Multiproduct Pricing with Core Goods and Side Goods

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Abstract

Many firms and organizations sell both core goods and side goods. Airports are a prime example. Runway and terminal capacity are the core goods, and car parking, car rentals, and other concession services are the side goods. It has been conjectured that airport operators limit the prices they charge for core goods in order to boost their revenues from selling side goods. If this is true it alleviates concerns about excessive market power and may preclude heavy-handed regulation. In this paper we investigate the conjecture using a general model in which consumers buy a side good only when they buy a core good. We derive several results for a monopoly. First, if consumers have identical preferences for the side good the monopolist prices the side good at marginal cost and extracts consumer's surplus through the core price. The monopolist profits indirectly from selling the side good because it boosts demand for the core good. Second, if demand for the side good rises the monopoly markup on the core good can increase, decrease, or remain unchanged depending on how the price elasticity of core-good demand changes as the demand curve shifts out. Third, if consumers are heterogeneous the monopolist can price the side good above or below marginal cost depending on how preferences for the core good and side good are correlated. We show that some of these results extend to a spatially differentiated duopoly.

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1 Introduction

Many firms sell a range of products or services that can be considered core goods or side goods. Core goods are purchased by all consumers and comprise the mainstay or heart of the firm’s business. Side goods are supplementary or optional. Some consumers may buy them when they buy the core goods, but they never buy the side goods alone.

Airports are a prime example of multiproduct businesses that sell core goods and side goods. The core goods are aeronautical services such as runway capacity, terminal capacity, and air traffic control.\(^1\) Side goods span a wide range of concession services such as car parking, car rentals, duty-free shopping and regular shopping, foreign exchange services, visitor and business services, and catering (Bezic and Vojvodic, 2009).\(^2\) Most people do not visit an airport just to consume these side services (Zhang and Czerny, 2012; Czerny, 2013). However, travellers may abstain from consuming any side services.

Railway stations are another transportation example with core and side businesses. The supply of railway infrastructure and operations are the core goods. Food and beverages, shopping, and tourist and travel services are the side businesses. Gas stations are a further example: in addition to their core product of gasoline, they often also sell ancillary goods such as windshield cleaning fluid, groceries, vehicle service, and repairs.

A general question regarding such establishments is how demand for their side goods affects their pricing of core goods. Some insights are found in the literature on multiproduct firms. A starting point is that core goods and side goods are complementary in the sense that an increase in the supply of one good, or a reduction in its price, increases the demand for the other good (Tirole, 1988). A firm selling both goods has an incentive to internalize the demand externality by reducing the price of each good. This motivation is heightened when consumers engage in multi-purpose or one-stop shopping so that a firm cannot sell anything unless it draws consumers to its establishment (or website) by offering an attractive package of products. Bliss (1988) considers a setting in which a monopolist sells multiple goods to one-stop shoppers subject to a reservation-utility constraint. Bliss shows that the monopolist charges Ramsey prices, and may sell some products below cost.

A number of authors have studied multiproduct pricing in competitive markets. Stahl (1982) notes that a supermarket selling complementary goods internalizes the demand externalities between the goods by reducing their prices. Other contributions along this line include Beggs (1994), Smith and Hay (2005), and Smith and Thomassen (2012). However, these studies do not consider a hierarchical structure in which consumers buy one good (or set of goods) only if they buy another.

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\(^1\) Airports also provide check-in, baggage handling, and boarding services for passengers. Charges for these services are often imposed through airport fees or taxes that are added to airline ticket prices.

\(^2\) Many airports derive more than 50 percent of their revenues from non-aeronautical concession services (ATRS, 2011). These services tend to be more profitable than aeronautical operations, and partly for this reason concession business plays an important role in airport privatization and regulation (Czerny, 2006; Yang and Zhang, 2011). As the Economist (2014) notes, airport retailing is becoming increasingly profitable. Retailers can use information on flight schedules and passenger characteristics to adjust the mix of products on display.
The relationship between the sale of side goods and the pricing of core goods is of particular interest for airports due to concerns about airport market power. Starkie (2001) has conjectured that airports may restrain the aeronautical charges they impose on airlines for using runways and terminals in order to boost traffic and expand the revenues they earn from non-aeronautical operations. If this is true, it may alleviate the need for heavy-handed airport regulation.

To our knowledge, Zhang and Zhang (1997) were the first to analyze the interdependence of aeronautical and non-aeronautical airport revenues. They showed that for a congested public airport it can be socially desirable to use concession revenues to cover fixed infrastructure costs for runways and terminals. A crucial assumption underlying their analysis is that passenger demand for flights does not depend on the supply of airport concession services. One argument to justify this independence assumption is that travelers typically buy tickets well before they buy concession services, and may be either unaware of concession prices or simply disregard them. A number of later studies have also made this independence assumption. One group of studies has shown that unregulated, private aeronautical charges are socially excessive despite the complementarity between aeronautical and non-aeronautical revenues (Zhang and Zhang, 2003; Oum et al., 2004; Zhang and Zhang, 2010; Yang and Zhang, 2011). The intuition for this result is that airport concession services reduce the socially optimal aeronautical charge more than the profit-maximizing charge because the private operator disregards the surplus consumers derive from concession services.

Another set of studies has relaxed the independence assumption. Czerny (2006) assumes that some individuals choose to fly only if airport concession services are available. This occurs if the utility they derive from flying falls short of the ticket price, but the combined utility from flying and consuming services exceeds the ticket price plus the outlay on the services. The sale of services (the side good) then has a positive demand effect on the core good. Czerny (2006) uses a numerical example to show that this effect can induce a monopoly airport to increase the core price. In a later paper, Czerny (2013) establishes that the monopoly aeronautical charge at a congested airport is socially excessive when concession services are priced at marginal costs.\(^3\) Bracaglia et al. (2014) have recently considered a competitive setting in which two spatially-differentiated airports compete for travelers. They find that supplying concession services can be a dominant strategy, but it causes equilibrium aeronautical charges and profits to fall, thereby putting the airports in a Prisoner’s Dilemma.

In this paper we build on the literature by using a simple microeconomic model to investigate Starkie’s (2001) conjecture. The model has several key properties. First, consumers buy the side good only if they also buy the core good. Second, consumers make decisions about buying the goods simultaneously rather than independently. The model thus features both the complementarity effect

\(^3\)Czerny (2013) also shows that profit-maximizing and welfare-maximizing airport charges can be identical if welfare-neutral concession services are sold. Concession services such as food and beverages are welfare neutral if airport sales come at the expense of other firms in the same jurisdiction that provide equally good and profitable substitutes. Welfare-neutral concession services are also considered by Zhang and Czerny (2012) and D’Alfonso et al. (2013).
and the demand effect of side-good supply on core-good demand and pricing. Third, and crucially, if consumers buy the side good, the amount they buy per unit of the core good is assumed to be independent of their total core-good purchases. Consequently, the ratio of side-good to core-good purchases is independent of the core-good price. As we elaborate later, this assumption is consistent with a tendency of travelers to make similar decisions about car parking, food and beverage consumption, and in-flight amenities on all their trips. We implement this assumption by expressing a consumer’s total utility as the sum of subutility functions for the core good and side good. By allowing general functional forms for these functions we can generate several commonly-used demand functions for the two goods that have closed-form solutions and display a range of properties.

We derive three general results for a monopoly. First, if consumers have identical preferences for the side good the monopolist prices the side good at marginal cost. This is because the firm can capture any consumer’s surplus from consumption of the side good through the core price. Thus, the complementarity effect of selling side goods on core-good pricing is absent because the profit margin on the side good is zero. However, the monopolist does profit from selling the side good indirectly because it boosts demand for the core good. A demand effect therefore does come into play.

Second, if demand for the side good increases the monopoly markup on the core good can increase, decrease, or remain unchanged depending on whether the price elasticity of demand for the core good decreases, increases, or stays the same in magnitude as the demand curve shifts out. A necessary condition for demand to become more elastic is that core-good demand be a strictly convex function of the core price. This condition is satisfied for some preference specifications, but not for those that generate linear demand functions. Importantly, this shows that the use of linear functions — which has been popular in the literature — can amount to assuming the answer to certain questions.

Third, if consumers differ in their core-good preferences, but have identical side-good preferences, the monopolist still prices the side good at marginal cost because it can extract consumer’s surplus most effectively through the core price. However, if consumers differ in their side-good preferences, the monopolist can price the side good above or below marginal cost depending on how preferences for the two goods are correlated. For example, if core and side preferences are positively correlated, consumers with large and inelastic core demands buy more of the side good than the average consumer. The monopolist then charges a positive markup on the side good in order to exploit their inelastic demands.

Finally, we show that most of the monopoly results generalize to a Hotelling duopoly market in which consumers are distributed along a line with firms located at either end. We also show that the demand effect of side good supply can be moderated by competition, and that firms can set core prices below the level they would set without the side good. Overall, however, our results raise doubts about Starkie’s (2001) conjecture in the airport setting that side-good revenues alleviate
concerns about market power.

The paper is organized as follows. Section 2 introduces the basic model of a monopolist that sells core and side goods to a representative consumer. Section 3 derives the monopolist’s profit-maximizing markups for the core and side goods. It also examines the case in which consumers either purchase a fixed amount of the side good, or nothing. Heterogenous preferences are analyzed in Section 4. A variant of the model with spatially differentiated duopolists is considered in Section 5. Section 6 concludes.

2 The Model

We first present a generic demand model in which core goods and side goods are treated symmetrically. We then impose a hierarchical structure in which the side good is consumed only if the core good is consumed as well.

2.1 General demand

Consider a monopolist and a representative individual with quasilinear preferences for a core good, a side good, and a numeraire. The monopolist sets a uniform price \( p \) for the core good, and a uniform price \( r \) for the side good. The consumer buys \( q \) of the core good, and \( s \) of the side good. Since income effects are absent, demand functions can be written as \( q(p, r) \) and \( s(p, r) \). Unless indicated otherwise, demand functions are assumed to be differentiable. The two goods are assumed to be complements so that \( \frac{\partial q}{\partial r} < 0 \) and \( \frac{\partial s}{\partial p} < 0 \). The core good has a constant marginal production cost of \( k \), and the side good a constant marginal production cost of \( c \). Fixed costs play no role in the analysis and are set to zero. Profits are therefore:

\[
\pi = (p - k) q(p, r) + (r - c) s(p, r). \tag{1}
\]

The monopolist chooses \( r \) and \( p \) to maximize (1). First-order conditions for a profit maximum are:

\[
\frac{\partial \pi}{\partial r} = (p - k) \frac{\partial q}{\partial r} + s + (r - c) \frac{\partial s}{\partial r} = 0, \tag{2}
\]

\[
\frac{\partial \pi}{\partial p} = q + (p - k) \frac{\partial q}{\partial p} + (r - c) \frac{\partial s}{\partial p} = 0. \tag{3}
\]

Consider eq. (3) for \( p \). The last term is the marginal change in profit from selling the side good due to a change in \( p \). Given \( \frac{\partial s}{\partial p} < 0 \), side-good profit decreases if \( r > c \). This encourages the monopolist to reduce \( p \), and corresponds to the complementarity effect mentioned in the introduction. The first two terms in (3) give the marginal change in profit from core-good sales when \( p \) is raised. Selling the side good tends to raise \( q \), but it also tends to increase the magnitude of \( \frac{\partial q}{\partial p} \). The sum of these two effects corresponds to the demand effect of selling the side good.
Depending on how the side good affects the own-price elasticity of demand for the core good, the
demand effect can be positive or negative. If it is positive, it works in the opposite direction to the
complementarity effect when \( r > c \). If it is negative, it reinforces the complementarity effect and
the profit-maximizing \( p \) decreases unambiguously.

Let \( \varepsilon_{ij} \) denote the elasticity of demand for good \( i \) with respect to the price of good \( j \). Equations
(2) and (3) can be solved to obtain implicit formulas for the profit-maximizing markups in terms
of the elasticities:

\[
\frac{r - c}{r} = \frac{\varepsilon_{sp} - \varepsilon_{qp}}{\varepsilon_{qp}\varepsilon_{sr} - \varepsilon_{qr}\varepsilon_{sp}},
\]

(4)

\[
\frac{p - k}{p} = \frac{\varepsilon_{qr} - \varepsilon_{sr}}{\varepsilon_{qp}\varepsilon_{sr} - \varepsilon_{qr}\varepsilon_{sp}}.
\]

(5)

These formulas were derived by Rohlfs (1979), and the right-hand sides are known as super-
elasticities. The common denominator, \( \varepsilon_{qp}\varepsilon_{sr} - \varepsilon_{qr}\varepsilon_{sp} \), is positive by the second-order condi-
tions for profit maximization. (Note that with no income effects, \( \partial s/\partial p = \partial q/\partial r \) which implies
\( \varepsilon_{sp} = (pq) / (rs) \varepsilon_{qr} \).) For brevity, the profit-maximizing markup on the side good will be called the
side markup, and the profit-maximizing markup on the core good will be called the core markup.

2.2 Hierarchical demands

The general-demand approach treats the core and side goods symmetrically, and therefore does not
embody an assumption that the side good can be consumed only if the core good is consumed as
well. One way to incorporate this hierarchical demand structure is to use a utility function of the
form

\[
U(q, s) = g(q) + \Psi(q, s),
\]

(6)

where \( g(q) \) and \( \Psi(q, s) \) are subutility functions. Function \( g(q) \) specifies utility from consuming
the core good by itself, and function \( \Psi(q, s) \) specifies utility from consuming the side good when
core-good consumption is \( q \). In the case of air travel, \( g(q) \) corresponds to utility from flying, and
\( \Psi(q, s) \) to utility from consuming goods or services at the airport or possibly during a flight. We
assume that \( g(q) \) is strictly increasing in \( q \) with \( g(0) = 0 \), while \( \Psi(q, s) \) is increasing in \( q \) and \( s \) and
satisfies \( \Psi(0, s) = \Psi(s, 0) = 0 \). Thus, utility can be derived from consuming just the core good but
not from consuming just the side good.

Underlying quasilinear preferences is the assumption that utility is a linear function of some
numeraire good, but concave with respect to consumption of the good or goods under study.\(^4\)
Consistent with this we will assume diminishing marginal utility from consumption of the core
good. However, the same may not be true for ancillary goods and services purchased in conjunction
with the core good. For example, frequent flyers may value parking, car rentals, food and beverages,
and other amenities as much as people who fly only rarely.\(^5\) Train commuters may value feeder bus

\(^4\)See Varian (1992, Section 10.3).

\(^5\)Indeed, frequent travelers may tire from the grind of travel and seek relief by consuming more, and/or more
service, park-and-ride facilities, fast-food breakfast service and so on in proportion to the number of commuting trips they make per month or year. Similar reasoning applies to amusement parks, vacation resorts, cruise ships, and other settings. When this is the case, it is reasonable to assume that, for given prices, side-good consumption is proportional to core-good consumption. This idea can be embodied in eq. (6) by assuming that subutility function \( \Psi(q, s) \) is homogeneous of degree one, and thus can be written as \( \Psi(q, s) = q \cdot h(s/q) \) where \( h(0) = 0 \). Function \( h(\cdot) \) is assumed to be concave so that \( h'' < 0 \), where a prime denotes a derivative. Utility function (6) then becomes:

\[
U(q, s) = g(q) + q \cdot h(s/q).
\]  

(7)

Utility function (7) is used for the rest of the paper.

3 Homogeneous Consumers

In this section we assume that consumers are identical so that the representative consumer model is applicable. The consumer’s utility maximization problem is:

\[
\max_{\{q, s\}} g(q) + q \cdot h(s/q) + m - p \cdot q - r \cdot s,
\]

(8)

where \( m \) denotes income. In what follows we assume that \( m \) is large enough that it does not affect consumption of the core or side good, and \( m \) is henceforth suppressed.

The first-order utility-maximization condition for \( s \) is \( \partial U / \partial s = r \). Given \( \partial U / \partial s = q \cdot h'(s/q) \), this yields:

\[
h'(s/q) = r.
\]

(9)

Define the inverse function \( y(r) \equiv (h')^{-1}(r) \), which is decreasing in \( r \). Equation (9) can then be written as

\[
s = q \cdot y(r).
\]

(10)

Eq. (10) stipulates that side-good consumption varies proportionally with core-good consumption, as intended. We will refer to the preferences underlying eq. (7) as proportional preferences, and the conditional demand function in (10) as proportional demand. These preferences will be implicit in the statement of all propositions and other formal statements in the paper.

For each unit consumed of the core good, the consumer earns a surplus from the side good of \( S(r) \equiv h(y(r)) - r \cdot y(r) \). Utility-maximization problem (8) can thus be rewritten:

\[
\max_q g(q) + (S(r) - p) \cdot q.
\]

(11)

The first-order condition for \( q \) is expensive, side goods.
\[ g'(q) + S(r) = p. \] 

(12)

The first term on the left-hand side of eq. (12) is the marginal utility derived from greater consumption of the core good. The second term is the surplus earned from consuming the side good. At the consumer’s optimum, the sum of the two benefits matches the price of the core good. Totally differentiating (10) and (12) one obtains the demand derivatives:

\[
\frac{\partial q}{\partial p} = \frac{1}{g'' q} < 0, \quad \frac{\partial s}{\partial p} = \frac{\partial q}{\partial r} = \frac{y}{g'' q} < 0, \quad \text{and} \quad \frac{\partial s}{\partial r} = \frac{y^2}{g'' q} + q y' < 0.
\] 

(13)

Consistent with the complementary nature of the two goods, an increase in price of either good reduces demand for both goods. Price elasticities of demand are:

\[
\varepsilon_{qp} = \varepsilon_{sp} = \frac{p}{g'' q} < 0, \quad \varepsilon_{qr} = \frac{ry}{g'' q} < 0, \quad \text{and} \quad \varepsilon_{sr} = r \left( \frac{y}{g'' q} + \frac{y'}{y} \right) < 0.
\] 

(14)

3.1 General monopoly pricing rules

Equation (14) reveals that, with proportional preferences, the core good and side good have the same elasticities with respect to \( p \): \( \varepsilon_{sp} = \varepsilon_{qp} \). It follows from eq. (4) that the side markup is zero. The core markup in eq. (5) simplifies to the standard formula for a monopolist selling only the core good:

\[
\frac{p - k}{p} = -\frac{1}{\varepsilon_{qp}}.
\] 

(15)

These two results are formalized in the following proposition:

**Proposition 1** If consumers are identical, a monopolist prices the side good at marginal cost and marks up the core good according to the standard inverse-elasticity rule.

The intuition underlying Prop. 1 is similar to that in Oi’s (1971) classic study of two-part pricing by a Disneyland monopolist. Oi’s monopolist charges an admission fee of \( T \) for the right to purchase rides at a price \( r \). Rides are produced at a constant marginal cost of \( c \). With identical consumers, the monopolist can extract all the consumer’s surplus with \( T \). The monopolist fully internalizes aggregate welfare (sum of producer’s and consumer’s surplus) derived from side good consumption, and hence prices output (rides) at marginal cost (i.e., \( r = c \)).

In our model the core good must be purchased in order to buy the side good. The core good is therefore analogous to admission, and the side good is analogous to rides. The analogy is inexact because core-good consumption is not fixed, but rather a function of \( p \) and \( r \). Nevertheless, consumer’s surplus from buying the side good translates into willingness to pay for the core good as is evident from the consumer’s first-order condition (12) for \( q \). The monopolist thus chooses to price the side good efficiently at marginal cost, and extracts consumer’s surplus as best it can.
with $p$. The monopolist earns no profit from the side good directly, so the complementarity effect underlying Starkie’s (2001) conjecture does not come into play so far.

Prop. 1 also harks back to Spence (1975) and Sheshinski (1976) who showed that a monopolist supplies optimal product quality if marginal and inframarginal consumers value quality equally. In our model the side good plays the role of quality because it increases willingness to pay for the core (or basic) good. Moreover, marginal and inframarginal valuations are clearly equal since side consumption is proportional to core-good consumption.6

Using eqs. (14) and (15), the profit-maximizing core price can be written

$$p = k - g''(q)q.$$  

The core price is a function of $q$ and the curvature of the core-good subutility function $g(\cdot)$. The core price does not depend directly on subutility function $h(\cdot)$. However, $p$ does depend on side-good preferences indirectly through their effect on $q$. Define

$$\omega(q) \equiv -q \cdot \frac{g'''}{g''},$$

which is a measure of the curvature (or convexity) of the inverse core demand and also describes the elasticity of the slope of the inverse demand (Aguirre et al., 2010). The effect of side-good demand on the core price can be formalized as:

**Proposition 2** If consumers are identical, and consumer’s surplus from the side good increases, the monopolist decreases the core markup if $\omega(q) > 1$, and increases the core markup if $\omega(q) < 1$.

Prop. 2 is proved in the appendix. It can be understood by noting that an increase in consumer’s surplus from the side good shifts out the demand curve for the core good. The profit-maximizing core price decreases if the price elasticity of core-good demand increases in magnitude at a given price, and vice versa. From (14) and (16), the price elasticity is

$$\varepsilon_{qp} = \frac{p}{g''q} = \frac{k - g''q}{g''q} = \frac{k}{g''q} - 1 = -\left(1 + \frac{k}{\|g''q\|}\right).$$

The elasticity increases in magnitude if $\|g''q\|$ decreases, which is equivalent to $\omega(q) > 1$. Starkie’s (2001) conjecture is therefore borne out even with no complementarity effect if (and only if) $\omega(q) > 1$, that is if (and only if) the demand effect of the side good causes core-good demand to become more price-elastic. Note that the measure of convexity must be strictly positive for this to hold (i.e., $\omega(q) > 0$) and hence the core demand must be a strictly convex function of the core price. This result is formalized as:

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6 As shown in Section 4, this is generally not the case with heterogeneous consumers.
Corollary 1 With homogeneous consumers, if consumer’s surplus from the side good increases, the monopolist decreases the core markup if the price elasticity of core-good demand increases in magnitude. A necessary condition for this is that the core-good demand curve be convex.

Four examples of (7) with different core-good and side-good preferences are now considered.

3.2 Examples

Power function preferences As a first illustration of (7), consider a power function for \( g(q) \), and a Cobb-Douglas function for \( qh(s/q) \):

\[
U(q, s) = uq^\gamma + vq^\alpha s^\beta, \tag{18}
\]

where parameters \( \alpha, \beta, \gamma, u, \) and \( v \) are all strictly positive, \( \alpha + \beta = 1 \), and \( \gamma < 1 \). Parameter \( u \) governs the scale of core-good demand, and parameter \( v \) governs the scale of side-good demand.\(^7\)

In the limit \( u \to 0 \), eq. (18) reduces to a conventional Cobb-Douglas utility function in which the core good has no primacy. Given \( g(q) = uq^\gamma \), \( \omega(q) = 2 - \gamma > 1 \). Hence, by Prop. 2 side-good supply reduces the monopoly core price.

To derive the core markup, consider first the consumer’s utility maximization problem. The first-order conditions are

\[
\frac{\partial U}{\partial s} = v\beta q^\alpha s^{\beta-1} = r, \tag{19}
\]

\[
\frac{\partial U}{\partial q} = u\gamma q^{\gamma-1} + v\alpha q^{\alpha-1}s^\beta = p. \tag{20}
\]

Eqs. (19) and (20) yield:

\[
s = \left( \frac{v\beta}{r} \right)^{\frac{1}{1-\beta}} q, \tag{21}
\]

\[
S(r) = v\alpha \left( \frac{v\beta}{r} \right)^{\frac{\beta}{1-\beta}}, \tag{22}
\]

and

\[
q = \left( \frac{p - S(r)}{u\gamma} \right)^{\frac{1}{\gamma-1}}, \tag{23}
\]

where, as earlier, \( S(r) \) is consumer’s surplus from the side good per unit consumed of the core good. To maintain finite demand in (23) we require \( S(r) < p \). Substituting (23) into (21) gives

\[
s = \left( \frac{v\beta}{r} \right)^{\frac{1}{1-\beta}} \left( \frac{p - S(r)}{u\gamma} \right)^{\frac{1}{\gamma-1}}. \tag{24}
\]

\(^7\) As shown below, price elasticities and profit-maximizing markups depend only on the ratio \( v/u \). Both parameters are retained in eq. (18) to allow for heterogeneity in individual demand functions later in the paper.
Price elasticities of demand are:

\[
\begin{align*}
\varepsilon_{qp} &= \varepsilon_{sp} = -\frac{p}{(1-\gamma)(p-S)}, \\
\varepsilon_{qr} &= \frac{-\beta}{(1-\gamma)(1-\beta)(p-S)}, \\
\varepsilon_{sr} &= -\frac{1}{1-\beta} \frac{-\beta}{(1-\gamma)(1-\beta)(p-S)}. 
\end{align*}
\]  

The side markup is zero as per Prop. 1. By eq. (15), the core markup is:

\[
\frac{p - k}{p} = \frac{(1-\gamma)k - (1-\gamma)S}{k - (1-\gamma)S},
\]  

where (with \( r = c \)) \( S = v\alpha(v\beta/c)^{\frac{\beta}{1-\beta}} \). As parameter \( v \) increases, \( S \) increases too and the core markup declines as per Prop. 2.

**Log-linear preferences** The second example of (7) features a log-linear specification for \( g(q) \):

\[
U(q,s) = (\ln \left( \frac{a}{q} \right) + 1) \frac{q}{b} + vq^{1-\beta}s^\beta,
\]  

where \( a > 0 \) and \( b > 0 \). With this function, \( \omega(q) = 1 \). Hence, by Prop. 2, the monopoly core price is unchanged by side-good supply. Eqs. (21) and (22) still apply for the side good. Solving the first-order condition \( \partial U/\partial q = p \), using (27), and applying (21), one obtains:

\[
q = ae^{-b(p-S(r))}.
\]

Price elasticities are:

\[
\begin{align*}
\varepsilon_{qp} &= \varepsilon_{sp} = -bp, \\
\varepsilon_{qr} &= -bv\beta \left( \frac{v\beta}{r} \right)^{\frac{\beta}{1-\beta}}, \\
\varepsilon_{sr} &= -\frac{1}{1-\beta} - bv\beta \left( \frac{v\beta}{r} \right)^{\frac{\beta}{1-\beta}}.
\end{align*}
\]

The price of the core good is \( p = k + 1/b \), and the core markup is:

\[
\frac{p - k}{p} = \frac{1}{1 + kb}.
\]

In contrast to the first example, the core markup does not depend on side-good demand. The independence of the core markup is apparent in eq. (28) from the fact that side-good surplus, \( S \),
inflates core-good demand by the same proportional factor for all values of $p$.

**Linear-quadratic preferences** A third example that will be used in later sections is the linear-quadratic utility function:

$$U(q, s) = a \cdot q - b \cdot \frac{q^2}{2} + q \left( e \cdot \frac{s}{q} - \frac{f}{2} \left( \frac{s}{q} \right)^2 \right),$$

where parameters $a$, $b$, $e$, and $f$ are all positive. Core-good utility is defined by parameters $a$ and $b$, and side-good utility by parameters $e$ and $f$. Parameter $a$ is the reservation price on core-good demand if the side good were not offered. If consumers incur a cost each time they consume the core good (e.g., a travel cost to reach an airport or train station), the cost can be deducted from parameter $a$. Parameter $a$ can therefore vary across consumers with their distance from the point of consumption, as will be assumed in Section 5.

With utility function (31), $\omega(q) = 0$. Side-good supply thus increases the monopoly core price by Prop. 2. Following the same derivation procedure as before, one obtains

$$S(r) = \frac{(e - r)^2}{2f},$$

$$q = \frac{1}{b} (a + S(r) - p),$$

$$s = \frac{e - r}{bf} (a + S(r) - p).$$

Equation (33) shows that the reservation price for the core good increases by $S$. Price elasticities work out to:

$$\varepsilon_{qp} = \varepsilon_{sp} = \frac{p}{a + S(r) - p},$$

$$\varepsilon_{qr} = \frac{r (e - r)}{f (a + S(r) - p)},$$

$$\varepsilon_{sr} = -\frac{r}{e - r} \frac{a + 3S(r) - p}{a + S(r) - p}.$$  (35)

Profit-maximizing prices are $r = c$, and $p = \frac{1}{2} \left( k + a + (e - c)^2 / (4f) \right)$. The core markup is:

$$\frac{p - k}{p} = \frac{a + S(c) - k}{a + S(c) + k}. $$

Unlike the first two examples, the core markup increases with side-good surplus.

Together, the first three examples show that selling the side good can induce a monopolist to increase, decrease, or leave unchanged the core-good price.
All-or-nothing side-good demand  The final example is a modified version of the linear-quadratic specification:

\[ U(q, s) = a \cdot q - b \cdot \frac{q^2}{2} + q \cdot 1_{\{s \geq \sigma\}} \cdot e \sigma, \tag{37} \]

where parameters \( a, b, e, \) and \( \sigma \) are positive, and \( 1_{\{x\}} \) is an indicator function equal to 1 if \( x \) is true and 0 otherwise. Eq. (37) stipulates that the consumer gains utility \( e \sigma \) from the side good if at least \( \sigma \) units of it are consumed, and no utility otherwise.\(^8\) For air travel this might describe a passenger who parks her car at the origin airport on each trip, rents a car at the destination airport on each trip, and so on. The side good is indivisible in the sense that a minimum quantity (\( \sigma \)) must be consumed to gain any benefit from it, whereas consuming more than the minimum is either impossible or pointless (e.g., paying for two parking spaces). Czerny (2006), Kratzsch and Sieg (2011), and D’Alfonso et al. (2013) adopt all-or-nothing specifications in their airport-pricing studies.\(^9\)

From (37) it is clear that if \( e > r \), the consumer buys \( \sigma \) units of the side good for each unit consumed of the core good. If \( e \leq r \), the consumer does not buy the side good. Thus, optimal consumption of the side good is \( s^* = q \cdot \sigma \cdot 1_{\{e > r\}} \), and total consumer’s surplus from the side good per unit of \( q \) is

\[ S(r) = \sigma \cdot (e - r)^+, \tag{38} \]

where \( x^+ = x \) if \( x \geq 0 \), and \( x^+ = 0 \) if \( x < 0 \). Substituting \( s^* \) into (37) yields:

\[ U(q, s^*) = a \cdot q - b \cdot \frac{q^2}{2} + q \cdot \sigma \cdot (e - r)^+. \tag{39} \]

Optimal core-good consumption is then

\[ q = \frac{1}{b} (a + S(r) - p). \tag{40} \]

Again, the consumer’s reservation price for the core good is inflated by \( S \). Price elasticities of demand are:

\[ \varepsilon_{qp} = \varepsilon_{sp} = \frac{-p}{a + S(r) - p}, \]

\[ \varepsilon_{qr} = \varepsilon_{sr} = \left\{ \begin{array}{ll} \frac{-\sigma r}{a + S(r) - p} & \text{for } r < e \\ 0 & \text{for } r > e \end{array} \right. . \]

Since \( \varepsilon_{qp} \varepsilon_{sr} - \varepsilon_{qr} \varepsilon_{sp} = 0 \), the superelasticity formulas (4) and (5) cannot be used to deduce profit-maximizing markups. Nevertheless, it is clear that the monopolist can profit from selling the side good if, and only if, \( e > c \). One solution is for the monopolist to sell the side good at marginal cost.

\(^8\)With a representative consumer it is unnecessary to include both parameters \( e \) and \( \sigma \) in (37), but doing so is useful when consumer heterogeneity is considered later.

\(^9\)Czerny (2006) and D’Alfonso et al. (2013) allow consumers to vary in their preferences for the side good: a complication we consider in Section 4.2.
The core markup is then:

\[ \frac{p - k}{p} = \frac{a + S(c) - k}{a + S(c) + k}. \]  

Eq. (41) is identical in structure to (36), and it has the same property that the core markup increases with \( S(c) \).

Eq. (41) is not the only profit-maximizing solution because core-good demand depends only on the full monetary cost, \( p + \sigma \), of purchasing the core and side goods as a package. The monopolist can set \( \sigma \) at any level not exceeding \( e \), and adjust \( p \) accordingly. This indeterminacy vanishes if consumers are heterogeneous as considered in the following section.

4 Heterogeneous Consumers

4.1 Continuous side-good demands

The analysis thus far treats consumers as identical even though they may differ widely in the amounts they consume of core and side goods as well as in their price elasticities of demand for each good. We have assumed that nonlinear pricing is impossible, and we will preclude third-degree price discrimination as well. These constraints complicate the monopolist’s problem because it can no longer capture all consumers’ surplus from the side good using only one price pair \( (p, \sigma) \). Consequently, pricing the side good at marginal cost is no longer profit-maximizing in general. Indeed, if individual demands for the two goods are correlated in certain ways, the monopolist chooses the markups to optimize the price differentials associated with consumers’ bundles of core- and side-good consumptions. For example, if consumers with price-inelastic core-good demands value the side good highly, the monopolist can take advantage of their inelastic demand by setting a high side markup.

To introduce consumer heterogeneity let \( \theta \) denote a consumer’s type, which can encompass such characteristics as parameters of the subutility functions considered in Section 3, transport costs incurred to reach the point of purchase, and so on. In what follows we will index quantities by a consumer’s type, \( \theta \), while suppressing other arguments to economize on writing. The utility maximization problem for consumer type \( \theta \) is:

\[
\max_{q(\theta)} g(\theta) + (S(\theta) - p) q(\theta).
\]

The first-order condition (12), \( g'(\theta) + S(\theta) - p = 0 \), implicitly determines \( q(\theta) \). Aggregate core-good demand for all consumer types, \( Q \), is:

\[
Q(p, \sigma) \equiv \int_{\theta} q(\theta) \phi(\theta) d\theta, \tag{42}
\]

where \( \phi(\theta) \) is the frequency distribution of types. Weighted in terms of core-good consumption,
average side-good consumption is

$$\bar{y} \equiv \frac{1}{Q} \int_\theta q(\theta) y(\theta) \phi(\theta) d\theta. \quad (43)$$

In place of (1), the monopolist’s profit function becomes

$$\pi = (p - k + (r - c) \bar{y}) Q. \quad (44)$$

First-order conditions for a profit maximum are:

$$\frac{\partial \pi}{\partial r} = (p - k) \frac{\partial Q}{\partial r} + \bar{y}Q + (r - c) \frac{\partial (\pi Q)}{\partial r} = 0, \quad (45)$$

$$\frac{\partial \pi}{\partial p} = Q + (p - k) \frac{\partial Q}{\partial p} + (r - c) \frac{\partial (\pi Q)}{\partial p} = 0. \quad (46)$$

A complementarity effect and a demand effect can be identified in eq. (46) in the same way as in eq. (3). However, with heterogeneous consumers total side-good consumption is, in general, no longer proportional to core consumption because \( \bar{y} \) depends on \( p \).

Eqs. (45) and (46) can be solved using routine algebra to obtain counterparts to eqs. (4) and (5):

$$\frac{r - c}{r} = \frac{\int_\theta \varepsilon_{sp}s(\theta)\phi(\theta)d\theta \cdot \int_\theta q(\theta)\phi(\theta)d\theta - \int_\theta \varepsilon_{qq}q(\theta)\phi(\theta)d\theta \cdot \int_\theta s(\theta)\phi(\theta)d\theta}{-\int_\theta \varepsilon_{sp}s(\theta)\phi(\theta)d\theta \cdot \int_\theta q(\theta)\phi(\theta)d\theta + \int_\theta \varepsilon_{qp}q(\theta)\phi(\theta)d\theta \cdot \int_\theta s(\theta)\phi(\theta)d\theta}, \quad (47)$$

$$\frac{p - k}{p} = \frac{\int_\theta \varepsilon_{qq}q(\theta)\phi(\theta)d\theta \cdot \int_\theta s(\theta)\phi(\theta)d\theta - \int_\theta \varepsilon_{sr}s(\theta)\phi(\theta)d\theta \cdot \int_\theta q(\theta)\phi(\theta)d\theta}{-\int_\theta \varepsilon_{qr}q(\theta)\phi(\theta)d\theta \cdot \int_\theta s(\theta)\phi(\theta)d\theta + \int_\theta \varepsilon_{qs}s(\theta)\phi(\theta)d\theta \cdot \int_\theta q(\theta)\phi(\theta)d\theta}. \quad (48)$$

Similar to eqs. (4) and (5), the common denominator of eqs. (47) and (48) is positive if core-good and side-good price elasticities are non-zero. It is clear from (47) and (48) that if none of the price elasticities depends on parameters that vary in the population, then the markups do not depend on the frequency distribution of those parameters. This result is formalized in the following proposition:

**Proposition 3** If none of the demand price elasticities depends on parameters that vary in the population, then the monopoly core and side markups do not depend on the frequency distribution of those parameters.

The elasticity formulas in subsection 3.2 reveal that in each of the four examples, the elasticities do not depend on one of the preference parameters. These instances are recorded in the fourth
column of Table 1. For example, with power-function preferences the elasticities do not depend on parameter \( u \).

<table>
<thead>
<tr>
<th>Preferences</th>
<th>Prop. 3</th>
<th>Coroll. 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Power</td>
<td>( uq^\gamma )</td>
<td>( vq^{-\beta}s^\beta )</td>
</tr>
<tr>
<td>Log-linear</td>
<td>((\ln(a/q) + 1) \frac{q}{b})</td>
<td>( vq^{-\beta}s^\beta )</td>
</tr>
<tr>
<td>Linear-quadratic</td>
<td>( aq - b^2q \frac{r}{2} )</td>
<td>( e \left(\frac{s}{a}\right)^2 )</td>
</tr>
<tr>
<td>All-or-nothing</td>
<td>( aq - b^2q \frac{r}{2} )</td>
<td>( 1_{(s\geq\sigma)} \cdot e\sigma )</td>
</tr>
</tbody>
</table>

Table 1: Summary of results with heterogeneous consumers

Since eqs. (47) and (48) are cumbersome to use we will work henceforth with the aggregate quantities \( Q \) and \( \bar{y} \) defined in eqs. (42) and (43). Eqs. (45) and (46) can be solved to obtain the following equations for the price markups:

\[
\frac{r - c}{r} = \frac{p}{Q} \left( \frac{\partial (\bar{y}Q)}{\partial p} \cdot Q - \frac{\partial Q}{\partial p} \cdot \bar{y}Q \right),
\]

\[
\frac{p - k}{p} = \frac{r}{Q} \left( \frac{\partial Q}{\partial r} \cdot \bar{y}Q - \frac{\partial (\bar{y}Q)}{\partial r} \cdot Q \right),
\]

with

\[
\Omega \equiv p \cdot r \cdot \left( \frac{\partial Q}{\partial p} \cdot \frac{\partial (\bar{y}Q)}{\partial r} - \frac{\partial (\bar{y}Q)}{\partial p} \cdot \frac{\partial Q}{\partial r} \right) = p \cdot r \cdot Q \left( \frac{\partial Q}{\partial p} \cdot \frac{\partial \bar{y}}{\partial r} - \frac{\partial Q}{\partial r} \cdot \frac{\partial \bar{y}}{\partial p} \right) > 0.
\]

It is useful to introduce two new variables:

\[
\hat{y}_r = \frac{\partial (\bar{y}Q)}{\partial r} = \bar{y} + Q \frac{\partial \bar{y}}{\partial Q} \frac{\partial r}{\partial Q},
\]

(52)

\[
\hat{y}_p = \frac{\partial (\bar{y}Q)}{\partial p} = \bar{y} + Q \frac{\partial \bar{y}}{\partial Q} \frac{\partial p}{\partial Q}.
\]

(53)

Quantity \( \hat{y}_r \) measures how a marginal change in \( r \) affects side-good sales relative to its effect on core-good sales. Quantity \( \hat{y}_p \) is defined in an analogous way with respect to \( p \). With homogeneous consumers, \( \hat{y}_p = \bar{y} \), but with heterogeneous consumers \( \hat{y}_p \) and \( \bar{y} \) differ because a change in \( p \) alters the quantity weights used to define \( \bar{y} \) as per eq. (43).

\(^{10}\) Columns 5 and 6 of Table 1 are discussed later.
Using eqs. (43), (52), and (53), eqs. (49) and (50) can be written
\[
\frac{r - c}{r} = \frac{p}{\Omega} \left( \bar{y}_p - \bar{y} \right) \frac{\partial Q}{\partial p} = \frac{p}{\Omega} Q \frac{\partial \bar{y}}{\partial p}, \quad (54)
\]
\[
\frac{p - k}{p} = \frac{r}{\Omega} \left( \bar{y} - \bar{y}_r \right) \frac{\partial Q}{\partial r} = \frac{r}{\Omega} Q \frac{\partial \bar{y}}{\partial r}. \quad (55)
\]

Eqs. (54) and (55) lead immediately to:

**Proposition 4** (i) The core markup is strictly positive. (ii) The side markup is strictly positive if \( \bar{y}_p < \bar{y} \), or equivalently if average side-good consumption is an increasing function of the core price. The side markup is negative if the opposite is true, and the side markup is zero if average side-good consumption does not depend on the core price.

Part (i) of Prop. 4 holds because \( \partial \bar{y}/\partial r < 0 \) in (55). Part (ii) follows directly from (54). Eq. (54) reveals that the side markup is zero if average side-good consumption, \( \bar{y} \), does not depend on \( p \). A sufficient condition for this to hold is that consumers have identical tastes for the side good. This result is recorded as:

**Corollary 2** If consumers have identical side-good preferences, the side markup is zero.

With power-function preferences as in eq. (18), side-good preferences are defined by parameters \( \nu \) and \( \beta \). If consumers have the same \( \nu \) and \( \beta \), then \( r = c \) by Corollary 2. This property is recorded in the first row of column 5 in Table 1. The other three examples are covered in the remaining rows of column 5.

To understand Corollary 2, note that if consumers value the side good equally the monopolist can extract their side-good surplus using the core price in the same way as with identical individuals. Starkie’s (2001) complementarity effect therefore comes into play only if average side-good consumption is an increasing function of \( p \). This will be the case if consumers with strong core-good preferences also have strong side-good preferences. Conversely, the side markup may be negative if preferences for the two goods are negatively correlated.

Figure 1 illustrates a case in which demands are positively correlated. The aggregate inverse demand curve for the core good shown includes side-good surplus (i.e., \( g'(\theta) + S(r, \theta) \)). Consumers are positioned along the horizontal axis in order of decreasing willingness to pay. If the side-good price is reduced from \( r' > c \) to a value \( r'' \) closer to \( c \), the inverse demand curve shifts upward. If valuations of core and side goods are positively correlated, the inverse demand curve also becomes steeper because consumers with high valuations of the core good also value the side good highly, and gain more surplus from the reduction in \( r \). The monopolist cannot capture the greater surplus

---

11 Note from eqs. (54) and (55) that the markups satisfy \( \frac{r - c}{r} = \frac{p}{\Omega} \frac{\partial \bar{y}}{\partial p} = \varepsilon_{\bar{y}c}/\varepsilon_{\bar{y}}, \) where \( \varepsilon_{\bar{y}z} \) is the elasticity of \( \bar{y} \) with respect to \( z \). The markups therefore satisfy a sort of inverse elasticity rule even though demands for the two goods are interdependent.
Figure 1: Core revenue gain (shaded area) when the side price is reduced from \( r' \) to \( r'' \) and the core quantity is given by \( Q \).

from all consumers by raising \( p \) (the shaded area depicts the corresponding revenue gain), since consumers with lower valuations (near \( Q \) in Figure 1) would reduce their core-good consumption. The monopolist therefore sets a positive markup on the side good to tap some of the ‘extra’ surplus from consumers with high valuations without pricing out those with lower valuations.

The monopolist’s pricing strategy can also be understood using eq. (16). If all consumers were of the same type, \( \theta \), the monopolist would set \( r = c \), and charge a core price of \( p = k + \left\| g''(\theta) \right\| q(\theta) \) as per eq. (16). The core price is higher the larger is \( q(\theta) \) (i.e., the larger is demand), and the larger is \( \left\| g''(\theta) \right\| \) (i.e., the less elastic is core demand). With heterogeneous consumer types, the monopolist cannot price discriminate directly, but it can do so indirectly by adjusting the side price and thus the price differentials associated with the individual bundles of core and side consumptions. If types with high \( q(\theta) \) and/or high \( \left\| g''(\theta) \right\| \) have high side-good demands, it is profitable to set \( r > c \). Conversely, if these types have low side-good demands, the monopolist is better off setting \( r < c \), and exploiting the high core-good demand consumers with a high core-good price.

4.2 All-or-nothing side-good demand

With all-or-nothing side-good demand it is reasonable to assume that some consumers do not buy the side good. The model is easily extended to accommodate this. As in the fourth example of Section 3, let \( \sigma \) denote fixed consumption of the side good and \( e \) utility per unit consumed. Consumers with \( e > r \) buy \( \sigma \) units of the side good for each unit of the core good. Consumers with \( e \leq r \) do not buy the side good. Let \( \hat{\theta} \) denote consumer characteristics other than \( e \). Consumers who buy the side good demand \( q(p, r; e, \hat{\theta}) \) of the core good. Consumers who do not buy the side
good demand a quantity \( q(p, r; e, \hat{\theta}) \). Since consumer’s surplus from the side good decreases to zero as \( r \) approaches \( e \) from below, the two core-good demand functions converge:

\[
\lim_{r \rightarrow e} q(p, r; e, \hat{\theta}) = \overline{q}(p; \hat{\theta}).
\]  

(56)

Let \( \phi(e, \hat{\theta}) \) denote the joint frequency distribution of \( e \) and \( \hat{\theta} \), \( \chi(e) \) the marginal distribution of \( e \), and \( \psi(\hat{\theta}|e) \) the distribution of \( \hat{\theta} \) conditional on \( e \). In addition, let \( \overline{Q}(p, r) \) denote aggregate core demand from consumers who only buy the core good. Define \( Q(p, r) \) analogously for consumers who buy both goods. In place of (44), the profit function becomes

\[
\pi = (p - k + (r - c) \overline{y}) (\overline{Q}(p, r) + Q(p, r))
\]  

(57)

with

\[
\overline{Q}(p, r) = \int_{e=0}^{e=r} \chi(e) \int_{\hat{\theta}} \psi(\hat{\theta}|e)\overline{q}(p; \hat{\theta})d\hat{\theta}de,
\]  

(58)

\[
Q(p, r) = \int_{e=r}^{\infty} \chi(e) \int_{\hat{\theta}} \psi(\hat{\theta}|e)q(p, r; e, \hat{\theta})d\hat{\theta}de,
\]  

(59)

\[
\overline{y} = \sigma \cdot \frac{Q}{\overline{Q} + Q}.
\]  

(60)

Note that \( \overline{y} < \sigma \), and \( \partial (\overline{Q} + Q) / \partial r = \sigma \partial Q / \partial p \) by (56). Furthermore, since individuals who buy the side good consume a fixed amount \( \sigma \), \( \partial q(p, r; e, \hat{\theta}) / \partial r = \sigma \cdot \partial q(p, r; e, \hat{\theta}) / \partial p \) for \( e > r \). Incremental side consumptions can then be written as

\[
\hat{y}_r = \sigma - \frac{\partial \overline{Q}}{\partial r} / \frac{\partial Q}{\partial r} > \sigma,
\]  

(61)

\[
\hat{y}_p = \sigma \cdot \frac{\partial \overline{Q}}{\partial p} / \frac{\partial (\overline{Q} + Q)}{\partial p} < \sigma,
\]  

(62)

where the inequality in (61) follows from \( \partial \overline{Q} / \partial r > 0 \) and \( \partial Q / \partial p < 0 \). Since \( \hat{y}_r > \sigma \) and \( \overline{y} < \sigma \) as per (60), the core markup is positive. Consistent with intuition, the monopolist is unwilling to capture all consumers’ surplus from the side good by pricing it at marginal cost, and raising \( p \), if it stands to lose a lot of core-good sales from price-sensitive consumers who buy only the core good. The monopolist thus marks up the side good.

The core and side markups can be derived from the first-order conditions for a profit maximum:

\[
\frac{\partial \pi}{\partial r} = \sigma \left( (p - k + (r - c) \sigma) \frac{\partial Q}{\partial p} + Q - (r - c) \frac{\partial \overline{Q}}{\partial r} \right) = 0,
\]  

(63)

\[
\frac{\partial \pi}{\partial p} = (p - k + (r - c) \sigma) \frac{\partial Q}{\partial p} + Q + (p - k) \frac{\partial \overline{Q}}{\partial p} + \overline{Q} = 0.
\]  

(64)
Eqs. (63) and (64) yield:

\[
\frac{r - c}{r} = \frac{Q \overline{Q}}{\Delta} \left( \frac{\partial Q}{\partial p} \frac{1}{Q} - \frac{\partial Q}{\partial p} \frac{1}{\overline{Q}} \right) = \frac{Q \overline{Q}}{p \Delta} \left( \frac{\partial Q}{\partial p} \frac{p}{Q} - \frac{\partial Q}{\partial p} \frac{p}{\overline{Q}} \right),
\]

(65)

\[
\frac{p - k}{p} = \frac{1}{\Delta} \left( (Q + \overline{Q}) \frac{\partial Q}{\partial r} + \sigma Q \frac{\partial Q}{\partial p} \right),
\]

(66)

where

\[
\Delta = \sigma \frac{\partial Q}{\partial p} \frac{\partial Q}{\partial p} - \partial (Q + \overline{Q}) \frac{\partial Q}{\partial p} \frac{\partial r}{\partial r} > 0.
\]

The core markup in (66) is positive since \( \partial Q / \partial r > 0 \) and \( \partial Q / \partial p < 0 \). Eq. (65) reveals that the side markup is positive if consumers who buy both goods have less elastic core demands than consumers who buy only the core good (i.e., if \( \partial Q / \partial p \cdot p / Q > \partial Q / \partial p \cdot p / \overline{Q} \)).

Altogether, this leads to:

**Proposition 5** With all-or-nothing side demands, the core markup is strictly positive. The side markup is positive if consumers who buy both goods have less elastic core demands than consumers who buy only the core good, and negative in the opposite case.

To this point we have assumed that the side good is available for sale. If it is not, the core good will be sold by itself at a stand-alone price. One may wonder how the stand-alone price compares with the core price when the side good is sold as well. To address this, let \((r^*, p^*)\) denote the monopoly prices when the side good is sold, and \(p\) denote the stand-alone core price. Comparisons are facilitated if either \(q(p, \hat{\theta})\) does not depend on \(\hat{\theta}\), or if \(\hat{\theta}\) and \(e\) are independently distributed. (Both conditions will be satisfied, of course, if \(e\) is the only parameter distinguishing consumers so that \(\hat{\theta}\) is empty.) Substituting eq. (63) into (64), one obtains an equation with \((r^*, p^*)\) that involves only \(Q(p, r)\):

\[
(p^* - k) \frac{\partial Q}{\partial p} + \overline{Q} = -(r^* - c) \frac{\partial Q}{\partial r}.
\]

(67)

If the side good is not sold, the left-hand side of (67) is zero at the profit maximum by (64). Since marginal profit is a decreasing function of \(p\) at a profit maximum, (67) implies \(Sgn(p^* - p) = Sgn(r^* - c)\). Thus, if the common sign is positive the monopolist prices the core good above its standalone price and also prices the side good above marginal cost. Conversely, if the common sign is negative, the monopolist prices the core good below the stand-alone price and prices the side good below cost. In either case, the two prices move in the same direction. To see why, note that an increase in side-good demand as measured by \(\sigma\) increases core-good demand for consumers with \(e > r\). If the profit-maximizing core price increases with \(q\) (i.e., if \(\omega(q) < 1\) for a majority of consumers), the monopolist wants to raise the core price and also price discriminate against consumers with \(e > r\) who buy the side good. The monopolist therefore raises \(p^*\) above \(p\), and sets \(r^*\) above marginal cost. The opposite is true if the profit-maximizing core price is a decreasing
function of $q$. The size of the potential deviations, $p^* - \bar{p}$ and $r^* - c$, will be illustrated numerically in the next subsection.

A final point to note is that if the side good is cheap to produce relative to customers’ average valuation of it (i.e., if $c$ is small compared to the typical $e$), the monopolist wants most consumers to buy the side good since it increases their willingness to pay for the core good by more than it costs to produce. The monopolist will therefore price the side good inexpensively.

### 4.3 Specific functional forms and numerical results

**Continuous side-good demand** Unlike the case of homogeneous consumers, analytical results are elusive with heterogeneous consumers. A few results are derived in the appendix and recorded in the last column of Table 1. In the case of log-linear preferences, the side markup is zero not only if consumers have the same side-good preferences as per Corollary 2, but also if they have the same $b$ regardless of the distributions of the other parameters. In the case of linear-quadratic preferences, $r > c$ either if consumers differ in $e$ alone or if they differ in $f$ alone. In both cases, consumers with high side-good demand (i.e., large $e$ or small $f$) also have higher and less price-elastic core-good demands. Consequently, the monopolist optimizes price differentials for different bundles of core and side good consumptions by pricing the side good above marginal cost as explained in the previous subsection.

If consumers differ in multiple dimensions, analysis is unwieldy. To obtain some insights we present two numerical examples featuring linear-quadratic preferences. For the first example we consider heterogeneity in parameters $b$ and $f$ which govern the scale of demand for the two goods. The other parameter values are set to $a = 1$, $e = 2$, $c = 1$, and $k = 0$.

**Parameters** $b$ and $f$ are varied in such a way that the mean values of $1/f$ and $1/b$ are held fixed at 1 so that mean demand functions do not change. With no heterogeneity in $b$ or $f$, the profit-maximizing solution is $r^* = 1$ and $p^* = 0.75$. If the side good were not offered, the profit-maximizing price would be $\bar{p} = 0.5$.

As shown in Table 1, heterogeneity in $b$ (or $1/b$) does not affect markups if other parameters are constant. It is readily shown that if $b$ and $f$ are independently distributed, heterogeneity in $b$ still does not affect markups. To allow for correlation between $b$ and $f$ we examine the polar cases of perfect correlation. We assume that $1/f$ is uniformly distributed over the interval $[1/f_M, 1/f_m]$, and $1/b$ is uniformly distributed over the interval $[1/b_M, 1/b_m]$. With the mean values of $1/f$ and $1/b$ fixed at 1, the maximum range of each variable is 2.

Figures 2 and 3 display the profit-maximizing prices when $b$ and $f$ are positively correlated. Consistent with the analytical results, $r^*$ rises as the range of $1/f$ increases. It falls slightly as the range of $1/b$ increases. The core price moves in the opposite direction. With low heterogeneity in $1/f$ and high heterogeneity in $1/b$ (i.e., on the upper side of the dotted line in Figure 3) the core

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12 Except for the difference $e - c$, these parameter value assignments are without loss of generality.

13 In the case of positive correlation, $b$ varies with $f$ according to the function $\frac{1}{b} = \frac{1}{b_m} + \frac{1}{f/f_m-1/f_M} \left( \frac{1}{b_m} - \frac{1}{b_M} \right)$. For negative correlation, the function is $\frac{1}{b} = \frac{1}{b_m} - \frac{1}{f/f_m-1/f_M} \left( \frac{1}{b_m} - \frac{1}{b_M} \right)$. 

20
price is slightly above the benchmark level of 0.75 with no heterogeneity. Elsewhere, the core price is below 0.75, and it reaches a minimum value of about 0.66. Thus, heterogeneity has a moderating influence on the monopolist’s exercise of market power in the core market although the core price remains well above the level $\overline{p} = 0.5$ that would obtain without the side good.

Figures 4 and 5 show the results with negative correlation. Similar to the case of positive correlation, the side price rises as side-good demand becomes more heterogeneous while being little affected by heterogeneity in core-good demand. The core price decreases with heterogeneity in either dimension, and it reaches a minimum value below 0.62. Thus, consumer heterogeneity has a slightly stronger moderating influence on the core markup with negative correlation than with positive correlation.

The pattern of results shown in Figures 2-5 is explained as follows. Consumers with small values of $f$ have large side-good demands and derive a large surplus from it. Consequently, they have a high reservation price for the core good and correspondingly low price elasticities of demand. The monopolist exploits this by raising the side-good price. This motive applies whether demands for the two goods are positively or negatively correlated. Because mean demands are held fixed, the core price tends to decrease as the side price increases.

As shown in Figure 3, one exception to this pattern occurs with positive correlation when consumers differ mainly in their core-good demands. Consumers with high side-good demand (small $f$) also have high core-good demand (small $b$) and thus contribute a lot to total demand. The monopolist takes advantage of their inelastic demands by raising $p$ slightly above the homogeneous-consumer level. The same pattern does not occur with negative correlation because consumers with high side-good demands have low core-good demands, and contribute less to the monopolist’s total profit. Put another way, consumers with high core-good demands have low side-good demands and relatively low reservation prices for the core good. Since they comprise an important consumer segment, the monopolist caters to them by setting a low core-good price that is lower than the lowest price with positive correlation.

**All-or-nothing side-good demand** The second example features all-or-nothing side demand with linear-quadratic preferences. Four of the parameters are set to the same values as in the first example: $a = 1$, $b = 1$, $c = 1$, and $k = 0$. Consumers are assumed to differ in parameter $e$, with $e$ distributed uniformly on the interval $[0, e_{Max}]$. Parameter $\sigma$ is the same for everyone. Parameters $e$ and $\sigma$ are both varied parametrically.

The results are displayed in Figure 6-8. With $e_{Max} = 1$, no consumer values the side good more than it costs to produce, and the side good is not sold. The (nominal) price is set to $r^* = c = 1$ (Figure 6), the core good is priced at its standalone value of $\overline{p} = 0.5$ (Figure 7), and no one buys the side good (Figure 8). As $e_{Max}$ increases, the monopolist raises both prices (recall that $\omega(q) = 0$ with linear-quadratic preferences) and the fraction of consumers who buy the side good rises too.
Figure 2: Positive correlation in $b$ and $f$. Price of side good ($r^*$)

Figure 3: Positive correlation in $b$ and $f$. Price of core good ($p^*$)
Figure 4: Negative correlation in $b$ and $f$. Price of side good ($r^*$)

Figure 5: Negative correlation in $b$ and $f$. Price of core good ($p^*$)
In the limit $\sigma \to 0$, the fraction of buyers converges to $1 - c/e_{Max}$.

Compared to the first example, the side price rises much further because willingness to pay is varied upwards rather than held fixed. However, the core price is restrained because of the high profits derived from the side good (the complementarity effect at work) as well as the fact that some consumers do not buy the side good.

5 Duopoly

We now introduce competition into the model by considering two firms a distance $2L$ apart. One firm (“the home firm”) is located at $x = 0$, and the other (“the rival”) is located at $x = 2L$. Consumers are distributed along the line between them, and incur a travel cost of $t$ per unit distance each time they purchase a unit of the core good. Home firm and rival choose prices simultaneously to maximize their respective profits. The home firm sets prices $(p, r)$, and the rival sets prices $(\hat{p}, \hat{r})$. A consumer at $x$ therefore pays a full price or generalized cost for the core good of $p + tx$ at the home firm, and $\hat{p} + t(2L - x)$ at the rival.

5.1 General analytical results

Consider the home firm’s choice of $(p, r)$ while taking $(\hat{p}, \hat{r})$ as given. The first step is to consider the market boundaries between the firms. Let $V(p + tx, r; \theta)$ denote the utility derived from patronizing the home firm by a consumer of type $\theta$ located at $x$. Define $V(\hat{p} + t(2L - x), \hat{r}; \theta)$ analogously for the rival. The market boundary between the firms, $x(\theta)$, is defined by the condition

$$V(p + tx(\theta), r; \theta) = V(\hat{p} + t(2L - x), \hat{r}; \theta). \quad (68)$$
Figure 7: Heterogeneity in $e$. Price of core good ($p^*$)

Figure 8: Heterogeneity in $e$. Fraction of users who buy side good
Differentiating (68) with respect to \( p \) one obtains:

\[
\frac{\partial V}{\partial p} (p + tx(\theta), r; \theta) \left( 1 + t \frac{\partial x(\theta)}{\partial p} \right) = - \frac{\partial V}{\partial p} (\hat{p} + t (2L - x), \hat{r}; \theta) \cdot t \frac{\partial x(\theta)}{\partial p}.
\]

Given \( \frac{\partial V}{\partial p} = -q(p + tx(\theta), r; \theta) \), this implies\(^{14}\)

\[
\frac{\partial x(\theta)}{\partial p} = - \frac{1}{t} \frac{q(p + tx(\theta), r; \theta)}{q(p + tx(\theta), r; \theta) + q(\hat{p} + t (2L - x), \hat{r}; \theta)}.
\]

Differentiating (68) with respect to \( r \), and using \( \frac{\partial V}{\partial r} = -s(p + tx(\theta), r; \theta) \), one obtains a corresponding derivative for the side price:

\[
\frac{\partial x(\theta)}{\partial r} = - \frac{1}{t} \frac{s(p + tx(\theta), r; \theta)}{q(p + tx(\theta), r; \theta) + q(\hat{p} + t (2L - x), \hat{r}; \theta)}.
\]

Let \( \phi(\theta, x) \) denote the joint frequency distribution of consumers according to type and location, and \( \psi(\theta|x) \) the frequency distribution of \( \theta \) conditional on \( x \). Furthermore, let \( z \) be a parameter or choice variable such as \( p \) or \( r \). The home firm’s profits are similar to (44) with

\[
Q(p, r) = \int_{\theta} \int_{x=0}^{x(\theta)} q(p + tx, r; \theta) \phi(\theta, x) \, dx \, d\theta,
\]

\[
\frac{\partial Q(p, r)}{\partial z} = \int_{\theta} \int_{x=0}^{x(\theta)} \frac{\partial q(\theta, x)}{\partial z} \phi(\theta, x) \, dx \, d\theta + \int_{\theta} \frac{\partial x(\theta)}{\partial z} q(p + tx(\theta), r; \theta) \psi(\theta|x(\theta)) \, d\theta,
\]

\[
\bar{y} = \frac{1}{Q} \int_{\theta} \int_{x=0}^{x(\theta)} s(p + tx, r; \theta) \phi(\theta, x) \, dx \, d\theta,
\]

Compared to eq. (42), eq. (71) includes an additional integral with respect to \( x \). The derivatives in (72) also include terms reflecting how a change in \( p \), \( r \), or some other parameter shifts the market boundaries for each consumer type. Eqs. (49) and (50) can be used to derive the home firm’s markups.

A modified version of Prop. 3 and Coroll. 2 carry over from monopoly to competition. The results are recorded as:

**Proposition 6** If none of the demand price elasticities or market boundaries depends on parameters that vary in the population, then the duopoly core and side markups do not depend on the frequency distribution of those parameters.

**Corollary 3** If consumers have identical side-good preferences, the duopoly side markup is zero.

Coroll. 3 is proved in the appendix. In order for the invariance result mentioned in Prop. 6 to hold, the market boundaries, \( x(\theta) \), and their derivatives, \( \partial x(\theta)/\partial p \) and \( \partial x(\theta)/\partial r \), must

\(^{14}\)In a symmetric equilibrium, considered below, quantities purchased from the home firm and rival are equal, and the right-hand side of eq. (69) reduces to \(-1/(2t)\) as in a standard Hotelling model with inelastic individual demands.
be invariant to parameters. In a symmetric equilibrium these conditions will be satisfied if the price-elasticity conditions are satisfied. All boundaries occur at the mid-point between the firms (i.e., \( x(\theta) = L \)). Moreover, the derivatives simplify to \( \partial x(\theta)/\partial p = -1/(2t) \) and \( \partial x(\theta)/\partial r = -y(\theta)/(2t) \). If the elasticities are invariant to parameter values, then so is \( y(\theta) \) and also \( \partial x(\theta)/\partial r \).

### 5.2 Uniform consumer locations and linear-quadratic preferences

We now assume that consumers are uniformly distributed on the line \( x \in [0, 2L] \), and have linear-quadratic preferences. We limit attention to symmetric equilibria in which home firm and rival set identical prices (i.e., \((p, r) = (^p, ^r)\)) so that market boundaries are \( x = L \) for all consumer types.

**Homogeneous side-good preferences** To begin, assume consumers differ only in their locations. Corollary 3 then applies, and the side markup is zero. Let \( T \equiv tL \) denote the transport cost for a consumer located at the market boundary. Let \( \tilde{a} \equiv a + S \) denote the reservation price for the core good, where \( S = (e - c)^2/(2f) \) is consumer’s surplus from the side good with \( r = c \). Using routine algebra, the symmetric equilibrium price is found to be

\[
p = \frac{\tilde{a} + k + 3T - \sqrt{((\tilde{a} - k)^2 - (2\tilde{a} - 2k - 13T)T)}}{2}.
\] (74)

With \( T = 0 \), eq. (74) reduces to the the Bertrand equilibrium result with \( p = k \). Parameter values are constrained by the condition that the consumer at the boundary purchases a nonnegative quantity (i.e., \( p + T \leq \tilde{a} \)). This condition translates to \( T \leq 2(\tilde{a} - k)/3 \). At the maximum value \( T = 2(\tilde{a} - k)/3 \), the equilibrium price is \( p = 2k/3 + \tilde{a}/3 \).

The effect of side-good demand on the core price can be measured using the derivative

\[
\frac{\partial p}{\partial S} = \frac{1}{2} \left( 1 - \frac{\tilde{a} - k - T}{\sqrt{(\tilde{a} - k)^2 - (2\tilde{a} - 2k - 13T)T}} \right).
\] (75)

Over the range \( T \in (0, 2(\tilde{a} - k)/3) \), the derivative in eq. (75) increases monotonically from 0 up to 3/7.

The duopoly equilibrium can be compared with monopoly for the same spatial market. The profit-maximizing monopoly price works out to\(^{16}\)

\[
p_{Mon} = \frac{\tilde{a} + k}{2} - \frac{T}{4}.
\] (76)

\(^{15}\)The equilibrium price reaches a maximum value at \( T = \frac{1 + \sqrt{27}}{14} (\tilde{a} - k) \), which is below \( 2(\tilde{a} - k)/3 \).

\(^{16}\)The same constraint \( T \leq 2(\tilde{a} - k)/3 \) applies as with duopoly, and with \( T = 2(\tilde{a} - k)/3 \) the monopoly price matches the duopoly price.
Equation (76) shows that the monopolist raises the core-good price by $0.50 for each $1 increase in side-good surplus. Therefore, the demand effect is weaker with duopoly than monopoly, and it gets weaker the closer the firms are to each other (i.e., the smaller is $L$).

Equations (74) and (76) are plotted in Figure 9 for $k = 0$, $a = 1$, $t = 1$, and two values of $S$: $S = 0.5$ and $S = 0.6$. With the higher value of $S$, the monopoly price rises uniformly by 0.05 for all values of $T$. By contrast, the duopoly price rises by an amount that increases with $T$, but remains below 0.05. In both cases the market range increases from $T = 1$ to $T = 1.1$.

**Heterogeneous side-good preferences** Now assume that parameter $f$ which governs the scale of side-good demand varies with location. To simplify the analysis, assume that all consumers have the same value of $e$, and also that all consumers at the same $x$ have the same $f$. Define $F = 1/f$. Assume that $F = \bar{F} + w(x - L/2)$, $x \in [0, L]$, with $w \in [-2\bar{F}/L, 2\bar{F}/L]$ so that $F \geq 0$ for all $x$. An equivalent density function applies for the rival with $x \in [L, 2L]$. If $w > 0$, side-good demand increases with distance from the firm. This is plausible if alternative shopping opportunities and services are plentiful near the firm, as might be the case for a central railway station. If $w < 0$, side-good demand decreases with distance, which is plausible for an airport in a distant suburb.

Analytical results are elusive with $w \neq 0$, so we proceed numerically. Similar to the earlier examples, parameter values are set to $a = 1$, $b = 1$, $e = 2$, $\bar{F} = 1$, $c = 1$, and $k = 0$. The location-related parameters are set to $L = 1$ and $t = 0.5$. Figure 10 shows the effects of varying parameter $w$ over its admissible range of $[-2, 2]$. For positive and moderate values of $w$, $r$ decreases with $w$ and $p$ increases. Although more remote consumers have a somewhat higher side-good demand than consumers close to the firm, they incur a higher transport cost. Remote consumers therefore have lower, and more price elastic, core-good demands and the firm discriminates in their favour by pricing the side good cheaply. For large values of $w$, however, remote consumers become less
price elastic and the pattern of prices reverses direction. To see why, note that a consumer located at \( x \) has a reservation price for the core good of

\[
a + \frac{F(e - r)^2}{2} - tx = a + \left( \bar{F} + w \left( L - \frac{x}{2} \right) \right) \frac{(e - r)^2}{2} - tx
\]

\[
= a + \left( \bar{F} - wL \right) \frac{(e - r)^2}{2} + \left( w \frac{(e - r)^2}{2} - t \right) x.
\]

When \( w \) is large enough that \( w(e - r)^2 / 2 - t > 0 \), the reservation price increases with \( x \).

For \( w < 0 \), the pattern of pricing is qualitatively the opposite to \( w > 0 \), with \( p \) first decreasing as \( w \) rises in magnitude and then increasing. The core price reaches a minimum value of \( p \simeq 0.437 \) with \( w \simeq -1 \). Since this is still above the benchmark value of \( p \simeq 0.349 \) that obtains if the side good is not offered, the demand effect is still positive.

Figure 11 displays the results of another experiment with \( w = -1 \) and the unit transport cost reduced to \( t = 0.25 \). Parameter \( e \) is varied from a low value of \( e = c = 1 \) (for which the side good is not sold) up to a high value of \( e = 3 \). Over the full range of \( e \) the side price rises, and the core price falls. This shows that Starkie’s (2001) conjecture can be borne out when consumers are heterogeneous in certain ways even though the conjecture is not fulfilled in the representative-consumer model with monopoly and homogeneous consumers.

6 Concluding Remarks

Many firms sell both core goods and side goods with interdependent demands. How does the sale of side goods affect pricing of the core goods? Does it moderate or enforce the exercise of market power
in the core business area? To investigate these questions we develop a general model that captures the complementarity and demand effects of side-good supply, and also embodies the assumption that individual demand for the side good is proportional to demand for the core good.

We show that if consumers have identical preferences for the side good a monopolist prices the side good at marginal cost so that the complementarity effect does not come into play. The demand effect can be positive, negative, or zero depending on whether the price elasticity of core demand decreases, increases, or remains unchanged in magnitude as the demand curve shifts out. We illustrate the three possibilities using commonly used demand functions — thereby demonstrating the importance of assumptions about functional forms. A few airport pricing studies have adopted linear functional forms that drive their conclusions about the effects of side-good supply on core markups.

The picture becomes more complicated when consumers differ in their side-good preferences. For example, if consumers with inelastic core good demands consume more of the side good than the average consumer, the complementarity effect comes into play because the monopolist takes advantage of their inelastic core demands by charging a positive side markup. In other cases the correlation between demands may be negative, and in yet other cases some individuals may not consume the side good at all.

We show that most of the monopoly results generalize to a Hotelling duopoly market in which consumers are distributed along a line with firms located at either end. In this setting core-good demand tends to decrease with distance from a firm because of transportation costs. Side-good demand can increase or decrease depending on the locations of substitutes. Altogether, our results raise doubts about Starkie’s (2001) conjecture that side-good revenues alleviate concerns about market power in the core business area.

There are several avenues for future research. Our analysis of competition was limited to
duopoly. This may be reasonable for airports, but an extension to multiple firms is desirable to describe other markets such as gasoline retailing. It is also of interest to consider firms in complementary relationships. In the case of air travel the airports at opposite ends of a route sell complementary services. The airport at the origin provides parking services, the airport at the destination provides car rentals, and both airports provide baggage handling. Another extension is to consider separate control of the core good and side good businesses through concession or other institutional arrangements. Finally, congestion or other negative externalities such as pollution are an important feature of some markets. Congestion pricing is a means of internalizing congestion externalities and the use of congestion fees at airports has been studied (e.g., Zhang and Zhang, 2010; Zhang and Czerny, 2012; Czerny, 2013; D’Alfonso et al., 2013). Since congestion externalities in air transport depend on airline market shares (e.g., Daniel, 1995; Brueckner, 2002, and Zhang and Zhang, 2006), the consideration of airline markets then becomes crucial for the analysis as well.
7 Notational Glossary

7.1 Latin characters

\(a\): reservation price of demand function for core good (linear demand)
\(b\): slope of inverse demand function for core good (linear demand)
\(c\): unit production cost of side good
\(e\): reservation price for side good (linear demand)
\(f\): slope of inverse demand function for side good (linear demand)
\(F\): \(1/f\)
\(g\): subutility function for core good (general demand function)
\(h\): subutility function for side good (proportional preferences)
\(k\): unit production cost of core good
\(L\): half the distance between firms in duopoly model
\(m\): income
\(p\): price of core good
\(q\): individual consumption of core good
\(Q\): aggregate consumption of core good
\(r\): price of side good
\(s\): individual consumption of side good
\(S\): consumer’s surplus from consumption of side good
\(t\): cost of transport per unit distance
\(u\): parameter of subutility function for core good
\(U\): utility function
\(v\): parameter of subutility function for side good
\(V\): indirect utility function
\(w\): slope of \(F\) with respect to \(x\)
\(x\): consumer location
\(y\): inverse function: \(y = (h')^{-1}\)

7.2 Greek characters

\(\alpha\): parameter of Cobb-Douglas subutility function for side good
\(\beta\): parameter of Cobb-Douglas subutility function for side good
\(\gamma\): parameter of power subutility function for core good
\(\Delta\): discriminant
\(\varepsilon_{ij}\): elasticity of demand for good \(i\) with respect to price of good \(j\)
\(\theta\): consumer type
\(\omega\): measures curvature of core demand
\(\pi\): profit
\( \phi \): density of consumer type
\( \chi(e) \): marginal density of \( e \)
\( \psi(\theta|e) \): density of \( \theta \) conditional on \( e \)
\( \Psi \): subutility function for side good (general preferences)
\( \Omega \): composite variable in markup formulas

8 Appendix

8.1 Proof of Proposition 2

Given \( r = c \), the first-order condition (3) for \( p \) simplifies to \((p - k)(\partial q/\partial p) + q = 0\). Given \( \partial q/\partial p = 1/g''(q) \) in eq. (13),

\[
p = k - g''(q)q.
\]

(77)
The first-order utility-maximization condition for \( q \) is eq. (12):

\[
g'(q) + S(r) = p.
\]

(78)

Totally differentiating eqs. (77) and (78) with respect to \( S \), while holding \( r = c \), yields the system of equations:

\[
\begin{bmatrix}
-g'' \cdot (\omega(q) - 1) & 1 \\
g''
\end{bmatrix}
\begin{bmatrix}
\frac{dq}{dS} \\
\frac{dp}{dS}
\end{bmatrix}
= \begin{bmatrix}
0 \\
-1
\end{bmatrix}.
\]

The second-order condition for \( p \) assures that the discriminant, \( \Delta = g'' \cdot (\omega(q) - 2) \), is strictly positive. The derivatives work out to

\[
\frac{dq}{dS} = \frac{1}{\Delta} > 0,
\]

\[
\frac{dp}{dS} = \frac{g'' \cdot (\omega(q) - 1)}{\Delta} = g'' \cdot (\omega(q) - 1) \cdot \frac{dq}{dS}.
\]

Given \( dq/dS > 0 \), \( dp/dS > 0 \) if \( \omega(q) < 1 \) which is equivalent to \( d\|qg''\|/dq > 0 \). Conversely, \( dp/dS < 0 \) if \( \omega(q) > 1 \). QED

8.2 Log-linear preferences

With log-linear preferences the markup eqs. (47) and (48) become

\[
\frac{r - c}{r} = \frac{1}{p\Delta} \left\{ \int_{\theta} \left( -b \left( \frac{\beta_v}{r} \right)^{1-\beta} q(\theta) \phi(\theta) d\theta \cdot \int_{\theta} q(\theta) \phi(\theta) d\theta \right) + \int_{\theta} bq(\theta) \phi(\theta) d\theta \cdot \int_{\theta} \left( \frac{\beta_v}{r} \right)^{1-\beta} q(\theta) \phi(\theta) d\theta \right\},
\]

(79)
\[
\frac{p - k}{p} = \frac{1}{p\Delta} \left\{ -\int_{\theta} q(\theta) \phi(\theta) d\theta \cdot \int_{\theta} \left( \frac{1}{1 - b} + b\nu \left( \frac{\beta v}{r} \right) \right) \left( \frac{\beta v}{r} \right)^{\frac{1}{r-\beta}} q(\theta) \phi(\theta) d\theta \right\},
\]
where
\[
\Delta \equiv \int_{\theta} q(\theta) \phi(\theta) d\theta \cdot \int_{\theta} \left( \frac{1}{1 - b} + b\nu \left( \frac{\beta v}{r} \right) \right) \left( \frac{\beta v}{r} \right)^{\frac{1}{r-\beta}} q(\theta) \phi(\theta) d\theta,
\]
and \(q = a \exp \left( -\beta p + b\nu \left( \frac{\beta v}{r} \right)^{\frac{1}{r-\beta}} \right)\). If consumers have a common \(b\), the numerator of (79) is zero and \(r = c\).

### 8.3 Linear-quadratic preferences

With linear-quadratic preferences the markup eqs. (47) and (48) are

\[
\frac{r - c}{r} = \frac{1}{\Delta} \left\{ \int_{\theta} q(\theta) \phi(\theta) d\theta \cdot \int_{\theta} \left( \frac{1}{2f(a-p)+e(r)^2} \right) q(\theta) \phi(\theta) d\theta \right\},
\]
and

\[
\frac{p - k}{p} = \frac{1}{\Delta} \left\{ \int_{\theta} q(\theta) \phi(\theta) d\theta \cdot \int_{\theta} \left( \frac{1}{e} + \frac{2f(a-p)+e(r)^2}{2f(a-p)+3e(r)^2} \right) q(\theta) \phi(\theta) d\theta \right\},
\]
where
\[
\Delta \equiv \int_{\theta} q(\theta) \phi(\theta) d\theta \cdot \int_{\theta} \left( \frac{1}{2f(a-p)+e(r)^2} \right) q(\theta) \phi(\theta) d\theta.
\]

\(q = \frac{1}{b} \left( a + \frac{(e-r)^2}{2f} - p \right) = \frac{f(a-p)+S}{b^2} \) and \(S = \frac{(e-r)^2}{2} \). Using this formula for \(q\), eqs. (81) and (82) can be written as

\[
\frac{r - c}{r} = \frac{1}{r} \int_{\theta} q(\theta) \phi(\theta) d\theta \cdot \int_{\theta} \left( \frac{f(a-p)+S}{b^2} \right) q(\theta) \phi(\theta) d\theta + \int_{\theta} \phi(\theta) d\theta \cdot \int_{\theta} \frac{e-r}{b} \frac{f(a-p)+S}{b^2} \phi(\theta) d\theta,
\]
and

\[
\frac{p - k}{p} = \int_{\theta} q(\theta) \phi(\theta) d\theta \cdot \int_{\theta} \left( \frac{e-r}{b} \right) \frac{f(a-p)+S}{b^2} \phi(\theta) d\theta + \int_{\theta} \phi(\theta) d\theta \cdot \int_{\theta} \frac{e-r}{b} \frac{f(a-p)+S}{b^2} \phi(\theta) d\theta.
\]
If consumers have the same \(a\) and \(b\), the numerator of (83) simplifies and

\[
Sgn \left( \frac{r-c}{r} \right) = \int_{e,f} \frac{(e-r)^3}{f^2} \phi(e,f) \, df \, de - \int_{e,f} \frac{e-r}{f} \phi(e,f) \, d\theta \int_{e,f} \frac{(e-r)^2}{f} \phi(e,f) \, df \, de.
\]

(85)

The right-hand side of (85) cannot be signed in general. If consumers differ only in \(e\), then

\[
Sgn \left( \frac{r-c}{r} \right) = \int_{e} (e-r)^3 \phi(e) \, de - \int_{e} (e-r) \phi(e) \, de \cdot \int_{e} (e-r)^2 \phi(e) \, de > 0.
\]

If consumers instead differ only in \(f\), then

\[
Sgn \left( \frac{r-c}{r} \right) = \int_{f} \frac{1}{f^2} \phi(f) \, df - \left( \int_{f} \frac{1}{f} \phi(f) \, df \right)^2 = \text{Var} (f^{-1}) > 0,
\]

(86)

where \(\text{Var} (f^{-1})\) is the variance of \(1/f\).

Let \(E (f^{-1})\) denote the mean value of \(1/f\) and \(CV (f^{-1}) \equiv \text{Var} (f^{-1}) / E (f^{-1})\) denote the coefficient of variation. The core markup simplifies to:

\[
\frac{p-k}{p} = \frac{(a-p+S \cdot E (f^{-1}))^2 + (3(a-p)+S \cdot E (f^{-1})) \cdot S \cdot CV (f^{-1})}{a-p+S \cdot E (f^{-1}) + 3 \cdot S \cdot CV (f^{-1})},
\]

(87)

which is a decreasing function of \(CV (f^{-1})\). Holding the mean value of \(1/f\) fixed, an increase in demand heterogeneity for the side good induces the monopolist to reduce the core markup. By raising the side markup as per (86), and reducing the core markup as per (87), the monopolist price discriminates against consumers with high side-good demand and correspondingly inelastic core-good demand.

### 8.4 Proof of Corollary 3

To economize on writing, define \(\hat{x} (\theta) = 2L - x (\theta)\) and let \(\hat{q} (\theta, \hat{x} (\theta))\) denote demand for the rival’s core good. Substituting (14) into eqs. (47) and (48) the markup equations become:

\[
\left( \int_{\theta} \int_{x=0}^{x(\theta)} \frac{y(\theta)}{g^2(\theta)} \phi(\theta, x) \, dx \, d\theta - \frac{1}{\tau} \int_{\theta} \int_{x=0}^{x(\theta)} \frac{y(\theta)q^2(\theta,x(\theta))}{q(\theta,x(\theta)) + q(\theta,x(\theta))} \psi(\theta|x(\theta)) \, d\theta \right)
\]

\[
\cdot \int_{\theta} \int_{x=0}^{x(\theta)} q(\theta, x) \phi(\theta, x) \, dx \, d\theta
\]

\[
- \left( \int_{\theta} \int_{x=0}^{x(\theta)} \frac{1}{g^2(\theta)} \phi(\theta, x) \, dx \, d\theta - \frac{1}{\tau} \int_{\theta} \int_{x=0}^{x(\theta)} \frac{q^2(\theta,x(\theta))}{q(\theta,x(\theta)) + q(\theta,x(\theta))} \psi(\theta|x(\theta)) \, d\theta \right)
\]

\[
\cdot \int_{\theta} \int_{x=0}^{x(\theta)} y(\theta) q(\theta, x) \phi(\theta, x) \, dx \, d\theta
\]

If \(y(\theta)\) is uncorrelated with both \(\frac{1}{g^2(\theta)}\) and \(q(\theta)\), the side markup is zero. Hence the side markup is zero if consumers have identical side-good preferences. QED.
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