Sequential Auctions of More Than Two Objects with Synergies

Pranjal Chandrakar, Manaswini Bhalla, Shubhabrata Das

January 18, 2020

Abstract

We examine a model of sequential English/Japanese auction in which, the objects being auctioned share synergetic relationships (complementarity and substitutability) between them. The objects being auctioned are divided into categories, which are defined as collections of substitutable items. Inter-category objects are treated as complements. Bidders demand one unit from each category and aim to create a bundle of inter-category/complementary objects. We solve for the optimal bidding strategies, equilibrium selling prices, bidders' profits, and the seller's revenue for this game. We discover that the present auction mechanism is efficient and that the order of sale doesn't affect the seller's total expected revenue. However, it may affect the selling prices of the objects being auctioned. Specifically, a decreasing trend in the selling prices is observed under certain conditions. We also discover the possibility of false bidding when certain assumptions of the model are violated. False bidding, if present, may increase the revenue of the seller but decrease bidders' profits.

JEL Classification Code: Auctions D440

Keywords: Sequential Auctions, Synergy, Substitutes, Complements, Categories

1 Introduction

Sequential auctions have frequently been used to sell several non-identical objects like artifacts, cattle, retail, construction projects, etc. In many of these auctions, the objects being auctioned are not entirely independent but share some simplistic relationships like complementarity or substitutability between them. Such relationships are very common in several real life applications of sequential auctions. For example, in the auction of the two construction projects, expertise gained in the first project can be useful in another. Hence, the two projects may be treated as complementary by the bidders. However, one of the popular and relatively newer examples of such auctions are those associated with sports tournaments.

Sequential auctions are used in many sports tournaments to sell the services of sports players. Some of the well known examples of such auctions are those affiliated to Indian Premier League (IPL)- a professional T20 Cricket tournament, Pro Kabaddi League (PKL) - a professional Kabaddi tournament, Premier Badminton League (PBL)- a professional badminton tournament, etc. Such tournaments are usually associated with team sports, which require a wide array of skills to win the game. Hence, the participating teams have to be proficient in every aspect of the game in order to be successful. Players participating in the tournament specialize in different skills and hence, categorized according to the values that they add to their teams. In order to be successful in the tournament, a team-management desires to have skilled players from each category in their team. Therefore, the complementarities between sports players with different specialities become an essential consideration for a team-management. Typically, several players are available in the auction who specialize in the same skill and hence, belong to the same category. Such players are treated as substitutes by the team-managers. Thus, for each aspect of the game, a team-management has several substitutable players to choose from. Hence, in this case, the number of objects available for sale is more than two.

Another example to explain the relevance of such an auction setting is the flower auction. Imagine a floriculturist who sells flowers of different breeds and colors via auctions. Flowers of several breeds like Roses and Lilies are available for the auction, with each breed available in several colors. Many florists participate in the auction and buy flowers of different kinds. They create various products like bouquets, flower baskets, etc. using the flowers, and sell them in the market. Florists consider many of the flower-breeds complementary because they need them together in their products. For example, a florist may consider both Roses and Lilies to be essential for his bouquet, and hence, treat them as complementary. However, in their product, florists may be willing to substitute between different colored flowers of the same breed. For example, they may be willing to substitute red-colored roses with white ones, etc. The florists aim to create a bundle of flowers which makes a good bouquet and can be sold in the market. With the help of the two examples above, we show that the sequential auction of more than two objects with complementary and substitutable relationships between them can be observed in diverse economic situations.

Complementary and substitutable relationships between the objects being auctioned are also known as synergies. When there are only two objects available for sale in an auction, only one of the two synergetic relationships is possible between them. However, in the presence of more than two objects, both types of relationships can co-occur, as mentioned in the examples above. In such an auction, bidders demand multiple objects and aim to create a bundle of complementary items when substitutes of each constituent of the bundle are also available for the auction. In this study, we try to understand the dynamics of such a sequential auction by solving for all possible equilibria.

Sequential first and second-price auctions, in the presence of identical objects, were first analyzed by [Milgrom & Weber, 1982]. Studies in the literature which attend to the sequential auctions of non-identical objects can be classified into two major streams, namely stochastically equivalent objects and heterogeneous objects. Some notable studies associated with stochastically equivalent objects are [Sørensen, 2006, Bernhardt & Scoones, 1994, Engelbrecht-Wiggans, 1994]. In the auctions of stochastically equivalent objects, bidders derive their valuations of non-identical objects from the same commonly known distribution. Therefore, the objects themselves are non-identical, but since their valuations are independently and identically distributed, they are stochastically equivalent. In such models, bidders valuations are private information throughout the game. Heterogenous objects have been studied by [Muramoto & Sano, 2016, Kittsteiner et al., 2004, Elmaghraby, 2003]. In the models involving the heterogeneous objects, bidders derive their signals from a publicly known distribution, but their signal themselves remains private information. Followed by this, a bidder calculates his valuations of all the non-identical objects using his private signal and a set of publicly known functions, each corresponding to an object. In our study, we use a combination of the two approaches mentioned above. Bidders derive one signal for each category of objects, and these signals are stochastically equivalent. To obtain the valuations of all the objects from a particular category, a bidder uses his private signal for that category as the argument to a set of publicly known functions. Each of these functions corresponds to an object from that category.

This paper also contributes to the literature of synergies in sequential auctions. The ways of modeling synergies in the literature can be divided into two major groups, namely deterministic and stochastic. A deterministic approach to model synergies evaluates the value of the bundle of objects by either multiplying or adding a constant to the sum of the values of individual objects [Branco, 1997, Menezes & Monteiro, 2003, Menezes & Monteiro, 2004, Leufkens et al., 2010]. This method of modeling synergies has been utilized with both heterogeneous and stochastically equivalent objects. Apart from the deterministic approaches, stochastic techniques have also been employed in the literature to model synergies [Jeitschko & Wolfstetter, 2002, De Silva et al., 2005, Jofre-Bonet & Pesendorfer, 2014]. In such models, a bidder draws his value of an object just before it is presented, i.e., bidders don't know their valuation of the second object while bidding for the first one. If the two objects being presented are complementary, then the winner of the first object draws his value of the second object from a distribution which stochastically dominates that of the first one. In other words, the winner of the first object is more likely to draw a higher value of the second object. However, the loser of the first object draws his valuation of the second item from the same distribution. Because of the method by which the stochastic synergies are modeled, they can only be used with stochastically equivalent objects. In our study, we use deterministic method to model complementary relationship between the inter-category objects. Specifically, in our model, the value of a bundle of complementary objects is obtained by multiplying the sum of individual values by a constant greater than one.

In summary, all the studies in the literature of sequential auctions analyze the effects of synergies in the presence of only two objects. To the best of our knowledge, our study is the first one to extend the literature by incorporating more than two objects into the model. Inclusion of more than two objects allows for both types of synergies to co-occur. Allowing for different kinds of synergies across the objects is what makes our study distinct from the rest of the literature.

In our model, we look at the sale of four non-identical objects with synergetic relationships between them. The objects are classified into two categories, where a category is defined as a collection of substitutable objects. Additionally, inter-category objects are treated as complements. Two risk-neutral bidders participate in the auction and demand one unit from each category. They aim to create a bundle of complementary (inter-category) objects, which brings them the maximum profit. The game takes place in two periods. In each period, both objects of a particular category are auctioned via a sequence of English auctions.

We assess all possible outcomes of the game. We find that the final outcome of the game depends on the bidders' preferences over substitutable objects. From the two available objects from a particular category, the object which offers more utility to a bidder is said to be ranked higher by him. In each category, bidders independently rank any of the two objects higher. If both bidders rank the same object higher in a particular category, we say that they compete in that category. We find that all possible outcomes of the game can be classified into four sets, depending on the presence or absence of the competition in the two categories. For each set of outcomes, we find out the optimal bidding strategy, equilibrium selling prices, bidders' profits and the seller's revenue. We find that bidders bid most aggressively when the competition is present in both categories. This leads to a high seller's revenue but low profits for the bidders. At the other extreme, when the competition is absent from both categories, all the objects are sold at their reserve prices. This outcome results in a low seller's revenue but high profits for the bidders. Furthermore, when bidders compete in only one of the two categories, their bidding behavior is moderately aggressive. Because of this, the seller's revenue and bidders profits are in between the two extreme outcomes previously described.

We also explore the possibility of any trend in the selling prices of the objects because of the order in which the objects are presented during the auction. We observe a decreasing trend in the selling prices when bidders compete in both categories. However, the total expected revenue of the seller remains independent of the order of sale in all possible outcomes. We discover that there is no possibility of an exchange of items between the bidders post-auction, and hence, conclude that the auction mechanism is efficient. Finally, we find that if certain assumptions of the models are violated, such an auction format can also encourage false bidding. We say that a bidder engages in false bidding when he bids positive amounts for an item that fetches him zero utility.

The paper is organized as follows. In section 2, we describe the mathematical model in detail. Section 3 comprises of our main results. We discuss the implications of our results in section 4, which is followed by the conclusion of the study in section 5.

2 The Model

The set of players $N = \{1, 2\}$ consists of two symmetric and risk neutral bidders. The set of objects $S = \{A_1, A_2, B_1, B_2\}$ contains four indivisible and non-identical objects. We define two subsets of S, namely $A = \{A_1, A_2\}$ and $B = \{B_1, B_2\}$, which are also called *categories* of objects. We define a category as a collection of substitutable items. Additionally, items from different categories are treated as complements. Bidders demand one unit from each category and aim to create a bundle of complementary (inter-category) objects, which maximizes their total expected profits. Here, we define profit as the difference between the utility obtained from a bundle of objects and the payments made to reserve it. Throughout the paper, we use superscript i to denote the bidders and subscript k to denote the objects. To put this auction setting into perspective, we proceed with an example.

Suppose a floriculturist/auctioneer wants to sell different collections of flowers via auction. There are two potential bidders 1 and 2, who compete for these flowers. Both bidders are florists, who create various products using different flowers that they buy in the auction, such as bouquets, flower baskets, etc. Assume that auctioneer has four different collections of flowers, which come from two breeds, namely Roses and Lilies. The collections of flowers are as follows: Red Roses (RR), White Roses (WR), Pink Lilies (PL) and Yellow Lilies (YL). Hence, the set S is described as $S = \{RR, WR, PL, YL\}$. We assume that each of the four collections of flowers is indivisible and hence cannot be divided into sub-parts and sold separately. In other words, a collection of flowers is treated as a unit throughout the auction. Both bidders consider flowers from the two breeds, i.e., Roses and Lilies, to be essential for a bouquet and treat them as complementary. Additionally, bidders consider flowers from the same breed as substitutes. For example, they treat RR and WR as substitutes. Similarly, PL and YL are also considered substitutable. Therefore, the categories are defined as follows Rose = $\{RR, WR\}$ and Lily = $\{PL, YL\}$. Bidders aim to create a bouquet using one kind of flowers from each breed. In other words, they demand only one collection/unit of flowers from each category. Since flowers are perishable goods, bidders intend to buy only a limited amount, which is required to make the desired products and hence have unit demand from each category[1].

The game takes place in two periods, i.e., $t \in \{1, 2\}$. In each period, all the objects from a particular category are sold via a sequence of English auctions. At $t = 1$, both objects from category A are auctioned, followed by category B objects at $t = 2$. Before the game begins (at $t = 0$), the seller releases an announcement which notifies the potential buyers about all the objects to be auctioned. In the context of our example, at $t = 0$, the auctioneer informs the two florists about the breed and color of the flowers to be auctioned at $t = 1, 2$. Specifically, both types of roses are auctioned at $t = 1$, and both types of lilies are auctioned at $t = 2$. After the announcement, bidders independently decide their relative rankings over the substitutable objects from both categories. Given two substitutable objects, we say that a bidder ranks one of them higher than the other when he derives a greater utility from it than its substitute. For example, if bidder i derives greater utility from object A_1 than its substitute, i.e., A_2, we say that he ranks A_1 higher than A_2. Bidders' relative rankings are common knowledge throughout the game. In the context of our example, the relative rankings being common knowledge can be interpreted as follows. Both bidders have been functioning in separate markets with distinct customers for a long time. Assume that bidders derive their rankings from their customers' preferences. Hence, each bidder has perfect information regarding his opponent's rankings. Relative rankings are not actual valuations. They are only indicators of preference relations over the same-category objects.

Given the relative rankings of the bidders, the sequence of sale O is determined as follows. In a particular category, if both bidders rank the same object higher, then that object is presented first. Otherwise, the order of sale remains (A_1, A_2) at $t = 1$ and (B_1, B_2) at $t = 2$. In any case, we denote the object being auctioned first and second at $t = 1$ by A_f and A_s respectively. Similarly, the first and second objects to be auctioned at $t = 2$ are denoted by B_f and B_s. The order of sale $O = (A_f, A_s, B_f, B_s)$ is also common knowledge throughout the game.

We denote the relative rankings of bidders i for category A and B objects by $r^i_A = (r^i_{A_f}, r^i_{A_s})$ and $r^i_B = (r^i_{B_f}, r^i_{B_s})$ respectively. Both r^i_A and r^i_B are two-dimensional binary vectors. A bidder's relative rank for a particular category reflects his preferences over the objects in that category. A bidder's rankings for both categories are collectively represented by $r^i = (r^i_{A_f}, r^i_{A_s}, r^i_{B_f}, r^i_{B_s})$. For example, if bidder 1 ranks object A_f higher than A_s in category A, then $r^1_A = (r^1_{A_f}, r^1_{A_s}) = (1, 0)$. Similarly, if he ranks B_s higher than B_f in category B, then $r^1_B = (0, 1)$. Collectively, the relative ranking vector for bidder 1, denoted by r^1, is as $r^1 = (1, 0, 0, 1)$.

At the beginning of each period, both bidders independently inspect the objects being auctioned in that period and receive private signals about their quality. They receive only one signal in each period, which is the same for both the objects being auctioned in that period. These signals are different from the public announcement that was made at

[1]The rational for demanding only one unit from each category is not the perishableness the but the composition of different flowers (breed and color) in a bouquet

$t = 0$. In the framework of our example, at the beginning of each period, both florists inspect the quality of the flowers being auctioned in that period. As a consequence of this inspection, they receive private signals about the quality of the flowers. We denote the signals received by bidder i at $t = 1$ and at $t = 2$ by x_A^i and x_B^i respectively, where $x_A^i, x_B^i \in [\delta_0, \delta_1]$. Here, δ_0 and δ_1 are common knowledge and $\delta_0, \delta_1 \in \mathbb{R}_+$, $\delta_1 > \delta_0$.

After observing the signal regarding the category A objects, i.e., x_A^i, bidder i calculates his valuation of category A objects as follows:

$$v_A^i = (v_k^i)_{k \in \{A_f, A_s\}} = ((1 - r_k^i)\delta_0 + r_k^i x_A^i)_{k \in \{A_f, A_s\}}, \ \forall i \in N \quad (1)$$

In words, a bidder's valuation of his lower-ranked object remains fixed at δ_0, whereas that of the higher-ranked object is $x_A^i \in [\delta_0, \delta_1]$, known privately to him. Mathematically, $r_k^i = 1$ implies that $v_k^i > \delta_0$, whereas $r_k^i = 0$ implies $v_k^i = \delta_0$. Bidder i's signal x_A^i can also be interpreted as his *type* at $t = 1$. At the beginning of $t = 1$, bidders are aware of their own rankings, their opponent's rankings, and the order of sale.

Actions taken by bidder i are nothing but the prices at which he decides to leave the auctions of A_f and A_s. We denote the actions taken by bidder i at $t = 1$ by $a_A^i = (a_{A_f}^i, a_{A_s}^i)$. Since all valuations are at least δ_0, which is common knowledge, the auctioneer starts the auction of every object at price δ_0 (reserve price). Therefore, a bidder has to pay at least δ_0 in order to buy an object. However, a bidder is allowed to stay inactive during an auction or bid zero. Hence, the set of actions can mathematically be described as:

$$a_A^i = (a_{A_f}^i, a_{A_s}^i) \in (\{0\} \cup [\delta_0, \infty)) \times (\{0\} \cup [\delta_0, \infty)), \ \forall i \in N \quad (2)$$

We say that bidder i participates in the auction of object k, when his bid is greater than or equal to δ_0, i.e. $a_k^i \geq \delta_0$, $\forall k \in \{A_f, A_s\}$. Outcome of the game at $t = 1$ depends on the actions taken by both bidders. It can be denoted with the help of three variables, i.e., $w_A^1 = (w_k^1)_{k \in \{A_f, A_s\}}$, $w_A^2 = (w_k^2)_{k \in \{A_f, A_s\}}$ and P_A. The outcome with respect to bidder i in category A, denoted by w_A^i, is described as follows:

$$w_A^i = (w_k^i)_{k \in \{A_f, A_s\}} = (I(a_k^i > a_k^{-i}))_{k \in \{A_f, A_s\}}, \ \forall i \in N \quad (3)$$

where $I(c)$ is an indicator random variable which takes value 1, when condition c holds. In words, $w_k^i = 1$ denotes that bidder i won object k. The other aspect of the outcome is the selling prices of the two objects. Since the objects are sold via a sequence of English auctions, a bidder pays only when he wins the object. Also, the price paid by the winner is the bid/action taken by his opponent. Mathematically, it can be described as follows:

$$P_A(a_A^1, a_A^2) = (P_k)_{k \in \{A_f, A_s\}} = (w_k^1 a_k^2 + w_k^2 a_k^1)_{k \in \{A_f, A_s\}} \quad (4)$$

At the beginning of $t = 2$, bidder i observes the second quality signal, i.e. x_B^i. He calculates his valuations of category B objects as follows:

$$v_B^i = (v_k^i)_{k \in \{B_f, B_s\}} = ((1 - r_k^i)\delta_0 + r_k^i x_B^i)_{k \in \{B_f, B_s\}}, \ \forall i \in N \quad (5)$$

At $t = 2$, bidder i's type is defined as a tuple, which consists of both of his private signals, i.e., $x^i = (x_A^i, x_B^i)$. At the beginning of the second period, the actions that were taken by both bidders in the first period, as well as the outcome of the game at $t = 1$, also become common knowledge. Just as before, bidder i's actions are the prices at which he leaves the auctions of objects B_f and B_s. Actions taken by bidder i at $t = 2$ are denoted by a_B^i, which are defined as follows:

$$a_B^i = (a_{B_f}^i, a_{B_s}^i) \in (\{0\} \cup [\delta_0, \infty)) \times (\{0\} \cup [\delta_0, \infty)), \ \forall i \in N \quad (6)$$

Outcome of the game at the end of $t = 2$ can be described in a similar fashion as that at $t = 1$. Explicitly, the outcome for bidder i at $t = 2$ is as follows

$$w^i_B = (w^i_k)_{k\in\{B_f,B_s\}} = (I(a^i_k > a^{-i}_k))_{k\in\{B_f,B_s\}},\ \forall i \in N \tag{7}$$

whereas selling prices can be described as

$$P_B(a^1_B, a^2_B) = (P_k)_{k\in\{B_f,B_s\}} = (w^1_k a^2_k + w^2_k a^1_k)_{k\in\{B_f,B_s\}} \tag{8}$$

We collectively denote the outcome for bidder i from both periods by w^i, where $w^i = (w^i_{A_f}, w^i_{A_s}, w^i_{B_f}, w^i_{B_s})$.

A strategy for bidder i is defined as a function from the set of types to the set of actions. We assume that bidding strategies are symmetric and invertible. A bidder is required to take actions in the auctions of all four objects, i.e., two objects in each period. Hence, the overall strategy of a bidder consists of four bidding functions, i.e. $b(x^i_A, x^i_B) = \big(b_{A_f}(x^i_A), b_{A_s}(x^i_A), b_{B_f}(x^i_A, x^i_B), b_{B_s}(x^i_A, x^i_B)\big)$. It should be noted that bidding functions of first period (b_{A_f} and b_{A_s}) do not depend on the signal of the second period x^i_B. Mathematically, bidding functions of first time period are described as:

$$b_k : [\delta_0, \delta_1] \to \{0\} \cup [\delta_0, \infty),\ \forall k \in \{A_f, A_s\} \tag{9}$$

In contrast, bidding functions of the second period depend on both signals. Mathematically, they can be described as:

$$b_k : [\delta_0, \delta_1] \times [\delta_0, \delta_1] \to \{0\} \cup [\delta_0, \infty),\ \forall k \in \{B_f, B_s\} \tag{10}$$

Next, we explain our approach to model substitutability between intra-category objects. As mentioned previously, a bidder demands only one unit from each category. In other words, if a bidder wins both items from a given category, he doesn't utilize his lower-ranked object at all. Therefore, having obtained the higher-ranked object, a bidder's utility from a lower-ranked object is zero. This assumption is enough to account for the substitutable relationship between intra-category objects. We model complementary relationships between inter-category objects using multiplicative factors. Specifically, a bidder's total utility from winning both of his higher-ranked objects is obtained by multiplying the sum of the individual values by a constant α, where $\alpha > 1$. If a bidder wins his higher-ranked object from only one of the two categories, his utility from this bundle is determined by multiplying the sum of individual values by a constant β where $1 < \beta < \alpha$. Bidders don't realize any synergy if both of their objects are lower-ranked ones, and hence, the value of the bundle is just the summation of individual values. Table 1 summarizes the utilities realized by bidder i, denoted by $u^i(\cdot)$, from various bundles, given that bidder i ranks A_f higher in category A, and B_f in category B, i.e. $r^i = (r^i_{A_f}, r^i_{A_s}, r^i_{B_f}, r^i_{B_s}) = (1, 0, 1, 0)$.

The utility of player i, depends on the outcome of the game i.e. w^i, which again depends on the actions of the players as shown in equations (3) and (7). Let $a^i = (a^i_{A_f}, a^i_{A_s}, a^i_{B_f}, a^i_{B_s})$ denote the actions/bids of bidder i in the auctions of all four objects. The gross utility structure of player i, described in Table 1, can mathematically be expressed as:

$$\begin{aligned} u^i(a^i, a^{-i}) &= (r^i_A \cdot w^i_A)(r^i_B \cdot w^i_B)\left[\alpha(x^i_A + x^i_B)\right] + (r^i_A \cdot w^i_A)(1 - r^i_B \cdot w^i_B)\left[\beta(x^i_A + \delta_0)\right] \\ &+ (1 - r^i_A \cdot w^i_A)(r^i_B \cdot w^i_B)\left[\beta(\delta_0 + x^i_B)\right] + (1 - r^i_A \cdot w^i_A)(1 - r^i_B \cdot w^i_B)\left[2\delta_0\right],\ \forall i \in N \end{aligned} \tag{11}$$

$w^i = (w^i_{A_f}, w^i_{A_s}, w^i_{B_f}, w^i_{B_s})$	$u^i(w^i)$ if $r^i = (1,0,1,0)$
$(1,0,1,0)$	$\alpha(x^1_A + x^1_B)$
$(1,0,0,1)$	$\beta(x^1_A + \delta_0)$
$(0,1,1,0)$	$\beta(\delta_0 + x^1_B)$
$(0,1,0,1)$	$2\delta_0$

Table 1: Utilities derived by bidder i from various outcomes (w^i) given his rankings (r^i)

Since the format of auctions is sequential English, bidder i pays only when he wins an object. In such cases, bidder i pays the amount equal to his opponent's bid. Therefore, payments of player i, denoted by ϕ^i, can mathematically be expressed as:

$$\phi^i(a^i, a^{-i}) = \sum_{k \in S} w^i_k a^{-i}_k, \ \forall i \in N \tag{12}$$

Finally, payoff of bidder i is obtained by subtracting his total payment from his total utility. We denote the payoff of bidder i by π^i, which can be written as:

$$\pi^i(a^i, a^{-i}) = u^i(a^i, a^{-i}) - \phi^i(a^i, a^{-i}), \ \forall i \in N \tag{13}$$

Next, we describe the belief structure of bidder i in both periods. We assume that the quality signals are independently and identically distributed in $[\delta_0, \delta_1]$ according to the density $f(\cdot)$ and distribution $F(\cdot)$. Consequently, the common prior p at $t = 0$ is given as $p = \prod_{i \in N, k \in \{A,B\}} f(x^i_k)$. Bidders *consistently* derive their beliefs regarding their opponent's signals from p using Bayes rule. We assume that $f(\cdot)$ is continuous and differentiable on its support $[\delta_0, \delta_1]$. At $t = 1$, bidder i knows his type x^i_A but believes that x^i_B, x^{-i}_A and x^{-i}_B are independently and identically distributed according to the density $f(\cdot)$. At $t = 2$, bidder i is aware of both x^i_A and x^i_B, but believes that x^{-i}_B is distributed according to the density $f(\cdot)$. Bidder i's belief about his opponent's signal at $t = 1$, i.e., x^{-i}_A has to be updated. This update in bidder i's belief about x^{-i}_A takes into account the information released by bidder $-i$'s actions at $t = 1$. Bidder i's beliefs about x^{-i}_A are updated according to the Bayes rule as follows:

$$\begin{aligned}\tilde{f}(x^{-i}_A) = r^{-i}_{A_f} &\left\{ w^{-i}_{A_f} \cdot \frac{f(x^{-i}_A).I\left[x^{-i}_A > b^{-1}_{A_f}\left(P_{A_f}\right)\right]}{\int_{b^{-1}_{A_f}(P_{A_f})}^{\delta_1} f(x^{-i}_A) dx^{-i}_A} + (1 - w^{-i}_{A_f}) \cdot b^{-1}_{A_f}\left(P_{A_f}\right) \right\} + \\ &r^{-i}_{A_s} \left\{ w^{-i}_{A_s} \cdot \frac{f(x^{-i}_A).I\left[x^{-i}_A > b^{-1}_{A_s}\left(P_{A_s}\right)\right]}{\int_{b^{-1}_{A_s}(P_{A_s})}^{\delta_1} f(x^{-i}_A) dx^{-i}_A} + (1 - w^{-i}_{A_s}) \cdot b^{-1}_{A_s}\left(P_{A_s}\right) \right\}\end{aligned} \tag{14}$$

The above expression consists of two terms, one of which is always zero, because bidder $-i$ ranks only one of A_f and A_s higher. The first expression becomes relevant when bidder $-i$ ranks A_f higher than A_s, i.e., $r^{-i}_{A_f} = 1$ and $r^{-i}_{A_s} = 0$. Bidder $-i$'s valuation of his lower-ranked object is always δ_0, which is independent of his type x^{-i}_A. Therefore, bidder $-i$'s overall profit from obtaining a lower-ranked object from category A does not depend on x^{-i}_A. Subsequently, bidder $-i$'s bid for his lower-ranked object is also independent of x^{-i}_A. Therefore, any information regarding bidder $-i$'s type, i.e., x^{-i}_A can only come from his bid of his higher-ranked object. Hence the first expression only depends on $b_{A_f}(\cdot)$, as

A_f is the higher-ranked object. If bidder $-i$ wins A_f, i.e., $w_{A_f}^{-i} = 1$, bidder i can only infer that $x_{A_f}^{-i}$ is higher than $b_{A_f}^{-1}\left(P_{A_f}\right)$, where P_{A_f} is the selling price of the object A_f. (Eq (4)). Consequently, bidder i updates his beliefs accordingly using Bayes rule as shown in the equation above. If bidder $-i$ loses his higher-ranked object, i.e., $w_{A_f}^{-i} = 0$, then $-i$'s bid becomes the common knowledge, and hence bidder i can exactly find out his opponent's signal to be equal to $b_{A_f}^{-1}\left(P_{A_f}\right)$, as described in the equation above. Similarly, the second term becomes relevant when bidder $-i$ ranks A_s higher than A_f.

We define that a strategy profile, denoted by $\mathbf{b}^* = (b^*(x^1), b^*(x^2))$, constitutes a subgame perfect Bayesian equilibrium, if for each player $i \in N$, each type x^i, and each possible action $b(x^i)$, the following inequality holds:

$$\mathbb{E}\left[\pi^i\left(\mathbf{b}^*\right)\right] \geq \mathbb{E}\left[\pi^i\left((b(x^i), b^*(x^{-i})\right)\right] \tag{15}$$

3 Results

In this section, we examine all possible situations which can arise as a result of the game described above. For each of these possible scenarios, we find out the bidders' optimal bidding strategies and their profits, equilibrium selling prices, and the seller's revenue.

3.1 Possible Classes of Games

Bidder i's relative ranking for all four objects is collectively represented by r^i, i.e. $r^i = (r^i_{A_f}, r^i_{A_s}, r^i_{B_f}, r^i_{B_s})$. It is a 4-dimensional binary vector and can take four different values. Mathematically:

$$r^i \in \{(0,1,0,1), (0,1,1,0), (1,0,0,1), (1,0,1,0)\}, \ \forall i \in \{1,2\}$$

Let (r^1, r^2) collectively denote the of relative rankings of all four objects of the two bidders. For example, if $r^1 = (1,0,1,0)$ and $r^2 = (0,1,0,1)$, then, $(r^1, r^2) = ((1,0,1,0),(0,1,0,1))$. Since bidders' relative rankings are independent of each other, the pair (r^1, r^2) can take 16 different values. But, out of the 16 possible values of (r^1, r^2), some values are not possible. This is because, if both bidders rank an object higher in a particular category, the auctioneer sells that object first. Therefore, it is not possible to have a situation in which, both bidders rank a particular object higher in any category, and it is presented second. For example, it is not possible to have $r^1 = r^2 = (0,1,0,1)$. This is because, having such a pair means that both bidders rank objects A_s and B_s higher, both of which are presented second at $t = 1$ and 2 respectively. The list of all values of (r^1, r^2), which cannot occur can be found in Table 4. We denote the set of possible values of (r^1, r^2) by R. Bidders' relative rankings influence their strategies and hence, also affect the outcome of the game. Therefore, each possible value of (r^1, r^2) can lead to a different outcome of the game. However, it is possible to cluster the elements of R, into a few subsets, to facilitate the further analysis of the game. Following definitions are useful to understand these subsets.

- Competition (C): "Competition" in a particular category occurs, when bidders' higher-ranked objects are the same in that category. For example, if both bidders rank object A_f higher, then we say that there is competition in category A.

Notation	Explanation
$N = \{1, 2\}$	Set of players
$S = \{A_1, A_2, B_1, B_2\}$	Set of objects
$A = \{A_1, A_2\}$	Set of category A objects
$B = \{B_1, B_2\}$	Set of category B objects
$t \in \{1, 2\}$	Time period
$O = \{\{A_f, A_s\}, \{B_f, B_S\}\}$	Order of sale
$r^i_A = (r^i_{A_f}, r^i_{A_s})$	Relative rankings of player i for category A objects
$r^i_B = (r^i_{B_f}, r^i_{B_s})$	Relative rankings of player i for category B objects
$r^i = (r^i_A, r^i_B)$	Relative rankings of player i for both categories
$\delta_0, \delta_1 \in \mathbb{R}_+, \delta_0 < \delta_1$	Real numbers in $\mathbb{R}_+$
$x^i_A \in (\delta_0, \delta_1)$	Signal and type of player i at $t = 1$
$x^i_B \in (\delta_0, \delta_1)$	Signal of player i at $t = 2$
$x^i = (x^i_A, x^i_B)$	Type of player i at $t = 2$
$f(\cdot), F(\cdot)$	Density and distribution of signals x^i_A and x^i_B
$v^i_A = (v^i_{A_f}, v^i_{A_s})$	Valuation of bidder i for category A objects
$v^i_B = (v^i_{B_f}, v^i_{B_s})$	Valuation of bidder i for category B objects
$v^i = (v^i_A, v^i_B)$	Valuation of bidder i for both categories
$a^i_A = (a^i_{A_f}, a^i_{A_s})$	Actions by player i at $t = 1$
$a^i_B = (a^i_{B_f}, a^i_{B_s})$	Actions by player i at $t = 2$
$a^i = (a^i_A, a^i_B)$	Actions by player i for both categories
$P_A = (P_{A_f}, P_{A_s})$	Selling prices of category A objects
$P_B = (P_{B_f}, P_{B_s})$	Selling prices of category B objects
$P = (P_A, P_B)$	Selling prices of objects of both categories
$w^i_A = (w^i_{A_f}, w^i_{A_s})$	Indicates objects won by bidder i in category A
$w^i_B = (w^i_{B_f}, w^i_{B_s})$	Indicates objects won by bidder i in category B
$w^i = (w^i_A, w^i_B)$	Indicates objects won by bidder i from both categories
$b_A = (b_{A_f}(x^i_A), b_{A_s}(x^i_A))$	Bidding functions for category A objects
$b_B = (b_{B_f}(x^i_A, x^i_B), b_{B_s}(x^i_A, x^i_B))$	Bidding functions for category B objects
$b = (b_A, b_B)$	Overall strategy of a bidder for the game
$\mathbf{b}^* = (b^*(x^1), b^*(x^2))$	Strategy profile
$u^i(a^i, a^{-i})$	Total utility of bidder i
$\phi^i(a^i, a^{-i})$	Sum of all the payments made by bidder i
$\pi^i(a^i, a^{-i})$	Payoff/Profit of bidder i

Table 2: Notations and Explanations

Class of the Game/Subset of R	Category A	Category B
NN	No Competition	No Competition
NC	No Competition	Competition
CN	Competition	No Competition
CC	Competition	Competition

Table 3: Possible classes of games

r^1 \ r^2	(0,1,0,1)	(0,1,1,0)	(1,0,0,1)	(1,0,1,0)
(0,1,0,1)	-	-	-	NN
(0,1,1,0)	-	-	NN	NC
(1,0,0,1)	-	NN	-	CN
(1,0,1,0)	NN	NC	CN	CC

Table 4: Relation between classes of games and players' relative rankings

- No competition (N): "No competition" in a particular category occurs, when bidders' higher-ranked objects are different in that category. For example, if bidder 1 ranks B_f higher, but bidder 2's higher-ranked object is B_s, then we say that there is no competition in category B.

Based on the definitions above, we define four subsets of R, namely NN, NC, CN, and CC. A brief description of these subsets can be found in Table 3. All possible values of (r^1, r^2) are classified into these four subsets, depending on the presence/absence of competition in categories A and B. For example, if $r^1 = (1, 0, 1, 0)$ and $r^2 = (0, 1, 1, 0)$, then bidders' higher-ranked objects are different in category A, but the same in category B. This value of (r^1, r^2) falls into the NC class of games, where there is no competition in category A but bidders compete in category B. Other values of (r^1, r^2) can also be classified in the same way. Table 4 describes all possible values of (r^1, r^2) and how they can be classified into different subsets.

Next, we argue that optimal bidding strategy for all values of (r^1, r^2) belonging to the same subset, are the same. This is because, all values of (r^1, r^2) within a subset differ from one another only in categories with no competition. For example, $((1, 0, 1, 0), (0, 1, 1, 0)) \in NC$ and also, $((0, 1, 1, 0), (1, 0, 1, 0)) \in NC$. These two values of (r^1, r^2) differ from each other only in bidders' preferences in category A. In the former, bidder 1 ranks A_f higher but bidder 2 ranks A_s higher, while in the latter, bidder 1 and 2's higher-ranked objects are A_s and A_f respectively. In both cases, bidders don't compete in category A, and the only difference in the relative rankings of bidders in the two situations is the identities of the objects. Since the bidders' signals are independently and identically distributed, the identities of the objects do not matter, and the two situations are equivalent. Hence, the optimal strategy for a game associated with one value of (r^1, r^2) can be extended to that associated with all the other values of (r^1, r^2) from the same subset. Because of this equivalence, we say that all values of (r^1, r^2) belonging to the same subset define a *class* of the game. We explore each of these classes separately in the next section.

3.2 Bidding Strategies and Equilibrium Selling Prices

3.2.1 NN class of games

In this section, we find the optimal bidding strategies and the equilibrium selling prices for all NN class games. In all realizations of the game of NN class, bidders' higher-ranked objects differ in both categories. Consequently, there is no competition in any of the two categories. The following lemma describes the optimal bidding functions in the NN class of games at $t = 2$.

Lemma 1. *In all realizations of the game of NN class, the optimal bidding functions of bidder i at $t = 2$, i.e. $b_B^*(x_A^i, x_B^i) = (b_k^*(x_A^i, x_B^i))_{k\in(B_f,B_s)}$ are given as follows:*

$$b_k^*\left(x_A^i, x_B^i|(r^1, r^2) \in NN\right) = r_k^i\delta_0,\ \forall k \in \{B_f, B_s\} \tag{16}$$

Proof. As mentioned previously, bidders demand only one unit from each category. Hence, the utility realized by a bidder from a lower-ranked object is zero, if he has already obtained the higher-ranked object from the same category. Since the auction of B_s is a simple English auction, the optimal bid is the same as the utility from the object. Suppose bidder i ranks object B_s higher and bidder $-i$'s higher-ranked object is B_f. Bidder i knows that if $-i$ wins object B_f, $-i$'s utility for object B_s will be zero. Consequently, bidder $-i$ will bid zero or will not participate in the auction of B_s. This event would enable bidder i to win B_s at minimum price δ_0. Hence, bidder i does not participate in the auction of B_f and ensures that $-i$ wins it at minimum price δ_0. Subsequently, bidder i wins his higher-ranked object B_s at δ_0. In summary, bidders bid δ_0 for their higher-ranked and zero for their lower-ranked object from category B as mentioned in the lemma. □

Corollary 2. *In all realizations of the game of NN class, the equilibrium selling prices at $t = 2$, i.e., $P_B = (P_{B_f}, P_{B_s})$ are given as follows:*

$$P_k\left(b_k^*(x_A^1, x_B^1), b_k^*(x_A^2, x_B^2)|(r^1, r^2) \in NN\right) = \delta_0,\ \forall k \in \{B_f, B_s\} \tag{17}$$

Lemma 3. *In all realizations of the game of NN class, the optimal bidding functions of bidder i at $t = 1$, i.e., $b_A^*(x_A^i) = (b_k^*(x_A^i))_{k\in(A_f,A_s)}$ are given as follows:*

$$b_k^*\left(x_A^i|(r^1, r^2) \in NN\right) = r_k^i\delta_0,\ \forall k \in \{A_f, A_s\} \tag{18}$$

Proof. At $t = 1$, both bidders know that they are going to win their higher-ranked objects from category B, irrespective of the results of category A auctions. Hence, category A and B auctions are independent in NN class of games. Therefore, the arguments given in the proof of Lemma 1 can be used to arrive at a similar result as that for $t = 1$. As given in the lemma, both bidders bid zero for their lower-ranked and δ_0 for their higher-ranked item. □

Corollary 4. *In all realizations of the game of NN class, selling prices at $t = 1$, i.e., $P_A = (P_{A_f}, P_{A_s})$ are given as follows:*

$$P_k\left(b_k^*(x_A^1), b_k^*(x_A^2)|(r^1, r^2) \in NN\right) = \delta_0,\ \forall k \in \{A_f, A_s\} \tag{19}$$

In an NN class game, all four objects are sold at reserve price δ_0 and both bidders win their higher-ranked objects from both categories. The following corollary gives the payoffs of each bidder for all realizations of the game of NN class.

Corollary 5. *In all realizations of the game of NN class, the overall payoff of bidder i is given as follows:*

$$\pi^i(\mathbf{b}^*) = \alpha(x_A^i + x_B^i) - 2\delta_0,\ \forall i \in N \tag{20}$$

3.2.2 NC class of games

In this section, we determine a bidder's optimal bidding strategy in the NC class of games. In any realization of the game of NC class, bidders rank different objects higher in category A but rank the same object higher in category B. As a result, bidders do not compete in category A but contest for their higher-ranked object in category B. We characterize the equilibria for such a game below.

Lemma 6. *In all realizations of the game of NC class, the optimal bidding functions of bidder i at* $t = 2$*, i.e.* $b^*_B(x^i_A, x^i_B) = (b^*_k(x^i_A, x^i_B))_{k\in(B_f,B_s)}$ *are given as follows:*

$$b^*_{B_f}\left(x^i_A, x^i_B | r^i_A \cdot w^i_A = 1, (r^1, r^2) \in NC\right) = \delta_0 + \alpha(x^i_A + x^i_B) - \beta(x^i_A + \delta_0) \tag{21}$$

$$b^*_{B_f}\left(x^i_A, x^i_B | r^i_A \cdot w^i_A = 0, (r^1, r^2) \in NC\right) = \beta(\delta_0 + x^i_B) - \delta_0 \tag{22}$$

$$b^*_{B_s}\left(x^i_A, x^i_B | w^i_{B_f} = 0, (r^1, r^2) \in NC\right) = \delta_0 \tag{23}$$

$$b^*_{B_s}\left(x^i_A, x^i_B | w^i_{B_f} = 1, (r^1, r^2) \in NC\right) = 0 \tag{24}$$

Proof. At $t = 2$, a bidder can either be a winner or a loser of his higher-ranked object from category A i.e. $r^i_A.w^i_A = 1$ or 0. Since each bidder's demand is only one unit, a bidder gets his lower-ranked object in case he loses his higher-ranked one in category A. The event that bidder i won his higher-ranked object from category A can be represented by the equation $r^i_A \cdot w^i_A = 1$. In case he loses his higher-ranked object from category A, $r^i_A \cdot w^i_A = 0$.

At $t = 2$, object B_f is auctioned first, as it is ranked higher by both bidders. Using backward induction, we consider the auction of object B_s first. When object B_s is auctioned, both bidders are aware of the outcome of object B_f's auction, i.e. $w^1_{B_f}$ and $w^2_{B_f}$ are common knowledge. Suppose bidder i won object B_f, i.e., $w^i_{B_f} = 1$ and $w^{-i}_{B_f} = 0$. Since bidder i's need of category B objects is satisfied, he bids zero in the auction of object B_s. Mathematically,

$$b^*_{B_s}\left(x^i_A, x^i_B | w^i_{B_f} = 1, (r^1, r^2) \in NC\right) = 0$$

consequently, bidder $-i$ need not go beyond δ_0 in order to win object B_s, i.e.

$$b^*_{B_s}\left(x^{-i}_A, x^{-i}_B | w^{-i}_{B_f} = 0, (r^1, r^2) \in NC\right) = \delta_0$$

We now consider the auction of the object B_f. Since both bidders rank B_f higher, they compete with each other in order to buy it. Optimal bidding function in the auction of B_f depends on the result of category A auctions. In other words, a bidder's bidding function depends on whether or not he won his higher-ranked object from category A. Hence, we find optimal bidding function for both such cases.

CaseI (Winner: $r^i_A \cdot w^i_A = 1$): Assume that bidder i wins his higher-ranked objects from category A, i.e. $r^i_A \cdot w^i_A = 1$, and pays p^i_A for it. If bidder i also wins his higher-ranked object from category B, i.e., B_f at price $p^i_{B_f}$, his profit $\pi^i(\cdot)$ will be given as:

$$\pi^i(\cdot | r^i_A \cdot w^i_A = 1, w^i_{B_f} = 1) = \alpha(x^i_A + x^i_B) - p^i_{B_f} - p^i_A$$

where $\alpha(x_A^i + x_B^i)$ is his total utility and p_A^i and $p_{B_f}^i$ are the prices he paid for higher-ranked objects from category A and B respectively. On the other hand, if he loses object B_f, he goes on to win object B_s and pays δ_0, hence his payoff is given as:

$$\pi^i(\cdot|r_A^i \cdot w_A^i = 1, w_{B_f}^i = 0) = \beta(x_A^i + \delta_0) - \delta_0 - p_A^i$$

where $\beta(x_A^i + \delta_0)$ is his total utility and $\delta_0 + p_A^i$ is the total price paid by him. Bidder i would stay in the auction of object B_f, until he is indifferent between the two payoffs, i.e.

$$\pi_{B_f}^i(\cdot|r_A^i \cdot w_A^i = 1, w_{B_f}^i = 1) = \pi_{B_f}^i(\cdot|r_A^i \cdot w_A^i = 1, w_{B_f}^i = 0)$$

solving the above equation gives:

$$p_{B_f}^i = b_{B_f}^*\left(x_A^i, x_B^i|r_A^i \cdot w_A^i = 1, (r^1, r^2) \in NC\right) = \delta_0 + \alpha(x_A^i + x_B^i) - \beta(x_A^i + \delta_0)$$

Case II (Loser: $r_A^i \cdot w_A^i = 0$): Suppose bidder i won his lower-ranked object from category A, i.e. $r_A^i \cdot w_A^i = 0$ and paid p_A^i. If bidder i wins his higher-ranked object from category B, i.e. B_f at price $p_{B_f}^i$, his profit is given as:

$$\pi^i(\cdot|r_A^i \cdot w_A^i = 0, w_{B_f}^i = 1) = \beta(\delta_0 + x_B^i) - p_A^i - p_{B_f}^i$$

where $\beta(\delta_0 + x_B^i)$ is the total utility and $p_A^i + p_{B_f}^i$ is the total payment made by bidder i. If bidder i loses B_f, his total profit is given as follows:

$$\pi^i(\cdot|r_A^i \cdot w_A^i = 0, w_{B_f}^i = 0) = 2\delta_0 - p_A^i - \delta_0$$

He will leave the auction of B_f when he becomes indifferent between the two situations, i.e.,

$$\pi_{B_f}^i(\cdot|r_A^i \cdot w_A^i = 0, w_{B_f}^i = 1) = \pi_{B_f}^i(\cdot|r_A^i \cdot w_A^i = 0, w_{B_f}^i = 0)$$

solving the above equation gives

$$p_{B_f}^i = b_{B_f}^*\left(x_A^i, x_B^i|r_A^i \cdot w_A^i = 0, (r^1, r^2) \in NC\right) = \beta(\delta_0 + x_B^i) - \delta_0$$

□

Corollary 7. *In all realizations of the game of NC class, selling prices at $t = 2$, i.e., $P_B = (P_{B_f}, P_{B_s})$ are given as follows:*

$$P_{B_f}\left(b_{B_f}^*(x_A^1, x_B^1), b_{B_f}^*(x_A^2, x_B^2)|(r^1, r^2) \in NC\right) = \min\left\{\delta_0 + \alpha(x_A^1 + x_B^1) - \beta(x_A^1 + \delta_0), \delta_0 + \alpha(x_A^2 + x_B^2) - \beta(x_A^2 + \delta_0)\right\} \quad (25)$$

$$P_{B_s}\left(b_{B_s}^*(x_A^1, x_B^1), b_{B_s}^*(x_A^2, x_B^2)|(r^1, r^2) \in NC\right) = \delta_0 \quad (26)$$

Lemma 8. *In all realizations of the game of NC class, the optimal bidding functions of bidder i at $t = 1$, i.e. $b_A^*(x_A^i) = (b_k^*(x_A^i))_{k\in(A_f, A_s)}$ are given as follows:*

$$b_k^*\left(x_A^i|(r^1, r^2) \in NC\right) = r_k^i\delta_0, \ \forall k \in \{A_f, A_s\} \quad (27)$$

Proof. Suppose bidder i's higher-ranked object is A_s and $-i$'s higher-ranked object is A_f. If $-i$ wins A_f, his utility for A_s will be zero. We show that, $-i$'s optimal bid in the auction of A_s will also be zero. A bidder makes a positive profit, only if his total payment for both the objects is lower than the value of the bundle. Hence, if a bidder pays more to reserve the first object, he would be spend less for the second. One can follow this line of argument to claim that bidder $-i$ may bid non-zero amount in the auction of A_s, in order to sabotage bidder i's purchasing ability at $t = 2$ (false bidding). However, we show that it is not feasible in the present model.

Suppose bidder $-i$ decides to engage in false bidding and takes up the price to p_{A_s}. At this price, if $-i$ wins A_s, his loss would be p_{A_s}. On the other hand, if $-i$ loses it, he successfully decreases the purchasing ability of bidder i by $p_{A_s} - \delta_0$, which would help $-i$ at $t = 2$. Suppose that the probability of bidder $-i$ losing A_s is q, when the price is p_{A_s}. Then, his expected profit from engaging in false bidding is $q(p_{A_s} - \delta_0) + (1 - q)(-\delta_0)$. At the beginning of the auction, i.e., when $p_{A_s} = \delta_0$, the expected profit is $(1 - q)(-\delta_0)$. If this expected profit is to be greater than zero, then $q > 1$, which is not possible. Hence, it is not profitable for bidder $-i$ to engage in false bidding. Bidder i knows this fact and lets bidder $-i$ win his higher-ranked object A_f. Bidder i achieves this by bidding zero in the auction of A_f. Consequently, bidder $-i$ wins A_f at price δ_0. Bidder $-i$ bids zero in the auction of A_s and bidder i wins A_s at price δ_0. In summary, both bidder bid δ_0 for their higher-ranked objects and zero for their lower-ranked one as described in the lemma. □

Corollary 9. *In all realizations of the game of NC class, selling prices at $t = 1$, $P_A = (P_{A_f}, P_{A_s})$ are given as follows:*

$$P_k\left(b_k^*(x_A^1), b_k^*(x_A^2) | (r^1, r^2) \in NN\right) = \delta_0,\ \forall k \in \{A_f, A_s\} \tag{28}$$

In an NC class game, both objects from category A are sold at reserve price δ_0. Whereas in category B, the object B_f is sold at a price higher than δ_0, but B_s's selling price remains δ_0. The following corollary describes the pay-off of each bidder in an NC game.

Corollary 10. *In all realizations of the game of NC class, the overall payoff of bidder i is given as follows:*

$$\pi^i\left(\mathbf{b}^* | w_{B_f}^i = 1, (r^1, r^2) \in NC\right) = \alpha(x_A^i + x_B^i) - \alpha(x_A^{-i} + x_B^{-i}) + \beta(x_A^{-i} + \delta_0) - 2\delta_0 \tag{29}$$

$$\pi^i\left(\mathbf{b}^* | w_{B_f}^i = 0, (r^1, r^2) \in NC\right) = \beta(x_A^i + \delta_0) - 2\delta_0 \tag{30}$$

3.2.3 CN class of games

In this section, we examine all realizations of the game of CN class and determine the optimal bidding strategy. In the CN class of games, bidders rank the same object higher in category A but different objects in category B. Hence, category B is devoid of any competition, whereas bidders compete in category A. We describe the optimal bidding functions at $t = 2$ in the following lemma.

Lemma 11. *In all realizations of the game of CN class, the optimal bidding functions of bidder i at $t = 2$, i.e. $b_B^*(x_A^i, x_B^i) = (b_k^*(x_A^i, x_B^i))_{k \in (B_f, B_s)}$ are given as follows:*

$$b_k^*\left(x_A^i, x_B^i | (r^1, r^2) \in CN\right) = r_k^i \delta_0,\ \forall k \in \{B_f, B_s\} \tag{31}$$

Proof. Since there is no competition, the proof is exactly same as that in Lemma 1. □

Corollary 12. *In all realizations of the game of CN class, selling prices at $t = 2$, $P_B = (P_{B_f}, P_{B_s})$ are given as follows:*

$$P_k\left(b_k^*(x_A^i, x_B^i), b_k^*(x_A^i, x_B^i)|(r^1, r^2) \in CN\right) = \delta_0,\ \forall k \in \{B_f, B_s\} \tag{32}$$

Lemma 13. *In all realizations of the game of CN class, the optimal bidding functions of bidder i at $t = 1$, i.e. $b_A^*(x_A^i) = (b_k^*(x_A^i))_{k\in(A_f,A_s)}$ are given as follows:*

$$b_{A_f}^*\left(x_A^i|r_{A_f}^i = 1, (r^1, r^2) \in CN\right) = \delta_0 + \alpha(x_A^i + \mathbb{E}\left[x_B^i\right]) - \beta(\mathbb{E}\left[x_B^i\right] + \delta_0) \tag{33}$$

$$b_{A_s}^*\left(x_A^i|w_{A_f}^i = 0, (r^1, r^2) \in CN\right) = \delta_0 \tag{34}$$

$$b_{A_s}^*\left(x_A^i|w_{A_f}^i = 1, (r^1, r^2) \in CN\right) = 0 \tag{35}$$

Proof. At $t = 1$, both bidders know that their higher-ranked objects from category B are different and they are going to win their higher-ranked objects at $t = 2$. Since both bidders win their higher-ranked objects from category B, irrespective of category A auctions (Lemma 11), outcomes of category A and B auctions are independent. In category A, object A_f is auctioned first as it is ranked higher by both bidders. However, using backward induction, we consider the auction of object A_s first. When object A_s is auctioned, both bidders are aware of the outcome of object A_f's auction, i.e., $w_{A_f}^1$ and $w_{A_f}^2$ are public knowledge. Suppose bidder i won object A_f i.e. $w_{A_f}^i = 1$ and $w_{A_f}^{-i} = 0$. Since bidder i's need of category A objects is satisfied, he bids zero in the auction of object A_s. Mathematically,

$$b_{A_s}^*\left(x_A^i|w_{A_f}^i = 1, (r^1, r^2) \in NC\right) = 0$$

consequently, bidder $-i$ need not go beyond δ_0 in order to win object A_s, i.e.

$$b_{A_s}^*\left(x_A^{-i}|w_{A_f}^{-i} = 0, (r^1, r^2) \in NC\right) = \delta_0$$

Next, we find out the optimal bidding function for the object A_f. According to our assumption made previously, bidders don't know their valuations of category B objects at $t = 1$. But they know that, in this case, they will win their higher-ranked object from category B at price δ_0. Suppose bidder i wins the auction of object A_f at price $p_{A_f}^i$, his expected payoff will be given as:

$$\mathbb{E}[\pi^i(\cdot|w_{A_f}^i = 1)] = \int_{\delta_0}^{\delta_1} \left(\alpha(x_A^i + x_B^i) - p_{A_f} - \delta_0\right) f(x_B^i)dx_B^i = \alpha(x_A^i + \mathbb{E}[x_B^i]) - p_{A_f}^i - \delta_0$$

On the other hand, if he loses the auction of A_f and goes on to win A_s, his expected profit will be given as:

$$\mathbb{E}[\pi^i(\cdot|w_{A_f}^i = 0)] = \int_{\delta_0}^{\delta_1} \left(\beta(x_B^i + \delta_0) - 2\delta_0\right) f(x_B^i)dx_B^i = \beta(\mathbb{E}[x_B^i] + \delta_0) - 2\delta_0$$

A bidder would stay in the auction of object A_f, until he becomes indifferent between these two situations. Mathematically,

$$\mathbb{E}[\pi^i(\cdot|w_{A_f}^i = 1)] = \mathbb{E}[\pi^i(\cdot|w_{A_f}^i = 0)]$$

solving the above equation gives:

$$p^i_{A_f} = b^*_{A_f}\left(x^i_A | r^i_{A_f} = 1, (r^1, r^2) \in CN\right) = \delta_0 + \alpha(x^i_A + \mathbb{E}\left[x^i_B\right]) - \beta(\mathbb{E}\left[x^i_B\right] + \delta_0)$$

□

Corollary 14. *In all realizations of the game of CN class, selling prices at $t = 1$, i.e., $P_A = (P_{A_f}, P_{A_s})$ are given as follows:*

$$P_{A_f}\left(b^*_{A_f}(x^1_A), b^*_{A_f}(x^2_A) | (r^1, r^2) \in CN\right) = \\ \min\left\{\delta_0 + \alpha(x^1_A + \mathbb{E}\left[x^1_B\right]) - \beta(\mathbb{E}\left[x^1_B\right] + \delta_0), \delta_0 + \alpha(x^2_A + \mathbb{E}\left[x^2_B\right]) - \beta(\mathbb{E}\left[x^2_B\right] + \delta_0)\right\} \tag{36}$$

$$P_{A_s}\left(b^*_{A_s}(x^1_A), b^*_{A_s}(x^2_A) | (r^1, r^2) \in CN\right) = \delta_0 \tag{37}$$

In a CN class game, both objects from category B are sold at reserve price δ_0. Whereas in category A, the first object is sold at a price higher than δ_0 but the selling price of the second object remains δ_0. The following corollary describes bidders' payoffs in a CN class game.

Corollary 15. *In all realizations of the game of CN class, the overall payoff of bidder i is given as follows:*

$$\pi^i\left(\mathbf{b}^* | w^i_{A_f} = 1, (r^1, r^2) \in CN\right) = \alpha(x^i_A + x^i_B) - \alpha(x^{-i}_A + \mathbb{E}\left[x^{-i}_B\right]) + \beta(\mathbb{E}\left[x^{-i}_B\right] + \delta_0) - 2\delta_0 \tag{38}$$

$$\pi^i\left(\mathbf{b}^* | w^i_{A_f} = 0, (r^1, r^2) \in CN\right) = \beta(\delta_0 + x^i_B) - 2\delta_0 \tag{39}$$

3.2.4 CC class of games

In this section, we explore the optimal bidding strategy and equilibrium selling prices in a CC class game. In this class of the game, bidders' relative rankings are the same in both categories. Therefore, higher-ranked objects are auctioned first in both categories, i.e., A_f in category A and B_f in category B. Using the backward induction, we first characterize the equilibrium bidding strategy for the category B auctions.

Lemma 16. *In all realizations of the game of CC class, the optimal bidding functions of bidder i at $t = 2$, i.e., $b^*_B(x^i_A, x^i_B) = (b^*_k(x^i_A, x^i_B))_{k\in(B_f, B_s)}$ are given as follows:*

$$b^*_{B_f}\left(x^i_A, x^i_B | w^i_{A_f} = 1, (r^1, r^2) \in CC\right) = \delta_0 + \alpha(x^i_A + x^i_B) - \beta(x^i_B + \delta_o) \tag{40}$$

$$b^*_{B_f}\left(x^i_A, x^i_B | w^i_{A_f} = 0, (r^1, r^2) \in CC\right) = \beta(\delta_0 + x^i_B) - \delta_0 \tag{41}$$

$$b^*_{B_s}\left(x^i_A, x^i_B | w^i_{B_f} = 0, (r^1, r^2) \in CC\right) = \delta_0 \tag{42}$$

$$b^*_{B_s}\left(x^i_A, x^i_B | w^i_{B_f} = 1, (r^1, r^2) \in CC\right) = 0 \tag{43}$$

Proof. First we point out that at $t = 2$, bidders are asymmetric in the CC class of game. This happens because only one of the two bidders wins the higher-ranked object from category A. This induces an asymmetry between them when they compete for category B objects. Therefore, the optimal strategies of the bidders depend upon the outcome in the first period. We consider the category B auctions first.

We derive the optimal bidding strategy for the object B_s first. Since it is the second object to be auctioned at $t = 2$, outcome of the auction of object B_f is common knowledge. Suppose bidder i does not win object B_f, i.e., $w^i_{B_f} = 0$ and $w^{-i}_{B_f} = 1$. Since bidder $-i$'s need of category B object has been satisfied, he bids zero in the auction of B_s. Please note that this argument holds irrespective of the outcome of the category A auctions. Hence, the following equation holds whether or not a bidder wins his higher-ranked object from category A, i.e., $w^{-i}_{A_f} = 0$ or 1.

$$b^*_{B_s}\left(x^{-i}_A, x^{-i}_B | w^{-i}_{B_f} = 1, (r^1, r^2) \in CC\right) = 0$$

Consequently, bidder i need not go beyond δ_0 in order to win object B_s. Here also, the argument is valid irrespective of the results of category A auctions. Hence,

$$b^*_{B_s}\left(x^i_A, x^i_B | w^i_{B_f} = 0, (r^1, r^2) \in CC\right) = \delta_0$$

We now find the optimal bidding strategies in the auction of object B_f. Results of category A auctions influence the bidding behavior in the auction of B_f as it is ranked higher by both bidders. We find the optimal bidding functions for the auction of B_f for the winner as well as the loser of the higher-ranked object from category A.

Case I (Winner of A_f): Suppose bidder i won object A_f at $t = 1$, i.e. $w^i_{A_f} = 1$ and paid $p^i_{A_f}$. If he also wins object B_f, i.e. $w^i_{B_f} = 1$ at price $p^i_{B_f}$, his total profit will be given as:

$$\pi^i\left(\cdot | w^i_{A_f} = 1, w^i_{B_f} = 1, (r^1, r^2) \in CC\right) = \alpha(x^i_A + x^i_B) - p^i_{A_f} - p^i_{B_f}$$

On the other hand, if he loses B_f, he wins object B_s, and pays δ_0. His payoff is given as:

$$\pi^i\left(\cdot | w^i_{A_f} = 1, w^i_{B_f} = 0, (r^1, r^2) \in CC\right) = \beta(x^i_A + \delta_0) - p^i_{A_f} - \delta_0$$

Bidder i would stay in the auction of object B_f, until he becomes indifferent between winning and losing, i.e.

$$\pi^i\left(\cdot | w^i_{A_f} = 1, w^i_{B_f} = 1, (r^1, r^2) \in CC\right) = \pi^i\left(\cdot | w^i_{A_f} = 1, w^i_{B_f} = 0, (r^1, r^2) \in CC\right)$$

Solving the above equation gives:

$$p^i_{B_f} = b^*_{B_f}\left(x^i_A, x^i_B | w^i_{A_f} = 1, (r^1, r^2) \in CC\right) = \delta_0 + \alpha(x^i_A + x^i_B) - \beta(x^i_B + \delta_o)$$

Case II (Loser of A_f): Suppose bidder i loses A_f, i.e., $w^i_{A_f} = 0$. Since there is unit demand from each category, he wins object A_s. Suppose he pays $p^i_{A_s}$ for it. If he wins B_f at a price $p^i_{B_f}$, his profit will be:

$$\pi^i\left(\cdot | w^i_{A_f} = 0, w^i_{B_f} = 1, (r^1, r^2) \in CC\right) = \beta(\delta_0 + x^i_B) - p^i_{A_s} - p^i_{B_f}$$

On the other hand, if he loses B_f, he wins his lower-ranked object B_s and his payoff is given as:

$$\pi^i\left(\cdot|w^i_{A_f}=0, w^i_{B_f}=0,(r^1,r^2)\in CC\right)=2\delta_0-p^i_{A_s}-\delta_0$$

Bidder i will stay in the auction of B_f, until he becomes indifferent between the two situations, i.e.

$$\pi^i\left(\cdot|w^i_{A_f}=0, w^i_{B_f}=1,(r^1,r^2)\in CC\right)=\pi^i\left(\cdot|w^i_{A_f}=0, w^i_{B_f}=0,(r^1,r^2)\in CC\right)$$

Solving the above equation gives:

$$p^i_{B_f}=b^*_{B_f}\left(x^i_A,x^i_B|w^i_{A_f}=0,(r^1,r^2)\in CC\right)=\beta(\delta_0+x^i_B)-\delta_0$$

□

Corollary 17. *In all realizations of the game of CC class, selling prices at $t=2$, $P_B=(P_{B_f},P_{B_s})$ are given as follows:*

$$P_{B_f}\left(b^*_{B_f}(x^1_A,x^1_B),b^*_{B_f}(x^2_A,x^2_B)|w^i_{A_f}=1,(r^1,r^2)\in CC\right)=$$
$$\min\left\{\delta_0+\alpha(x^i_A+x^i_B)-\beta(x^i_B+\delta_o),\beta(\delta_0+x^{-i}_B)-\delta_0\right\} \quad (44)$$

$$P_{B_s}\left(b^*_{B_s}(v^1),b^*_{B_s}(v^2)|(r^1,r^2)\in CC\right)=\delta_0 \quad (45)$$

Now we direct our attention to the category A auctions. While bidding for category A objects, bidders evaluate not only their immediate payoffs from category A but also the effects of the outcome of category A auctions on category B. The following lemma describes the optimal bidding strategy for the higher-ranked object from category A.

Lemma 18. *In all realizations of the game of CC class, the optimal bidding functions of bidder i at $t=1$, i.e. $b^*_A(x^i_A)=(b^*_A(x^i_A))_{k\in(A_f,A_s)}$ are given as follows:*

$$b^*_{A_f}\left(x^i_A|(r^1,r^2)\in CC\right)=\Gamma_1(x^i_A)+\Gamma_2(x^i_A)-\frac{\Gamma_3(x^i_A)}{1-F\left(x^i_A\right)} \quad (46)$$

$$b^*_{A_s}\left(x^i_A|w^i_{A_f}=0,(r^1,r^2)\in CC\right)=\delta_0 \quad (47)$$

$$b^*_{A_s}\left(x^i_A|w^i_{A_f}=1,(r^1,r^2)\in CC\right)=0 \quad (48)$$

where

$$\Gamma_1(x^i_A)=\iint_{C_{WW}}\left[\alpha(x^i_A+x^i_B)-\beta(\delta_0+x^{-i}_B)+\delta_0\right]f(x^i_B)f(x^{-i}_B)dx^i_Bdx^{-i}_B \quad (49)$$

$$\Gamma_2(x^i_A)=\iint_{C_{WL}}\left[\beta(x^i_A+\delta_0)-\delta_0\right]f(x^i_B)f(x^{-i}_B)dx^i_Bdx^{-i}_B \quad (50)$$

$$\Gamma_3(x^i_A)=\iiint_{C_{LW}}\left[\beta(\delta_0+x^i_B)-\alpha(x^{-i}_A+x^{-i}_B)+\beta(x^{-i}_A+\delta_0)-2\delta_0\right]f(x^{-i}_A)f(x^i_B)f(x^{-i}_B)dx^{-i}_Adx^i_Bdx^{-i}_B \quad (51)$$

$$C_{WW}:\alpha(x^i_A+x^i_B)-\beta(x^i_A+\delta_0)+\delta_0>\beta(\delta_0+x^{-i}_B)-\delta_0 \quad (52)$$

$$C_{WL}:\alpha(x^i_A+x^i_B)-\beta(x^i_A+\delta_0)+\delta_0<\beta(\delta_0+x^{-i}_B)-\delta_0 \quad (53)$$

$$C_{LW}:x^{-i}_A>x^i_A;\beta(\delta_0+x^{-i}_A)-\delta_0>\delta_0+\alpha(x^{-i}_A+x^{-i}_B)-\beta(x^{-i}_A+\delta_0) \quad (54)$$

Proof. At $t = 1$, object A_f is auctioned first as it is ranked higher by both bidders. Using backward induction, we consider the auction of object A_s first. When object A_s is auctioned, both bidders are aware of the outcome of object A_f's auction, i.e., $w^1_{A_f}$ and $w^2_{A_f}$ are common knowledge. Suppose bidder i won object A_f i.e. $w^i_{A_f} = 1$ and $w^{-i}_{A_f} = 0$. Since bidder i's need of category A object is satisfied, he bids zero in the auction of object A_s. He does not engage in false bidding as it is not favorable as shown in Lemma 8. Mathematically,

$$b^*_{A_s}\left(x^i_A|w^i_{A_f} = 1, (r^1, r^2) \in CC\right) = 0$$

Consequently, bidder $-i$ need not go beyond δ_0 in order to win A_s. Therefore:

$$b^*_{A_s}\left(x^{-i}_A|w^{-i}_{A_f} = 0, (r^1, r^2) \in CC\right) = \delta_0$$

Next, we consider the auction of object A_f. Assume that bidder i wins the higher-ranked object from category A and pays $p^i_{A_f}$. There are two possibilities at $t = 2$. Bidder i can either win or lose the higher-ranked object from category B. If he also wins higher-ranked object from category B, his payoff is given as:

$$\pi^i\left(\cdot|w^i_{A_f} = 1, w^i_{B_f} = 1\right) = \alpha(x^i_A + x^i_B) - p^i_{A_f} - \beta(\delta_0 + x^{-i}_B) + \delta_0$$

In this equation, $\alpha(x^i_A + x^i_B)$ is the total utility of bidder i by winning both of his higher-ranked objects. And, $\beta(\delta_0 + x^{-i}_B) - \delta_0$ is the price paid by bidder i for B_f (Eq. 40) as it is the losing bid. Bidder i wins the higher-ranked object from category B when he bids higher than his opponent in the auction of B_f, i.e,

$$\alpha(x^i_A + x^i_B) - \beta(x^i_A + \delta_0) + \delta_0 > \beta(\delta_0 + x^{-i}_B) - \delta_0$$

We denote this condition by C_{WW} and it can be obtained from Eq 39 and 40. Therefore, the expected pay-off from winning the higher-ranked objects from both categories can be described as:

$$\mathbb{E}\left[\pi^i\left(\cdot|w^i_{A_f} = 1, w^i_{B_f} = 1\right)\right] = \frac{\iint_{C_{WW}}\left[\alpha(x^i_A + x^i_B) - p_{A_f} - \beta(\delta_0 + x^{-i}_B) + \delta_0\right] f(x^i_B) f(x^{-i}_B) dx^i_B dx^{-i}_B}{\iint_{C_{WW}} f(x^i_B) f(x^{-i}_B) dx^i_B dx^{-i}_B}$$

Where denominator of the above equation denotes the probability of the bidder i winning the higher-ranked objects from both categories. Let this probability be denoted by P_{WW}.

Consider the other situation in which, bidder i wins the higher-ranked object only from category A. In such a situation, he wins the lower-ranked object from category B and pays δ_0 (Eq. 41) for it. Therefore, his pay-off is given by the following expression:

$$\pi^i\left(\cdot|w^i_{A_f} = 1, w^i_{B_f} = 0\right) = \beta(x^i_A + \delta_0) - p^i_{A_f} - \delta_0$$

Here $\beta(x^i_A + \delta_0)$ is the total utility obtained by bidder i and $p^i_{A_f}$ and δ_0 are the total payments made by him. Bidder i loses B_f when his bid is lower than that of his opponent, i.e., the following condition holds:

$$\alpha(x^i_A + x^i_B) - \beta(x^i_A + \delta_0) + \delta_0 < \beta(\delta_0 + x^{-i}_B) - \delta_0$$

We denote this condition by C_{WL} and it can be obtained from Eq 39 and 40. Therefore, the expected pay-off of bidder i, when he wins the higher-ranked object only from category A, is given by the following expression:

$$\mathbb{E}\left[\pi^i\left(\cdot|w^i_{A_f}=1,w^i_{B_f}=0\right)\right]=\frac{\iint_{C_{WL}}\left[\beta(x^i_A+\delta_0)-p^i_{A_f}-\delta_0\right]f(x^i_B)f(x^{-i}_B)dx^i_Bdx^{-i}_B}{\iint_{C_{WL}}f(x^i_B)f(x^{-i}_B)dx^i_Bdx^{-i}_B}$$

Where denominator of the above equation denotes the probability of bidder i winning the higher-ranked object from category A but not from B. Let this probability be denoted by P_{WL}.

Next, we consider the two possibilities that arise when bidder i loses A_f. Here again, bidder i can either win or lose his higher-ranked object from category B. We first consider the situation in which he loses the higher-ranked object from category A but wins from category B. In such a scenario, his expected pay-off is given by the following expression:

$$\pi^i\left(\cdot|w^i_{A_f}=0,w^i_{B_f}=1\right)=\beta(\delta_0+x^i_B)-\alpha(x^{-i}_A+x^{-i}_B)+\beta(x^{-i}_A+\delta_0)-2\delta_0$$

Here $\beta(\delta_0+x^i_B)$ is the total utility obtained by him. The total payments made by bidder i for A_s and B_f are δ_0 and $\delta_0+\alpha(x^{-i}_A+x^{-i}_B)-\beta(x^{-i}_A+\delta_0)$ respectively. Bidder i finds himself in this situation, when the following two conditions hold:

$$x^{-i}_A>x^i_A$$

$$\beta(\delta_0+x^{-i}_A)-\delta_0>\delta_0+\alpha(x^{-i}_A+x^{-i}_B)-\beta(x^{-i}_A+\delta_0)$$

We collectively represent these two conditions as C_{LW}. Given C_{LW}, the expected profit of bidder i from winning the higher-ranked object only from category B can be represented by the following expression:

$$\mathbb{E}\left[\pi^i\left(\cdot|w^i_{A_f}=0,w^i_{B_f}=1\right)\right]=$$
$$\frac{\iiint_{C_{LW}}\left[\beta(\delta_0+x^i_B)-\alpha(x^{-i}_A+x^{-i}_B)+\beta(x^{-i}_A+\delta_0)-2\delta_0\right]\tilde{f}(x^{-i}_A)f(x^i_B)f(x^{-i}_B)dx^{-i}_Adx^i_Bdx^{-i}_B}{\iiint_{C_{LW}}\tilde{f}(x^{-i}_A)f(x^i_B)f(x^{-i}_B)dx^{-i}_Adx^i_Bdx^{-i}_B}$$

The denominator of the above equation denotes the probability of bidder i losing the higher-ranked object from category A but winning that from category B. Let this probability be denoted by P_{LW}. Please note that the density of x^{-i}_A used in the above equation is the updated according to the Baye's rule as described in Eq 14.

When bidder i loses both of his higher-ranked objects, his payoff remains zero, as his overall utility is the same as the total payments made by him, i.e.

$$\pi^i\left(\cdot|w^i_{A_f}=0,w^i_{B_f}=0\right)=0$$

Therefore, his expected profit also remains zero, i.e.

$$\mathbb{E}\left[\pi^i\left(\cdot|w^i_{A_f}=0,w^i_{B_f}=0\right)\right]=0$$

Let the probability of bidder i losing both of his higher-ranked object be denoted by P_{LL}.

Bidder i would leave the auction of A_f, when he becomes indifferent between winning and losing:

$$\mathbb{E}\left[\pi^i\left(\cdot|w^i_{A_f}=1, w^i_{B_f}=1\right)\right]\cdot P_{WW}+\mathbb{E}\left[\pi^i\left(\cdot|w^i_{A_f}=1, w^i_{B_f}=0\right)\right]\cdot P_{WL}=$$
$$\mathbb{E}\left[\pi^i\left(\cdot|w^i_{A_f}=0, w^i_{B_f}=1\right)\right]\cdot P_{LW}+\mathbb{E}\left[\pi^i\left(\cdot|w^i_{A_f}=0, w^i_{B_f}=0\right)\right]\cdot P_{LL}$$

Using the respective values in the above expression:

$$\iint_{C_{WW}}\left[\alpha(x^i_A+x^i_B)-p^i_{A_f}-\beta(\delta_0+x^{-i}_B)+\delta_0\right]f(x^i_B)f(x^{-i}_B)dx^i_Bdx^{-i}_B+$$
$$\iint_{C_{WL}}\left[\beta(x^i_A+\delta_0)-p^i_{A_f}-\delta_0\right]f(x^i_B)f(x^{-i}_B)dx^i_Bdx^{-i}_B=$$
$$\iiint_{C_{LW}}\left[\beta(\delta_0+x^i_B)-\alpha(x^{-i}_A+x^{-i}_B)+\beta(x^{-i}_A+\delta_0)-2\delta_0\right]\tilde{f}(x^{-i}_A)f(x^i_B)f(x^{-i}_B)dx^{-i}_Adx^i_Bdx^{-i}_B$$

where $\tilde{f}(\cdot)$ is the updated density of x^{-i}_A i.e.

$$\tilde{f}(x^{-i}_A)=\frac{f(x^{-i}_A)}{\int_{x^i_A}^{\delta_1}f(x^{-i}_A)dx^{-i}_A}$$

hence, the previous expression can be written as

$$\iint_{C_{WW}}\left[\alpha(x^i_A+x^i_B)-\beta(\delta_0+x^{-i}_B)+\delta_0\right]f(x^i_B)f(x^{-i}_B)dx^i_Bdx^{-i}_B+$$
$$\iint_{C_{WL}}\left[\beta(x^i_A+\delta_0)-\delta_0\right]f(x^i_B)f(x^{-i}_B)dx^i_Bdx^{-i}_B-p^i_{A_f}\iint_{C_{WW}+C_{WL}}f(x^i_B)f(x^{-i}_B)dx^i_Bdx^{-i}_B=$$
$$\frac{1}{1-F\left(x^i_A\right)}\left[\iiint_{C_{LW}}\left[\beta(\delta_0+x^i_B)-\alpha(x^{-i}_A+x^{-i}_B)+\beta(x^{-i}_A+\delta_0)-2\delta_0\right]f(x^{-i}_A)f(x^i_B)f(x^{-i}_B)dx^{-i}_Adx^i_Bdx^{-i}_B\right]$$

Since

$$\iint_{C_{WW}+C_{WL}}f(x^i_B)f(x^{-i}_B)dx^i_Bdx^{-i}_B=1$$

the optimal bid can be found out as

$$p^i_{A_f}=b^*_{A_f}\left(x^i_A|(r^1,r^2)\in CC\right)=\Gamma_1(x^i_A)+\Gamma_2(x^i_A)-\frac{\Gamma_3(x^i_A)}{1-F\left(x^i_A\right)}$$

where $\Gamma_1(x^i_A)$, $\Gamma_2(x^i_A)$ and $\Gamma_3(x^i_A)$ are as described in Eq 48, 49 and 50. □

Corollary 19. *In all realizations of the game of CC class, selling prices at* $t=1$, $P_A=(P_{A_f},P_{A_s})$ *are given as follows:*

$$P_{A_f}\left(b^*_{A_f}(x^1_A),b^*_{A_f}(x^2_A)|(r^1,r^2)\in CC\right)=$$
$$\min\left\{\Gamma_1(x^1_A)+\Gamma_2(x^1_A)-\frac{\Gamma_3(x^1_A)}{1-F\left(x^1_A\right)},\Gamma_1(x^2_A)+\Gamma_2(x^2_A)-\frac{\Gamma_3(x^2_A)}{1-F\left(x^2_A\right)}\right\} \quad (55)$$

$$P_{A_s}\left(b^*_{A_s}(x^1_A),b^*_{A_s}(x^2_A)|(r^1,r^2)\in CC\right)=\delta_0 \quad (56)$$

In the realization of the game of CC class, higher-ranked objects from both categories are sold at prices higher than δ_0, whereas lower-ranked objects are sold at δ_0. The following corollary describes the expected payoffs of each bidder in a CC class game.

Corollary 20. *In all realizations of the games of CC class, the overall payoff of bidder i is given as follows:*

$$\pi^i\left(\mathbf{b}^*|w^i_{A_f}=1, w^i_{B_f}=1, (r^1,r^2)\in CN\right) = \alpha(x^i_A+x^i_B)-\Gamma_1(x^{-i}_A)-\Gamma_2(x^{-i}_A)+\frac{\Gamma_3(x^{-i}_A)}{1-F\left(x^{-i}_A\right)}- \beta(\delta_0+x^{-i}_B)+\delta_0 \quad (57)$$

$$\pi^i\left(\mathbf{b}^*|w^i_{A_f}=1, w^i_{B_f}=0, (r^1,r^2)\in CN\right) = \beta(x^i_A+\delta_0)-\Gamma_1(x^{-i}_A)-\Gamma_2(x^{-i}_A)+ \frac{\Gamma_3(x^{-i}_A)}{1-F\left(x^{-i}_A\right)}-\delta_0 \quad (58)$$

$$\pi^i\left(\mathbf{b}^*|w^i_{A_f}=0, w^i_{B_f}=1, (r^1,r^2)\in CN\right) = \beta(\delta_0+x^i_B)-\alpha(x^{-i}_A+x^{-i}_B)+\beta(x^{-i}_B+\delta_o)-2\delta_0 \quad (59)$$

$$\pi^i\left(\mathbf{b}^*|w^i_{A_f}=0, w^i_{B_f}=0, (r^1,r^2)\in CN\right) = 0 \quad (60)$$

3.3 Impact of the Order of Sale

In our model, the seller decides the order of sale based on the bidders' relative rankings. As mentioned previously, if an object is ranked higher by both bidders, the seller presents it first. Next, we argue that this indeed is the revenue maximizing order in which, the seller should present the objects.

In any category, if the seller first presents an object which is ranked lower by both bidders, it may remain unsold. This is because a lower-ranked object always provides the minimum utility to a bidder. Hence, a rational bidder will always want to buy his higher-ranked object first. A bidder participates in the auction of his lower-ranked object, only when he fails to reserve the higher-ranked one. Hence, if an object that is ranked lower by both the bidders is presented first, it may fetch zero bids and remain unsold. The auctioneer anticipates this and always presents the higher-ranked object first. Hence, the seller is already optimizing his revenue as far as the order of sale within a category is concerned. Next we explore the possibility of optimizing the seller's revenue by interchanging the order in which categories are presented.

In the model, we assume that the bidders derive their signals from a distribution $F(\cdot)$ at both time periods. Therefore, from the seller's point of view, bidders' valuations for their higher-ranked objects from both categories are independent and identically distributed random variables. Hence, the seller's expected revenue is not affected by the order in which categories are presented during the auction.

3.4 Efficiency

In the context of a single-unit allocation, an auction mechanism is efficient if the object being sold is allocated to the bidder who values it the most. However, the situation is quite complicated in the present model. Several restrictions like unit demand from each category, synergetic relationships between the objects, etc. make it difficult to define efficiency. Hence, we resort to the definition of Pareto optimality in order to evaluate the efficiency of the mechanism. We say that the present auction mechanism is efficient if a post-auction exchange of the objects between the bidders cannot benefit both of them. In this post-auction exchange of objects, bidders can either swap their entire bundles or make an exchange between the same-category objects. Since the amounts paid by the bidders in the auction are their sunk cost, it is sufficient to analyze only the change in their utilities to determine whether ax exchange would take place or not. In other words, if it is possible to increase both bidders' utilities by any type of post-auction exchange, then the auction mechanism is not efficient. Using this definition of efficiency, we show that the present auction mechanism is efficient for all four classes of the game.

In the outcome of an NN class game, the utilities of both bidders are at their maximum. Hence, any exchange will only decrease their utilities. Therefore, the outcome of an NN class game is efficient. As the result of an NC class game, both bidders win their higher-ranked objects from category A, but only one bidder wins his higher-ranked object from category B. Suppose that bidder i won the higher-ranked object from category B. Since both bidders win their higher-ranked objects from category A, making an exchange between the category A objects will result in a decrease in both bidders' utilities, and hence, such an exchange will not take place. Exchanging category B objects will increase the utility of bidder $-i$, but will decrease that of the bidder i, and hence, such an exchange will also not occur. Also, exchanging the entire bundle is not possible since it will drop bidder i's utility from the maximum to the minimum. A similar line of arguments can be given in the case of the CN class of games.

In the case of the CC class of games, two types of outcomes are possible. In the first type, only one bidder wins the higher-ranked objects from both categories, whereas in the second, each bidder wins his higher-ranked object from one of the two categories. In the first type of outcome, it is trivial to observe that any exchange of objects will decrease the utility of the bidder who won the higher-ranked objects from both categories. Hence, any type of exchange is not possible in the first type of outcome. In the second type, exchanging objects from a single category will result in the minimum utility to one bidder and the maximum to the other, and hence, will not occur. Next we argue that exchanging the entire bundle is also not feasible. Suppose bidder i won the higher-ranked object from category A and bidder $-i$ won that from category B, and hence, their utilities are given as $\beta(x^i_A + \delta_0)$ and $\beta(\delta_0 + x^{-i}_B)$ respectively. Whereas, if the entire bundle is exchanged, bidders i and $-i$' utilities will be given as $\beta(\delta_0 + x^i_B)$ and $\beta(x^{-i}_A + \delta_0)$ respectively. Since the bidding function for the higher-ranked object at $t = 1$ is symmetric and strictly increasing, it can be concluded that $x^i_A > x^{-i}_A$. Also, given the results of category A auctions, it is possible for bidder $-i$ to win the higher-ranked object from category B only when $x^{-i}_B > x^i_B$. The above two conditions can be expressed simultaneously as

$$x^i_A - x^{-i}_A > 0 > x^i_B - x^{-i}_B$$

$$x^i_A - x^i_B > x^{-i}_A - x^i_B > x^{-i}_A - x^{-i}_B$$

$$(x^i_A + \delta_0) - (x^i_B + \delta_0) > x^{-i}_A - x^i_B > (x^{-i}_A + \delta_0) - (x^{-i}_B + \delta_0)$$

since $\beta > 1$,

$$\beta(x^i_A + \delta_0) - \beta(x^i_B + \delta_0) > \beta(x^{-i}_A - x^i_B) > \beta(x^{-i}_A + \delta_0) - \beta(x^{-i}_B + \delta_0)$$

Therefore, if $x_A^{-i} > x_B^i$, then:

$$\beta(x_A^i + \delta_0) - \beta(x_B^i + \delta_0) > 0$$

which means that bidder i's utility before the exchange is more than that after it. Hence, the exchange of the bundles will not occur. On the other hand, if $x_A^{-i} < x_B^i$, then:

$$\beta(x_A^{-i} + \delta_0) - \beta(x_B^{-i} + \delta_0) < 0$$

which means that the change in the utility for bidder $-i$ is negative from the exchange, and hence, such an exchange will also not take place. In conclusion, the auction mechanism studied in the present study is efficient for all four classes of the game.

4 Discussion

In this section, we discuss our results obtained previously and their implications to get more insights into the problem. We start with interpreting the bidding strategies.

4.1 Interpretations of Bidding Strategies

In this subsection, we try to interpret and give intuitive explanations of the optimal bidding strategies of all 4 classes of the game. In any realization of the NN class game, bidder don't compete in any category. Hence, the optimal bidding strategy requires them to bid the minimum, which is δ_0. As a result, both bidders win their higher-ranked objects from both categories.

In an NC class game, bidders compete in category B but not in A. Hence, the optimal bidding strategy of bidder i requires him to bid the minimum, i.e. δ_0, in the auction of the higher-ranked object from category A. As a result, both bidders win their higher-ranked objects from category A. Bidder i's optimal biding function in the auction of the higher-ranked object from category B depends on his signals from both periods. This is because of the complementary relationship between the inter-category objects. The prescribed strategy is to stay in the auction of higher-ranked object from category B, until its price reaches $\delta_0 + \alpha(x_A^i + x_B^i) - \beta(x_A^i + \delta_0)$. This expression can be perceived as a summation of two components. The first component is δ_0, which is the minimum amount every bidder has to pay in order to reserve any object (higher or lower-ranked). The second component, which is $\alpha(x_A^i + x_B^i) - \beta(x_A^i + \delta_0)$, can be recognized as the difference between the utilities in the two possible scenarios. $\alpha(x_A^i + x_B^i)$ is the utility that bidders i realizes when he wins his higher-ranked object from both categories. Likewise, $\beta(x_A^i + \delta_0)$ is the utility realized by bidder i, when he loses his higher-ranked object from category B. Hence, if this difference in utilities is taken away from bidder i, he becomes indifferent between the two situations. Therefore, in the presence of competition, bidder i is willing to sacrifice this extra utility in order to get his higher-ranked object from category B.

In a CN class game, bidders compete in category A but not in B. While bidding for category A objects, both bidders are aware that they would win their higher-ranked objects from category B. However, bidders are not aware of their actual valuations for category B objects at $t = 1$. Therefore, bidder i uses the expected values of x_B^i while bidding for the higher-ranked object from category A. Optimal bidding functions in a CN class game can be interpreted in exactly the same manner as those in NC class games.

In a CC type game, bidders compete in both categories. Consequently, when bidders get to category B auctions, they are asymmetric. This asymmetry arises owing to the fact that only one of the two bidders gets the higher-ranked object from category A. As a result, bidders use different strategies in category B auctions. The optimal bidding strategy of a bidder who lost his higher-ranked object from category A (let bidder i), is to stay in the auction of object B_f till its price reaches $\beta(\delta_0 + x_B^i) - \delta_0$. We rewrite this expression as $\delta_0 + \beta(\delta_0 + x_B^i) - 2\delta_0$ in order to be able to provide a better interpretation. This expression can be broken down to two components with first component being δ_0. This is the base price all bidders have to pay in order to reserve any object. The second component, i.e., $\beta(\delta_0 + x_B^i) - 2\delta_0$ can be seen as a difference between the utilities realized in two situations. The utility realized by bidder i, if he wins his higher-ranked object from category B, is $\beta(\delta_0 + x_B^i)$, whereas he realizes the utility of $2\delta_0$ if he also loses the higher-ranked object from category B. Bidder i is willing to sacrifice this difference in utilities in order to reserve the higher-ranked object from category B. In other words, loss of this extra utility makes bidder i indifferent between the two situations. A distinguishing feature of this expression is that it is independent of x_A^i. This demonstrates that bidder i's strategy in category B auction is independent of his first signal x_A^i, because he is not able to explore the complementarities between the objects. The optimal strategy of a bidder who won his higher-ranked object from category A (let bidder $-i$) is to stay in the auction of the higher-ranked object from category B, until the price reaches $\delta_0 + \alpha(x_A^{-i} + x_B^{-i}) - \beta(x_A^{-i} + \delta_0)$. This is exactly the same expression as that of an NC type game, and hence can be interpreted in the same way as before.

Bidder i's optimal bidding strategy for the higher-ranked object from category A requires him to stay in the auction until the price reaches $\Gamma_1(x_A^i) + \Gamma_2(x_A^i) - \frac{\Gamma_3(x_A^i)}{1-F\left(\frac{x_A^i-\delta_0}{\delta_1}\right)}$. This expression is very complicated and hence difficult to interpret.

4.2 Comparison Between Bids Across Game Types

In this section, we compare the optimal strategies for the higher-ranked objects in category A and B across game types. We assume that both x_A^i and x_B^i are uniformly distributed between $\delta_0 = 1$ and $\delta_1 = 2$. Synergy parameters α and β were chosen to be 1.25 and 1.1 respectively. Figure 1 compares the optimal bidding functions of bidder i for the higher-ranked object from category A as function of x_A^i in different classes of the game. As shown in the figure, category A bids are constant at $\delta_0 = 1$ in NN and NC class games for all values of x_A^i (no competition). Optimal bidding strategy increases linearly with x_A^i in a CN type game. The optimal bidding strategy for the higher-ranked object in category A is given as follows:

$$b^*_{A_f}\left(x_A^i|(r^1, r^2) \in CC\right) = 0.264682 + 1.22061 \cdot x_A^i + 0.00538636 \cdot \left(x_A^i\right)^2 - 0.000306818 \cdot \left(x_A^i\right)^3$$

Although, it is a cubic equation in x_A^i, the coefficients of square and cube terms are very small. Therefore, its graph looks like a straight line in Figure 1. However, as visible from the figure, a bidder bids most aggressively for his higher-ranked object from category A in a CC class game for a given value of x_A^i. This indicates the presence of intense competition in a CC class game. Intuitively, while bidding for the higher-ranked object from category A in a CC class game, bidders not only incorporates the outcome of category A auctions, but also the effects it will have on the outcome of category B auctions. In other words,

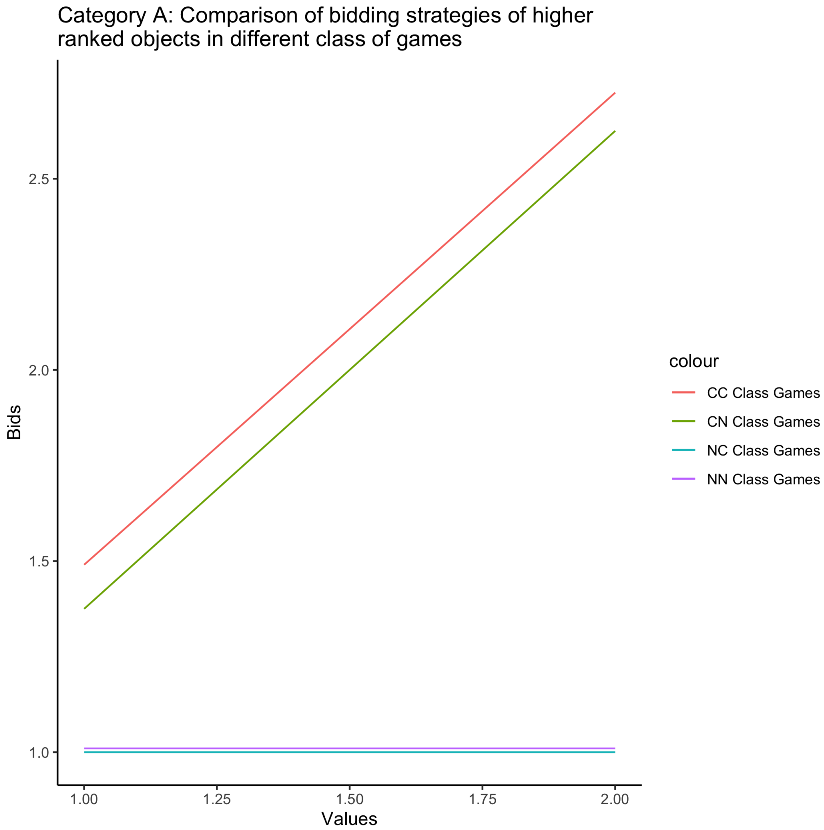

Figure 1: Category A Bids

while bidding for the higher-ranked object from category A, a bidder also competes for the advantage that he gets in category B auctions by winning in category A.

Now, we turn our attention to the comparison of the optimal bidding strategies for the higher-ranked object from category B across different game types. Figure 2 illustrates how the optimal bidding functions of bidder i for the higher-ranked object from category B varies with x^i_B across different classes of the game. Optimal strategies in the NN and CN class games are constant at $\delta_0 = 1$ (no competition). Winner of the higher-ranked object from category A, in a CC class game, bids for the higher-ranked from category B in the same way as each bidder does in the NC class game. This is because in an NC class game, each bidder enters into category B auctions as a winner owing to the absence of any competition in category A. These two strategies increase linearly with x^i_B. The optimal bidding function of a bidder who lost the higher-ranked object in category A, in a CC class game, also increases linearly with x^i_B, but it is always lesser than that of the winner, for the same value of x^i_B. This shows that, in a CC class game, winning the higher-ranked object from category A gives an advantage to a bidder, while bidding for the higher-ranked object from category B.

4.3 Ranking of Bidders' Expected Profits and Seller's Expected Revenue

In this section, we analyze the bidders' ex-ante expected profits and the seller's ex-ante expected revenue. Different classes of the game have different values of these two variables depending on the level of competition present. In an NN class game, bidders

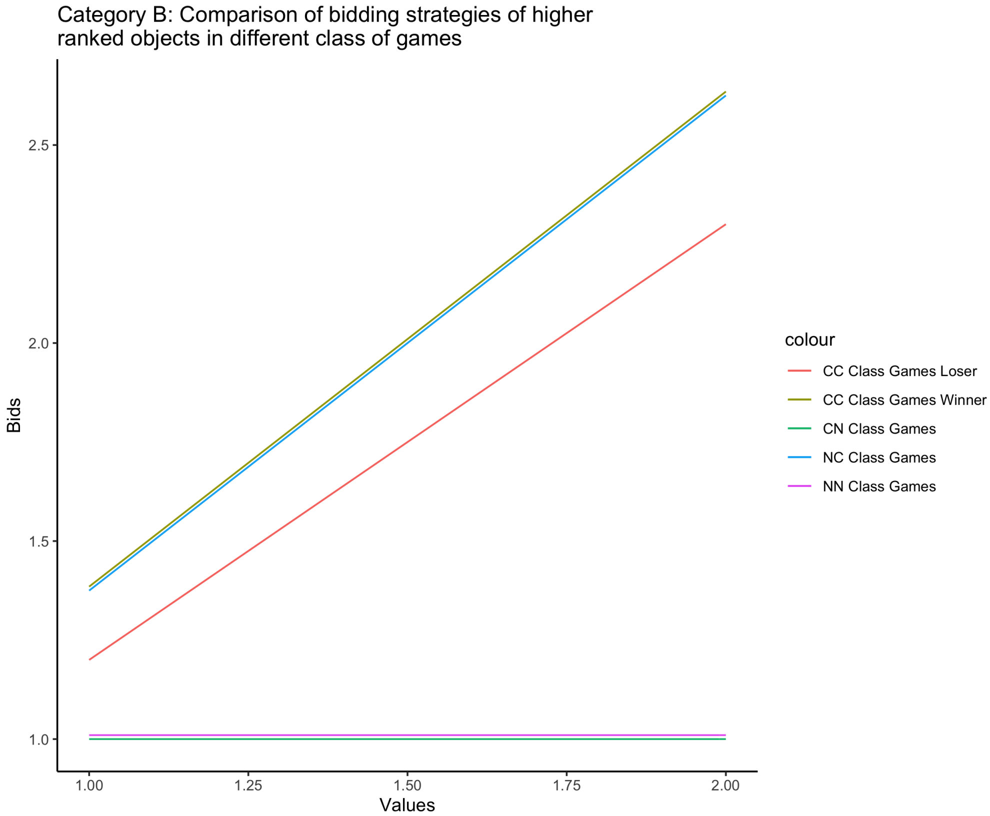

Figure 2: Category B Bids

don't compete at all, and hence, they get their higher-ranked objects from both categories at the minimum price. Therefore, a bidder's expected profit from an NN class game is the highest, whereas the expected revenue of the seller is the lowest. In a CC class game, bidders' higher-ranked objects are the same in both categories. This leads to intense competition between them, which subsequently leads to the their lowest expected profit. The presence of intense competition benefits the seller and his expected revenue is the maximum from a CC class game. In the NC and CN classes of the game, bidders compete in only one of the two categories. Since the valuations for both categories' objects are stochastically equivalent, bidders' expected profits are equal for these two categories. This stochastic equivalency also leads to the equal expected revenue for the seller from both of these classes of the game. The level of competition in a CN or NC class game is neither as low as that an NN class game nor as high as that a CC class game. Because of this, a bidder's expected profit is higher than that of a CC but lower than that from an NN class of the game. The expected revenue of the seller follows the opposite order, i.e., it is higher than that of an NN class game but lower than that of a CC class game. In summary, a bidder's expected profits from all four classes of the game can be ranked as follows:

$$P_{N,N} > P_{N,C} = P_{C,N} > P_{C,C}$$

Correspondingly, the seller's expected revenue across the four classes of the game can be ranked as follows:

$$R_{C,C} > R_{C,N} = R_{N,C} > R_{N,N}$$

4.4 Possibility of Price Trends

Next, we explore the possibility of any price trend in the model, keeping the order of sale fixed. We say that a price trend is observed in an auction format, if an object gets a higher/lower selling price purely because of its position in the order of sale.

Within a category, the selling price of one object is always δ_0, while that of the other object is either equal or more than δ_0. This entirely depends on whether bidders compete in that category or not, and is unaffected by order of sale within a category. In other words, no intra-category price trends are observed.

While comparing inter-category price trends, it is sufficient to only compare the prices of higher-ranked objects, since the lower-ranked objects are always sold at δ_0. In an NN class game, selling prices of all the objects are δ_0, and we see no price trends at all. In an NC class game, selling prices of both objects from category A remain δ_0, but the higher-ranked object from category B gets a competitive price. However, this increase in the selling price is purely because of the competition in category B. Similarly, in a CN type game, selling prices decrease, but again, this happens due to the competition in category A. In all three of these game types, the order in which categories are presented, does not affect the selling prices. However, in a CC class game, the higher-ranked object from category A gets a higher selling price than that from category B, as evident from the bidding functions from figures 1 and 2. Here, the competition is present in both categories, bidders draw signals from the same distribution, and yet, the higher-ranked object from category A fetches a higher selling price. This leads to a decrease in selling prices resulting purely because of the order in which categories are presented. Therefore, the order of sale affects the selling prices only in a CC class game. However, this doesn't affect the overall revenue of the seller as the sum of the revenues from both categories is independent of the order in which they are presented.

4.5 Possibility of False Bidding

We say that a bidder engages in false bidding, when he chooses to bid up the prices of the objects to harm the other bidders. In our model, we show that there is no possibility of false bidding (Lemma 8). However, our proof relies on the assumption that a bidder's value for his lower-ranked object is always the minimum. If this assumption doesn't hold, a bidder will have an incentive to sabotage the other bidder's purchasing ability or engage in false bidding.

5 Conclusions

In this study, we attempted to model a variant of a sequential English auction, which involves selling four objects with synergetic relationships. We introduced the idea of categories of objects, which are nothing but the collection of substitutable items, to better approach the problem. We found out that in the presence of two bidders, this auction can have several different outcomes which can be grouped into four classes. These classes of the game are characterized by the presence or absence of competition in the two categories, which subsequently influences the optimal strategies of bidders. We found out the optimal bidding strategies and equilibrium selling prices of all the objects for all four classes of the game. We discovered that in such an auction format, many objects are always sold at their reserve price, some objects are sold at competitive prices, while some objects can get exceptionally high selling prices. We find that the auction format is efficient as there is no possibility of any post-auction exchange of objects that benefit both bidders. Our analysis reveals that, when bidders compete in both categories, the expected selling price of the higher-ranked object from category A is higher than that from category B. Hence, decreasing selling prices are observed in a CC class game. We also observed that the total expected revenue of the seller doesn't change with the change in the order of sale. We discovered that there is a possibility of false bidding in such an auction format, if certain assumptions of the model are violated.

References

[Bernhardt & Scoones, 1994] Bernhardt, D. & Scoones, D. (1994). A note on sequential auctions. *The American economic review*, 84(3), 653–657.

[Branco, 1997] Branco, F. (1997). Sequential auctions with synergies: an example. *Economics Letters*, 54(2), 159–163.

[De Silva et al., 2005] De Silva, D. G., Jeitschko, T. D., & Kosmopoulou, G. (2005). Stochastic synergies in sequential auctions. *International Journal of Industrial Organization*, 23(3), 183–201.

[Elmaghraby, 2003] Elmaghraby, W. (2003). The importance of ordering in sequential auctions. *Management Science*, 49(5), 673–682.

[Engelbrecht-Wiggans, 1994] Engelbrecht-Wiggans, R. (1994). Sequential auctions of stochastically equivalent objects. *Economics Letters*, 44(1), 87–90.

[Jeitschko & Wolfstetter, 2002] Jeitschko, T. D. & Wolfstetter, E. (2002). Scale economies and the dynamics of recurring auctions. *Economic Inquiry*, 40(3), 403–414.

[Jofre-Bonet & Pesendorfer, 2014] Jofre-Bonet, M. & Pesendorfer, M. (2014). Optimal sequential auctions. *International Journal of Industrial Organization*, 33, 61–71.

[Kittsteiner et al., 2004] Kittsteiner, T., Nikutta, J., & Winter, E. (2004). Declining valuations in sequential auctions. *International Journal of Game Theory*, 33(1), 89–106.

[Leufkens et al., 2010] Leufkens, K., Peeters, R., & Vermeulen, D. (2010). Sequential auctions with synergies: The paradox of positive synergies. *Economics Letters*, 109(3), 139–141.

[Menezes & Monteiro, 2003] Menezes, F. M. & Monteiro, P. K. (2003). Synergies and price trends in sequential auctions. *Review of Economic Design*, 8(1), 85–98.

[Menezes & Monteiro, 2004] Menezes, F. M. & Monteiro, P. K. (2004). Auctions with synergies and asymmetric buyers. *Economics Letters*, 85(2), 287–294.

[Milgrom & Weber, 1982] Milgrom, P. R. & Weber, R. J. (1982). A Theory of Auctions and Competitive Bidding. *Econometrica*, 50(5), 1089–1122.

[Muramoto & Sano, 2016] Muramoto, A. & Sano, R. (2016). Sequential auctions of heterogeneous objects. *Economics Letters*, 149, 49–51.

[Sørensen, 2006] Sørensen, S. T. (2006). Sequential auctions for stochastically equivalent complementary objects. *Economics Letters*, 91(3), 337–342.