C'_o(\rho_1) = \int_{\rho_1}^{\rho_o} \eta \frac{N}{s} f_1(\omega) \omega d\omega + \frac{\eta}{1 + \eta} \rho_1 \alpha \frac{N}{s} \bar{F}_1(\rho_1). \]  

(25)

Total travel cost at this no policy equilibrium is

\[ TC'_o = \int_{\rho_1}^{\rho_o} C'_o(\omega) f_1(\omega) N d\omega. \]  

(26)

Similarly, under the system optimal equilibrium, queuing should be eliminated and schedule delay cost should be minimized. Therefore, commuters with larger \( \rho_1 \) should arrive closer to the desired arrival time, \( t^* \). For a commuter with \( \rho_1 \), travel cost is

\[ C'_{so}(\rho_1) = \frac{\eta}{1 + \eta} \rho_1 \alpha \frac{N}{s} F(\rho_1). \]  

(27)

Total travel cost at this system optimal equilibrium is

\[ TC'_{so} = \int_{\rho_1}^{\rho_o} C'_{so}(\omega) f_1(\omega) N d\omega. \]  

(28)

In cases (i), i.e., the average case, from a long term perspective, the total travel cost is an expectation, which is given by

\[ TC'_{av} = \int_{\rho_1}^{\rho_o} \eta \frac{N}{s} \frac{1}{2} f_1(\omega) N d\omega. \]  

(29)

In case (ii), i.e., the worst case, for a specific commuter with \( \rho_1 \), travel cost is

\[ C'_w(\rho_1) = \frac{\eta}{1 + \eta} \rho_1 \alpha \frac{N}{s} F_1(\rho_1). \]  

(30)

Further, total travel cost is
\[ TC'_w = \int_{\rho_1}^{\rho_1} C'_w(\omega) f_1(\omega) N d\omega. \]  

**Proposition 5.** For all the cases, the total travel costs satisfies
\[ TC'_{so} < TC'_{av} < TC'_w. \]

**Proof.** See Appendix C. □

It can be shown that \( TC'_{so} < TC'_{o} \), however, it worth mentioning that \( TC'_w \) is not necessarily less than \( TC'_{o} \). This is because, in the original equilibrium or the socially optimal equilibrium, commuters with a larger \( \rho_1 \) will arrive at the destination closer to \( t^* \). However, in the considered worst case, commuters with a larger \( \rho_1 \) arrives farther away from \( t^* \), and experience relatively large schedule penalties which hurts the system a lot.

**Proposition 6.** Total travel costs in the system optimum and the worst case satisfy
\[ \frac{TC'_w}{TC'_{so}} \leq \frac{\bar{\rho}_1}{\tilde{\rho}_1}. \]

**Proof.** See Appendix D. □

To further look at the efficiency loss due to user heterogeneity, we look at a special distributions of \( \rho_1 \), i.e., a uniform distribution
\[ f_1(\rho_1) = \begin{cases} \frac{1}{\bar{\rho}_1 - \tilde{\rho}_1} & \rho_1 \in [\tilde{\rho}_1, \bar{\rho}_1] \\ 0 & \text{otherwise} \end{cases}, \]
\[ F_1(x) = \frac{x - \tilde{\rho}_1}{\bar{\rho}_1 - \tilde{\rho}_1}. \]

**Proposition 7.** When \( \rho_1 \sim U[\tilde{\rho}_1, \bar{\rho}_1] \), we have
\[ \frac{TC'_{so}}{TC'_{o}} = \frac{3}{2}; \quad \frac{TC'_{av}}{TC'_{o}} < \frac{3}{4}; \quad \frac{TC'_{w}}{TC'_{o}} < 1; \quad \frac{TC'_w}{TC'_{so}} < 2. \]

**Proof.** When \( \rho_1 \sim U[\tilde{\rho}_1, \bar{\rho}_1] \), from Eqs.(26), (28), (29) and (31), it can be shown that
\[
\frac{TC'_so}{TC'_o} = 1; \quad \frac{TC'_av}{TC'_o} = \frac{1}{2} \left( \frac{\bar{p}_1 + p_1}{\bar{p}_1 + 2p_1} \right); \quad \frac{TC'_w}{TC'_o} = \frac{1}{2} \left( \frac{2\bar{p}_1 + p_1}{\bar{p}_1 + p_1} \right); \quad \frac{TC'_w}{TC'_so} = \frac{2\bar{p}_1 + p_1}{\bar{p}_1 + p_1}.
\]  
(36)

With Eq.(21), one can verify that
\[
\frac{\bar{p}_1}{\rho_1} \uparrow \Rightarrow \frac{TC'_av}{TC'_o} \uparrow; \quad \frac{TC'_w}{TC'_o} \uparrow; \quad \frac{TC'_w}{TC'_so} \uparrow.
\]  
(37)

Since \(1 < \frac{\bar{p}_1}{\rho_1} < \infty\), with Eq.(36), one can verify Eq.(35) holds. □

From Proposition 7, we see that when the population’s schedule delay penalty is relatively uniform distributed, the reservation system can still reduce total travel cost even if \(\frac{\bar{p}_1}{\rho_1} \to \infty\). Further, under the current user heterogeneity type, the efficiency of the reservation system decreases compared to that under the proportional heterogeneity, i.e., the first three ratios in Eq.(35) is higher than that in Eq.(20). This is because in the original equilibrium under the considered non-proportional heterogeneity, commuters’ arrival is ordered in a preferable way and schedule delay cost is minimized, which means this equilibrium is efficient in terms of minimizing schedule delay cost and the potential for the reservation system to reduce travel cost is smaller. Another point we would like to highlight is that under both types of heterogeneity, the total travel cost in the worst case is always at most twice of that in the best case or the socially optimal case.

5. Concluding Remarks

This paper conducts an economic analysis of the highway reservation system which will allocate the highway space to potential users during different time intervals and accept or reject user travel requests based on the highway capacity and the number of reservations made. We show that since the system manager can reject reservation request beyond highway capacity, traffic congestion can be relieved. Also, this paper evaluates the efficiency of the reservation system for morning commute. The factors affecting the performance of the reservation system is discussed, and efficiency loss due to these factors are examined.

In reality, travel demand over time would not be deterministic. Since all travel demand should make reservations, under a reservation system, demand uncertainty would be reduced and traffic efficiency can be improved. Future research may explore the efficiency gains in the aspect of reducing demand uncertainty. Another possible extension would be taking into account another user heterogeneity type under which the ratio of
commuters’ early arrival penalty to late arrival penalty can vary. Besides, we would also like to discuss practical issues related to the reservation system for implementation.

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Appendix A. Proof of Proposition 2.

Under the user heterogeneity setting in section 4.2.1, the commuters can be rearranged in the order of decreasing $\alpha$, the $m$-th commuter’s VOT is then

$$\alpha(m) = F^{-1}\left(1 - \frac{m}{N}\right).$$  \hfill (38)

where $m \in [0, N]$, $\alpha(0) = \bar{\alpha}$ and $\alpha(N) = \underline{\alpha}$. The total travel cost at the no policy equilibrium given in Eq.(11) then can be rewritten as

$$TC_o = \int_0^N \rho \alpha(m) \frac{N}{s} \, dm.$$  \hfill (39)

At the system optimum, the $m$-th commuter would arrive at destination either at $t^* - \frac{\rho}{\rho_1} \frac{m}{s}$ or $t^* + \frac{\rho}{\rho_2} \frac{N-m}{s}$. Total travel cost given in Eq.(13) can be rewritten as

$$TC_{so} = \int_0^N \rho \alpha(m) \frac{m}{s} \, dm.$$  \hfill (40)

At the worst case, the $m$-th commuter would arrive at destination either at $t^* - \frac{\rho}{\rho_1} \frac{N-m}{s}$ or $t^* + \frac{\rho}{\rho_2} \frac{N-m}{s}$. Total travel cost given in Eq.(16) can be rewritten as

$$TC_w = \int_0^N \rho \alpha(m) \frac{N-m}{s} \, dm.$$  \hfill (41)

In addition, the total travel cost in the average case given in Eq.(14) can be rewritten as

$$TC_{av} = \int_0^N \rho \alpha(m) \frac{\frac{1}{2}N}{s} \, dm.$$  \hfill (42)

From Eqs.(39)-(42), it can be easily shown that $TC_{so} < TC_o$, $TC_w < TC_o$ and $TC_{av} < TC_o$, since $m \leq N$, $N-m \leq N$ and these two inequalities strictly hold for some $m$, and $\frac{1}{2}N < N$. Since $TC_{av} = \frac{1}{2}TC_o$, and $TC_w + TC_{so} = TC_o$, to prove the proposition, it suffices to show that $TC_{so} < TC_w$. From Eqs.(40) and (41), we see that

$$TC_{so} = \int_0^{\frac{1}{2}N} \rho \alpha(m) \frac{m}{s} + \rho \alpha(N-m) \frac{N-m}{s} \, dm$$

$$TC_w = \int_0^{\frac{1}{2}N} \rho \alpha(m) \frac{N-m}{s} + \rho \alpha(N-m) \frac{m}{s} \, dm.$$  \hfill (43)
For $m \leq \frac{1}{2}N$, one can verify that
\[
\rho \alpha \left( m \frac{m}{s} + \rho \alpha \left( N - m \right) \frac{N - m}{s} \right) \leq \rho \alpha \left( m \frac{N - m}{s} + \rho \alpha \left( N - m \right) \frac{m}{s} \right).
\tag{44}
\]

Note that the inequality in Eq.(44) strictly holds for some $m$, then with Eq.(43), we conclude that $TC_{so} < TC_w$. This completes the proof. □

**Appendix B. Proof of Proposition 3.**

From Eqs.(40) and (41), we know that
\[
TC_{so} \geq \rho \alpha \int_0^N \frac{m^2}{s^2} dm = \rho \alpha \frac{N^2}{2}.
\tag{45}
\]
\[
TC_w \leq \rho \alpha \int_0^N \frac{N - m}{s} dm = \rho \alpha \frac{N^2}{2}.
\tag{46}
\]
From Eqs.(45) and (46), we conclude that
\[
\frac{TC_w}{TC_{so}} \leq \frac{\alpha}{\alpha}.
\tag{47}
\]
This completes the proof. □

**Appendix C. Proof of Proposition 5.**

In section 4.2.2, the commuters are rearranged in the order of decreasing $\rho_1$, the $m$-th commuter will either arrive at their destination at $t^* - \frac{m}{1 + \eta \frac{m}{s}}$ or $t^* + \frac{1}{1 + \eta \frac{m}{s}}$. The total travel cost at the no policy equilibrium given in Eq.(26) then can be rewritten as
\[
TC'_o = \frac{\eta}{1 + \eta} \alpha \int_0^N \left( \rho_1 \left( m \right) \frac{m}{s} + \int_0^m \rho_1 \left( x \right) \frac{1}{s} dx \right) dm.
\tag{48}
\]
At the system optimum, the $m$-th commuter would arrive at destination either at $t^* - \frac{m}{1 + \eta \frac{m}{s}}$ or $t^* + \frac{1}{1 + \eta \frac{m}{s}}$. Total travel cost given in Eq.(28) can be rewritten as
\[
TC'_{so} = \frac{\eta}{1 + \eta} \alpha \int_0^N \rho_1 \left( m \right) \frac{m}{s} dm.
\tag{49}
\]
At the worst case, the $m$-th commuter would arrive at destination either at $t^* - \frac{N-m}{1 + \eta \frac{N-m}{s}}$ or $t^* + \frac{1}{1 + \eta \frac{N-m}{s}}$. Total travel cost given in Eq.(31) can be rewritten as
\[
TC'_{w} = \frac{\eta}{1 + \eta} \alpha \int_0^N \rho_1 \left( m \right) \frac{N-m}{s} dm.
\tag{50}
\]
In addition, the total travel cost in the average case given in Eq.(29) can be rewritten as
\[ \text{TC}'_{av} = \frac{n}{1+n} \alpha \int_0^N \rho_i \left( \frac{1}{2} \frac{N}{s} \right) dm. \]  
(51)

With Eqs.(49) (50) and (51), the rest proof is similar with Proof of Proposition 2. □

Appendix D. Proof of Proposition 6.

From Eqs.(49) and (50), we know that

\[ \text{TC}'_{so} \geq \frac{n}{1+n} \alpha \rho_i \int_0^N \frac{m}{s} dm = \frac{n}{1+n} \alpha \rho_i \frac{1}{2} \frac{N^2}{s}. \]  
(52)

\[ \text{TC}_w \leq \frac{n}{1+n} \alpha \rho_i \int_0^N \frac{N-m}{s} dm = \frac{n}{1+n} \alpha \rho_i \frac{1}{2} \frac{N^2}{s}. \]  
(53)

From Eqs.(52) and (53), we conclude that

\[ \frac{\text{TC}_w}{\text{TC}'_{so}} \leq \frac{\rho_i}{\rho_i}. \]  
(54)

This completes the proof. □

Reference


