Can traffic lights achieve the same results as tolls?

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Abstract
This paper studies the extent to which traffic lights can provide an alternative to road pricing in a simple network with two routes connecting one origin destination pair. I distinguish between the case in which the main purpose of the traffic lights is to regulate the intersection and the case in which the sole objective of the traffic light is to affect route choice. For this last case, I show that road pricing performs at least as good as traffic lights. For the network in which the traffic lights regulate the intersection of the two routes, I show that the implementation of a flow-dependent signal setting allows to render road pricing superfluous.

Inverse Stackelberg game, road pricing, signal settings

1 Introduction
One of the main constraints in the optimization of networks, is the socially suboptimal behavior of drivers. This suboptimal behavior can manifest itself in an excessive amount of drivers on the road, or in an inefficient distribution of vehicles over alternative routes. In this paper, the focus will be solely on this second type of inefficiency.

The main reason behind this inefficiency is an externality often present in transportation: when a driver makes a trip on a congested road this driver thereby also slows down the other users of this road. As the driver does not take into account this effect on other users when deciding on which road to take, an inefficient situation arises.

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The solution proposed by Pigou (1920) and Knight (1924) is to internalize the externality by introducing road pricing. Travellers are then confronted with their marginal social cost, rather than their average cost, and will consequently make the socially optimal route choice.

While the theory on road pricing is well developed, its implementation is hampered by the limited political and social acceptability. This paper therefore pushes forward traffic lights as an alternative to road pricing.

Already in 1974, Allsop pointed out that signal settings can affect route choice. Indeed, a change in the signal settings influences the average cost of the routes, leading to a change in route choice. This insight has generated two approaches in the literature to optimize a network with traffic lights: (1) the iterative procedure and (2) the global optimization approach (Cantarella et al. (1991)).

The iterative procedure iteratively solves the signal setting problem for a fixed flow pattern and the assignment problem for fixed signal settings until two successive flow patterns or signal settings converge (Allsop and Charlesworth (1977), Gartner et al. (1980), Cantarella et al. (1991)). As this procedure solves the signal setting problem and the assignment problem separately, techniques to solve the independent problems can be applied. As such, large networks can be tackled. With a simple example, however, Dickson (1981) showed that the total network cost can increase during the iterative procedure. The procedure does thus not necessarily lead to the optimal solution.

When the global optimization approach is applied, some network objective function is optimized while taking into account the equilibrium route choice behaviour of the drivers. The global optimization problem can be modelled as a bilevel programming problem (Yang and Yagar (1995)) or as a Stackelberg game (Fisk (1984)).

In Evers and Proost (2013) a Stackelberg approach is used to optimize an intersection of two congested routes connecting one OD-pair that is regulated by traffic lights. In Section 4 of this paper, I extend this network to include road pricing and develop a framework that allows to reach the same results with traffic lights only. To this end, I will use the inverse Stackelberg approach (Olsder (2009)).

Besides focussing on a network in which the traffic lights’ primary objective is to regulate an intersection, this paper will also compare the performance of traffic lights and road pricing for a network in which traffic lights are installed with as a sole objective to influence route choice. For this net-
work, I will show that road pricing obtains results that are at least as good as the minimal costs traffic lights can achieve.

The remainder of this paper is organized as follows: in Section 2 the different networks are described. In Section 3 the relative performance of road pricing and traffic lights is compared for the network with two parallel routes. In Section 4 I focus on the network in which the main purpose of the traffic lights is to avoid collisions. Section 5 offers a discussion and Section 6 concludes this paper.

2 Basic set-up: network, demand, equilibrium conditions

Per time unit N users wish to travel from a single origin (A) to a single destination (B). The drivers can choose between two alternative routes indexed by \( i \in \{1, 2\} \). Let \( f_1 \) (\( f_2 \)) be the minimum travel time from A to B via Route 1 (Route 2). Both routes are congestible, and the congestion is represented by assuming that the variable time is an increasing linear function of the number of users, \( X_i \), that travels on this route. The route’s sensitivity to congestion is denoted by \( a_i \).

![Figure 1: A network without intersection](image)

In this paper, I distinguish between two different networks. In the first network (Figure 1) the capacity of road CB is such that the merge of Route 1 and Route 2 drivers can occur without hindrance. In that case, no additional cost is incurred at point C and the average cost when using Route \( i \) equals:

\[
AC_i = f_i + a_i X_i
\]  

(1)
In the second network (Figure 2) the intersection is regulated by traffic lights. In that case, the two routes can not have simultaneous right of way, and the drivers on both routes will thus experience an expected traffic light waiting time cost \((T_1(c, r), T_2(c, r))\).

![Figure 2: A network with an intersection](image)

The private cost of a user who takes Route \(i\) equals the sum of the fixed and variable time cost and the expected waiting time cost at the traffic light \((T_i(r, c))\):

\[
AC_i = f_i + a_iX_i + T_i(r, c) \quad (2)
\]

In this paper, \(r_2\) is the red time for Route 2. The duration of the sum of the red and the green phase is the cycle time ‘\(c\)’, which, to simplify matters, is held fixed. Hence, it follows that including intergreen time in the analysis is not relevant and will thus be ignored. The red time for Route 1 will be \((c - r_2)\).

It is clear that the expected traffic light waiting cost functions are increasing in the red time and decreasing in the green time \((\frac{\partial T_1(c, r_2)}{\partial r_2} < 0, \frac{\partial T_2(c, r_2)}{\partial r_2} > 0)\). When it is always red \((r_2 = c\) for Route 2 and \(r_2 = 0\) for Route 1) the expected traffic light waiting cost is infinitely high. For simplicity, I will assume that the queue that builds up during the red time dissipates at an infinitely high speed during the green time. In that case the expected traffic light functions take the following form for \(0 < r_2 < c\):

\[
T_1(c, r_2) = \frac{(c - r_2)^2}{2c} \quad (3)
\]

\[
T_2(c, r_2) = \frac{r_2^2}{2c} \quad (4)
\]
All drivers are identical and try to minimize their expected travel cost. The equilibrium concept used in this paper is known as the user equilibrium. It was first introduced by Bernstein and Smith (1994) and used by e.g. De Palma and Nesterov (1998). In the user equilibrium no arbitrarily small portion of drivers on a route can lower its private cost by deviating to another route that connects the same origin destination pair\(^1\). When the private cost function are continuous, the user equilibrium reduces to the Wardrop equilibrium (Wardrop (1952)). With two routes, the Wardrop equilibrium will either be all drivers on Route 1, or all drivers on Route 2, or a division of drivers over the the two routes such that the average cost on both routes is equal.

3 A network without intersection

In this section I will compare the minimal total cost when route choice is influenced by either road pricing or traffic lights for a network in which traffic lights are not strictly necessary, in the sense that there are no conflicting traffic streams.

3.1 Road pricing to influence route choice

First I will determine the total minimal cost when road pricing is applied to the network. Suppose that in the network in Figure 3 a toll is levied on Route 1. When demand is inelastic, the inefficiency caused by the socially suboptimal behaviour of drivers only manifests itself in a socially suboptimal route-choice. Therefore, a toll on only one of the two routes suffices to account for this suboptimality, even when both routes are congested.

Depending on the toll value, the Wardrop equilibrium\(^2\) will either be \(X_1 = N\), \(X_2 = N\) or \(0 < X_1 < N\) such that the average cost is equal on both routes. The total cost for this network equals the sum of the average cost of all Route 1 and Route 2-drivers minus the toll revenue, i.e. \((a_1X_1+f_1+\tau)X_1+(a_2X_2+f_2)X_2-\tau X_1\). If in the equilibrium all drivers take Route 1, the total cost thus equals \((f_1 + a_1 N) N\), if all drivers take Route 2, the total cost is \((f_2 + a_2 N) N\), and when all drivers divide themselves over

\(^1\)See Section 4.2 for a formal definition.
\(^2\)Both equation (1) and (2) are continuous and so the equilibrium distribution of vehicles will be a Wardrop equilibrium.
Figure 3: Road pricing to influence route choice

the two routes, the total cost equals

\[
(f_1 + \frac{a_1(a_2N + f_2 - f_1)}{a_1 + a_2} + \frac{a_2\tau}{a_1 + a_2}) N - \tau X_1^e
\]

(Appendix A).

3.2 Traffic lights to influence route choice

I will compare these results with the total cost when, instead of road pricing, a traffic light is installed on Route 1 to influence route choice (Figure 4). In Appendix B the total cost for the different user equilibria is derived (Table 1).

Figure 4: A traffic light to influence route choice

If the user equilibrium is such that all drivers take Route 1, then the total cost when a traffic light is installed, is equal to the total cost when road pricing is applied. If in the user equilibrium all drivers take Route 2, then the total cost is also equal in both cases. If in the user equilibrium...
the drivers divide themselves over the two routes, then the total cost when road pricing is applied is as least as low as when a traffic light is installed. Indeed, for every \( f_1 = T_1 \), \( \left( f_1 + \frac{a_1(a_2N+f_2-f_1)}{a_1+a_2} + \frac{a_2T_1}{a_1+a_2} \right) N \) is always larger than or equal to \( \left( f_1 + \frac{a_1(a_2N+f_2-f_1)}{a_1+a_2} + \frac{a_2T_1}{a_1+a_2} \right) N - \tau X_1^e \) because \( X_1^e > 0 \) and \( \tau \geq 0 \).

Given the above assertions for the different user equilibria and the observation that both instruments are equally flexible in directing the equilibrium route choice, it is clear that road pricing performs better than the traffic light. This result is intuitive. While influencing route choice, road pricing does not affect the total cost. A traffic light can affect the route choice to the same extent, but this comes at a cost, namely the increase of the total cost.

Even though road pricing performs better than traffic lights, the use of traffic lights to influence route choice can still be preferable for specific cases. Take the case in which the lowest cost occurs when all drivers take Route 2. A well chosen toll can force all drivers to use Route 2. The installation of a well implemented traffic light on Route 1 to, for example, allow pedestrians to cross the street more safely, has the same effect as the toll levy. This second measure will nevertheless be accepted more easily.

4 A network with intersection

In this section, I focus on a network in which a traffic light is essential for reasons of traffic safety. For this network, I first determine the minimal cost that can be attained when road pricing is applied. Subsequently, I show that when road pricing can not be levied and only fixed signal settings can be implemented the performance of the network will be equally as good or worse. Thereupon I determine the conditions the signal settings have to satisfy to obtain the same results as road pricing. This is illustrated with a
4.1 The optimal solution when road pricing is applied

To determine the optimal solution of the network in Figure 5 on which road pricing can be applied, I will model this problem as a Stackelberg game (Von Stackelberg (1934)) in which the traffic authority is the leader, and the drivers are the follower.

Figure 5: Road pricing applied to a network with an intersection

The traffic authority who controls the signal settings and toll, can either give always green to Route 1, or give always green to Route 2 or implement an alternating signal setting. The toll is levied on Route 1, and can be positive or negative. In deciding upon his optimal control (optimal in the sense that it minimizes total cost), the traffic authority will take into account the reaction of the drivers. The drivers can react to a certain signal and toll-combination by all together taking Route 1, or they can all take Route 2, or they can divide themselves over the two routes in equilibrium.
To determine the optimal solution, I will use backward induction. If $r_2 = 0$, the Wardrop equilibrium is $X_2 = N$. As a consequence, all combinations of variables for which $r_2 = 0$ and $0 < X_1 \leq N$ are not part of the feasible set. Following the same reasoning, if $r = c$ the only feasible flow variable is $X_1 = N$. As road pricing does not affect the objective function, the value of the objective function will be the same for solutions $(r_2 = c, X_1 = N, \tau > 0)$ and $(r_2 = c, X_1 = N, \tau = 0)$ and for solutions $(r_2 = 0, X_2 = N, \tau > 0)$ and $(r_2 = 0, X_2 = N, \tau = 0)$. The implementation of road pricing brings along costs$^3$. Therefore, I only retain candidate solutions $(r_2 = 0, X_2 = N, \tau = 0)$ and $(r_2 = c, X_1 = N, \tau = 0)$.

For $0 < r_2 < c$, the objective function will only be lower than the objective function of the two previous candidate solutions if $0 < X_1 < N^4$.

$^3$For simplicity, this has not been included in the objective function

$^4$Indeed, for a signal setting $r \in ]0, c[$, $(f_1 + a_1 N + T_1 (c, r_2)) N$ is always larger than $(f_1 + a_1 N) N$ and $(f_2 + a_2 N + T_2 (c, r_2)) N$ is always larger than $(f_2 + a_2 N) N$. 

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As a consequence, the third candidate solution will be the solution of the following minimization problem conditional on this solution being the user equilibrium. In this optimization problem $a_i$ represents the sensitivity to congestion of route $i$, $X_i$ equals the flow on route $i$, $f_i$ stands for the minimal time and resource cost of route $i$, $\tau$ is the value of the toll levied on Route 1 and $T_i(c, r)$ is the expected waiting time cost on route $i$ at the intersection.

$$\min_{r, X_1, X_2, \tau} \left( f_1 + a_1 X_1 + T_1(c, r) + \tau \right) X_1 + \left( f_2 + a_2 X_2 + T_2(c, r) \right) X_2 - \tau X_1$$

(5)

s.t.

$$f_1 + a_1 X_1 + T_1(c, r) + \tau = f_2 + a_2 X_2 + T_2(c, r)$$

(6)

$$X_1 + X_2 = N$$

(7)

$$X_1 > 0$$

(8)

$$X_2 > 0$$

(9)

$$r_2 > 0$$

(10)

$$r_2 < c$$

(11)

For simplicity I will assume that the parameter values are as such that the total cost function is convex ($2(a_1 + a_2)N - c \geq 0$). Then the solution of this minimization problem is the global minimum. Furthermore, I will assume that the parameter values are as such that $0 < r^*_2 < c$ and $0 < X^*_1 < N$.

Solving the first order conditions of the Lagrangian function associated with this optimization problem, yields the optimal flow pattern, toll value and signal setting (Appendix C). The optimal distribution of vehicles ($X^{int}$) can be found where the marginal cost curves of both routes intersect:

$$f_1 + 2a_1 X_1 + T_1(c, r) = f_2 + 2a_2 X_2 + T_2(c, r)$$

(12)

The optimal toll ($\tau_{int}$) equals the difference between the marginal external congestion cost of Route 1 and the marginal external congestion cost of Route 2.

$$\tau = a_1 X_1 - a_2 X_2$$

(13)

The optimal signal setting ($r^{int}_2$) is determined by the following equation:

$$\frac{dT_1(c, r)}{dr_2} X_1 = -\frac{dT_2(c, r)}{dr_2} X_2$$

(14)
It can be shown that the total cost associated with this interior solution \( TC^{int} \) can be lower than \( (a_1 N + f_1) N \) and \( (a_2 N + f_2) \). Furthermore, it can be shown that if the parameter values are as such that the interior solution has the lowest cost, then both routes are used in equilibrium when the signal setting is \( r_2^{int} \) and the toll value equals \( r^{int} \). As a consequence, the interior solution is a candidate solution.

Depending on the parameter values, the desired choice for the traffic authority will thus be one of the three following combinations: either Route 1 always has green, no toll is levied and all drivers take Route 1 or Route 2 receives always green, no toll is levied and all drivers take Route 2 or the optimal alternating signal setting \( (r_2^{int}) \) is implemented, the optimal toll \( (r^{int}) \) is levied and the drivers are distributed over the two routes such that \( MC_1 = MC_2 \).

Remark that the total costs associated with these solutions are the lowest attainable in this network if there are to be \( N \) drivers per time unit going from A to B. The network itself is after all only restricted by the fact that the signal settings have to be such that collisions are avoided. The lowest costs for the network are thus associated with the solutions of the following optimization problem

\[
\min_{r_2, X_1} (f_1 + a_1 X_1 + T_1 (c, r_2)) X_1 + (f_2 + a_2 (N - X_1) + T_2 (c, r_2)) (N - X_1)
\]

\[\text{s.t.}\]

\[0 \leq X_1 \leq N\]  \hspace{1cm} (16)

\[0 \leq r_2 \leq c\]  \hspace{1cm} (17)

An inspection of the corner solutions already provides two candidate solutions: \( (r_2 = c, X_1 = N) \) and \( (r_2 = 0, X_1 = 0) \). The total cost associated with these solutions is \( (a_1 N + f_1) N \) and \( (a_2 N + f_2) N \) respectively. The third candidate solution is the interior solution \( (0 < r_2 < c \text{ and } 0 < X_1 < N) \). In Appendix D it is shown that the flow pattern of the optimal interior solution \( (X_1^*) \) is determined by \( MC_1 = MC_2 \). Remark that the flow distribution when the optimal toll is levied equals the socially optimal flow pattern. This implies that, for this network, road pricing allows to completely control route choice. The optimal interior signal setting \( (r_2^*) \) is determined by

\[
\frac{dT_1(c,r_2)}{dr_2} X_1 = -\frac{dT_2(c,r_2)}{dr_2} X_2,
\]

which equals equation (14). Given that \( X_1^* \) equals \( X_1^{int} \), \( r_2^* \) is equal to \( r_2^{int} \) and the toll value does not affect the value
of the objective function, the total cost will equal $TC^{\text{int}}$.

4.2 The optimal solution when road pricing can not be applied

I showed that, for the network in Figure 5, road pricing allows to reach the socially optimal result. Is it possible to obtain the same result, for the same network, but without road pricing?

In Figure 4.2 the different options for the traffic authority and the possible reactions of the drivers are shown when the traffic authority can implement fixed signal settings. From Evers and Proost (2013), I know that, depending on the parameter values, the minimal cost for the network is $(a_2 N + f_2) N$, $(a_1 N + f_1) N$ or $(a_1 a_2 (a_1 + a_2) N + 2 a_1 (a_1 + a_2) f_2 - f_1 + a_1 a_2 c + f_1) N$. The associated optimal signal settings are respectively $r_2 = 0$, $r_2 = c$ and $r_2 = \frac{a_2 c}{a_1 + a_2}$.\footnote{That is, flow-independent.}

When the parameter values are such that $(a_1 N + f_1) N$ is the lowest cost attainable with road pricing, then the simple implementation of $r_2 = c$ will thus do the job. When $(a_2 N + f_2) N$ is the minimum, then the signal setting $r_2 = 0$ provides this solution. When $TC^{\text{int}}$ is the minimal cost, then the strategy to obtain this cost is not so straightforward. Indeed, if the traffic authority implements $r_2 = 0$ or $r_2 = c$, then the total cost equals $(a_2 N + f_2) N$ respectively $(a_1 N + f_1) N$. These cost are by assumption larger than $TC^{\text{int}}$. The minimal cost when the traffic authority implements the optimal alternating signal setting $r_2 = \frac{a_2 c}{a_1 + a_2}$ is also at least as large as $TC^{\text{int}}$. This can be seen by realizing that $TC^{\text{int}}$ is the solution of an unconstrained optimization problem. The minimal cost associated with $r_2 = \frac{a_2 c}{a_1 + a_2}$ is the solution of the same optimization problem, but constrained by the equilibrium reaction of the driver.

This observation indicates that the restricting aspect in obtaining the lowest cost is the route choice behaviour of the drivers. If the traffic authority could exert more control over the route choice behaviour of the drivers, the lowest cost, would be attainable. The inverse Stackelberg game approach allows to exert more control over the choices of the follower.

\footnote{Remark that the optimal alternating signal setting $r_2 = \frac{a_2 c}{a_1 + a_2}$ only equals $r^*_2$ for those parameter values for which the flow that solves $AC_1 = AC_2$ also solves $MC_1 = MC_2$.}
The inverse Stackelberg game is an extension of the basic Stackelberg game. In the basic Stackelberg game, the leader chooses an action after which the follower determines his optimal response. In the inverse Stackelberg game, the leader action is generalized from making a direct decision to determining a function that maps the followers’ decision space into the leader’s decision space. Here this means that rather than implementing a fixed signal setting, the signal setting now has to be a function of the flow pattern. In this way, the traffic authority can manipulate the signal setting so as to enforce the user equilibrium to coincide with the system optimum. This approach can be seen as a combination of anticipatory and responsive signal control. The traffic authority reacts to the specific distribution of vehicles over the two routes, but the reaction is such that a previously determined goal is achieved.

The traffic authority needs to implement a function that maps all the
possible distributions over the two routes into a signal setting such that the user equilibrium coincides with the socially optimal distribution of vehicles over the two routes. The constraints this function has to satisfy will thus depend on the properties of the user equilibrium. Therefore, I will first give a formal definition of the user equilibrium.

**Definition 1.** A feasible flow pattern \( f^* \) is a user equilibrium if for any OD-pair and all routes connecting this OD-pair

\[
AC_r(f^*) \leq \lim_{\epsilon \to 0} \inf \{ AC_s(f^* + \alpha 1_r - \alpha 1_s) : 0 < \alpha < \min(\epsilon, f_s) \}
\]

with \( s \) any route connecting the same OD pair as route \( r \). In this definition \( 1_i \) denotes the vector with a '1' in position \( i \) and a '0' elsewhere.

To obtain the social optimum, the traffic authority will implement the socially optimal signal setting \((r_2^*)\) when the flow pattern is socially optimal. So, for the socially optimal flow pattern to be a user equilibrium the following constraints have to be satisfied.

\[
AC_1(X_1^*, r_2^*) \leq \lim_{\epsilon \to 0} \inf \{ AC_2(X_2^* + \alpha 1_r) : 0 < \alpha < \min(\epsilon, f_2^*) \}
\]

\[
AC_2(X_2^*, r_2^*) \leq \lim_{\epsilon \to 0} \inf \{ AC_1(X_1^* + \alpha 1_r) : 0 < \alpha < \min(\epsilon, f_1^*) \}
\]

It is clear that \( AC_1(X_1^*, r_2^*) \) and \( AC_2(X_2^*, r_2^*) \) are generally not equal. Equation (19) and equation (20) thus indicate that the function that maps the flow distribution in a signal setting has to be discontinuous.

When the flow is not in equilibrium due to e.g. a shock, the drivers will change routes to minimize their private cost. This process has to converge to the equilibrium flow. As a consequence, the function also has to satisfy the following constraints:

\[
AC_1(X_1) < AC_2 \quad \forall X_1 < X_1^*
\]

\[
AC_1(X_1) > AC_2 \quad \forall X_1 > X_1^*
\]

Many functions can satisfy these constraints, so to keep things as simple as possible I propose a fixed signal setting \( r_2^h \) for all \( X_1 > X_1^* \) and a fixed signal setting \( r_2^l \) for all \( X_1 < X_1^* \), which I will fully determine.

I will focus first on the area right of the social optimum. With fixed signal settings, undersaturated traffic conditions and linear congestion \( AC_1 \)
Figure 6: The average cost in the optimum is different on both routes.

is linearly increasing in $X_1$. So any $r^h$ satisfying $AC_2(X^*_2, r^h_2) \leq AC_1(X^*_1, r^h_2)$ will also satisfy equation (20). With $AC_1$ linearly increasing in $X_1$ and $AC_2$ linearly decreasing in $X_2$, any $r^h$ satisfying $AC_1(X^*_1, r^h_2) > AC_2(X^*_2, r^h_2)$ will also satisfy equation (24). The constraints that thus determine a feasible $r^h_2$ can be written as follows:

$$f_1 + a_1 X^*_1 + \frac{(c - r^h_2)^2}{2c} \geq a_2 X^*_2 + f_2 + \frac{(r^*_2)^2}{2c}$$  \hspace{1cm} (23)

$$f_1 + a_1 X^*_1 + \frac{(c - r^h_2)^2}{2c} > a_2 X^*_2 + f_2 + \frac{(r^*_2)^2}{2c}$$  \hspace{1cm} (24)

Equation (23) reduces further to:

$$r^h_2 \leq c - \sqrt{\left( f_2 - f_1 + (a_2 - a_1) X^*_1 + a_2 N + \frac{(r^*_2)^2}{2c} \right) \frac{2c}{c}}$$  \hspace{1cm} (25)

and equation (24) reduces to:

$$r^h_2 < f_1 - f_2 - (a_2 - a_1) X^*_1 - a_2 N + \frac{c}{2}$$  \hspace{1cm} (26)
Remark that if $AC_2(X_2^*, r_2^*) > AC_1(X_1^*, r_2^*)$, then equation (25) implies equation (26) for all $X_1 > X_1^*$.

Indeed, if there exists a $r_2^h$ for which $AC_1(X_1^*, r_2^h) \geq AC_2(X_2^*, r_2^h)$ and $AC_2(X_2^*, r_2^h) > AC_1(X_1^*, r_2^h)$, then $AC_1(X_1^*, r_2^h) > AC_1(X_2^*, r_2^h)$, which implies that $r_2^h < r^*$. As a consequence, $AC_2(X_1^*, r_2^h) < AC_2(X_2^*, r_2^*)$, and $AC_1(X_1^*, r_2^h) > AC_2(X_1^*, r_2^*)$.

On the other hand, if $AC_2(X_2^*, r_2^*) < AC_1(X_1^*, r_2^*)$, then equation (26) implies equation (25).

From Figure 7 is clear that for $r_2^*$ equation (23) and equation (24) are satisfied. If $r_2^h$ increases, then the average cost curve of Route 1 shifts down ($AC_1'$ in Figure 7) and the average cost curve of Route 2 shifts up ($AC_2'$ in Figure 7). Figure 7 shows that there exists a $r_2^h$ for which equation (24) is violated while equation (23) is still satisfied. For $r_2^h < r_2^*$, both equations (23) and (24) are always satisfied.

![Figure 7](image)

**Figure 7:** If $AC_2^* < AC_1^*$, then equation (26) implies equation (25).

Next, I will analyse the area left of the social optimum. Also here, there are two constraints that have to be satisfied.

With fixed signal settings $AC_2$ is linearly decreasing in $X_1$ and equation
(19) is always satisfied when $AC_1(X^*_1, r^*_2) \leq AC_2(X^*_2, r^*_2)$ is satisfied. Equation (21) always holds for any $r^*_2$ that satisfies $AC_1(X^*_1, r^*_2) < AC_2(X^*_2, r^*_2)$. A feasible $r^*_2$ will thus satisfy the following constraints:

$$a_2X^*_2 + f_2 + \frac{(r^*_2)^2}{2c} \geq f_1 + a_1X^*_1 + \frac{(c - r^*_2)^2}{2c}$$

(27)

$$a_2X^*_2 + f_2 + \frac{(r^*_2)^2}{2c} > f_1 + a_1X^*_1 + \frac{(c - r^*_2)^2}{2c}$$

(28)

Equation (27) reduces to

$$r^*_2 \geq \sqrt{\left(f_1 - f_2 + (a_1 + a_2)X^*_1 - a_2N + \frac{(c - r^*_2)^2}{2c}\right) \frac{2c}{2c}}$$

(29)

and equation (28) reduces to

$$r^*_2 > f_1 - f_2 + \frac{(r^*_2)^2}{2c} + (a_1 + a_2)X^*_1 - a_2N + \frac{c}{2}$$

(30)

If $AC_1(X^*_1, r^*_2) > AC_2(X^*_2, r^*_2)$, then equation (29) implies equation (30) and if $AC_2(X^*_2, r^*_2) > AC_1(X^*_1, r^*_2)$, then equation (30) implies equation (29).

One might wonder why I did not use the inverse Stackelberg approach in Section 4.1. Remark however, that I showed that the results obtained with road pricing were equal to the socially optimal solution, which could not be improved by a flow-dependent signal setting, because the socially optimal solution is the result of the optimization problem in which the signal setting and the flow can be chosen freely.

### 4.3 A numerical example

Suppose there are 30000 drivers per time unit (N) that want to go from A to B. The minimal time and resource cost equals 22 euro for Route 1 ($f_1$), and 3 euro for Route 2 ($f_2$). The parameters $a_1$ and $a_2$ are 0.003 and 0.0005 euro time units per driver respectively. The total duration of red and green time (c) equals 50 time units.

For these parameter values, the total cost when the traffic authority implements the optimal alternating signal setting and the optimal toll is lower than the total cost when the traffic authority implements either $r_2 = 0$ or $r_2 = c$ (see Table 2). The traffic authority will thus implement a signal setting that varies with the flow distribution to obtain this cost without having
Table 2: When road pricing is applied, the optimal result is obtained when both routes are used in the equilibrium.

<table>
<thead>
<tr>
<th>Route configuration</th>
<th>Total cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>((r_2 = 0; \tau = 0))</td>
<td>1,110,000</td>
</tr>
<tr>
<td>((r_2 = 50; \tau = 0))</td>
<td>2,790,000</td>
</tr>
<tr>
<td>((r_2^{int} = 7,5; \tau^{int} = 0,75))</td>
<td>1,056,000</td>
</tr>
</tbody>
</table>

The optimal red time for Route 2 equals 7.5 time units. The optimal flow on Route 1 is 4500 vehicles per time unit, and thus 25500 vehicles per time unit on Route 2. For this optimal combination of red time and flow distribution, \(AC_2(25500; 7.5)\) is larger than \(AC_1(4500; 7.5)\). Consequently, the constraint that determines \(r_2^h\) for all \(X_1 > X_1^*\) is given by equation (25). For the given parameter values, this equation reduces to:

\[
0 < r_2^h \leq 1.72 \tag{31}
\]

The constraint that determines \(r_2^l\) for all \(X_1 < X_1^*\) is given by equation (30). Here, this equation reduces to:

\[
c > r_2^l > 6.75 \tag{32}
\]

The leader function \(f(X_1)\) can then be determined as follows:

\[
f(X_1) = \begin{cases} 
10 & \text{if } X_1 < 4500 \\
7.5 & \text{if } X_1 = 4500 \\
1 & \text{if } X_1 > 4500 
\end{cases}
\]

Remark that even though for all the flow patterns left of the equilibrium \((X_1 < X_1^*)\) there are not enough Route 1 drivers compared to the amount of Route 1 drivers in the equilibrium, equation (32) does not specify that the red time for Route 1 has to decrease for all \(X_1 < X_1^*\) (compared to the red time for Route 1 in the equilibrium). A deviation to Route 2 from the equilibrium is thus not necessarily punished by an increase in the red time for Route 2. This is because the other cost components are such that even for \(7.5 \geq r_2^l > 6.75\) equation (19) and equation (21) are still satisfied.

The different total costs in Table 3 show that \(f(X_1)\) indeed allows to obtain the same result as road pricing. The optimal flow-independent alternating signal setting, by contrast, has a higher total cost.
\[
\begin{array}{ccc}
\frac{r_2}{r_1} = \frac{\alpha_2 c}{\alpha_1 + \alpha_2} & f(X_1) & (r_2^{int} = 7.5; \tau^{int} = 0, 75) \\
\hline
\text{Total cost} & 1,056,120 & 1,056,000 \\
& 1,056,000 & 1,056,000 \\
\end{array}
\]

Table 3: Contrary to the optimal fixed signal setting, the flow dependent signal setting allows to obtain the same result as road pricing.

Figure 8: Shape of the average cost functions for a feasible flow-dependent signal setting

With \(f(X_1)\) the flow dependent signal setting, the average cost curve of Route 2 is given by:

\[
AC_2(X_2) = \begin{cases} 
23 + 0.0005X_2 & \text{if } X_1 < 4500 \\
35.31 & \text{if } X_1 = 4500 \\
22.01 + 0.0005X_2 & \text{if } X_1 > 4500 
\end{cases}
\]

The average cost curve of Route 1 equals:

\[
AC_1(X_1) = \begin{cases} 
19 + 0.003X_1 & \text{if } X_1 < 4500 \\
34.56 & \text{if } X_1 = 4500 \\
27.01 + 0.003X_1 & \text{if } X_1 > 4500 
\end{cases}
\]

Remark that for all \(X_1 > 4500\) \(AC_1\) is always greater than 35.31 and \(AC_1\) is always greater than \(AC_2\). Similarly, for all \(X_1 < 4500\), \(AC_2\) is always
greater than 34.56 and $AC_2$ is always greater than $AC_1$.

5 Discussion

In this paper, I have investigated the potential of traffic lights in providing an alternative to the politically sensitive and therefore hard to implement road pricing. In the previous section I have shown that for a network in which traffic lights regulate the intersection of two routes connecting one OD-pair, a flow dependent signal setting can provide the same results as road pricing. However, for traffic lights to provide a feasible alternative, it does not only have to provide the same results, but it also has to be socially acceptable.

There is a clear difference in perception between road pricing and traffic lights. Road pricing is generally perceived as a negative measure. When road pricing is introduced, people tend to focus on the fact that today they have to pay for something which yesterday was free of charge. Hereby either ignoring that road pricing is a neutral operation\(^7\) or distrusting the efficiency of the authorities.

Traffic lights, on the other hand, are generally perceived as a valid government intervention. When they regulate an intersection, their benefit, in terms of safety, is clearly visible. And even when there is no clear safety benefit (e.g. ramp metering), traffic lights are still easily accepted.

Another argument in favor of traffic lights to influence route choice regards the visibility of both measures. Suppose that the minimal costs are obtained when both routes are used in the equilibrium. If the difference between the average costs in the optimal solution and the average cost when the optimal fixed signal setting is implemented\(^8\) is small, then the drivers will probably not notice that the signal settings have been adapted. However, if road pricing is implemented to obtain the same result, it will immediately be noticed.

\(^7\)The toll revenues can be used to provide public goods and services.

\(^8\)I assume that if the flow dependent signal setting is not implemented the traffic light is regulated by anticipatory control.
6 Conclusion

In this paper, I showed that for a network with two parallel routes the installation of traffic lights does not allow to reach the same result as road pricing when in the optimum both routes are used.

For a different network in which traffic lights regulate the intersection of two routes connecting one OD pair, I showed that the minimal cost that can be obtained by a combination of road pricing and fixed signal settings, can only be obtained by fixed signal settings alone if in the optimum only one route is used. For the case in which in the optimum both routes are used, I derived the conditions a signal setting has to satisfy to be able to obtain the same result. With a numerical example, this paper showed that it is possible to find a flow-dependent signal setting that can render road pricing redundant.

A possible extension to this paper would then deal with the question if for all possible specifications of the average cost functions there exists a flow-dependent signal setting that allows to reach the same result as road pricing.
A Computation of the total cost for a network without intersection with road pricing

Figure 9: Road pricing to influence route choice

Suppose that in the network in Figure 9 a toll is levied on Route 1. Then all drivers will take Route 1 in equilibrium if \( f_2 > a_1 N + f_1 + \tau \). If however, \( f_1 + \tau > a_2 N + f_2 \) then all drivers will take Route 2 in equilibrium. Finally, if \( f_2 \leq a_1 N + f_1 + \tau \) and \( f_1 + \tau \leq a_2 N + f_2 \), then the drivers will use both routes in equilibrium. When both routes are used in equilibrium, then the average cost of the two routes has to be equal.

\[
f_1 + a_1 X_1 + \tau = a_2 X_2 + f_2 \tag{33}
\]

Substituting this constraint in the total cost function, and using \( X_1 + X_2 = N \), the total cost takes the following form:

\[
\left( f_1 + \frac{a_1(a_2 N + f_2 - f_1)}{a_1 + a_2} + \frac{a_2 \tau}{a_1 + a_2} \right) N - \tau \left( \frac{a_2 N + f_2 - f_1 - \tau}{a_1 + a_2} \right) \tag{34}
\]
Computation of the total cost for a network without intersection with traffic lights

Figure 10: A traffic light to influence route choice

Consider the network in Figure 10. All drivers will take Route 1 in the equilibrium if \( f_2 > a_1 N + f_1 + T_1 \). If, however, \( f_1 + T_1 > a_2 N + f_2 \) then all drivers will take Route 2 in the equilibrium. Finally, if \( f_2 \leq a_1 N + f_1 + T_1 \) and \( f_1 + T_1 \leq a_2 N + f_2 \), then the drivers will use both routes in the equilibrium. If all drivers take Route 1, then the total cost is the sum of the minimal time cost, the congestion cost and the traffic light waiting cost for every driver. In this case, the total cost can be minimized by setting the traffic light always to green. The total cost then equals \((a_1 N + f_1)N\). On Route 2 there is no traffic light, so the average cost will be the sum of the minimal time cost and the congestion cost. If all drivers take Route 2 in equilibrium, the total cost will thus be \((f_2 + a_2 N)N\). When both routes are used in equilibrium, the average cost of both routes has to be equal:

\[
f_1 + a_1 X_1 + T_1 = a_2 X_2 + f_2
\]  

(35)

Substituting this equation and \( X_1 + X_2 = N \) in \((f_1 + a_1 X_1 + T_1(c,r)) X_1 + (f_2 + a_2 X_2) X_2\), I obtain:

\[
\left( f_1 + \frac{a_1(a_2 N + f_2 - f_1)}{a_1 + a_2} + \frac{a_2 T_1}{a_1 + a_2} \right) N
\]  

(36)
C Optimization of the network with road pricing and traffic lights

\[
\min_{r, X_1, X_2, \tau} (f_1 + a_1 X_1 + T_1 (c, r_2) + \tau) X_1 + (f_2 + a_2 X_2 + T_2 (c, r_2)) X_2 - \tau X_1
\]

(37)

s.t.

\[
f_1 + a_1 X_1 + T_1 (c, r_2) + \tau = f_2 + a_2 X_2 + T_2 (c, r_2)
\]

(38)

\[
X_1 + X_2 = N
\]

(39)

\[
X_1 > 0
\]

(40)

\[
X_2 > 0
\]

(41)

\[
r_2 > 0
\]

(42)

\[
r_2 < c
\]

(43)

The associated Langragian is given by:

\[
L = (f_1 + a_1 X_1 + T_1 + \tau) X_1 + (f_2 + a_2 X_2 + T_2) X_2 - \tau X_1 + \lambda_1 (N - X_1 - X_2) + \lambda_2 (f_2 - f_1 - a_1 X_1 - T_1 + T_2) + \lambda_3 X_1
\]

(44)

The FOC are the following:

\[
\frac{dL}{dX_1} = a_1 X_1 + f_1 + a_1 X_1 + T_1 (c, r_2) - \lambda_1 - \lambda_2 a_1 = 0
\]

(45)

\[
\frac{dL}{dX_2} = a_2 X_2 + f_2 + a_2 X_2 + T_2 (c, r_2) - \lambda_1 - \lambda_2 a_2 = 0
\]

(46)

\[
\frac{dL}{d\lambda_1} = N - X_1 - X_2 = 0
\]

(47)

\[
\frac{dL}{d\lambda_2} = f_2 + a_2 X_2 + T_2 (c, r) - f_1 - a_1 X_1 - T_1 (c, r) - \tau = 0
\]

(48)

\[
\frac{dL}{d\tau} = -\lambda_2 = 0
\]

(49)

\[
\frac{dL}{dr_2} = \frac{dT_1 (c, r_2)}{dr_2} X_1 + \frac{dT_2 (c, r_2)}{dr_2} X_2 + \lambda_2 \left( \frac{dT_2 (c, r_2)}{dr_2} - \frac{dT_1 (c, r_2)}{dr_2} \right) = 0
\]

(50)

24
Combining $\lambda_2 = 0$ and the first order conditions for $X_1$ and $X_2$, I see that in the optimum $MC_1 = MC_2$.

If I combine $\lambda_2 = 0$, the first order conditions for $X_1$ and $X_2$ and the first order condition for $\lambda_2$, I obtain the toll:

$$\tau = a_1 X_1 - a_2 X_2$$  \hspace{1cm} (51)

Finally, if I combine $\lambda_2 = 0$ and the first order condition for $r_2$, I find that the optimal signal setting is determined by the following equation:

$$\frac{dT_1 (c, r_2)}{dr_2} X_1 = - \frac{dT_2 (c, r_2)}{dr_2} X_2$$  \hspace{1cm} (52)

The optimal variables are then:

$$\tau = \frac{(f_2 - f_1 - \frac{c}{2} ) (a_1 + a_2) N + a_2 N c}{2 (a_1 + a_2) N - c}$$  \hspace{1cm} (53)

$$r = \frac{(f_2 - f_1 - \frac{c}{2} + 2a_2 N) c}{2 (a_1 + a_2) N - c}$$  \hspace{1cm} (54)

$$X_1^e = \frac{(f_2 - f_1 - \frac{c}{2} + 2a_2 N) N}{2 (a_1 + a_2) N - c}$$  \hspace{1cm} (55)

And the minimal cost equals:

$$TC^{int} = \frac{N (c^2 + 4c(f_1 - f_2) - 4(4a_2 N f_1 - (f_1 - f_2)^2 + 4a_1 N (a_2 N + f_2))}{8(c - 2(a_1 + a_2)N)}$$  \hspace{1cm} (56)
D Unconstrained optimization of the network with traffic lights

\[
\begin{align*}
\min_{r_2, X_1} & \quad (f_1 + a_1 X_1 + T_1 (c, r_2)) X_1 + (f_2 + a_2 (N - X_1) + T_2 (c, r_2)) (N - X_1) \\
\text{s.t.} & \quad 0 < X_1 < N \\
& \quad 0 < r_2 < c
\end{align*}
\]

The FOC are given by:

\[
\frac{dTC}{dX_1} = 2a_1 X_1 + \phi_1 + T_1 (c, r_2) - 2a_1 (N - X_1) - \phi_2 - T_2 (c, r_2) = 0 \tag{60}
\]

\[
\frac{dTC}{dr} = -\frac{c - r}{c} X_1 + \frac{r}{c} (N - X_1) = 0 \tag{61}
\]

Remark that equation (60) equals:

\[
MC_1 - MC_2 = 0 \tag{62}
\]

The optimal variables are given by:

\[
r_2 = \frac{(f_2 - f_1 - \frac{c}{2} + 2a_2 N) c}{2 (a_1 + a_2) N - c} \tag{63}
\]

\[
X_1 = \frac{(f_2 - f_1 - \frac{c}{2} + 2a_2 N) N}{2 (a_1 + a_2) N - c} \tag{64}
\]
References


Wardrop, J. (1952). Some theoretical aspects of road traffic research. In *Proceedings of Institution of Civil Engineers*, number 0.