Investment decisions in situation of risk

Bernard Lapeyre,* Emile Quinet†

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Abstract

This text links two research fields. The first one is cost-benefit analysis (CBA) of investment projects. CBA has received a lot of attention in the economics literature, from both the conceptual and practical standpoints. Many scientific books and articles, as well as practical guidelines, have been devoted to this subject. Similarly, almost all countries have issued guidelines on how to implement cost–benefit analysis. However, while these texts pay great attention to how to estimate yearly benefits, to take into account consumers’ surplus or externalities, they do not insist much on how these yearly benefits can be used to derive prioritisation rules. The basic criterium is the Net Present Value; but little is said on how to use NPV to choose among several alternatives or to decide when to implement a project.

Another field of research is the inclusion of risk in economic calculus. This very vivid field has put a lot of light on taking into account the systemic risk, according to which the surpluses generated by the project are correlated with the economic growth. Important insights have been given to the way of reckoning the average expectation of NPV of a project, and one of the basic results is that discounting should be made according to the so-called extended Ramsey rule, the extensions of it lead to many debates and progress. But this field of research has almost not addressed the above-mentioned practical issue of the rules of decision for a project.

The communication aims at filling this gap, and trying to assess decision rules about when, and whether, to implement a project in presence of systematic risk.

Using methods derived of real options theory, we find that, in the case of Brownian stochastic processes, the decision rule can be expressed as a threshold value of the First Year Advantage/Cost ratio, and that this threshold value is depending on each of the parameters appearing in the relations defining those Brownian processes. This threshold can be expressed in a closed form including the means, standard deviations and correlations of the stochastic variables. Simulations with sensible current values of these parameters show that the most important ones are the three standard values of the processes, and especially the parameter of the construction cost of the project. Some extensions are explored. Other are suggested for further research.

JEL : C61, D61, D81, D92, G12, H54.

*Cermics, Ecole des Ponts ParisTech
†Paris Scool of Economics, Ecole des ponts ParisTech
This text aims at providing a stone in the field of investment decision rules under risk. Its specific input is to build a bridge between two streams up to now without much contact: CBA and its traditional criteria which pay little attention to risk, especially systematic risk, and modern theory of discount rate which focuses on systemic risk in its recent developments but has up to now no large input in CBA practice.

CBA mainly aims at estimating monetary values of the effects of an investment. In its major developments, and even more in its implementation directives, the bulk of the effort is directed towards how to reckon the benefits but there has been little attention to the indicators able to hierarchize the projects, and even less on how to take into consideration situations of risk and uncertainty, though, everybody agrees, are of paramount importance.

On the other side, risk is having an increasing consideration in modern theories of discounting. But it is striking that the main results deal on how to compare effects taking place at various instants of time, and how to estimate correctly a Net Present Value (NPV) in presence of risk, but not to set rules for decision making.

This text should be a first step in the direction of putting together these up to now rather separate fields.

A first section briefly recalls the condition in which CBA is implemented and stresses on the poor role of hierarchization rules in general and risk in these rules. The second section reminds the main basic lines of discounting in presence of uncertainty. The third one shows how it is possible to fill this gap and presents in a simple case a closed formula giving the decision rule for the implementation of an investment. The fourth one presents simulations drawn from the previous formula, using sensible parameters in the range of what is encountered in current transport infrastructure projects, and shows that some up to now not noticed play an important role. The fifth one concludes by presenting possible further research directions.

CBA practice misregards uncertainty  

CBA has received a lot of attention in the economics literature, from both the conceptual and practical standpoints. Many scientific books and articles, as well as practical guidelines, have been devoted to this subject. Some books deal exclusively with this topic, and all textbooks devoted to transport economics - the sector that will be taken as an example in this paper - have a chapter on it. Similarly, almost all countries, at least in Europe, have issued guidelines on how to implement cost-benefit analysis (CBA). However, while these texts pay great attention to how to estimate yearly benefits, to take into account consumers’ surplus or externalities, they do not insist much on how these yearly benefits can be used to derive prioritisation rules. On the theoretical grounds, the most evoked criteria are:

- the net present value (NPV):

\[ NPV = \sum_{t=t_0}^{T} \frac{a(t) - c(t)}{(1 + j)^t} - \frac{I}{(1 + j)^{t_0}} + \frac{RV}{(1 + j)^T}, \]

where \( T \) is the appraisal period, \( j \) is the discount rate, \( a(t) \) and \( c(t) \) are the yearly benefits.

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1It builds on B. Lapeyre and E. Quinet ¡¡Choix des investissements et prise en compte du risque systémique¿¿. Annexe to the report ¡¡Evaluation socio-économique des investissements publics¿¿ (Quinet, 2013)

2Let us quote, without being exhaustive, de Rus (2009), Florio (2007) and Glaister and Layar (1994)

3For instance, Button (2010), Small and Verhoef (2009) and Quinet and Vikerman (2004)
and costs of the project, $I$ is the cost of building the project (with a proper discounting procedure if implementation takes several years) and $RV$ is the residual value;

- the net present value per euro spent ($NPV/E$);
- in the case of public-private participation, a variant of $NPV/E$ is the net present value per public euro spent ($NPV/PuE$);
- the internal rate of return ($IRR$), which is the discount rate that makes the $NPV$ equal to zero;
- the first-year benefit-cost ratio ($FYBCR$):

$$FYBCR = \frac{a(t_0) - c(t_0)}{I}.$$ 

But literature says very few things on how to use them. The more classical theoretical results are that the reference criterion is the NPV. IRR is considered to have much less solid bases, and can lead to contradictions with the previous NPV criterion (see for instance Florio (2007)). It is generally acknowledged that the rule for optimizing the year of implementation of the investment is, under some mild hypotheses which will be recalled in the following section, that the $FYBCR$ is equal to the discount rate. Furthermore, a debate takes place about the use of $NPV/E$ and $NPV/PuE$: several authors consider that it is the right indicator to hierarchize several investments, especially in the case of public budget constraint, while other advocate for more complicated procedures based on linear programming.

On the ground of practical recommendations, the situation is similar: very little space is devoted to the criteria and their uses. Generally, an array of indicators are proposed, but without stating exactly the purpose of each and their virtues and drawbacks. For instance, the excellent and well-documented HEATCO report (and especially Odgaard, Kelly and Laird (2005)), which records and surveys the guidelines for CBA of European countries, cannot devote more than five pages out of 72 to the indicators, the rest of the text dealing mainly with how to calculate the yearly benefits and costs. In the PIARC report [PIARC, 2004], which surveys the procedures of road investment appraisal throughout the world, only three pages out of 85 focus on the indicators and, like the previously quoted texts, explain that these indicators are used for different purposes depending on the country, without clearly justifying the choices made by each country. The situation is the same at the level of most countries. In France, the most comprehensive national CBA guideline (Ministère de l’Equipement, 2004) devotes less than two pages out of 50 to the subject of the indicators, and furthermore without clearly saying which one is relevant to answer which question.

In the UK [Transport Analysis Guidance, 2010] which describes the overall CBA methodology, devotes two pages out of 14 to the indicators, mainly filled with describing them. The overall impression of this non-exhaustive survey seems to be that several tools have been designed, but without saying for which task and how they can be used. Furthermore, none of them pay much attention to risk and uncertainty. Apart from the well known optimism bias, which is fought through expertise methods, the most elaborated recommendations deal with scenarios where the most elaborated procedures consist of putting probabilities on scenarios.
On the theoretical side uncertainty has been addressed by several authors using dynamic optimisation methods through option values (see [Henry, 1974]) or Bellman or Pontryagin procedures (see [Pindyck, 1991]), and many specific studies have used similar procedures among which a seminal one is [Samuelson, 1965, McKean, 1965]. But they never have been implemented in practical directives; furthermore these studies almost take into account the specific risk coming from the project, and not the links between the risk of the project and the system risk.

Discount rate theory and risk  On the other side, risk and especially system risk is presently a major research theme on discounting theory. It is the basis of the present ideas on discounting the future and has been enhanced by the debate on sustainable development and the far future, for instance the Stern-Nordhaus debate. The theory has been developed (See for instance [Gollier, 2011],[Arrow et al., 2012],[Groom et al., 2007], [Barro, 2006],[Weitzman, 2012]).

Starting from the seminal work of Ramsey [Ramsey, 1928], research has developed in the recent years to take into account systematic risk. In order to lay the bases of this work, let us recall the foundations of this theory, building on [Gollier, 2011].

Let us start with a collective utility function, supposed to be separable in time and introduce a rate of impatience \( \delta \):

\[
  u(c_0, c_1, \ldots, c_j, \ldots) = u(c_0) + e^{-\delta}u(c_1) + \cdots + e^{-\delta t}u(c_t) + \ldots
\]

We will now introduce risk progressively through three steps of increasing complexity.

Let us first assume that there is no risk at all. The optimisation of the collective function implies that two marginal changes in consumptions of different times, \( a(0) \) and \( a(t) \), are equivalent:

\[
  a(0) \times u'(c_0) = a(t) \times e^{-\delta} \times u'(c_t).
\]

Introducing rates of change between the two marginal variations \( r_0 \), which is the discount rate, we have:

\[
  e^{r_0 t} = \frac{a(t)}{a(0)} = e^{-\delta} \frac{u'(c(0))}{u'(c(t))}
\]

Let us now make the simplest assumption on the growth of consumption, i.e. the rate of growth is constant over time or:

\[
  c(t) = c(0)e^{\mu t}.
\]

It is easy to see that in such conditions and under the classical assumption that \( u(c) \) is proportionnal to \( 1/c^{\gamma-1} \):

\[
  e^{r_0 t} = \frac{a(t)}{a(0)} = e^{-\delta} \frac{u'(c(0))}{u'(c(t))} = e^{\delta+\gamma \mu}.
\]

Then:

\[
  r = \delta + \gamma \mu.
\]

This is the classical Ramsey formula without risk.

Then in a second step, let us assume that the growth of consumption is at risk: the consumption level at time \( t \) is not known, just its probability distribution. In (1), the expression \( u'(t) \) is
replaced by its expectation, and the relation is transformed and becomes:

\[ r = \delta - \frac{1}{t} \log \left( \frac{E(u'(c(t)))}{u'(c(0))} \right). \]

If we make the simple but easy to manipulate assumption that the path of consumption \( c \) is given by

\[ d \log (c(t)) = \mu dt + \sigma_1 dW^1_t. \]

where \((W^1_t, t \geq 0)\) is a Wiener process, then simple manipulations show that the discount rate becomes:

\[ r = \delta + \mu \gamma - \frac{1}{2} \gamma^2 \sigma^2_1. \]

The third and last step of this classical construction consists in considering the situation where not only the consumption is risky, but also the variation of this consumption. Let us for instance think to an investment which procures advantages in year \( t \), but this advantage is not certain, it is endowed with a probability distribution;

Equivalence between \( a(0) \) and \( a(t) \) implies that:

\[ a(0) \times u'(c(0)) = e^{-\delta t}E[a(t)u'(c(t))]. \]

Suppose that the discount rate \( r \) is defined by:

\[ e^{rt} = \frac{E(a(t))}{a(0)}. \tag{2} \]

Introducing this expression in 2, and after standard manipulations, it appears that:

\[ e^{rt} = e^{rf} \frac{E[a(t)u'(c(t))]}{E[a(t)]E[u'(c(t))]} \]

If we assume that the random walk of \( a(t) \) is given, as \( c(t) \), by

\[ d \log(a(t)) = g dt + \sigma_2 dW^2_t, \]

where \((W^2_t, t \geq 0)\) is another brownien motion, correlated with \((W^1_t, t \geq 0)\) and such that\(^4\)
\( d < W^1, W^2 >_t = \beta t \), it is easy to show that:

\[ r = r_f + \gamma \sigma^2_1 \beta, \]

This standard presentation has been refined, especially to take into account the fact that, when calibrated on past values of statistical series, it provides very small risk premium, a situation which arises also in financial markets, and is known as the equity premium puzzle; in a similar way, statistical test show that the past series of GDP do not follow a Wiener process. Research has been made in order to sophisticate the random walk, for instance including possible rare catastrophes; see for instance [Barro, 2006, Gollier, 2008, Weitzman, 2012]. But these works do not systematically deal with the issue of decision rules in CBA, the point which is now addressed; in this aim we will stick to the hypothesis of a Wiener process, leaving other assumptions for further research.

\(^4\)Note that this implies that \( \beta \) can be identified as the correlation coefficient of the regression of \( a(t) \) against \( c(t) \)
Decision rules when using a random model  In the following we will grasp the problem at the root. Let us assume that the GDP at time $t$ is $Y(t)$, and that the collective utility function is:

$$U(Y(t), t \geq 0) = \int_0^{+\infty} e^{-\delta t} u(Y(t)) dt,$$

where $u(y) = -Y(t)^{-\gamma + 1}/(\gamma - 1)$, for a $\gamma > 1$.

Let us now consider a possible marginal-investment implemented at time $T$, whose marginal-cost is $I(T)$, and which provides increases in future $Y(t)$ values by a marginal-amount $a(t)$. The so-called Net Present Value ($NPV$) of this investment is the marginal-change in $U$ induced by this investment. $NPV$ is a function of $T$ which can be defined by

$$NPV(T) = \int_T^{+\infty} a(t)Y(t)^{-\gamma}e^{-\delta t} dt - I(T)Y(T)^{-\gamma}e^{-\delta T}.$$

The question is now: should the project be implemented, and when?

The answer is known in the case of certainty, and we will recall it with the assumption that the 3 functions $Y(t), a(t)$ and $I(t)$ are exponential ones:

$$d\log(Y(t)) = \mu dt,$$
$$d\log(a(t)) = g dt,$$
$$d\log(I(t)) = k dt.$$

Then

$$NPV(T) = \int_T^{+\infty} a(0)e^{gt}Y(0)^{-\gamma}e^{-\gamma\mu t}e^{-\delta t} dt - I(0)Y(0)^{-\gamma}e^{kT}e^{-\gamma\mu T}e^{-\delta T}$$

or:

$$NPV(T) = \int_T^{+\infty} a(0)Y(0)^{-\gamma}e^{(g-\gamma\mu-\delta)t} dt - I(0)Y(0)^{-\gamma}e^{(k-\gamma\mu-\delta)T}$$

Let us note that this expression is finite only if $g < \delta + \gamma \mu$. When this condition is fulfilled, it has an interior-extremum, for a $T$ such that:

$$\frac{a(T)}{I(T)} = \delta + \gamma \mu - k.$$  \hspace{1cm} (3)

And it is a maximum if the second derivative is negative, which is equivalent here to $g > k$. If this condition is fulfilled, then $NPV(T)$ is positive.

Then the decision rule is: wait until (3) is fulfilled; if $g < k$, then implement the investment at $T$. Otherwise do not implement\footnote{If the growths are not exponential, the condition (3) is an extremum, but not necessary a global one, neither a maximum.}. Let us note that we encounter the discount rate $r_0$, previously defined:

$$r_0 = \delta + \gamma \mu - k.$$  

The rule is that the implementation should take place when the immediate rate of return is equal to the discount rate $r_0$, the one which takes place in case of no risk.
Let us now go to the risky situation. We assume that three function of times $a(t), Y(t)$ and $I(t)$ are random. In this case it is necessary to use the informations given by $a(t), Y(t)$ and $I(t)$ in order to determine the optimal time of the investment. A deterministic time cannot give an optimal decision time when randomness is introduced.

We need to determine an optimal stopping time and will use optimal stopping theory for this. See appendix A, page 16 for a short presentation of a result for a case where we have an explicit representation of the optimal stopping time. We will prove in appendix B that our multidimensional model can be reduce to this simple case.

We assume that the logarithm of $a(t), Y(t)$ and $I(t)$ are Brownian motions:

$$
d \log(Y(t)) = \mu dt + \sigma_1 dW_1(t),
\quad d \log(a(t)) = \gamma dt + \sigma_2 dW_2(t),
\quad d \log(I(t)) = \delta dt + \sigma_3 dW_3(t).
$$

Moreover, we suppose that these Brownian motions are correlated. To be more specific, $\bar{W}_1^t, \bar{W}_2^t, \bar{W}_3^t$ being 3 independant Brownian motions, we assume that:

\begin{align*}
W_1^t &= \bar{W}_1^t \\
W_2^t &= \rho \bar{W}_1^t + \sqrt{1 - \rho^2} \bar{W}_2^t \\
W_3^t &= \rho' \bar{W}_1^t + \sqrt{1 - (\rho')^2} \bar{W}_3^t
\end{align*}

Now we have to maximise $E(NPV(\tau))$ where:

$$
NPV(\tau) = \int_\tau^{+\infty} e^{-\delta t} a(t) Y(t)^{-\gamma} dt - e^{-\delta \tau} I(\tau) Y(\tau)^{-\gamma}.
$$

among all the stopping times $\tau$ of the filtration $\mathcal{F}_t = \sigma(Y_s, a_s, I_s, s \leq t)^6$.

Using strong Markov property for the process $(a, I, Y)$, we obtain:

$$
E \left( \int_\tau^{+\infty} e^{-\delta t} a(t) Y(t)^{-\gamma} dt \bigg| \mathcal{F}_\tau \right) = E_{\mathcal{F}_\tau, a(\tau), I(\tau), Y(\tau)} \left( \int_\tau^{+\infty} e^{-\delta t} a(t) Y(t)^{-\gamma} dt \right)
$$

$$
= e^{-\delta \tau} a(\tau) Y(\tau)^{-\gamma} E \left( \int_0^{+\infty} e^{-\delta s + \gamma \sigma_1 \bar{W}_1^s} - e^{-\gamma \sigma_1 \bar{W}_2^s} - e^{-\gamma \sigma_1 \bar{W}_3^s} ds \right).
$$

The last integral can be computed. Taking into account the correlation structure of the Brownian motions $(W^2, W^1)$, we get (assuming $\delta_1 > 0$):

$$
E \left( \int_\tau^{+\infty} e^{-\delta t} a(t) Y(t)^{-\gamma} dt \bigg| \mathcal{F}_\tau \right) = \frac{e^{-\delta \tau} a(\tau) Y(\tau)^{-\gamma}}{\delta_1},
$$

where:

$$
\delta_1 = \delta + \gamma \mu - g - \frac{1}{2} \bar{\sigma}_2^2.
$$

with

$$
\bar{\sigma}_2^2 = \sigma_2^2 + \gamma^2 \sigma_1^2 - 2\gamma \rho \sigma_1 \sigma_2
$$

---

$^6$ $\mathcal{F}_t$ is a mathematical object which capture the information given by the trajectories of $a, Y$ and $I$, up to time $t$. A stopping time does not anticipate on the future information of the trajectory in the decision of stopping.
So we have obtained a simplified formula for the expectation of the NPV:

$$\mathbb{E} (NPV(\tau)) = \mathbb{E} \left( e^{-\delta Y(\tau)^{-\gamma} \left[ \frac{a(\tau)}{\delta_1} - I(\tau) \right]} \right).$$  \hfill (4)

The + added in the formula denote the positive part of a number and is needed as we always have the possibility to not invest in the project at any time.

Now, let us denote by $u(a_0, Y_0, I_0)$ the value function of the optimal stopping problem:

$$u(a_0, Y_0, I_0) = \sup_{\tau, F_t \text{t.a.}} \mathbb{E}(NPV(\tau_+)).$$

A classical result of this theory (see for instance [Van Moerbeke, 1975]) allows us to identify an optimal stopping time as:

$$\tau_{\text{opt}} = \inf \left\{ t \geq 0, u(a(t), Y_t, I_t) = Y_t^{-\gamma} \left[ \frac{a(t)}{\delta_1} - I(t) \right]_+ \right\}.$$  \hfill (5)

Moreover the value function $u$ (see appendix B for details) can be rewritten as:

$$u(a_0, I_0, Y_0) = Y_0^{-\gamma} \bar{u}(a_0, I_0)$$

with:

$$\bar{u}(a_0, I_0) = \sup_{\tau, F_t \text{t.a.}} \tilde{\mathbb{E}} \left( e^{-\delta_2 T} \left( \frac{a_0}{\delta_1} e^{\mu_2 T + \tilde{\sigma}_2 \tilde{W}_T^2} - I_0 \right)_+ \right),$$

where

$$\begin{align*}
\bar{\sigma}_2^2 &= \sigma_2^2 + \gamma^2 \sigma_1^2 - 2 \rho \gamma \sigma_1 \sigma_2 \\
\bar{\delta}_1 &= \delta + \gamma \mu - g - \frac{1}{2} \bar{\sigma}_2^2 \\
\bar{\sigma}_3^2 &= \sigma_3^2 + \gamma^2 \sigma_1^2 - 2 \rho \gamma \sigma_1 \sigma_3 \\
\bar{\delta}_2 &= \delta + \gamma \mu - k - \frac{1}{2} \bar{\sigma}_3^2 \\
\bar{\sigma}_2^2 &= \sigma_2^2 + \sigma_3^2 - 2 \rho \rho_1 \sigma_2 \sigma_3 \\
\rho_3 &= \frac{(\sigma_3 \rho_1 - \gamma \sigma_1)(\rho \sigma_2 - \rho_1 \sigma_3) - \sigma_3^2 (1 - \rho_1^2)}{\bar{\sigma}_2 \bar{\sigma}_3} \\
\mu_2 &= g - k + \rho_3 \bar{\sigma}_2 \bar{\sigma}_3
\end{align*}$$

But the function $\bar{u}$ can be explicitly computed (see appendix A for details and references). So we define an optimal stopping time $\tau^*$ by setting:

$$\tau^* = \inf \left\{ t \geq 0, a(t) \geq I_t \frac{x^*(\sigma)}{I_0} \right\},$$

where:

$$x^*(\sigma) = \sqrt{\mu^2 + 2j \sigma^2 - \mu} \left( \frac{\sqrt{\mu^2 + 2j \sigma^2 - \mu}}{\sqrt{\mu^2 + 2j \sigma^2 - \mu} - \rho \sigma_2} I_0 (j - \mu - \sigma^2 / 2) \right)$$

with:

$$\begin{align*}
j &= \delta_2 \\
C &= I_0 \\
\mu &= \mu_2 \\
\sigma &= \bar{\sigma}_2
\end{align*}$$

8
**Teachings from simulations**  The previous relation is not easy to grasp. It is not clear what are the important parameters, and even in what sense they play a role. The recourse to simulation is intended to make things more understandable; these simulations are achieved through sensible estimates of the parameters of the previous formula, drawn from real situations.

With this objective, we have been able to gather, through the precious collaboration of SE-TRA, yearly data of traffic on about 550 road sections which have benefitted from an investment, for a period of 16 years (1992-2008), and a few rail sections. From these traffic series, a rough calculation has been made to get yearly surplus series: it has been assumed that the surplus indexes are the product of the traffic evolution indexes and the value of time evolution (it has been assumed that value of time elasticity to GDP is 0.7). A calibration of each of these series against GDP have been made. Relations between the cost of construction and GDP are drawn from [Becker et al., 2013].

The general characteristics of the data and the results of the calibrations are described in the table 1.

<table>
<thead>
<tr>
<th></th>
<th>Yearly growth rate on a 16 years period</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>GDP</strong></td>
<td></td>
</tr>
<tr>
<td>Mean yearly growth rate</td>
<td>0.15</td>
</tr>
<tr>
<td>Standard deviation of the growth rate</td>
<td>0.02</td>
</tr>
<tr>
<td><strong>Construction cost</strong></td>
<td></td>
</tr>
<tr>
<td>Mean yearly growth rate</td>
<td>0.00</td>
</tr>
<tr>
<td>Standard deviation of the growth rate</td>
<td>0.04</td>
</tr>
<tr>
<td>Correlation with GDP</td>
<td>0.50</td>
</tr>
<tr>
<td><strong>Surplus</strong></td>
<td></td>
</tr>
<tr>
<td>Mean yearly growth rate</td>
<td></td>
</tr>
<tr>
<td>Average</td>
<td>0.03</td>
</tr>
<tr>
<td>5% percentile</td>
<td>0.00</td>
</tr>
<tr>
<td>95% percentile</td>
<td>0.07</td>
</tr>
<tr>
<td>Standard deviation of the growth rate</td>
<td></td>
</tr>
<tr>
<td>Average</td>
<td>0.07</td>
</tr>
<tr>
<td>5% percentile</td>
<td>0.01</td>
</tr>
<tr>
<td>95% percentile</td>
<td>0.50</td>
</tr>
<tr>
<td>Correlation with GDP</td>
<td></td>
</tr>
<tr>
<td>Average</td>
<td>0.50</td>
</tr>
<tr>
<td>5% percentile</td>
<td>-0.20</td>
</tr>
<tr>
<td>95% percentile</td>
<td>0.80</td>
</tr>
</tbody>
</table>

Table 1: Characteristics of the data (covering 554 sections of road and 16 years from 1992 to 2008)

Putting these values in the relations giving the discount rates leads to risk premium very low: the discount rate $r_0$ (whose definition is $r_0 = \delta + \mu \gamma - k$) is 3.5%. The discount rate $r_f$ (whose definition is $r_f = r_0 - \gamma^2 \sigma^2 / 2$) is 3.39%, and the risk premium $\phi$ appearing in:

$$r = r_f + \beta \phi$$
is only: 0.01%; it is quite small and the risk should be quite negligible. It is another version of the well-known equity premium puzzle already mentioned. There are two ways to overcome it. The first one is to use other random walks than the pure Wiener process; this way is appealing as it is clear that at least the GDP, and also many other series such as surpluses, do not follow this process; the frequency of catastrophic events is higher than what happens in normal distributions. Another way is to stick to normal distributions and to use values different from those given by the historical analysis; this way has the advantage to keep using Brownian move, which is much simpler to deal with and often leads to closed formulae; this way is used in finance where in Black-Scholes applications, the parameters used are those deduced from the equity price and not from the standard deviation of its moves.

In the following, we use this second direction, and use values of parameters which provide sensible values for the risk premium. These sensible values cannot come from observation as, contrarily to the finance markets where it is possible to observe the dividends of equities, it is not possible to observe surpluses. The expertise on the equity premium is that it is around 1 to 2% (see [Quinet, 2013]). The following simulation are based on the values of parameters given by table 2 and lead to a risk premium of 1.7% and discount rates \( r_0 = 0.05 \) and \( r_f = 0.025 \).

<table>
<thead>
<tr>
<th>Nature</th>
<th>Symbol</th>
<th>GDP</th>
</tr>
</thead>
<tbody>
<tr>
<td>Trend</td>
<td>( \mu )</td>
<td>0.015 et 0.005</td>
</tr>
<tr>
<td>Standard deviation</td>
<td>( \sigma_1 )</td>
<td>de 0 à 0, 10</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Nature</th>
<th>Symbol</th>
<th>Surplus</th>
</tr>
</thead>
<tbody>
<tr>
<td>Trend</td>
<td>( g )</td>
<td>0.034 et 0.02</td>
</tr>
<tr>
<td>Standard deviation</td>
<td>( \sigma_2 )</td>
<td>de 0 à 0, 10</td>
</tr>
<tr>
<td>Correlation with GDP</td>
<td>( \rho )</td>
<td>de (-1) à +1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Nature</th>
<th>Symbol</th>
<th>Construction cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>Trend</td>
<td>( k )</td>
<td>0 et 0.1</td>
</tr>
<tr>
<td>Standard deviation</td>
<td>( \sigma_3 )</td>
<td>de 0 à 0, 10</td>
</tr>
<tr>
<td>Correlation with GDP</td>
<td>( \rho )</td>
<td>de (-1) à +1</td>
</tr>
</tbody>
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Table 2: Values of parameters used in simulations

On these bases, comparisons will be made between on one side the optimal immediate rate of return and one the other side the different discount rates which has been enlightened in section 2:

- the discount rate without risk: \( r_0 = \delta + \mu \gamma - k \),
- the discount rate with risk on GDP: \( r_f = r_0 - \gamma^2 \sigma_1^2 / 2 \),
- the discount rate with risk on GDP and on the future effect: \( r = r_f + \gamma \beta \sigma_1^2 = r_f + \beta \phi \).
Let us note that this last one is different whether the effect is on surplus or on cost. The simulations are achieved on situations with an increasing complexity. First if we assume no risk (the standard deviations of all three series all zero), without surprise, the optimal rate of return must be equal to \( r_0 \).

When, second step, risk is introduced in GDP and just in GDP, it is normal to compare the optimal value to the discount rate with risk \( r_f \); simulations are reported in figure 1. They show that the corresponding ratio is higher than 1 but in a very small proportion: approximating the optimal ratio by \( r_f \) is quite acceptable.

![Figure 1: Alea just on GDP; Dependance of the ratio :(optimal rate/discount rate \( r_f \)) with the standard deviation of GDP](image1)

Let us now introduce risk on surpluses but not on construction costs. The results are shown on figure 2, where comparisons of optimal rate are made first with \( r_f \) and second with the \( r = r_f + \beta \phi \) relative to surpluses (i.e. introducing in the formula the value of \( \beta \), and noted \( r_{\text{surplus}} \)). No simple interpretation are apparent.

![Figure 2: Risk on GDP and surplus, no risk on construction cost. Ratio : optimal rate to \( r_{\text{surplus}} \) against standard deviation of surplus](image2)

On the contrary, the ratio between optimal rate and \( r_f \) leads to more simple result as shown in figure 3. It appears that the ratio is regularly increasing with the standard deviation of surplus.
and is -almost-not sensitive to the correlation between surplus and GDP. The growth with the standard deviation is not much sensible to the GDP parameters. The following graph indicates how the surface is changed when parameters of GDP are changed (growth of GDP is 0.015 in the first graph and 0.005 in the second one; standard deviation is 0.075 is the same on both).

**Figure 3:** Risk on GDP and surplus, no risk on construction cost; ratio : (optimal rate / discount rate \( r_f \)) against surplus standard deviation

It appears that the ratio remains insensitive to correlation between GDP and surplus, but is very sensitive to the rate of growth of GDP.

**Figure 4:** Risk on GDP and surplus, no risk on construction cost; ratio : (optimal rate / discount rate \( r_f \)) against surplus standard deviation as in Figure 2 but with a lower GDP growth rate

When, going further, we introduce alea on costs, we get the following graph for the ratio between optimal rate and discount rate \( r_f \):
We observe the same shape, but the starting value is higher than unity. In order to explain this higher value, let us draw the same sort of graph, but with the ratio between the optimal immediate rate and the discount rate for costs:

\[ r_{\text{cost}} = r_f + \gamma \sigma_1 \sigma_3 \rho_1 = r_f + \phi \beta_{\text{cost}}. \]

The result is given in figure 6.

In this graph, the values of the parameters for GDP and cost are the default values of table 2. Simulations with other values for GDP and surplus show the same shape. But the effect of construction cost is varied: the ratio between optimal immediate rate and discount rate of
construction cost is roughly 1 when the trend of construction cost is zero or close to zero and
the standard deviation is “small”; this ratio changes and decreases when the trend and standard
deviation of construction costs are larger. Nevertheless even in those situations, the optimal
immediate rate of return grows with the standard deviation of surplus as in figure 5, and the
correlation between surplus and GDP plays almost no role.

It is also interesting to compare the optimal rule to the rule which would be to optimise the
expectation of NPV among given time, eluding the future information. This last strategy boils
down to maximise:

$$\max_T \{E(NPV(T))\}$$

where:

$$E(NPV(T)) = E \left( e^{-\delta T} Y(T)^{-\gamma} \left[ \frac{a(T)}{\delta_1} - I(T) \right] \right)$$

and $\delta_1 = \delta + \mu - g - (\gamma \sigma_1^2 + \sigma_2^2 - 2\gamma \rho \sigma_1 \sigma_2)/2$. Introducing the expressions of $a(t)$, $Y(t)$ and
$I(t)$ in this relation and taking the expectation leads to:

$$E(NPV(T)) = Y(0)^{-\gamma} \left( \frac{a(0)}{\delta_1} e^{(-\delta-\gamma \mu + g + (\gamma \sigma_1^2 + \sigma_2^2 - 2\gamma \rho \sigma_1 \sigma_2)/2)T} - I(0) e^{(-\delta-\gamma \mu + k + (\gamma \sigma_1^2 + \sigma_2^2 - 2\gamma \rho \sigma_1 \sigma_3)/2)T} \right).$$

Getting the derivative to zero and simplifying leads to:

$$a(0) e^{(\sigma_2^2/2 - \rho \sigma_1 \sigma_2)T} = I(0) e^{(k + \sigma_3^2/2 - \rho_1 \gamma \sigma_1 \sigma_3)T} (\delta + \gamma \mu - k - \sigma_3^2/2 - \gamma^2 \sigma_1^2/2 + \rho_1 \gamma \sigma_1 \sigma_3).$$

Or through the expression of $a(T)$ and $I(T)$:

$$\frac{E(a(T))}{E(I(T))} = e^{\gamma \sigma_1 (\rho \sigma_2 - \rho_1 \sigma_3)T} (\delta + \gamma \mu - k - \sigma_3^2/2 - \gamma^2 \sigma_1^2/2 + \rho_1 \gamma \sigma_1 \sigma_3)$$

$$= e^{\gamma \sigma_1 (\rho \sigma_2 - \rho_1 \sigma_3)T} \left( r_f - k - \sigma_3^2/2 + \phi \beta_{cost} \right).$$

This formula gives the rule to apply if the decision on the date of implementation should be
taken right now. Simulations show that the gap between this rule and the optimal rule amounts
to at least 20% for parameters around the default values, and that the optimal rule is more severe
than the other one: taking into account the future information leads to a postponement of the
decision.

Let us now gather the whole of these conclusions as practical rules: when the construction
cost’ rate of growth is zero or “low” and its standard deviation is “low”, the optimal immediate
rate of return is equal to the discount rate of construction cost when the standard deviation of
surplus is zero. When the standard deviation is not zero, the optimal rate increses but slowly
and remains very close to the previous one, and almost does not vary with the correlation be-
tween GDP and surplus. The same applies when the construction cost’ rate of growth and/or
standard deviation are not “low” except that the optimal immediate rate of return can be rather
different from the discount rate of construction cost. Let us note that the parameters of con-
struction cost are similar for all infrastructure projects of a given sector, for instance transport.
Then it stems from these results that the optimal rate is “roughly” the same for all projects of
this sector.
**Conclusion**  This text aims at providing a stone in the field of investment decision rules under risk. It specific input is to build a bridge between two streams up to now without much contact: CBA and its traditional criteria which pay little attention to risk, especially systematic risk, and modern theory of discount rate which focuses on systemic risk in its recent developments but has up to now no large input in CBA practice.

The specific situation addressed for this purpose is the case of investment whose surpluses and costs are random variables as well as GDP, these three processes following linked Wiener processes. Under such assumptions, it has been possible to find decision rules based on a closed formula which expresses the value of the immediate rate of return which triggers the implementation of the investment. This expression generalises the usual rule well known in the certain case. As it is not easy to see the implications of the formula which is rather complex, simulations have shown various interesting features.

- First, the simulations allow to check some desirable properties of the result: for instance, when random variables’ standard deviations are set to zero, the formula comes back to the certain world rule: the threshold for the immediate rate of return is equal to the interest rate $r_0$. Similarly when only the GDP is at random, the surpluses and the cost of the investment being certain, the threshold is the discount rate at risk $r_f$, which is lower than the basic discount rate: risk on GDP induces to anticipate the implementation. On the same vein, when just the surpluses are at random, while the GDP and cost are certain, the threshold is larger than the threshold in situation of certainty which is the discount rate: risk induces a delay in the implementation of the project. All those conclusions are in accordance with expectations.

- In the general case when the three series follow Wiener processes, the optimal immediate rate of return is equal to the discount rate of construction cost in the case where the standard deviation of surplus is zero. When it is not the case, the optimal rate increases with the standard deviation of the surplus, and almost does not vary with the correlation between GDP and surplus.

This result casts interest on a often ignored parameter, namely the cost of construction and how it varies according to the conjuncture.

Those results should be confirmed and enlarged through further research in several directions, for instance:

- First to deal more explicitly with the premium puzzle. This extension should take into account other processes than the Wiener ones; unfortunately it is highly probable that no closed result can be found and that numerical simulations are necessary

- Second to study other decision situations than just to do or not to do and when a single project, for instance to study situations where there is choice between two alternative and mutually exclusive projects

- Third to study the situation where the implementation does not take place once at all, but gradually over time.
A classical case

In this appendix, we remind a classical result which is useful in our computation. This optimal stopping problem look like to the computation of a perpetual American put option in the Black-Scholes model. The first appearance of this problem can be dated back to [McKean, 1965] in an appendix of a paper by Samuelson [Samuelson, 1965]. This computation is also useful when working on real options (see, for instance, the chapter 5 of [Dixit and Pindyck, 1994] for a proof).

A model for the random evolution

We assume that the process $X$ is given as the solution of
\[
d\log(X_t) = \mu dt + \sigma dW_t,
\]

- where $(W_t, t \geq 0)$ is a Wiener process.
- $\sigma$ is a real parameter which tune the size of the randomness. $\sigma$ is called the volatility in financial models.
- It can be useful to introduce $\mu' = \mu + \sigma^2/2$ as, using Itô calculus we have
\[
dX_t = X_t (\mu' dt + \sigma dW_t), X_0 = x.
\]

Computing the expectation of the NVT

The $NPV$ is defined as:
\[
NPV(T) = \int_T^{+\infty} e^{-jt}X_t dt - Ce^{-jT},
\]

where $C$ is a real parameter (which can be interpreted as the price at $T$ of the project).

In this simple model, it is easy to compute the expectation of the $NPV$ explicitly:
\[
\mathbb{E} (NPV(T)) = \mathbb{E} \left( \int_T^{+\infty} e^{-jt}X_t dt - Ce^{-jT} \right) = \int_T^{+\infty} e^{-jt}\mathbb{E}(X_t) dt - C e^{-jT}.
\]

But as $\log(X_t)$ is a Gaussian random variable, $\mathbb{E}(X_t)$ can be computed easily, which lead to a formula for $V(T, x) = \mathbb{E} (NPV(T))$:
\[
V(T, x) = \frac{xe^{-jT}}{j - \mu - \sigma^2/2} - Ce^{-jT}.
\]

Moreover the (strong) Markovian property of $X$ allows to prove that:
\[
\mathbb{E} (NPV(\tau)_+) = \mathbb{E} (V(\tau, X_\tau)_+).
\]

This explicit expression simplify the optimisation problem which follows.
Optimising among all stopping times  We need to find a stopping time $\tau$ maximising:

$$E(NPV(\tau)_+) = E(V(\tau, X_\tau)_+) = E(e^{-j\tau} \left( \frac{X_\tau}{j - \mu - \sigma^2/2} - C \right)_+) .$$

This optimal stopping problem can be solved using a variant of the result of [McKean, 1965], and we obtain an explicit representation of an optimal stopping time $\tau_{opt}$ as:

$$\tau_{opt} = \inf \{ t \geq 0, X_t \geq x^*(\sigma) \} .$$

(6)

where:

$$x^*(\sigma) = \frac{\sqrt{\mu^2 + 2j\sigma^2 - \mu}}{\sqrt{\mu^2 + 2j\sigma^2 - \mu - \sigma^2}} C(j - \mu - \sigma^2/2) .$$

Note that an expansion in $\sigma$ of $x^*(\sigma)$ show that $x^*(\sigma)$ tends to $x^*_0 = Cj$ when $\sigma$ tends to 0. Moreover putting $x^*_0$ in (6) define a deterministic time which is optimal in the case $\sigma = 0$.

We can also check that $x^*(\sigma) > x^*_0$ when $\sigma > 0$: in this case, randomness postpone the decision time.

B Computing the value function of the optimal decision problem

We denote by $u(a_0, Y_0, I_0)$ the value function of the optimal stopping problem:

$$u(a_0, Y_0, I_0) = \sup_{\tau, F_t} E(NPV(\tau)_+) .$$

Using Girsanov theorem (see C for a reminder), we will give, in this appendix, an explicit formula for $u$, using the formula given for a one dimensional case in appendix A.

First, let us define $L_T$ by:

$$L_T = \frac{I(T)Y(T)^{-\gamma}}{E(I(T)Y(T)^{-\gamma})} .$$

For $T$ a given real positive number, $L_T \geq 0$ and $E(L_T) = 1$. Moreover a simple computation, taking into account the correlation between $W^1$ and $W^3$, gives :

$$E\left( (I(T)Y(T)^{-\gamma} \right) = I_0 Y_0^{-\gamma} e^{(k-\gamma\mu)T+\frac{1}{2}\theta_3^2T} ,$$

where $\theta_3^2 = \theta_3^2 + \gamma^2\theta_1^2 - 2\rho_I\gamma\theta_1\theta_3$. So

$$L_T = e^{\sigma_3\theta_3^2 - \gamma\sigma_1\theta_1^2 - \frac{1}{2}\theta_3^2} .$$

Now note that when $\tau$ is a finite stopping time, $E(\ NPV(\tau)_+) \text{ can be rewritten as:}$

$$E(\ NPV(\tau)_+) = Y_0^{-\gamma} E\left( L_\tau e^{-\delta_2\tau} \left( \frac{1}{\delta_1} A(\tau) - I_0 \right)_+ \right)$$

This restriction is necessary as decisions have to be taken, using only past (and not future!) informations.
where \( \delta_2 = \delta - k + \gamma \mu - \frac{1}{2} \sigma_3^2 \) and \( A \) the process:

\[
A(t) = \frac{a(t)}{(I(t)/I_0)} = a_0 e^{(g - k) t + \sigma_2 W_t^2 - \sigma_3 W_t^3}.
\]

Now, let \( \tilde{\sigma}_2 = \sqrt{\sigma_2^2 + \sigma_3^2 - 2 \rho \rho_I \sigma_2 \sigma_3} \) and

\[
\tilde{W}_t^2 = \frac{\sigma_2 W_t^2 - \sigma_3 W_t^3}{\tilde{\sigma}_2},
\]

then \( A \) can be written as:

\[
A(t) = a_0 e^{(g - k) t + \tilde{\sigma}_2 \tilde{W}_t^2},
\]

where \( \tilde{W}^2 \) is a standard Brownian motion. Moreover, let us define a new Brownian \( \tilde{W}^3 \) by:

\[
\tilde{W}_t^3 = \frac{\sigma_3 W_t^3 - \gamma \sigma_1 W_t^1}{\tilde{\sigma}_3^2},
\]

so \( L_T \) can be rewritten as

\[
L_T = e^{\tilde{\sigma}_3 W_T^3 - \frac{1}{2} \tilde{\sigma}_3^2 T}.
\]

But a simple computation shows that:

\[
\mathbb{E} \left( \tilde{W}_t^3 \tilde{W}_t^2 \right) = \rho_3 t,
\]

where

\[
\rho_3 = \frac{(\sigma_3 \rho_I - \gamma \sigma_1)(\rho \sigma_2 - \rho_I \sigma_3) - \sigma_3^2 (1 - \rho_I^2)}{\tilde{\sigma}_3^2 \sigma_3}.
\]

And we obtain, using an argument relying on the Gaussian character of the vector of processes \((\tilde{W}^3, \tilde{W}^2)\) that:

\[
\tilde{W}_t^3 = \rho \tilde{W}_t^2 + \sqrt{1 - \rho^2} \tilde{W}_t^1
\]

where \( \tilde{W}^1 \) is an independent of \( \tilde{W}^2 \), Brownian motion. Girsanov theorem (see appendix C) prove that if \( T \) is a finite real number we can define a probability \( \tilde{P} \) on \( \sigma(\mathcal{F}_t, t \leq T) \) by setting, for every positive or integrable \( \mathcal{F}_T \) measurable random variable \( X \):

\[
\mathbb{E} (X) = \mathbb{E} (L_T X).
\]

Moreover under this probability \( \tilde{P} \), \((\tilde{W}_t^2, \tilde{W}_t^1, t \leq T)\) where:

\[
\tilde{W}_t^2 = \tilde{W}_t^2 - \rho_3 \tilde{\sigma}_3 t,
\]

and independent standard Brownian motions. Now let us consider a stopping time \( \tau \) bounded by \( T \) and denote \( \mu_2 = g - k + \rho_3 \tilde{\sigma}_2 \tilde{\sigma}_3 \):

\[
\mathbb{E} (NPV(\tau)_+) = Y_0 e^{-\beta T} \mathbb{E} \left[ L_T e^{-\beta T} \left( \frac{a_0}{\delta_1} e^{\mu_2 \tau + \tilde{\sigma}_2 \tilde{W}_T^2} - I_0 \right)_+ \right] \\
= Y_0 e^{-\beta T} \mathbb{E} \left[ L_T e^{-\beta T} \left( \frac{a_0}{\delta_1} e^{\mu_2 \tau + \tilde{\sigma}_2 \tilde{W}_T^2} - I_0 \right)_+ \right] \\
= Y_0 e^{-\beta T} \mathbb{E} \left( e^{-\beta T} \left( \frac{a_0}{\delta_1} e^{\mu_2 \tau + \tilde{\sigma}_2 \tilde{W}_T^2} - I_0 \right)_+ \right).
\]

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This last expression prove that we can rewrite $u(a_0, I_0, Y_0) = \sup_{\tau \text{.a.}} E (NPV(\tau)_+) as:

$$u(a_0, I_0, Y_0) = Y_0 - \gamma \bar{u}(a_0, I_0).$$

where

$$\bar{u}(a_0, I_0) = \sup_{\tau \text{.a.}} E \left( e^{-\delta_2 T} \left( \frac{a_0}{\delta_1} e^{\mu_2 \tau + \sigma_2 \tilde{W}^2_\tau} - I_0 \right)_+ \right).$$

From this we deduce that we are able to compute $\bar{u}$ using the elementary case presented in the appendix A if we choose for the parameters $(j, C, \mu, \sigma, X_0)$ those computed using the following formulas:

$$\begin{align*}
\delta_2 &= \sigma_2^2 + \gamma^2 \sigma_1^2 - 2 \rho \gamma \sigma_1 \sigma_2 \\
\delta_1 &= \delta + \gamma \mu - g - \frac{1}{2} \sigma_2^2 \\
\sigma_3^2 &= \sigma_3^2 + \gamma^2 \sigma_1^2 - 2 \rho \gamma \sigma_1 \sigma_3 \\
\delta_3 &= \delta + \gamma \mu - k - \frac{1}{2} \sigma_3^2 \\
\bar{\sigma}_2^2 &= \sigma_2^2 + \sigma_3^2 - 2 \rho \rho_1 \sigma_2 \sigma_3 \\
\rho_3 &= \frac{(\sigma_3 \rho_1 - \gamma \sigma_1)(\rho \sigma_2 - \rho_1 \sigma_3) - \sigma_1^2 (1 - \rho_1^2)}{\bar{\sigma}_2 \bar{\sigma}_3} \\
\mu_2 &= g - k + \rho_3 \bar{\sigma}_2 \bar{\sigma}_3 \\
j &= \delta_2 \\
C &= I_0 \\
\mu &= \mu_2 \\
\sigma &= \bar{\sigma}_2
\end{align*}$$

So we can express the boundary $x^*(\sigma)$ corresponding to this model using the formula (taking into account that $C = I_0$):

$$x^*(\sigma) = \frac{\sqrt{\mu^2 + 2 j \sigma^2} - \mu}{\sqrt{\mu^2 + 2 j \sigma^2} - \mu - \sigma^2 / 2} I_0 (j - \mu - \sigma^2 / 2)$$

and the optimal stopping time is expressed as:

$$\tau^* = \inf \left\{ t \geq 0, u(a(t), Y_t, I_t) = Y_t - \gamma \left( \frac{a(t)}{\delta_1} - I(t) \right) \right\} = \inf \left\{ t \geq 0, \bar{u}(a(t), I_t) = \frac{a(t)}{\delta_1} - I(t) \right\}.$$  

Finally, considering the special form of $\bar{u}$, this stopping time can be rewritten as:

$$\tau^* = \inf \left\{ t \geq 0, a(t) \geq I_t \frac{x^*(\sigma)}{I_0} \right\}. \quad (7)$$

### C  On the Girsanov theorem

In this appendix we recall a version of the Girsanov theorem suited to our problem.
The Girsanov theorem

Let \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})\) be a probability space endowed with a filtration \(\mathcal{F}\) which is the natural filtration of a standard Brownian motion \((B_t)_{0 \leq t \leq T}\), indexed by the space interval \([0, T]\). The following result is known as Girsanov theorem (for a proof see [Karatzas and Shreve, 1988]).

- Let \((\theta_t)_{0 \leq t \leq T}\) be an adapted process such that \(\int_0^T \theta_s^2 ds < \infty\) a.s.. Moreover we assume that the process \((L_t)_{0 \leq t \leq T}\) defined by
  \[
  L_t = \exp \left( - \int_0^t \theta_s dB_s - \frac{1}{2} \int_0^t \theta_s^2 ds \right)
  \]
  is a martingale on the space interval \([0, T]\) (which is equivalent to \(\mathbb{E}(L_T) = 1\).

Then, under a new probability \(\mathbb{P}^{(L)}\) defined by, if \(A\) is an \(\mathcal{F}_T\)-measurable set :

\[
\mathbb{P}^{(L_T)}(A) = \mathbb{E}(L_T 1_A),
\]

the process \((W_t)_{0 \leq t \leq T}\) defined by \(W_t = B_t + \int_0^t \theta_s ds\) is a standard \((\mathcal{F}_t)\)-Brownian motion.

Remark  
Note that if \(\theta\) is a deterministic (non random) and bounded function of time, it is easy to check that \(\mathbb{E}(L_T) = 1\) (because of Brownian properties of the stochastic integral), so \((L_t)_{0 \leq t \leq T}\) is a martingale and Girsanov theorem can be used.

In this work we use this theorem in a very simple case where \(\theta\) does not depend on \(t\).
References


