Bargaining Between Collaborators of a Stochastic Project

Z. Eddie Ning *

December 25, 2019

Abstract

Some projects require collaboration between two firms. The expected return from such alliance can change over time due to evolving market conditions or arrival of new information. In such cases, when do firms agree to collaborate, and how do they split the profit? To address these questions, the paper studies a continuous-time model of bargaining with a stochastic surplus. The paper shows that the existence of outside options creates a hold-up problem that leads to inefficiencies. Firms act too impatiently, causing both the probability of collaborating and its timing to be sub-optimal. Increasing the frequency of counteroffers in bargaining, which has the effect of balancing their bargaining power, improves efficiency by reducing the hold-up. More importantly, the paper finds that a more balanced bargaining power can lead to Pareto improvement. The model illustrates the effect of outside options and bargaining power on firms' decision to collaborate, and shows how potential collaborators should (not) bargain.

Keywords: collaboration, bargaining, alliance, joint decision-making, continuous-time game.

*Cheung Kong Graduate School of Business. Email: zhaoning@ckgsb.edu.cn. I'm indebted to J. Miguel Villas-Boas, Yuichiro Kamada, Brett Green, Ganesh Iyer, and Philipp Strack for their input. All errors are my own.

1 Introduction

Firms often collaborate for development and marketing of products and services. Each year, more than 2,000 alliances are formed, and this number has continued to rise rapidly (Steinhilber 2013). These alliances are formed for a variety of purposes, ranging from joint product development, co-advertising campaigns, to global distribution partnerships. These collaborative efforts allow firms to combine complementary resources (dAspremont and Jacquemin 1988), access technology and markets for new business opportunities (Hamel 1991), and achieve economies of scale (Gomes-Casseres 1997). While past research has examined behaviors and performances after firms join alliances, this paper is interested in firms' strategies *before* they reach an agreement to collaborate. The paper asks the following questions: when do firms agree to collaborate and how do they agree to split the profit if (1) the potential return from collaborating evolves stochastically over time and (2) firms reach agreement through bargaining.

Evolving Return on Collaboration

The potential return from a collaboration between two firms can change over time due to changing market conditions or technologies. For example, in July of 2018, Walmart and Microsoft announced a strategic partnership to "accelerate Walmarts digital transformation in retail" (Walmart, 2018) with use of cloud and artificial intelligence technology (Chatterjee, 2018). While the two firms had working relationships before, the need for a larger scale partnership arise amid multiple trends. First, Walmart had been moving aggressively away from brick-and-mortar and into online retail to capture the demand of younger shoppers (Meyersohn, 2018). The advancement in technology also increased Walmart's need for digital transformation. In an interview with Forbes, Microsoft's lead for retail strategy commented on the timing of the deal: "The cloud is now the key to their ability to bring huge volumes of data together at speeds they never could imagine before-and it's essential for leading companies like Walmart to seize the opportunity at this particular point in time to do this." (Evans 2018) Furthermore, the competitive tension between Walmart and Amazon heightened in 2017 when Amazon entered into offline retail by acquiring Whole Foods. This makes collaborating with Microsoft, a competitor of Amazon in the cloud computing market, more attractive. In a separate example, Amazon and Dish Network explored a potential partnership in 2017 to create their own wireless networks. A proprietary wireless network became attractive to Amazon as more Americans had shifted to mobile Internet and smart appliances (Fung 2017). This shift in technology and consumer preferences opened up opportunities for collaboration that would not have been profitable in earlier years.

The expected return on collaboration can also change with the arrival of information.

Companies often conduct market research or trials before committing to a major decision. The data collected from those trials updates a firm's belief about the attractiveness of a partnership. The process of matching between collaborators also reveal information that influence the viability of an alliance. In 2014, Apple and IBM formed a partnership to create enterprise apps, but their conversation started years prior. The two firms dedicated teams to work with each other to find complementarities before finally establishing the partnership (Cook and Rometty, 2014). In 2016, after the successes of many smaller scale collaborations, Red Bull and GoPro entered into a global content partnership. When asked about the partnership, the CEO of GoPro described the two brands as "extremely compatible and collaborative", and stated that "the feedback [GoPro gets] from [Red Bull] is phenomenal." (Beer 2016).

The idea of an evolving surplus from collaborating also shows up in contract settlement. In 2010, Starbucks breached its exclusive contract with Kraft in order to enter the coffee pods market. The contract dispute lasted three years, during which time Starbucks enjoyed success in the pods market. The eventual settlement resulted in Starbucks paying $2.75 million to Kraft (Shonk 2018). The amount could be very different had the market performed differently (Shonk 2018). Last but not least, a common tactic in negotiation is to work with the opponent in finding new "win-win" solutions (see, e.g., Bazerman et al. 1985 and De Dreu et al. 2000). Such efforts have uncertain impacts on the total surplus to be shared.

Timing of Collaboration

If the expected return on a project changes over time, then the timing of that project's implementation becomes important. This problem is well understood if the project only requires a single firm to implement. Literature on real options shows that the decision maker should delay implementing it until its expected return reaches a threshold, or take an outside option when the surplus reaches a lower threshold (see, e.g., Dixit 1993). Various papers have studied such decisions in contexts such as R&D funding (e.g., Roberts and Weitzman 1981), capital investments (e.g., Dixit and Pindyck 1994), and consumer search (e.g., Branco et al. 2010).

However, if a project requires collaboration between two firms, the outcome becomes less clear. In order for the project to be implemented, the two partners must simultaneously agree to do so. On the flip side, each potential partner can unilaterally decide to walk away and take an outside option. The problem is further complicated if one considers how firms split the return on collaboration. In most partnerships, neither firm has pricing power over the other. This encourages us to consider the impact of bargaining on whether and when collaboration takes place. If a firm wants to implement the project at a given time, it has to either make an offer that the other party is willing to accept, or accept the other party's offer.

Bargaining Between (Potential) Collaborators

If two firms has an opportunity to collaborate, and the return on collaboration evolves stochastically, when do they reach an agreement to collaborate? How do they agree to split the profit from their alliance? Is the outcome efficient? How does bargaining power affect their decisions to collaborate?

To answer these questions, this paper studies a model of bargaining with a stochastic surplus. Two firms face the opportunity to collaborate on a project with a return that follows Brownian motion. At each moment, one firm is the proposer and the other firm is the responder. If the proposer wants to collaborate, it can propose a spot offer to the responder. The responder then decides whether to accept the offer, reject and wait, or quit and take the outside option. Firms switch their roles as proposer and responder after some time, so the previous responder becomes the new proposer and can make counteroffers.[1]

A common feature of sequential bargaining games is that the bargaining power is determined by who gets to make the offer and when. In this model, bargaining power is governed by the rate of switching between the proposer and the responder. I refer to this rate as the frequency of counteroffers. If counteroffers are more frequent, the two parties are more balanced in their power. If counteroffers arrive less frequently, the bargaining power favors the current proposer more because the responder has to wait longer before it can make offers.

Changing the frequency of counteroffers also has the effect of varying the procedures by which firms bargain. As the frequency becomes infinitely low, the game approaches a repeated-offers model, where one firm makes all the offers. As the frequency becomes infinitely high, the game approaches the continuous-time limit of alternating-offers game. Intermediate levels of frequency represent a class of symmetric bargaining procedures between the two extremes.

Main Findings

The paper shows that increasing the frequency of counteroffers can be mutually beneficial by improving the ex-ante chance of collaboration as well as making the timing of such collaboration to be more efficient. This suggest that both parties prefer a more balanced bargaining power because it helps fostering collaboration. Bucklin and Sengupta (1993) finds empirically that reducing power imbalances between partners can improve the effectiveness of a co-marketing alliance. This paper extends their conclusion by suggesting that reducing

[1] This model of bargaining can also be thought of as an alternating-offers model, with no commitment between when an offer is made and when the next counter-offer arrives.

power imbalances may also improve the efficiency of the formation of such alliances in the first place.

When firms have outside options, the equilibrium outcome is efficient. Firms both collaborate too early and take outside options too early, compared to what is socially optimal. The ex-ante probability of collaboration is also sub-optimal. Such inefficiency is a result of a hold-up problem faced by the responder. When the expected surplus from collaborating is low, waiting can be seen as a relationship-specific investment. The cost of waiting is the delaying of outside option. The return on that investment, however, is determined by future bargaining. Thus, the responder, who has less bargaining power, under-invests in that relationship. The responder's decision to take outside option earlier than socially optimal means that there is a higher chance that the collaboration breaks down. In response, the proposer is willing to make more generous offers early so that firm reach agreement to collaborate "prematurely". A higher frequency of counteroffers balances bargaining power, which mitigates the hold-up problem. Increasing the frequency of counteroffers always leas to a more socially efficient outcome. Thus in this model, increasing the frequency of counteroffers has two effects. It not only redistribute the bargaining power but also expands the total welfare of the two firms.

Both firms benefit from a more balanced bargaining power if the expansion of total welfare dominates the redistribution effect. The paper finds that, when the initial size of the surplus is not too big or too small, a low frequency of counteroffers may be Pareto dominated by a high frequency of counteroffers. This phenomenon does not exist if the return on collaboration is constant or if there is no outside option. Those cases do not exhibit the hold-up problem, so varying the frequency of counteroffers does not affect total utility, only the split of that utility.

The paper is organized as follows. After literature review, Section 2 presents the model. Section 3 presents the benchmark where firms do not have outside options. Section 4 solves the equilibrium with outside options and discusses the effect of bargaining power on social and Pareto efficiency. Section 5 discusses the model's limitations and explores the case of asymmetric firms. Section 6 concludes the paper.

Literature Review

In the economics and management literature, theoretical works on alliance or collaboration often focus on firms' strategic decision after joining alliance, and the inefficiencies that arise. For example, Kamien et al. (1992) and Chen and Ross (2000) study the competitive effects of R&D joint ventures and capacity sharing alliances. They show that these agreements can reduce competition and hurt consumers. Other papers examine the free-riding problem in alliances. Bonatti and Hörner (2011) study individuals' dynamic effort choices in teams that undertake projects with uncertain duration and outcome. They find efforts decrease over time but free riding can be mitigated by deadlines. Amaldoss et al. (2000), Amaldoss and Staelin (2010), and Bhaskaran and Krishnan (2009) study resource-commitment decisions for joint product developments and explore structural solutions, such as profit-sharing agreements, to under-investments. In the above papers, collaboration is inefficient due to players' actions after joining alliance. In comparison, the current paper studies the bargaining problem in the time period before firms agree to collaborate. Less focus has been placed on factors that lead to the formation of alliances. Cai and Raju (2016) study market entry as an alliance. They discuss how market size and market competitiveness determines when entry as an alliance is more profitable than entering independently. Bloch (1995) studies the formation of association in an oligopoly through a bargaining game. There are major differences in the role of bargaining between Bloch (1995) and the current paper. In Bloch (1995), multiple firms bargain over who belongs to the association, but not how to split the surplus generated from the association. The idea of an evolving return on collaboration is also absent in the aforementioned papers.

The model closely relates to bargaining models with stochastic payoffs or proposing orders. The most related works are Merlo and Wilson (1995, 1998). They present a discrete-time model of bargaining with random pie and random order of proposing, and prove the existence and uniqueness of a stationary SPE payoff. However, they do not consider outside options. Daley and Green (2018), Ishii et al. (2018), and Ning (2019) study continuous-time bargaining with a stochastic pie but do not allow for counteroffers. In Ortner (2019), the right to propose is stochastic, but the total surplus is fixed. In contrast, this paper allows for both stochastic surplus and a random order of proposing. Another related paper is Frankel (1998), in which players can exert effort to expand the size of the pie, but under-invest due to the hold-up problem.

2 The Model

Two risk-neutral firms, i and j, can collaborate on a project. They bargain over whether to collaborate and how to split the returns from their alliance. The expected return from the project at time t, x_t, is observable to both firms and is assumed to follow a Brownian motion $dx_t = \sigma dW_t$ with volatility $\sigma \geq 0$ and initial position x_0, where W_t is a Wiener process. Below I present micro motivations for a stochastic return in which a Brownian motion can be derived as a limiting case.

2.1 Stochastic Surplus

A hypothetical firm is considering whether to enter a new market. To operate in the new market, the firm has to collaborate with a partner. The alliance may be needed to co-develop the product, engage in co-marketing campaign, or provide distribution and localized service.

Evolving Consumer Preference

The potential profit from the alliance depends on consumer preference in the market, which evolves over time. For example, consider a Hotelling line of length l. A mass of consumers is located at $z_t \in [0, l]$ at time t, and z_t takes a random walk with reflecting boundaries at 0 and l. There exists a competitive fringe at location 0 with price normalized to 0, and the alliance can enter at location l. If the collaborators enter at time t, the highest price they can charge to make a sale is $p_t = 2z_t - l$, which is a random walk with reflecting boundaries at l and $-l$. As $l \to \infty$, the price approaches a Brownian motion.

Arrival of Information

Suppose that the new product provides a value of v_t to consumers, where v_t follows a random walk with variance σ^2. However, the firm does not observe the true value of v_t, but can obtain signals by conducting market research. Firms have a normal prior with mean $\hat{v}_0$ and variance $\hat{\rho}_0$. At each moment, the two firms receive a signal of v_t with a normal error of variance η^2, and updates the posterior mean $\hat{v}_t$ and variance $\hat{\rho}_t$ using Bayes' rule. The signal S_t accumulates as $dS_t = v_t dt + \eta dW_t$, where W_t is a Wiener process. By the Kalman-Bucy filter (see Ruymgaart and Soong 1988, ch.4), the posterior mean $\hat{v}_t$ follows $d\hat{v}_t = (\hat{\rho}_t/\eta)dB_t$ for some Wiener process B_t, and posterior variance follows $\frac{d\hat{\rho}_t}{dt} = -\hat{\rho}_t^2/\eta^2 + \sigma^2$. The posterior variance $\hat{\rho}_t$ approaches $\sigma\eta$ asymptotically over time. If $\hat{\rho}_0 = \sigma\eta$, then $\hat{v}_t$ is a Brownian motion with variance $\sigma^2\eta^2$.[2]

Matching

The success of the alliance depends on how well the firm matches with its partner. The two companies discover their match over time as they build relationship and explore the potential alliance. There is a mass of attributes important for the success of the project. The end product provides more value to consumers if the two collaborators match on more attributes. Assume the ex-ante probability of match on each attribute is 50%. Let dt be the size of each attribute. If the collaborators match on an attribute, it delivers $z_t = +\sigma\sqrt{dt}$ value to consumers, whereas a mismatched attribute delivers $-\sigma\sqrt{dt}$, so that we have $\mathbb{E}[z_t] = 0$

[2]Note that one can model with fixed consumer preference as well. Suppose consumers values the product at either v_h or v_l, and firms receive signals of the true state and update their belief $\hat{\pi}_t$ that the state is v_h. Then one can show that the log-likelihood of the posterior, $\log(\frac{\hat{\pi}_t}{1-\hat{\pi}_t})$, follows a Brownian motion. The surplus from forming the alliance at time t is $\hat{\pi}_t(v_h - v_l)$.

and $Var[z_t] = \sigma^2 dt$. The expected value of the product after observing t attributes can be written as $x_t = x_0 + \sum_0^t z_s$, where x_0 is the prior. As the mass of total attributes approaches infinity and dt approaches 0, the expected product value x_t becomes a Brownian motion.

2.2 Bargaining Game

Firms cannot write contingency contracts on when they will collaborate and how they will split the profit. Instead, at the moment when they agree to collaborate, firms must agree on how they will split the profit.

Extensive Form

The game is played in continuous time. The order of movement at time t is determined by a recognition process f_t. We call firm i the *Proposer* at time t if $f_t = i$ and the *Responder* at time t if $f_t \neq i$. At time t, the Proposer can propose a split of x_t if it wants to collaborate, and the Responder decides whether to accept the proposal. The roles are switched upon the arrival of a Poisson process with rate λ.[3] Before the next switch, the roles remain the same and the Proposer can make repeated offers. Thus, λ captures the speed at which counteroffers arrive. Let (Σ, $\mathcal{F}$, P) be the probability space that supports the Wiener process W_t and the Poisson counting process, and F = $(\mathcal{F}_t)_{t\in[0,\infty)}$ be the filtration process satisfying the usual assumptions.

The game is played as follows. At time t, the surplus x_t and the identity of the Proposer f_t are realized. Upon the realization, the Proposer can choose to proposes an offer. Note that the Proposer can choose not to make an offer if it does not want to form the alliance at time t. Let p_t denote the amount offered to the Responder, and denote $p_t = -\infty$ if the Proposer does not make an offer, since making an unacceptable offer is equivalent to not making one. Given an offer p_t, the Responder chooses whether to accept, reject and continue, or take the outside option. If the offer is accepted, the alliance is formed and firms split the return as agreed. If the Responder chooses the outside option, then firms can no longer collaborate. The game ends and both receive their outside options. If the Responder chooses to reject the offer but continue, the game proceeds with new realizations of x_t and f_t. Figure 1 illustrates the game graphically.

Utility

Firms are symmetric with discount rate r. Upon reaching an agreement, the Proposer receives utility of $x_t - p_t$ and the Responder receives utility of p_t. Firms have the same outside option $\omega \geq 0$.

[3]Assuming WLOG that $f_0 = 1$, then $f_t = f_0 + N(t) \bmod 2$, where $\{N(t), t \geq 0\}$ is a Poisson counting process with rate λ.

Figure 1

Let a_t be an indicator function for whether the Responder agrees to the Proposer's offer at time t, and q_t be an indicator function for whether Responder takes the outside option at time t. Then the game ends at $\tau = \inf\{t|a_t = 1 \text{ or } q_t = 1\}$. If firms reach agreement at time τ, i.e., $a_\tau = 1$, then the expected utility of firm i at time t is defined as:

$$u_{i,t} = e^{-r(\tau-t)}[\overbrace{\mathbb{1}\{f_\tau = i\}(x_\tau - p_\tau)}^{\text{Proposer utility at }\tau} + \overbrace{\mathbb{1}\{f_\tau \neq i\}p_\tau}^{\text{Responder utility at }\tau}]$$

If firms take outside options at time τ, i.e., $a_\tau = 0$ and $q_\tau = 1$, then the expected utility of firm i at time t is:

$$u_{i,t} = e^{-r(\tau-t)}\omega$$

Firms receive zero utility if they never take a stopping action, i.e., $\tau = \infty$ or $\max(a_\tau, q_\tau) = 0$.

Strategy

We study subgame-perfect equilibrium with pure strategies. Additionally, we focus on equilibrium with symmetric and stationary strategies. That is, equilibrium strategies only depend on the state, x, and the role of each firm (i.e., Proposer or Responder). Thus, an equilibrium strategy profile can be described by:

1. A proposing strategy $p_t = p(x_t) : \mathcal{R} \mapsto \mathcal{R}$ for the Proposer at time t.

2. An agreement strategy $a_t = a(x_t, p_t) : \mathcal{R}^2 \mapsto \{0, 1\}$ for the Responder at time t.

3. A quitting strategy $q_t = q(x_t) : \mathcal{R} \mapsto \{0, 1\}$ for the Responder at time t.

Outcome

An equilibrium outcome can be described by $\{U(x), V(x), A, Q\}$, where:

- Value function $U(x)$ is the expected utility of the Proposer.
- Value function $V(x)$ is the expected utility of the Responder.
- $A = \{x \mid a(x, p(x)) = 1\}$ is the set of states in which firms reach agreement to collaborate.
- $Q = \{x \mid q(x) = 1\}$ is the set of states in which firms take outside options.[4]

Suppose the game is played in discrete time with an infinitesimal time interval dt. If $x_t \in A$, then the Responder must be indifferent between taking the offer and waiting till next period, otherwise the Proposer should lower its offer. This implies that for all $x \notin Q$, the Responder's utility is the value of continuing to next period:

$$V(x) = e^{-rdt}\ \mathbb{E}[\mathbb{1}\{f_{t+dt} = f_t\}V(x + dx) + \mathbb{1}\{f_{t+dt} \neq f_t\}U(x + dx)] \tag{1}$$

The probability that the counter-offer arrives in dt is $\lambda dt + o(dt)$, and the probability that it arrives more than once is of $o(dt)$. Applying Ito's Lemma to $\mathbb{E}[V(x+dx)]$ and $\mathbb{E}[U(x+dx)]$, one can get:

$$V(x) = (1 - rdt)\ \left\{(1 - \lambda dt)\ \left[V(x) + \frac{\sigma^2}{2}V''(x)dt\right] + \lambda dt\ \left[U(x) + \frac{\sigma^2}{2}U''(x)dt\right]\right\} + o(dt) \tag{2}$$

which after simplification and taking $dt \to 0$ becomes:

$$(r + \lambda)V(x) = \frac{\sigma^2}{2}V''(x) + \lambda U(x) \qquad \forall x \notin Q \tag{3}$$

Similarly, if no agreement is reached at time t and the Responder does not quit, then the Proposer's expected utility is its continuation value, which means that

$$(r + \lambda)U(x) = \frac{\sigma^2}{2}U''(x) + \lambda V(x) \qquad \forall x \notin (A \cup Q) \tag{4}$$

[4]By the definition of utility, firms receive payoffs of 0 if they neither reach agreement nor quit at τ. This effectively restrict A and Q to be closed sets. Without the restriction to closed sets, one can create alternative equilibrium outcomes with agreement region $A \setminus Z_1$ and quitting region $Q \setminus Z_2$, where Z_1 and Z_2 are sets of measure 0. See Ortner (2019) for a similar restriction.

The above two conditions pin down the equilibrium outcome[5], which is presented in the following sections.

In equilibrium, the sum of the firms' value functions have to exceed the current return on collaboration at all time. Intuitively, if the sum of their utilities by following their equilibrium strategies is less than the return from collaborating right now, then they can profitably deviate by reaching an agreement (or taking their respective outside options) immediately.

Lemma 1. *$(U + V)(x) \geq x$ for all $x \in \mathcal{R}$.*

2.3 Frequency of Counteroffers and Bargaining Power

The arrival rate λ captures the frequency by which offers are countered. When the Poisson event arrives, the previous Responder becomes the new Proposer and can make counteroffers. The firms remain in their new roles until the next Poisson event arrives. Varying the parameter λ is analogous to varying the bargaining procedure. A game with a higher λ features more frequent counteroffers. If $\lambda \to \infty$, the game approaches the continuous-time limit of the alternating-offers paradigm of Rubinstein (1982). On the other hand, as $\lambda \to 0$, all offers are made by the initial Proposer. Thus the game approaches the repeated-offers paradigm studied by Fudenberg et al. (1985) and Gul et al. (1986) (but without asymmetric information in this case).

In sequential bargaining games, bargaining power between two symmetric players is determined by who has the ability to propose; hence, in this game, the choice of λ. This point can be illustrated by examining the static case. Assume $\sigma = 0$ so that the surplus generated from the project is fixed over time. Also assume $\omega = 0$, so the outside option is irrelevant. There is a unique equilibrium outcome with immeidate agreement. Plugging $\sigma = 0$ into equations (3) shows that the expected utility for the Responder if it rejects the first offer is $\frac{\lambda}{r+\lambda}U(x_0)$. Thus, the Responder accepts if and only if the offer is at least this amount, and the Proposer offers this amount immediately. The Proposer's share of the pie is $\frac{r+\lambda}{r+2\lambda}$, and the Responder's share is $\frac{\lambda}{r+2\lambda}$. As λ decreases, the Proposer gets a larger share of the pie. Less frequent counteroffers translates to higher bargaining power for the Proposer. Conversely, as λ increases towards infinity, allowing more frequent counteroffers, the Proposer loses its

[5] Simon and Stinchcombe (1989) illustrate the problems with subgame-perfection in continuous time. Because "waiting until next moment" is undefined in continuous time, there can exist SPE outcomes in continuous time that do not correspond to anything in discrete time. If our motive for studying continuous-time game is to capture the limit of discrete-time games, then these outcomes are not of any interest. Other papers have dealt with this issue by defining new equilibrium concepts. In this paper, this problem is addressed by restricting equilibrium outcomes to those that satisfy equation (3). This is sufficient for the model to produce a unique equilibrium outcome that correspond to the limit of discrete-time outcomes.

advantage and the profit split approaches being even. It is worth noting that this equilibrium is analogous to the symmetric equilibrium from Rubinstein (1982). Particularly, if we define $\delta = \frac{\lambda}{r+\lambda}$, then the Proposer's share is $\frac{1}{1+\delta}$ and the Responder's share is $\frac{\delta}{1+\delta}$, as in Rubinstein (1982). This equivalence no longer holds when the size of the surplus is stochastic.

To model bargaining sequentially, one faces the problem of choosing the "right" game. Fudenberg et al. (1985) point out two issues: "[f]irst, because the results depend on the extensive form, one needs to argue that the chosen specification is ... a good approximation to the extensive forms actually played. Second, even if one particular extensive form were used in almost all bargaining, the analysis is incomplete because it has not ... begun to address the questions of why that extensive form is used." This view is echoed by Sutton (1986). This paper tries to address these concerns from a specific angle. First, by varying the frequency of counteroffers, λ, one can examine how the equilibrium outcome is affected by the frequency of counteroffers. Second, it provides a meaningful comparison between different frequencies. Section 4 shows that, when firms have outside options, the equilibrium outcome under a higher λ can dominate the outcome under a lower λ in Pareto efficiency. In such cases, both firms and any social planner should prefer to bargain with more frequent counteroffers. This finding helps to explain why a particular bargaining procedure should (not) be used.

3 Benchmark: No Outside Options

This section presents the benchmark where firms do not have outside options. This means that they have to collaborate with each other at some point. What is unknown is the timing of their agreement and the split that they agree to.

Socially Efficient Outcome

For comparison, we first derive the socially efficient outcome. A social planner should delay the project until its return reaches threshold $\overline{x_s}$. The social value function, denoted as $W_s(x)$, must satisfies the HJB equation $rW_s(x) = \frac{\sigma^2}{2}W''(x)$ for $x \leq \overline{x_s}$. The solution is of the form

$$W_s(x) = C_1 e^{\sqrt{\frac{2r}{\sigma^2}}x} + C_2 e^{-\sqrt{\frac{2r}{\sigma^2}}x} \tag{5}$$

with $C_2 = 0$, because the value function has to approach 0 as $x \to -\infty$.

The threshold $\overline{x_s}$ has to satisfy $W_s(\overline{x_s}) = \overline{x_s}$ and $W_s'(\overline{x_s}) = 1$. The second condition is referred to as smooth-pasting and guarantees the stopping time to be optimal (see, e.g., Dixit 1993). Solving these two conditions gives the socially efficient threshold $\overline{x_s} = \sqrt{\frac{\sigma^2}{2r}}$ and the social value function $W_s(x) = \sqrt{\frac{\sigma^2}{2r}} e^{\sqrt{\frac{2r}{\sigma^2}}(x - \overline{x_s})} = \sqrt{\frac{\sigma^2}{2r}} e^{\sqrt{\frac{2r}{\sigma^2}}x - 1}$ for $x \leq \overline{x_s}$..

Equilibrium Outcome

The following Lemma shows that the equilibrium outcome must also be governed by a threshold. Firms agree to collaborate once the return reaches this threshold. Intuitively, if firms agree to collaborate at some $\overline{x}$, then there is some split such that both firms are willing to stop. Then for any $x > \overline{x}$, there should also exist a split such that both firms are willing to stop.

Lemma 2. *In equilibrium, $\exists\, \overline{x} \geq 0$ such that firms collaborate for $x_t \geq \overline{x}$.*

To characterize an equilibrium outcome, one only needs to solve for the agreement threshold, $\overline{x}$, the value functions for the Proposer, $U(x)$, and for the Responder, $V(x)$.

Proposition 1 shows that the agreement threshold in equilibrium must be equal to the socially efficient threshold. Thus the outcome must socially optimal. The intuition is that the two firms can "coordinate" a profitable deviation. If the agreement threshold is not socially optimal, then there exist a different threshold that improves total welfare, so at least one firm must benefit from such deviation. Bargaining then allows that firm to transfer some of the efficiency gain to the other firm, so that both firms can benefit. The only agreement threshold that firms cannot mutually benefit from deviating is the socially optimal one. When the two firms do not have outside options, they must form an alliance, and their timing of collaboration will be efficient. The bargaining procedure can only influence how the return from the collaboration is split. Merlo and Wilson (1995, 1998) find a similar result in a discrete time bargaining model with random cake size and proposing order. They prove the existence and uniqueness of a stationary SPE payoff in such games. We characterize this payoff below.

Knowing the agreement threshold, one can find proposal rules that implements such threshold. The proposal rule is unique in the agreement region, which implies that the equilibrium outcome is unique.[6]

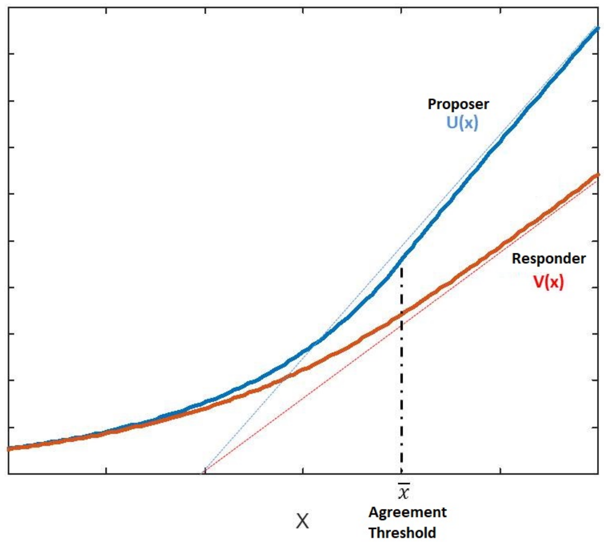
Proposition 1 (No Outside Option)**.** *Firms collaborate when the project's return reaches $\overline{x} = \overline{x_s} = \sqrt{\frac{\sigma^2}{2r}}$. Upon agreement at $\overline{x}$, the Responder receives a share of $\frac{\lambda}{r+2\lambda} + \frac{1}{4}\Big[\frac{r}{r+2\lambda} - \left(\frac{r}{r+2\lambda}\right)^{1.5}$*

Closed-form solutions of the value functions $U(x)$ and $V(x)$ are presented in Appendix A.1. Figure 2 depicts the equilibrium outcome graphically. The Proposer's value function is

[6] The proposal rule can be different for the non-agreement region, but such multiplicity in strategy do not affect the equilibrium outcome. If an offer is rejected, the size of that offer becomes irrelevant.

always strictly above the Responder's, which reflects the Proposer's higher bargaining power. Also note that the Responder's utility is strictly above its share in the static game discussed in Section 2.3, which is $\frac{\lambda}{r+2\lambda}$.

Figure 2: Equilibrium Outcome without Outside Option



Because the agreement threshold does not depend on λ, the choice of λ does not affect total welfare but only how it is distributed. One can easily verify that $U(x)$ decreases with λ and $V(x)$ increases with λ for all x_0. Thus a higher frequency makes the bargaining power more balanced, as expected.

Corollary 2 (For intermediate x_0)**.** *Total ex-ante utility does not change with λ. The Proposer's ex-ante utility strictly increases with λ, and the Responder's ex-ante utility strictly decreases with λ. As $\lambda \to \infty$, each firm receives half of the return on collaboration. As $\lambda \to 0$, the Proposer receives the entire return.*

4 Outside Options and the Hold-Up Problem

If firms have outside options, then they do not have to collaborate with each other. In this case, we show that the outcome is no longer socially efficient. More importantly, the degree of inefficiency is determined by the frequency of counteroffers, λ. A higher λ makes firms act more patiently and leads to more efficient timing. Furthermore, the equilibrium outcome under a higher λ can Pareto dominate the outcome under a lower λ, suggesting that there can be mutual gain from a more balanced bargaining power.

Socially Efficient Outcome

A social planner with discount rate r and an outside option of size 2ω implements the project if the surplus exceeds the threshold $\overline{x_s}$, and takes the outside option if the surplus falls below the threshold $\underline{x_s}$. The social planner's value function has the same form as equation (5):

$$W_s(x) = \gamma_1 e^{\sqrt{\frac{2r}{\sigma^2}}x} + \gamma_2 e^{-\sqrt{\frac{2r}{\sigma^2}}x}$$

The thresholds have to satisfy:

$$\begin{cases} W_s(\overline{x}) = \overline{x} & W_s(\underline{x}) = 2\omega \\ W_s'(\overline{x}) = 1 & W_s'(\underline{x}) = 0 \end{cases} \tag{6}$$

where the first two are value-matching conditions and the last two are smooth-pasting conditions. Together they imply

$$\overline{x_s} = \sqrt{\frac{\sigma^2}{2r} + 4\omega^2} \quad \text{and} \quad \underline{x_s} = \overline{x_s} - \sqrt{\frac{\sigma^2}{2r}} \log\Big(\frac{\sqrt{\frac{\sigma^2}{2r}} + \sqrt{\frac{\sigma^2}{2r} + 4\omega^2}}{2\omega}\Big) \tag{7}$$

The value function for $\underline{x_s} < x < \overline{x_s}$ is:

$$W_s(x) = \frac{1}{2}\Big(\overline{x_s} + \sqrt{\frac{\sigma^2}{2r}}\Big)e^{\sqrt{\frac{2r}{\sigma^2}}(x-\overline{x_s})} + \frac{1}{2}\Big(\overline{x_s} - \sqrt{\frac{\sigma^2}{2r}}\Big)e^{-\sqrt{\frac{2r}{\sigma^2}}(x-\overline{x_s})} \tag{8}$$

Equilibrium Outcome

As in the case without outside option, there exists an agreement threshold $\overline{x}$. This threshold has be higher than the sum of outside options for the collaboration to be profitable. If must also not exceed the socially efficient thresholds.

Lemma 3. *In equilibrium, $\exists$ an agreement threshold $\overline{x}$ such that $\overline{x_s} \geq \overline{x} \geq 2\omega$. Firms agree to collaborate for $x_t \geq \overline{x}$.*

But because the Responder may opt out for the outside option, to characterize an equilibrium outcome we have to specify the states in which that happens. The following result says those states can be described by a quitting threshold $\underline{x}$. The quitting threshold has to be higher than the socially efficient threshold $\underline{x_s}$.

Lemma 4. *In equilibrium, $\exists$ a quitting threshold $\underline{x}$ such that $\underline{x} \geq \underline{x_s}$. The Responder opt out for $x_t \leq \underline{x}$.*

In the previous section when there is no outside option, firms collaborate with certainty and their timing of collaboration is efficient. This is no longer true when outside options exist. In equilibrium, firms reach agreement too early, opting for outside options too early, and their ex-ante probability of collaborating is sub-optimal. Proposition 3 describes how $\overline{x}$, $\underline{x}$, and the ex-ante probability of collaborating changes with λ.

Proposition 3. *Let* $\underline{\lambda} = \frac{4r^2\omega^2+\sqrt{16r^4\omega^4+8\sigma^2r^3\omega^2}}{2\sigma^2}$*:*

1. *If* $\lambda \leq \underline{\lambda}$*, then* $\underline{x} \geq \overline{x} = 2\omega$*. Firms collaborate or take outside options immediately at* $t = 0$*.*

2. *If* $\lambda > \underline{\lambda}$*, then* $\overline{x} > \underline{x}$*. The agreement threshold* $\overline{x}$ *strictly increases in* λ *and the quitting threshold* $\underline{x}$ *strictly decreases in* λ*.*

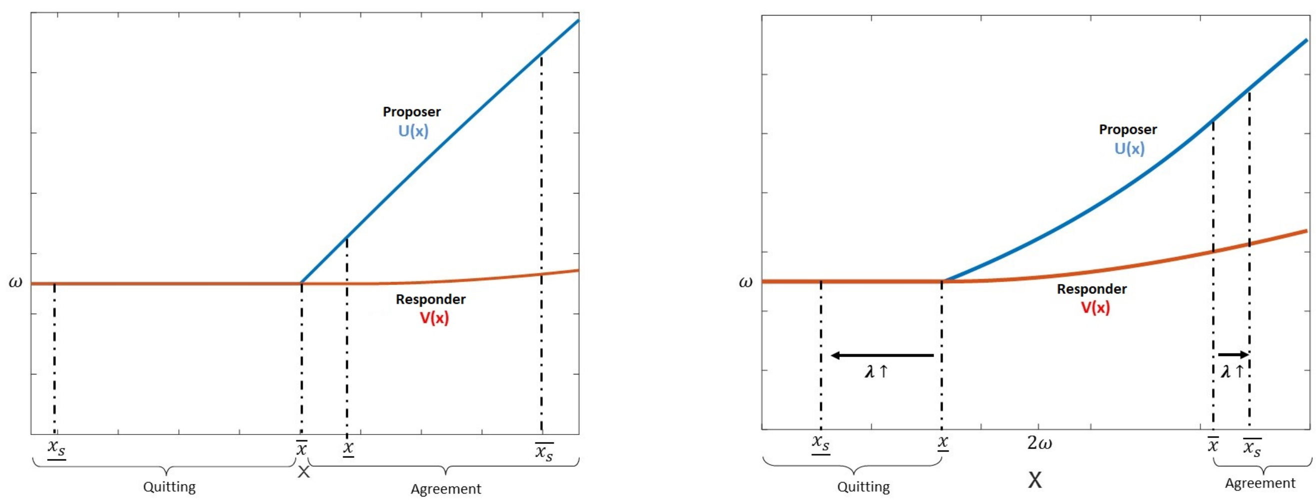
3. *There exists a cutoff* $\tilde{x}(\lambda)$ *such that the ex-ante probability of collaboration increases in* λ *for* $x_0 < \tilde{x}(\lambda)$ *and decreases for* $x_0 > \tilde{x}(\lambda)$*.*

The closed-form expression of firms' value functions are in Appendix A.2. Figure 3 depicts the equilibrium outcome and compares it to the socially efficient outcome. Figure 3(a) shows the case of $\lambda \leq \underline{\lambda}$. In this case, $\underline{x} \geq \overline{x}$, so that the game ends immediately in all states. The alliance is formed if and only if $x_0 \geq 2\omega$. Figure 3(b) shows the case of $\lambda > \underline{\lambda}$. Firms wait until the project's return reaches either $\underline{x}$ or $\overline{x}$. However, these thresholds are not socially optimal. Because these thresholds are reached earlier than the socially efficient thresholds, firms end up acting to impatiently. Both $\overline{x}$ and $\underline{x}$ move towards the socially efficient thresholds as λ increases. The ex-ante probability of agreement can either increase or decrease in λ depending on the initial position, x_0.

Bargaining Power and the Hold-up Problem

The equilibrium is inefficient because the Responder takes the outside option too early. We can interpret this as the result of a hold-up problem due to the Responder's lack of bargaining power. Consider the Responder's decision between taking the outside option and waiting for one more "period". The Responder weighs its cost and benefit from waiting. The cost is the delaying of outside option, and the benefit comes from the probability of collaborating at a future time when the return is higher. However, the size of this benefit depends on the result of bargaining, and the Responder's current lack of proposing right means an expected disadvantage in future bargaining. If we view the decision to delay outside option as a relationship-specific investment, then the Responder under-invests because it incurs half of the total cost but captures less than half of the total gain from collaborating. Given that the Responder quits earlier, the Proposer has to make an agreeable offer earlier to avoid the higher risk of breakdown. Note that a lower agreement threshold is efficient conditional on the quitting threshold, so the inefficiency comes from the early quitting decision only.

Figure 3: Equilibrium Outcome with Outside Option



(a) $\lambda \leq \underline{\lambda}$ (b) $\lambda > \underline{\lambda}$

The severity of this hold-up problem changes with bargaining power. Because an uneven bargaining power causes the weaker player to "under-invest" and act too impatiently, a higher frequency of counteroffers, which balances the bargaining power, alleviates this problem. In the extreme case, if firms are required to split the return on collaboration evenly, then their timing of agreement and quitting would be socially efficient.

Corollary 4 (Effect of λ on Social Efficiency)**.** *Ex-ante welfare* $U(x_0)+V(x_0)$ *increases in* λ*. As* $\lambda \to \infty$*, the agreement and quitting thresholds approach the socially efficient thresholds, and the utility of each firm approaches one-half of the socially efficient total welfare.*

In this model with a stochastic return and outside options, increasing the frequency of counteroffers has two effects. It redistributes total welfare between the two firms as before. But additionally, it expands the total welfare to be shared between the two firms. This fact means that a higher frequency of counteroffers may not necessarily be detrimental to the Proposer. Although with less bargaining power, the Proposer gets a smaller share of the pie, but the total size of the pie becomes larger. If the increased efficiency outweighs the loss of bargaining power, then both firms prefers to bargain under the higher λ. Proposition 5 outlines the conditions under which this happens.

We say that λ_1 *is Pareto Dominated by* λ_2 if the equilibrium outcome under frequency λ_1 is Pareto dominated by the equilibrium outcome under frequency λ_2.

Proposition 5 (Effect of λ on Pareto efficiency)**.**

1. *For intermediate x_0, there exists $\tilde{\lambda}$ such that any $\lambda \le \tilde{\lambda}$ is Pareto dominated by some $\lambda' > \tilde{\lambda}$.*

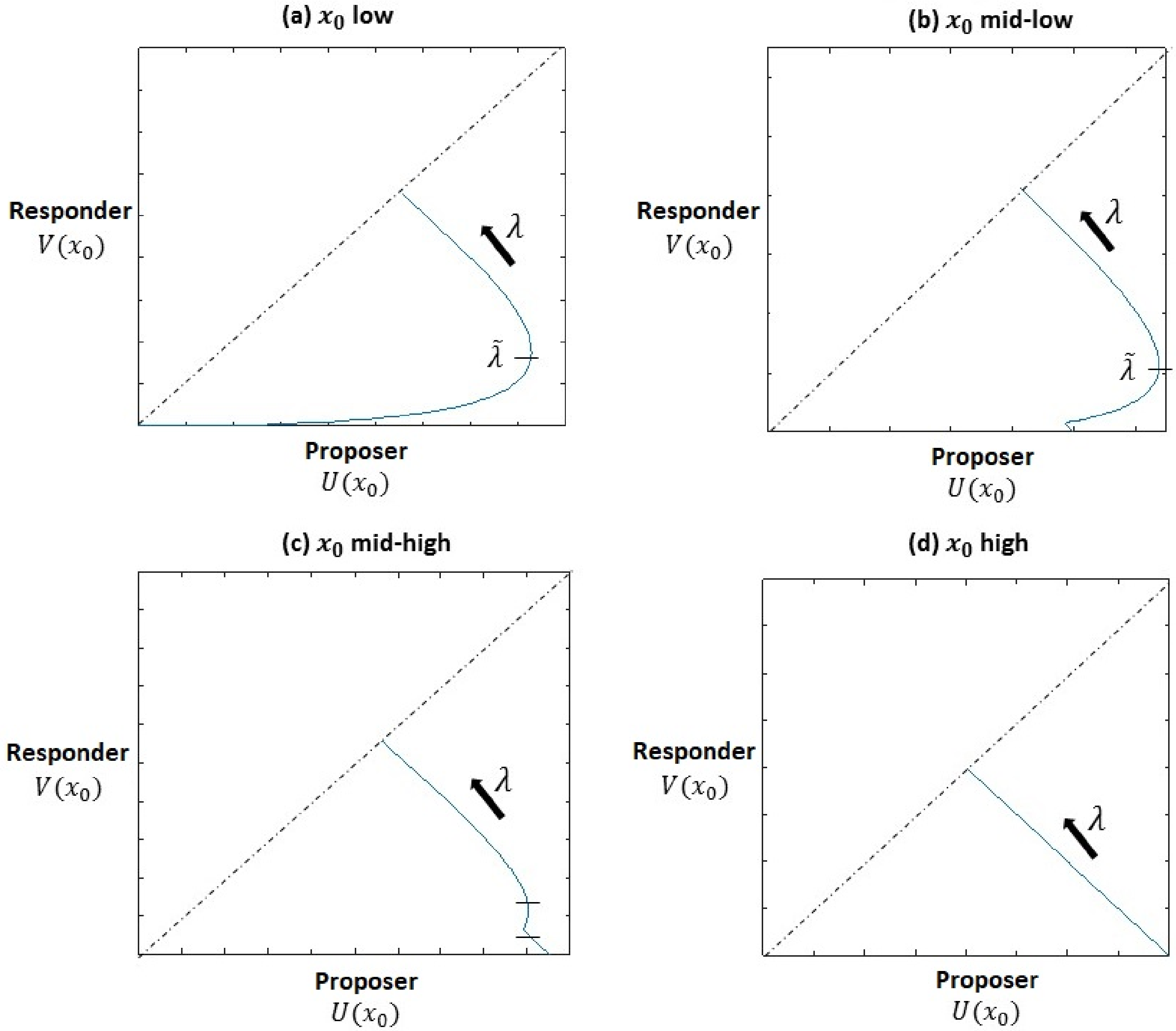
2. *For any λ, there exists some x_0 such that λ is Pareto dominated by some $\lambda' > \lambda$.*

Proposition 5(1) states that there is an intermediate range of initial values such that all low λ are Pareto dominated. Both firms benefit from more frequent counteroffers if the frequency of counter-offers is below some threshold. The Proposer benefits from a loss of bargaining power because a more efficient timing of collaboration outweighs the negative effect of giving up a bigger share to its opponent. Proposition 5(2) states that no choice of λ is immune to Pareto improvement. Regardless of the level of λ, there is always some initial value such that firms can mutually benefit from an even higher frequency of counteroffers. Note that if the initial value of the return on collaboration is too low ($x_0 < \underline{x_s}$) or too high ($x_0 > \overline{x_s}$), then by Proposition (3), the game stops immediately. In these cases, the frequency of counteroffers does not affect Pareto efficiency. Total welfare is not impacted by λ, only the distribution is.

Numerical Example

For $\sigma^2 = 0.9$, $r = 0.05$, and $\omega = 0.5$, figure 4 traces the ex-ante utilities of the two firms as functions of λ. The initial value, x_0, increases from 4(a) to 4(d). In 4(a) and 4(b), all λ smaller than some $\tilde{\lambda}$ are Pareto dominated, and all $\lambda \ge \tilde{\lambda}$ are on the Pareto frontier. In figure 4(c) with a higher initial value, some middle levels of λ are dominated, whereas both high and low λ's are on the frontier. In 4(d), the initial value is above $\overline{x_s}$ so that all choices of λ are efficient, since firms collaborate immediately regardless of λ.

Figure 4: Ex-ante utilities as functions of λ for initial values from low to high



5 Extensions

5.1 Allowing Proposer to Quit

In the base model, only the Responder is allowed to quit. In equilibrium, the Proposer's expected utility is always above the Responder's, which means that Proposer does not have an incentive to quit. Thus the equilibrium outcome from the base model must still be equilibrium when the Proposer is allowed to quit. However, other equilibria exist if the Proposer is allowed to quit. For example, there exists trivial equilibria where firms quit simultaneously, which can arbitrarily raise the quitting threshold.[7] We can show that the equilibrium outcome from the base model is the most efficient equilibrium outcome when the Proposer can quit for the outside option.

The intuition is straightforward. The inefficiency arises because the Responder, who expects to get less from collaborating, leaves for the outside option too early. Allowing the Proposer to quit can only exacerbate this problem. Consider a sub-game in which the Proposer takes the outside option if its proposal is rejected. Then the Responder is willing to accept any offer weakly above ω, the outside option. Thus the Responder's utility in this sub-game is exactly ω, the lowest possible utility in equilibrium. Giving the Proposer the ability of threat can only reduce the Responder's bargaining power, leading to less efficient outcomes.

Proposition 6. *The equilibrium outcome from Proposition 3 is the most efficient equilibrium outcome when the Proposer can quit.*

5.2 Asymmetric Firms

In reality, collaborations often involve two firms of different sizes, power, or outside opportunities. Given the connection between bargaining power and collaborative outcome, a natural extension is to consider firms with asymmetric power. However, an extended model where each firm has a different frequency of counteroffer and discount rate is difficult for a few reasons. First, there are now four sub-players to analyze: firm 1 as the Proposer, firm 1 as the Responder, firm 2 as the Proposer, and firm 2 as the Responder. For example, consider discounting till the next counteroffer. In the classic alternating-offers bargaining model, a player discounts its utility at the next counteroffer by its discount factor, regardless of who the Proposer is. However, with different frequencies of counteroffers, the time until next counteroffer depends on who the Proposer is, so the four sub-players discount differently as well. Another difficulty is the bargaining power now depends on multiple factors. In the symmetric model, λ serves as a proxy for bargaining power. But in an asymmetric model, bargaining power at time t is determined by relative sizes of λ_1, λ_2, r_1, r_2, as well as f_t, the identity of the Proposer at time t.

[7] If the quitting decisions are made simultaneously, quitting is always (weakly) optimal if the opponent is quitting at the same time. So, quitting at any state can be supported in an equilibrium. Sequential order does not eliminate this problem. For example, take any random threshold $\underline{x} < \omega$, and let both firms quit for all $x < \underline{x}$. Then there's no profitable deviation in these states regardless the order of quitting. Because the state evolves continuously, the second mover is indifferent because the opponent will quit in the "next instant" regardless. The state x_t cannot jump out of the quitting region. Thus any choice of $\underline{x} < \omega$ can be supported in an equilibrium in this fashion.

In this section, we study a simplified version of the model by directly parameterizing bargaining power. Let ω_i, r_i, and α_i denote firm i's outside option, discount rate, and bargaining power, respectively. If firms agree to collaborate at time t, they split the return according to the Nash bargaining solution:

$$\begin{aligned} \underset{p_1,p_2}{\text{argmax}} \quad & (p_1-\omega_1)^{\alpha_1}(p_2-\omega_2)^{\alpha_2} \\ \text{s.t.} \quad & p_1+p_2=x_t \end{aligned}$$

We can normalize $\omega_1+\omega_2$ and $\alpha_1+\alpha_2$ both to 1 WLOG. Let ω and α denote firm 1's outside option and bargaining power. We can also assume WLOG that $\alpha \geq \omega$. Thus firm 1 is the stronger firm here, corresponding to the Proposer in the base model.

At each time t, both firms decide simultaneously whether to collaborate. Collaboration happens at time t if and only if both firms choose to collaborate. Upon collaboration, firm 1 receives $\omega+\alpha(x-\omega-(1-\omega))$, and firm 2 receives $(1-\omega)+(1-\alpha)(x-\omega-(1-\omega))$. If no agreement at time t, firm 2 decides whether to end the process by opting out. Again this assumption that only the weaker party can actively quit is used to avoid trivial equilibria caused by simultaneous quitting. Similarly to Section 5.1, one can show that this gives the most efficient equilibrium outcome if both firms are allowed to quit.

Consider the special case of $\alpha=\omega$. Firm 1 always gets α share of total utility and firm 2 always gets $1-\alpha$ share of total utility, whether the game ends with collaboration or outside options. Note that the social planner's solution from equation (7) is invariant to the same normalization on x and ω (σ is the standard deviation on instantaneous change in x, so has to be normalized by the same factor as x). This means that the two firms' incentives are perfectly aligned, and the socially optimal thresholds maximizes both firms' utility. The equilibrium outcome is efficient. However, if $\alpha>\omega$, firm 2 gets $1-\omega$ share of the outside option, but $1-\alpha<1-\omega$ share of the excess return on collaboration. This translates to a higher cost and a smaller gain from delaying, which pushes firms 2 to take the outside option earlier. Changing bargaining power also affects Pareto efficiency. For example, $\alpha=\omega$ Pareto dominates $\alpha=1$ for $x_0 \in (\underline{x_s}, 1)$. If $\alpha=1$, then firm 2 opts out immediately, so both firms receive their outside options. But at $\alpha=\omega$, both firms are better off with α and $1-\alpha$ share of the socially efficient payoff.

We say that α is Pareto dominated if there exists α' such that the equilibrium outcome under α' Pareto dominates the equilibrium outcome under α. We can prove the following results, which mirror the main findings from the base model.

Proposition 7. *If $\alpha>\omega$, then*

1. *The equilibrium quitting threshold is higher than the socially efficient quitting threshold.*

2. *For intermediate x_0, there exists $\overline{\alpha}$ such that $\forall \alpha > \overline{\alpha}$ are Pareto dominated.*

3. *For any α, there exists x_0 such that α is Pareto dominated.*

Numerical Example

Let $U_i(x)$ denote firm i's value function. For states in which the game continues, the value functions must satisfy the HJB equation $rU_i(x) = \frac{\sigma^2}{2}U_i''(x)$. The solution is of the form

$$U_i(x) = C_i e^{\sqrt{\frac{2r}{\sigma^2}}x} + D_i e^{-\sqrt{\frac{2r}{\sigma^2}}x} \tag{9}$$

Let $\overline{x}$ and $\underline{x}$ denote the agreement and quitting thresholds. Besides the normal value-matching conditions, a solution to $\overline{x}$ has to satisfy:

$$[U_1'(\overline{x}) - \alpha][U_2'(\overline{x}) - (1-\alpha)] = 0$$
$$U_1'(\overline{x}) + U_2'(\overline{x}) \geq 1$$

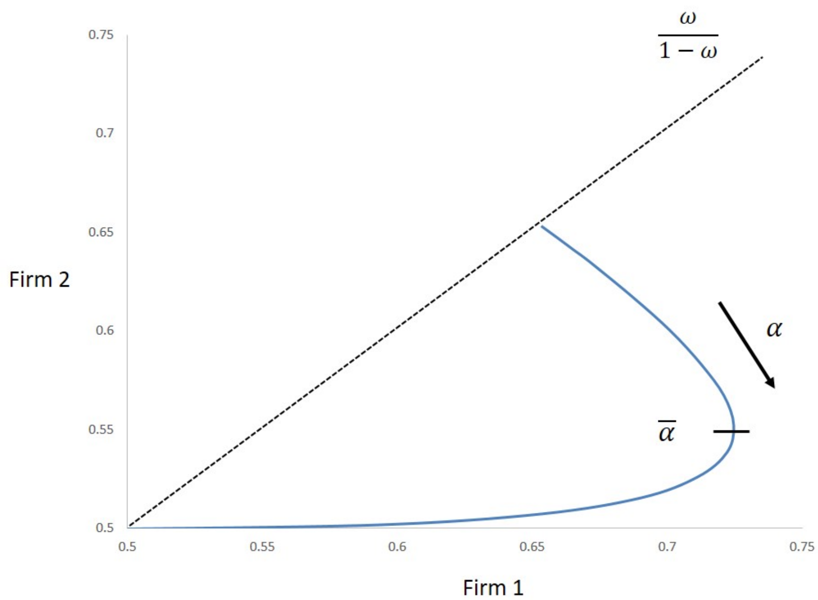
The first condition ensures that $\overline{x}$ is the optimal stopping point for at least one of the firms. The second condition ensures that the other firm does not have incentive to delay beyond $\overline{x}$.

For $\sigma^2 = 0.9$, $r = 0.05$, $\omega = 0.5$, and $x_0 = 0$, figure 5 traces the ex-ante utilities of the two firms as functions of α. Note that the graph closely resembles figure 4(a). There is a threshold $\overline{\alpha}$ such that any α bigger than that leads to a Pareto-inferior outcome. Thus a decrease in firm 1's bargaining power can be mutually beneficial by improving the timing of their collaboration.

6 Conclusion

In this paper, two firms decides when to collaborate on a project with an evolving return. They bargain over how to split the return from their collaboration, and can only collaborate when an agreement is reached. The paper investigates the effect of bargaining power on the timing and efficiency of alliance formation. Two symmetric firms with time discounting and outside options bargain over a surplus that follows a random walk. One firm makes repeated-offers to its opponent, and they switch roles following a Poisson process. The frequency of counteroffers controls their relative bargaining power, as a lower frequency favors the current proposer. The paper shows that, when there is no outside option, firms collaborate with certainty and with a socially efficient delay. However, when firms have outside options, the outcome is no longer efficient. Agreement to collaborate is reached too early and firms opt

Figure 5: Ex-ante utilities as functions of α



out for outside options too early. The inefficiency arises from a hold-up problem faced by the weaker party. A higher frequency of counteroffers balances the bargaining power and reduces the severity of the hold-up. The increase in social efficiency can outweigh the loss of bargaining power for the stronger party. As a result, bargaining with a more balanced bargaining power can lead to a Pareto improvement.

This paper provides theoretical insights on how bargaining power and procedure affects the formation of collaborative partnerships, and how potential collaborators should or should not bargain. However, this paper remains agnostic about how the bargaining procedure is selected. I assume that a frequency of counteroffers is exogenously set. Future research might explore what happens if the selection is done endogenously by the parties involved.

References

Amaldoss W, Meyer R, Raju JS, Rapoport A (2000). Collaborating to Compete. *Marketing Science*, 19(2), 105-126.

Amaldoss W, Staelin R (2010). Cross-Function and Same-Function Alliances: How Does Alliance Structure Affect the Behavior of Partnering Firms? *Management Science*, 56(2), 302-317.

Bazerman M, Magliozzi T, Neale M (1985). Integrative Bargaining in a Competitive Market. *Organizational Behavior and Human Decision Processes*, 35(3), 294-313.

Beer J (2016). Why GoPro Just Signed A New Exclusive Content Partnership with Red Bull. *The Fast Company*, https://www.fastcompany.com/3060159/why-gopro-just-signed-a-new-exclusive-content-partnership-with-red-bull.

Bhaskaran SR, Krishnan V (2009). Effort, Revenue, and Cost Sharing Mechanisms for Collaborative New Product Development. *Management Science*, 55(7), 1152-1169.

Bloch F (1995). Endogenous Structures of Association in Oligopolies. *RAND Journal of Economics*, 26(3), 537-556.

Branco F, Sun M, Villas-Boas JM (2010). Optimal search for product information. *Management Science*, 58(11), 2037-2056.

Bonati A, H́orner J (2011). Collaborating. *American Economic Review*, 101(2), 632-633.

Bucklin LP, Sengupta S (1993). Organizing Successful Co-Marketing Alliances. *Journal of Marketing*, 57(2), 32-46.

Cai CJ, Raju JS (2016). Keeping Your Enemies Closer: When Market Entry as an Alliance with Your Competitor Makes Sense. *Marketing Science*, 35(5), 743-755.

Chatterjee R (2018). Walmart, Microsoft in Partnership to Use Cloud Tech. *Reuters.com*, https://www.reuters.com/article/us-walmart-microsoft/walmart-microsoft-in-partnership-to-use-cloud-tech-idUSKBN1K70EB.

Chen Z, Ross TW (2000). Strategic Alliances, Shared Facilities, and Entry Deterrence. *RAND Journal of Economics*, 31(2), 326-344.

Cook T, Rometty V (2014). Interview by Josh Lipton. *Closing Bell* [Television Broadcast], CNBC, July 15, 2014.

Daley B, Green B (2019). Bargaining and News. *American Economic Review*, forthcoming.

dAspremont C, Jacquemin A (1988). Cooperative and noncooperative R&D in duopoly with spillovers. *American Economic Review*, 78(5), 1133-1137.

De Dreu CK, Weingart LR, Kwon S (2000). Influence of social motives on integrative negotiation: a meta-analytic review and test of two theories. *Journal of Personality and Social Psychology*, 78(5), 889-905.

Dixit A (1993) *The Art of Smooth Pasting*. Harwood Academic.

Dixit A, Pindyck R (1994). *Investment Under Uncertainty*. Princeton University Press, Princeton, NJ.

Evans B (2018). Walmart CIO: We Picked Microsoft For Huge Cloud Deal To Accelerate Digital Transformation. *Forbes.com*, https://www.forbes.com/sites/bobevans1/2018/07/18/walmart-cio-we-picked-microsoft-for-huge-cloud-deal-to-accelerate-digital-transformation/#1dc60f97491c.

Fudenberg D, Levine D, and Tirole J (1985). Infinite-Horizon Models of Bargaining with One-Sided Incomplete Information. *Game Theoretical Models of Bargaining*. Cambridge University Press.

Fung B (2017). Why Amazon might want a deal with Dish. *The Washington Post*, https://www.washingtonpost.com/news/the-switch/wp/2017/07/07/why-amazon-might-want-a-deal-with-dish.

Frankel D (1998). Creative Bargaining. *Games and Economic Behavior*, 23, 43-53.

Gomes-Casseres B (1997). Alliance strategies of small firms. *Small Business Economics*, 9(1), 33-44.

Gul F, Sonnenschein H, Wilson R (1986). Foundations of dynamic monopoly and the Coase conjecture. *Journal of Economic Theory*, 39(1), 155190.

Hamel G (1991). Competition for competence and inter-partner learning within international strategic alliances. *Strategic Management Journal*, 12, 83-103.

Ishii Y, Öry A, Vigier A (2019). Competing for Talent. *Cowles Foundation Discussion Paper No.2119*.

Kamien MI, Muller E, Zang I (1992). Research Joint Ventures and R&D Cartels. *American Economic Review*, 82(5), 1293-1306.

Merlo A, Wilson C (1995). A Stochastic Model of Sequential Bargaining with Complete Information. *Econometrica*, 63(2), 371-399.

Merlo A, Wilson C (1998). Efficient delays in a stochastic model of bargaining. *Economic Theory*, 11, 39-55.

Meyersohn N (2018). Walmart Figures Out Its Amazon Strategy. So Why is the Stock Down 13%?. *CNN.com*, https://money.cnn.com/2018/05/16/news/companies/walmart-stock-jet-amazon-whole-foods/index.html?iid=EL.

Ning ZE (2019). List Price and Discount in a Stochastic Selling Process. *Working Paper*.

Roberts K, Weitzman ML (1981). Funding criteria for research, development, and exploration projects. *Econometrica*, 49, 1261-1288.

Rubinstein A (1982). Perfect equilibrium in a bargaining model. *Econometrica*, 50(1), 97109.

Ruymgaart PA, Soong TT (1988). *Mathematics of Kalman-Bucy Filtering*. Springer-Verlag KG, Berlin, Germany.

Shonk K (2018). Examples of Negotiation in Business: Starbucks and Krafts Conflict Management. *Harvard Law School Blog*, https://www.pon.harvard.edu/daily/business-negotiations/the-starbucks-kraft-dispute-in-business-negotiations-prepare-for-problems.

Ortner J (2019). A Continuous-Time Model of Bilateral Bargaining. *Games and Economic Behavior*, 113, 720-733.

Walmart (2018). Walmart Establishes Strategic Partnership with Microsoft to Further Accelerate Digital Innovation in Retail. *Walmart.com*, https://corporate.walmart.com/newsroom/2018/07/17/walmart-establishes-strategic-partnership-with-microsoft-to-further-accelerate-digital-innovation-in-retail.

Appendix A Derivation of Value Functions

If $x \in A$, then the Proposer offers the Responder $p(x)$ and the Responder accepts immediately, so $U(x) = 1 - p(x)$ and $V(x) = p(x)$. If $x \in Q$, then the Responder quits immediately, so $U(x) = V(x) = \omega$. If $x \in \mathcal{R} \setminus (\mathcal{A} \cup \mathcal{Q})$, we have:

$$\begin{aligned} (r+\lambda)U(x) &= \frac{\sigma^2}{2}U''(x) + \lambda V(x) \\ (r+\lambda)V(x) &= \frac{\sigma^2}{2}V''(x) + \lambda U(x) \end{aligned} \tag{10}$$

Adding and subtracting the two equations produces:

$$\begin{aligned} (U+V)(x) &= \frac{\sigma^2}{2r}(U+V)''(x) \\ (U-V)(x) &= \frac{\sigma^2}{2r+4\lambda}(U-V)''(x) \end{aligned} \tag{11}$$

The solution to these two differential equations is of the form:

$$\begin{aligned} (U+V)(x) &= \alpha_+ e^{\sqrt{\frac{2r}{\sigma^2}}x} + \beta_+ e^{-\sqrt{\frac{2r}{\sigma^2}}x} \\ (U-V)(x) &= \alpha_- e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}x} + \beta_- e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}x} \end{aligned} \tag{12}$$

for some coefficients α_+, α_-, β_+, and β_-.

A.1 Without Outside Option

Firms' value functions for $x \geq \overline{x}$ must satisfy equation (3) and their sum must be x:

$$\begin{aligned} U(x) + V(x) &= x \\ (r+\lambda)V(x) &= \frac{\sigma^2}{2}V''(x) + \lambda U(x) \end{aligned} \tag{13}$$

Combining the two, one gets:

$$V(x) = \frac{\lambda}{r+2\lambda}x + \frac{\sigma^2}{2r+4\lambda}V''(x)$$

which has solution in the form:

$$V(x) = \frac{\lambda}{r+2\lambda}x + \gamma_1 e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}x} + \gamma_2 e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}x} \qquad \forall x \geq \overline{x} \tag{14}$$

for some coefficients γ_1 and γ_2. The Responder can never get more than the full surplus or agree to accept a negative amount, so $0 < V(x) \leq x \; \forall x$; therefore $\gamma_1 = 0$, otherwise this is violated for x large enough.

For states below $\overline{x}$, firms' value functions must follow equations (12). Note that $(U +$

$V)(x)$ captures the social value function, and $(U-V)(x)$ captures the advantage of being the Proposer. As $x \to -\infty$, the social value must approaches zero, which implies $\beta_+ = 0$. Similarly, as social value approaches zero, the difference between the two firms has to approach zero, which implies $\beta_- = 0$. Thus we have a simpler version:

$$\begin{aligned} (U+V)(x) &= \alpha_+ e^{\sqrt{\frac{2r}{\sigma^2}}x} \qquad \forall x < \overline{x} \\ (U-V)(x) &= \alpha_- e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}x} \qquad \forall x < \overline{x} \end{aligned} \tag{15}$$

Following the proof of Proposition 1, we get $\alpha_+ = \sqrt{\frac{\sigma^2}{2r}} e^{-\sqrt{\frac{2r}{\sigma^2}}(\overline{x_s})} = \sqrt{\frac{\sigma^2}{2r}} e^{-1}$, $\alpha_- = \left(\frac{1}{2} - \frac{\lambda}{r+2\lambda}\right)\left(\sqrt{\frac{\sigma^2}{2r}} + \sqrt{\frac{\sigma^2}{2r+4\lambda}}\right) e^{-\sqrt{\frac{2r+4\lambda}{2r}}}$, and $\gamma_2 = \frac{1}{2}\left(\frac{1}{2} - \frac{\lambda}{r+2\lambda}\right)\left(\sqrt{\frac{\sigma^2}{2r}} - \sqrt{\frac{\sigma^2}{2r+4\lambda}}\right) e^{\sqrt{\frac{2r+4\lambda}{2r}}}$. This pins down the closed-form solutions of the equilibrium value functions:

$$U(x) = \begin{cases} \frac{1}{2}\left(e^{\sqrt{\frac{2r}{\sigma^2}}(x-\overline{x})} + \left(\frac{1}{2} - \frac{\lambda}{r+2\lambda}\right)\left(\sqrt{\frac{\sigma^2}{2r}} + \sqrt{\frac{\sigma^2}{2r+4\lambda}}\right) e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}(x-\overline{x})}\right) & \forall x < \sqrt{\frac{\sigma^2}{2r}} \\ \frac{r+\lambda}{r+2\lambda}x - \frac{1}{2}\left(\frac{1}{2} - \frac{\lambda}{r+2\lambda}\right)\left(\sqrt{\frac{\sigma^2}{2r}} - \sqrt{\frac{\sigma^2}{2r+4\lambda}}\right) e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}(\overline{x}-x)} & \forall x \geq \sqrt{\frac{\sigma^2}{2r}} \end{cases} \tag{16}$$

$$V(x) = \begin{cases} \frac{1}{2}\left(e^{\sqrt{\frac{2r}{\sigma^2}}(x-\overline{x})} - \left(\frac{1}{2} - \frac{\lambda}{r+2\lambda}\right)\left(\sqrt{\frac{\sigma^2}{2r}} + \sqrt{\frac{\sigma^2}{2r+4\lambda}}\right) e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}(x-\overline{x})}\right) & \forall x < \sqrt{\frac{\sigma^2}{2r}} \\ \frac{\lambda}{r+2\lambda}x + \frac{1}{2}\left(\frac{1}{2} - \frac{\lambda}{r+2\lambda}\right)\left(\sqrt{\frac{\sigma^2}{2r}} - \sqrt{\frac{\sigma^2}{2r+4\lambda}}\right) e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}(\overline{x}-x)} & \forall x \geq \sqrt{\frac{\sigma^2}{2r}} \end{cases} \tag{17}$$

where $\overline{x} = \sqrt{\frac{\sigma^2}{2r}}$.

A.2 With Outside Option

Firms' value functions for $x \geq \overline{x}$ must satisfy equation (3) and their sum must be x::

$$\begin{aligned} U(x) + V(x) &= x \\ (r+\lambda)V(x) &= \frac{\sigma^2}{2}V''(x) + \lambda U(x) \end{aligned} \tag{18}$$

Combining the two, one gets:

$$V(x) = \frac{\lambda}{r+2\lambda}x + \gamma_1 e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}x} + \gamma_2 e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}x} \qquad \forall x \geq \overline{x} \tag{19}$$

for some coefficients γ_1 and γ_2. The Responder can never get more than the full surplus or agree to accept negative amount, so $0 < V(x) \leq x \; \forall x$; thus $\gamma_1 = 0$, otherwise this is violated for x large enough. So $V(x) = \frac{\lambda}{r+2\lambda}x + \gamma_2 e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}x}$ for some γ_2.

In the case of $\underline{x} \geq \overline{x}$, γ_2 can be solved using the conditions $V(\underline{x}) = \omega$ and $V'(\underline{x}) = 0$. See proof of Proposition 3 for details. The Responder's utility is

$$V(x) = \frac{\lambda}{r+2\lambda}x + \frac{\lambda}{r+2\lambda}\sqrt{\frac{\sigma^2}{2r+4\lambda}} e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}(\underline{x}-x)} \quad \text{for} \quad x \geq \underline{x} \quad \text{and} \quad V(x) = \omega \quad \text{for} \quad x < \underline{x}$$

The Proposer's utility is

$$U(x) = x - V(x) \text{ for } x \geq 2\omega, \quad \text{and } U(x) = \omega \text{ for } x \leq 2\omega$$

In the case of $\underline{x} < \overline{x}$. There is a region of waiting between $\underline{x}$ and $\overline{x}$. For $\underline{x} < x < \overline{x}$, the value functions follow equations (12):

$$\begin{aligned} (U+V)(x) &= \alpha_+ e^{\sqrt{\frac{2r}{\sigma^2}}x} + \beta_+ e^{-\sqrt{\frac{2r}{\sigma^2}}x} \\ (U-V)(x) &= \alpha_- e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}x} + \beta_- e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}x} \end{aligned} \tag{20}$$

For $x < \underline{x}$, the Responder quits and both firms receive $U(x) = V(x) = \omega$.

So the closed-form expressions of equilibrium value functions are:

$$U(x) = \begin{cases} \frac{r+\lambda}{r+2\lambda}x - \gamma_2 e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}x} & \text{for } x \geq \overline{x} \\ \frac{1}{2}\alpha_+ e^{\sqrt{\frac{2r}{\sigma^2}}x} + \frac{1}{2}\beta_+ e^{-\sqrt{\frac{2r}{\sigma^2}}x} + \frac{1}{2}\alpha_- e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}x} + \frac{1}{2}\beta_- e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}x} & \text{for } \underline{x} < x < \overline{x} \\ \omega & \text{for } x \leq \underline{x} \end{cases} \tag{21}$$

and

$$V(x) = \begin{cases} \frac{\lambda}{r+2\lambda}x + \gamma_2 e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}x} & \text{for } x \geq \overline{x} \\ \frac{1}{2}\alpha_+ e^{\sqrt{\frac{2r}{\sigma^2}}x} + \frac{1}{2}\beta_+ e^{-\sqrt{\frac{2r}{\sigma^2}}x} - \frac{1}{2}\alpha_- e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}x} - \frac{1}{2}\beta_- e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}x} & \text{for } \underline{x} < x < \overline{x} \\ \omega & \text{for } x \leq \underline{x} \end{cases} \tag{22}$$

with coefficients

$$\begin{cases} \alpha_+ = \frac{1}{2}(\overline{x} + \sqrt{\frac{\sigma^2}{2r}})e^{-\sqrt{\frac{2r}{\sigma^2}}\overline{x}} \\ \beta_+ = \frac{1}{2}(\overline{x} - \sqrt{\frac{\sigma^2}{2r}})e^{\sqrt{\frac{2r}{\sigma^2}}\overline{x}} \\ \alpha_- = \frac{1}{2}\frac{r}{r+2\lambda}(\overline{x} + \sqrt{\frac{\sigma^2}{2r+4\lambda}})e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}\overline{x}} \\ \beta_- = -\frac{1}{2}\frac{r}{r+2\lambda}(\overline{x} + \sqrt{\frac{\sigma^2}{2r+4\lambda}})e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}(2\underline{x}-\overline{x})} \\ \gamma_2 = \frac{1}{4}\frac{r}{r+2\lambda}(\overline{x} + \sqrt{\frac{\sigma^2}{2r+4\lambda}})e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}(2\underline{x}-\overline{x})} + \frac{1}{4}\frac{r}{r+2\lambda}(\overline{x} - \sqrt{\frac{\sigma^2}{2r+4\lambda}})e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}\overline{x}} \end{cases} \tag{23}$$

from equations (24) in the proof of Proposition 3, and $\overline{x}$ is the solution to the implicit function.

$$F(\overline{x}, \lambda) = \sqrt{\frac{r}{r+2\lambda}}\Big(\overline{x} + \sqrt{\frac{\sigma^2}{2r+4\lambda}}\Big)\Big(\frac{2\omega + \sqrt{4\omega^2 + \frac{\sigma^2}{2r} - \overline{x}^2}}{\overline{x} + \sqrt{\frac{\sigma^2}{2r}}}\Big)^{\sqrt{\frac{r+2\lambda}{r}}} - \sqrt{4\omega^2 + \frac{\sigma^2}{2r} - \overline{x}^2} = 0$$

Appendix B Proofs

Proof of Lemma 1

Assume in equilibrium there is a state x such that $U(x)+V(x)<x$, then the Responder must accept all offers $p \geq V(x)$. Then the Proposer can propose a $p=V(x)$, and receives immediate payment of $x-V(x)$, which is greater than $U(x)$, a contradiction.

Proof of Lemma 2

First, one can show that there must $\exists x$ s.t. $a(x,p(x))=1$ in interval $(\underline{m},\infty)$ for any $\underline{m}$. Suppose they never reach agreement, then $U(x)+V(x)=0<x$ for $x>0$, a contradiction to Lemma 1. Suppose there exists $\overline{m}$ such that firms do not reach agreement for all $x>\overline{m}$. Then their value functions must be decreasing as x increases beyond $\overline{m}$. Thus for $x>\overline{m}$, we have $U(x)+V(x)<U(\overline{m})+V(\overline{m})=\overline{m}<x$, again a contradiction to Lemma 1.

Now define $\overline{x}=inf\{x \mid a(x,p(x))=1\}$. We want to show that, if $a(x',p(x'))=1$ and $a(x'',p(x''))=1$ for some $x'<x''$, then $\nexists$ a open set $Z\subset(x',x'')$ s.t. $a(x,p(x))=0$ $\forall x\in Z$. This means that, firms cannot disagree on an open set between two agreement states. Suppose not, define $x_l=\sup\{x<Z \mid a(x,p(x))=1\}$ and $x_r=\inf\{x>Z \mid a(x,p(x))=1\}$. For any $x\in Z$, firms delay until the state reaches x_l or x_r. The probability of reaching x_l first is $\frac{x_r-x}{x_r-x_l}$. Thus without time discounting ($r=0$), the sum of firms value functions must be $U(x)+V(x)=\frac{x_r-x}{x_r-x_l}[U(x_l)+V(x_l)]+\frac{x-x_l}{x_r-x_l}[U(x_r)+V(x_r)]=\frac{x_r-x}{x_r-x_l}x_l+\frac{x-x_l}{x_r-x_l}x_r=x$. If $r>0$, then we must have $U(x)+V(x)<x$, a contradiction to Lemma 1.

By the utility definition, $A=\{x|\ a(x,p(x))=1\}$ is a closed set. Thus firms must reach agreement for all $x>\overline{x}$.

Proof of Proposition 1

We know that $(U+V)(\overline{x})=\overline{x}$. To prove $\overline{x}=\overline{x_s}$, we only need to show that $(U+V)'(x)=1$, because $\overline{x_x}$ is the unique value that satisfies both conditions.

First, we have $\lim_{x\to\overline{x}^+}(U+V)'(x)=1$, because $(U+V)(x)=x$ for $x\geq\overline{x}$. If $\lim_{x\to\overline{x}^-}(U+V)'(x)>1$, then $\exists x<\overline{x}$ such that $(U+V)(x)<x$, a contradiction to Lemma 1. If $\lim_{x\to\overline{x}^-}(U+V)'(x)<1$, then we must have either $\lim_{x\to\overline{x}^-}U'(x)<\lim_{x\to\overline{x}^+}U'(x)$ or $\lim_{x\to\overline{x}^-}V'(x)<\lim_{x\to\overline{x}^+}V'(x)$. In either case, the firm with the convex kink at $\overline{x}$ can profitably deviate by delaying agreement for an infinitesimal dt. This follows from standard proof of the smooth-pasting condition in optimal stopping problem (see Dixit 1993 for details). This implies that $\lim_{x\to\overline{x}^-}(U+V)'(x)=1$, and thus $(U+V)'(\overline{x})=1$. This proves that $\overline{x}=\overline{x_s}=\sqrt{\frac{\sigma^2}{2r}}$.

Given $\lim_{x\to\overline{x}^-}(U+V)'(x)=\lim_{x\to\overline{x}^+}(U+V)'(x)$ and that neither firm's value function can have a convex kink at $\overline{x}$, we have:

$$\begin{cases}\lim_{x\to\overline{x}^+}V(x)=\lim_{x\to\overline{x}^-}V(x)\\ \lim_{x\to\overline{x}^+}V'(x)=\lim_{x\to\overline{x}^-}V'(x)\end{cases}$$

Because $\overline{x}=\overline{x_s}$, we have $(U+V)(x)=W_s(x)=\sqrt{\frac{\sigma^2}{2r}}e^{\sqrt{\frac{2r}{\sigma^2}}x-1}$, which implies $\alpha_+=\sqrt{\frac{\sigma^2}{2r}}e^{-1}$

in equation (15). Plugging in equations (14), (15), we get:

$$\begin{cases} \frac{1}{2}\left(\sqrt{\frac{\sigma^2}{2r}}e^{-1}e^{\sqrt{\frac{2r}{\sigma^2}}\overline{x}}\right) - \frac{\alpha_-}{2}e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}\overline{x}} & = \frac{\lambda}{r+2\lambda}\overline{x} + \gamma_2 e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}\overline{x}} \\ \frac{1}{2} - \sqrt{\frac{2r+4\lambda}{\sigma^2}}\frac{\alpha_-}{2}e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}\overline{x}} & = \frac{\lambda}{r+2\lambda} - \sqrt{\frac{2r+4\lambda}{\sigma^2}}\gamma_2 e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}\overline{x}} \end{cases}$$

Solving the system of equations gives:

$$\begin{cases} \alpha_- & = \left(\frac{1}{2} - \frac{\lambda}{r+2\lambda}\right)\left(\sqrt{\frac{\sigma^2}{2r}} + \sqrt{\frac{\sigma^2}{2r+4\lambda}}\right)e^{-\sqrt{\frac{2r+4\lambda}{2r}}} \\ \gamma_2 & = \frac{1}{2}\left(\frac{1}{2} - \frac{\lambda}{r+2\lambda}\right)\left(\sqrt{\frac{\sigma^2}{2r}} - \sqrt{\frac{\sigma^2}{2r+4\lambda}}\right)e^{\sqrt{\frac{2r+4\lambda}{2r}}} \end{cases}$$

Plugging in coefficients α_+, α_-, and γ_2 into equations (14) and (15) gives value functions $U(x)$ and $V(x)$. Particularly, for $x \geq \overline{x}$, $V(x) = \frac{\lambda}{r+2\lambda}x + \frac{1}{2}\left(\frac{1}{2} - \frac{\lambda}{r+2\lambda}\right)\left(\sqrt{\frac{\sigma^2}{2r}} - \sqrt{\frac{\sigma^2}{2r+4\lambda}}\right)e^{\sqrt{\frac{2r+4\lambda}{2r}}(\overline{x}-x)}$. This gives $V(\overline{x} = \sqrt{\frac{\sigma^2}{2r}}) = \frac{\lambda}{r+2\lambda}\overline{x} + \frac{1}{4}\left[\frac{r}{r+2\lambda} - \left(\frac{r}{r+2\lambda}\right)^{1.5}\right]\overline{x}$.

Proof of Corollary 2

Because $\overline{x} = \overline{x_s}$, the equilibrium outcome must be socially efficient regardless of λ. Taking derivatives of $U(x)$ and $V(x)$ from equations (16) and (17) with respect to λ proves the rest. Particularly, for $x < \overline{x}$, the terms $\left(\frac{1}{2} - \frac{\lambda}{r+2\lambda}\right)$, $\left(\sqrt{\frac{\sigma^2}{2r}} + \sqrt{\frac{\sigma^2}{2r+4\lambda}}\right)$, and $e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}(x-\overline{x})}$ all decrease in λ, thus $\frac{dU(x)}{d\lambda} < 0$ and $\frac{dV(x)}{d\lambda} > 0$.

Proof of Lemma 3

The proof that firms reach agreement for all x above a threshold $\overline{x}$ is identical to the proof of Lemma 2.

Additionally we need to prove that $\overline{x} \leq \overline{x}_s$. Suppose instead $\overline{x} > \overline{x_s}$, then for $x \in (\overline{x_s}, \overline{x})$, we have $U(x) + V(x) \geq x$ by Lemma 1. Also, the total payoff in equilibrium cannot exceed the socially efficient payoff, which is x for $x > \overline{x_s}$. Thus we must have $U(x) + V(x) = x$ for $x \in (\overline{x_s}, \overline{x})$. However, given the form of $U(x)$ and $V(x)$ in equations (12), no parameters can satisfy $U(x) + V(x) = x$ in an open interval, a contradiction.

Proof of Lemma 4

First, we show that there must $\exists x$ s.t. $q(x) = 1$ in intervals $(-\infty, \overline{m})$ for all $\overline{m}$. Suppose the Responder never quits, then $V(x)$ approaches 0 as $x \to -\infty$. Then the Responder can profitably deviate by quitting. Suppose there exists a lowest state such that the Responder quits. Denote that state as $\underline{m}$, so the Responder does not quit for all $x < \underline{m}$. Then again $V(x)$ approaches 0 as $x \to -\infty$, for which the Responder can deviate by quitting.

Now define $\underline{x} = sup\{x \mid q(x) = 1\}$. We show that the Responder cannot choose to continue on any open set between two quitting states. Suppose not, define $x_l = \sup\{x < Z \mid q(x) = 1\}$ and $x_r = \inf\{x > Z \mid q(x) = 1\}$. The Responder gets utility of ω at x_l and x_r. If an agreement is reached in this interval, then the Proposer must make an offer such that the Repsonder is indifferent between agreeing or not. Thus for $x_l < x < x_r$, the Responder must gets utility smaller than ω due to discounting. The Responder can profitably deviate by quitting in this interval.

Third, we need to prove that $\underline{x} \geq \underline{x_s}$. Suppose instead $\underline{x} < \underline{x_s}$. For states $x < \underline{x}$, the socially efficient outcome is quitting, and the equilibrium outcome cannot have a higher payoff than the socially efficient payoff. Thus $U(x) + V(x) \leq 2\omega$. Because the Responder does not quit in the region $(\underline{x}, \underline{x_s})$, we must have $V(x) \geq \omega$ in this region. This implies $U(x) \leq \omega$ in $(\underline{x}, \underline{x_s})$. Then there must exists some $x' > \underline{x}$ such that $U(x') < V(x')$ and $U'(x') < V'(x')$. This implies that $(U - V)(x') < 0$ and $(U - V)'(x') < 0$.

By equations (18), if $(U - V)(x') < 0$ then $(U - V)''(x') < 0$, which implies that $(U - V)'(x) < 0$ and $(U - V)(x) < 0$ for $\underline{x} < x < \overline{x}$. Thus we must have $U'_-(\overline{x}) < V'_-(\overline{x})$ and $U(\overline{x}) < V(\overline{x})$. Because $(U + V)(x) = x$ for $x \geq \overline{x}$, we must have $(U + V)'_-(\overline{x}) \leq 1$, otherwise there exists some $x < \overline{x}$ such that $(U + V)(x) < x$, a contradiction to Lemma 1. Thus $U'_-(\overline{x}) < \frac{1}{2}$.

Because $U(\overline{x}) < V(\overline{x})$, γ_2 from equation (19) must be positive. Then by equations (18) and (19), we have $U'_+(\overline{x}) > V'_+(\overline{x})$. Because $(U + V)(x) = x$ for $x \geq \overline{x}$, we can conclude that $U'_+(x) > \frac{1}{2}$.

Because $U'_-(\overline{x}) < \frac{1}{2}$ and $U'_+(x) > \frac{1}{2}$, there is a convex kink on $U(x)$ at $\overline{x}$. Then the Proposer can profitably deviate by delaying the trade for a small dt. Thus we cannot have $\underline{x} < \underline{x_s}$ in equilibrium, concluding the proof.

Proof of Proposition 3

We solve for the equilibrium outcome in two cases.

Case 1: $\underline{x} \geq \overline{x}$:

In this case, x_0 is either greater than $\overline{x}$ or smaller than $\underline{x}$, so the game always end immediately. Firms trade immediately for $x_0 \geq 2\omega$, and the Responder quit immediately for $x_0 < 2\omega$. We know that $V(x) = \omega$ for $x \leq \underline{x}$. Then we have $V'_-(\underline{x}) = 0$. In order for the quitting threshold to be optimal, we need to have $V'_+(\underline{x}) = 0$, otherwise the Responder can profitably deviate by delaying quitting for time dt. Thus we have

$$V(\underline{x}) = \omega \quad \textbf{and} \quad V'(\underline{x}) = 0$$

The form of $V(x)$ is given in equation (19) in Appendix A.2. Plugging in equation (19) with $\gamma_1 = 0$, we get:

$$\begin{cases} \frac{\lambda}{r+2\lambda}\underline{x} + \gamma_2 e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}\underline{x}} = \omega \\ \frac{\lambda}{r+2\lambda} - \sqrt{\frac{2r+4\lambda}{\sigma^2}}\gamma_2 e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}\underline{x}} = 0 \end{cases}$$

Solving the two equations, we find that the solution exists if and only if $\lambda \leq \underline{\lambda} = \frac{2r^2\omega^2 + \sqrt{4r^4\omega^4 + \sigma^2 r^3\omega^2}}{\sigma^2}$. The solutions are:

$$\begin{cases} \underline{x} = \frac{r+2\lambda}{\lambda}\omega - \sqrt{\frac{\sigma^2}{2r+4\lambda}} \\ \gamma_2 = \frac{\lambda}{r+2\lambda}\sqrt{\frac{\sigma^2}{2r+4\lambda}} e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}\underline{x}} \end{cases}$$

The Responder's utility is $V(x) = \frac{\lambda}{r+2\lambda}x + \frac{\lambda}{r+2\lambda}\sqrt{\frac{\sigma^2}{2r+4\lambda}} e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}(\underline{x}-x)}$ for $x \geq \underline{x}$ and $V(x) = \omega$ for $x < \underline{x}$. The Proposer's utility is $U(x) = x - V(x)$ for $x \geq \underline{x}$, $U(x) = x - \omega$ for $2\omega < x < \underline{x}$, and $U(x) = \omega$ for $x \leq 2\omega$.

Case 2: $\underline{x} < \overline{x}$:
An equilibrium outcome has to satisfy the following four value-matching conditions:

$$\begin{cases} (U+V)(\underline{x}) = 2\omega \\ (U-V)(\underline{x}) = 0 \\ (U+V)(\overline{x}) = \overline{x} \\ (U-V)(\overline{x}) = \overline{x} - 2V(\overline{x}) = \frac{r}{r+2\lambda}\underline{x} - 2\gamma_2 e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}\overline{x}} \end{cases}$$

An equilibrium outcome also has to satisfy the following three conditions:

1. $V'(\underline{x}) = 0$ or $(U+V)'(\underline{x}) = (U-V)'(\underline{x})$ for optimality of the quitting threshold.
2. $(U+V)'(\overline{x}) = 1$. We know that $(U+V)'_+(\overline{x}) = 1$ since $(U+V)(x) = x$ for $x > \overline{x}$. If $(U+V)'_-(\overline{x}) > 1$, then one of the firm has a convex kink on his/her value function at $\overline{x}$, and can profitably deviate and delaying trade for dt. If $(U+V)'_-(\overline{x}) < 1$, then $(U+V)(x) < x$ for some $x < \overline{x}$, a contradiction to Lemma 1.
3. $(U-V)'_-(\overline{x}) = (U-V)'_+(\overline{x})$. We know that $(U+V)'_-(\overline{x}) = (U+V)'_+(\overline{x}) = 1$. Also, there cannot be convex kink for either $U(x)$ or $V(x)$ at $\overline{x}$, otherwise a profitable deviation exists. Thus $U'_-(\overline{x}) = U'_+(\overline{x})$ and $V'_-(\overline{x}) = V'_+(\overline{x})$.

Using the forms of $(U+V)(x)$ and $(U-V)(x)$ from equations (12) for the region $\underline{x} < x < \overline{x}$, the above seven conditions constitute the following system of equations:

$$\begin{cases} \alpha_+ e^{\sqrt{\frac{2r}{\sigma^2}}\underline{x}} = 2\omega \\ \alpha_- e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}\underline{x}} + \beta_- e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}\underline{x}} = 0 \\ \alpha_+ e^{\sqrt{\frac{2r}{\sigma^2}}\overline{x}} + \beta_+ e^{-\sqrt{\frac{2r}{\sigma^2}}\overline{x}} = \overline{x} \\ \alpha_- e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}\overline{x}} + \beta_- e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}\overline{x}} = \frac{r}{r+2\lambda}\overline{x} - 2\gamma_2 e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}\overline{x}} \\ \sqrt{\frac{2r}{\sigma^2}}(\alpha_+ e^{\sqrt{\frac{2r}{\sigma^2}}\underline{x}}) = \sqrt{\frac{2r+4\lambda}{\sigma^2}}(\alpha_- e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}\overline{x}} + \beta_- e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}\overline{x}}) \\ \alpha_+ e^{\sqrt{\frac{2r}{\sigma^2}}\overline{x}} - \beta_+ e^{-\sqrt{\frac{2r}{\sigma^2}}\overline{x}} = \sqrt{\frac{\sigma^2}{2r}} \\ \alpha_- e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}\overline{x}} - \beta_- e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}\overline{x}} = \frac{r}{r+2\lambda}\sqrt{\frac{\sigma^2}{2r+4\lambda}} + 2\gamma_2 e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}\overline{x}} \end{cases}$$

Solving the system of equations, we get the following implicit equation of $\overline{x}$ and λ:

$$F(\overline{x},\lambda) = \sqrt{\frac{r}{r+2\lambda}}\Big(\overline{x} + \sqrt{\frac{\sigma^2}{2r+4\lambda}}\Big)\Big(\frac{2\omega + \sqrt{4\omega^2 + \frac{\sigma^2}{2r} - \overline{x}^2}}{\overline{x} + \sqrt{\frac{\sigma^2}{2r}}}\Big)^{\sqrt{\frac{r+2\lambda}{r}}} - \sqrt{4\omega^2 + \frac{\sigma^2}{2r} - \overline{x}^2} = 0$$

By Lemma 3, $2\omega \le \overline{x} \le \overline{x_s}$. Here we use a Lemma proved in the Online Appendix.

Lemma 5. *The equation $F(\overline{x},\lambda) = 0$ has a unique solution $\overline{x}(\lambda)$ in the range of $\overline{x} \in (2\omega, \overline{x_s})$. The solution $\overline{x}(\lambda)$ is increasing in λ for $\lambda \ge \underline{\lambda}$, with $\overline{x}(\underline{\lambda}) = 2\omega$, and $\overline{x} \to \overline{x_s}$ as $\lambda \to \infty$.*

By Lemma 5, the equilibrium outcome is unique. If $\lambda \leq \underline{\lambda}$, then the equilibrium outcome is in case 1. If $\lambda > \underline{\lambda}$, then the equilibrium outcome is in case 2. There exists a unique $\overline{x}$ as a function of λ in the range of $(2\omega, \overline{x_s})$. We can solve the rest of the parameters as functions of λ and $\overline{x}$. The solutions are:

$$\begin{cases} \alpha_+ = \frac{1}{2}(\overline{x} + \sqrt{\frac{\sigma^2}{2r}})e^{-\sqrt{\frac{2r}{\sigma^2}}\overline{x}} \\ \beta_+ = \frac{1}{2}(\overline{x} - \sqrt{\frac{\sigma^2}{2r}})e^{\sqrt{\frac{2r}{\sigma^2}}\overline{x}} \\ \underline{x} = \overline{x} - \sqrt{\frac{\sigma^2}{2r}}\log\frac{2\omega - \sqrt{4\omega^2 + \frac{\sigma^2}{2r} - \overline{x}^2}}{\overline{x} - \sqrt{\frac{\sigma^2}{2r}}} \\ \alpha_- = \frac{1}{2}\frac{r}{r+2\lambda}(\overline{x} + \sqrt{\frac{\sigma^2}{2r+4\lambda}})e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}\overline{x}} \\ \beta_- = -\frac{1}{2}\frac{r}{r+2\lambda}(\overline{x} + \sqrt{\frac{\sigma^2}{2r+4\lambda}})e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}(2\underline{x}-\overline{x})} \\ \gamma_2 = \frac{1}{4}\frac{r}{r+2\lambda}(\overline{x} + \sqrt{\frac{\sigma^2}{2r+4\lambda}})e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}(2\underline{x}-\overline{x})} + \frac{1}{4}\frac{r}{r+2\lambda}(\overline{x} - \sqrt{\frac{\sigma^2}{2r+4\lambda}})e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}\overline{x}} \end{cases} \tag{24}$$

For Proposition 3.2: The only thing left to prove is that $\frac{d\underline{x}}{d\lambda} < 0$ for $\lambda > \underline{\lambda}$. Because $\frac{d\overline{x}}{d\lambda} > 0$, we just need to prove $\frac{d\underline{x}}{d\overline{x}} < 0$. From equations (24), we have: $\underline{x} = \overline{x} - \sqrt{\frac{\sigma^2}{2r}}\log\frac{2\omega - \sqrt{4\omega^2 + \frac{\sigma^2}{2r} - \overline{x}^2}}{\overline{x} - \sqrt{\frac{\sigma^2}{2r}}} = \overline{x} - \sqrt{\frac{\sigma^2}{2r}}\log\frac{\overline{x} + \sqrt{\frac{\sigma^2}{2r}}}{2\omega + \sqrt{4\omega^2 + \frac{\sigma^2}{2r} - \overline{x}^2}}$. Then taking derivative of $\underline{x}$ with respect to $\overline{x}$, we get:

$$\frac{d\overline{x}}{d\underline{x}} = \frac{\overline{x}\big(2\omega\sqrt{4\omega^2 - \overline{x}^2 + \frac{\sigma^2}{2r}} - \overline{x}(\sqrt{\frac{\sigma^2}{2r}} + \overline{x}) + 4\omega^2\big)}{(\sqrt{\frac{\sigma^2}{2r}} + \overline{x})\sqrt{4\omega^2 - \overline{x}^2 + \frac{\sigma^2}{2r}}\big(2\omega + \sqrt{4\omega^2 - \overline{x}^2 + \frac{\sigma^2}{2r}}\big)}$$

Given that the denominator is strictly positive, the sign of $\frac{d\overline{x}}{d\underline{x}}$ depends on the sign of $2\omega\sqrt{4\omega^2 - \overline{x}^2 + \frac{\sigma^2}{2r}} - \overline{x}(\sqrt{\frac{\sigma^2}{2r}} + \overline{x}) + 4\omega^2 = 2\omega\sqrt{4\omega^2 - \overline{x}^2 + \frac{\sigma^2}{2r}} - \overline{x}\sqrt{\frac{\sigma^2}{2r}} + 4\omega^2 - \overline{x}^2$. Given that $\overline{x} > 2\omega$, we have $\sqrt{\frac{\sigma^2}{2r}} > \sqrt{4\omega^2 - \overline{x}^2 + \frac{\sigma^2}{2r}}$, thus $2\omega\sqrt{4\omega^2 - \overline{x}^2 + \frac{\sigma^2}{2r}} - \overline{x}\sqrt{\frac{\sigma^2}{2r}} + 4\omega^2 - \overline{x}^2 < 0$.

One can check that $\underline{x} = \underline{x_s}$ when $\overline{x} = \overline{x_s}$. So $\underline{x}$ approaches $\underline{x_s}$ as $\lambda \to \infty$.

For Proposition 3.3, note that the ex-ante probability of alliance is only non-trivial for $x_0 \in (\underline{x}, \overline{x})$, so we only need to focus on this case. Because x_t follows a Brownian motion, the ex-ante probability of reaching $\overline{x}$ before $\underline{x}$ can be written as

$$\mu = \frac{x_0 - \underline{x}}{\overline{x}_0\underline{x}}$$

Then

$$\frac{d\mu}{d\lambda} = \frac{(\overline{x} - \underline{x})(-\frac{d\underline{x}}{d\lambda}) - (x_0 - \underline{x})(\frac{d\overline{x}}{d\lambda} - \frac{d\underline{x}}{d\lambda})}{(\overline{x} - \underline{x})^2} = \frac{(\overline{x} - x_0)(-\frac{d\underline{x}}{d\lambda}) - (x_0 - \underline{x})\frac{d\overline{x}}{d\lambda}}{(\overline{x} - \underline{x})^2}$$

Let $\tilde{x}(\lambda) = \frac{\overline{x}(-\frac{d\underline{x}}{d\lambda}) + \underline{x}\frac{d\overline{x}}{d\lambda}}{\frac{d\overline{x}}{d\lambda} - \frac{d\underline{x}}{d\lambda}) - (x_0 - \underline{x})}$. Since $\frac{d\overline{x}}{d\lambda} > 0$ and $-\frac{d\underline{x}}{d\lambda} > 0$, then $\frac{d\mu}{d\lambda} > 0$ for $x_0 < \tilde{x}(\lambda)$ and $\frac{d\mu}{d\lambda} < 0$ for $x_0 > \tilde{x}(\lambda)$.

Proof of Corollary 4

If $\lambda \leq \underline{\lambda}$, then game ends immediately, so $(U+V)(x) = \max\{x_0, 2\omega\}$. Ex-ante welfare is not affected by λ.

Suppose $\lambda > \underline{\lambda}$. From Appendix A.2, we have:

$$(U+V)(x) = \frac{1}{2}(\overline{x} + \sqrt{\frac{\sigma^2}{2r}})e^{\sqrt{\frac{2r}{\sigma^2}}(x-\overline{x})} + \frac{1}{2}(\overline{x} - \sqrt{\frac{\sigma^2}{2r}})e^{\sqrt{\frac{2r}{\sigma^2}}(\overline{x}-x)}$$

for $\underline{x} < x < \overline{x}$. The derivative with respect to λ is:

$$\frac{d(U+V)(x)}{d\lambda} = \frac{d(U+V)(x)}{d\overline{x}}\frac{d\overline{x}}{d\lambda} = \frac{1}{2}\sqrt{\frac{2r}{\sigma^2}}\overline{x}(e^{\sqrt{\frac{2r}{\sigma^2}}(\overline{x}-x)} - e^{-\sqrt{\frac{2r}{\sigma^2}}(\overline{x}-x)})\frac{d\overline{x}}{d\lambda}$$

Because $\frac{1}{2}\sqrt{\frac{2r}{\sigma^2}}\overline{x}(e^{\sqrt{\frac{2r}{\sigma^2}}(\overline{x}-x)} - e^{-\sqrt{\frac{2r}{\sigma^2}}(\overline{x}-x)}) > 0$ and $\frac{d\overline{x}}{d\lambda} > 0$ from Lemma 5, we can conclude that $\frac{d(U+V)(x)}{d\lambda} > 0$. Thus if $x_0 \in (\underline{x}, \overline{x})$, then the ex-ante welfare is strictly increasing in λ for $\lambda < \underline{\lambda}$. If $x_0 \notin (\underline{x}, \overline{x})$, then the game ends immediately and the ex-ante welfare is not affected by λ.

As $\lambda \to \infty$, $\overline{x} \to \overline{x_s}$ by Lemma 5. As $\overline{x} \to \overline{x_s}$, we have $\underline{x} = \overline{x} - \sqrt{\frac{\sigma^2}{2r}} \log \frac{2\omega - \sqrt{4\omega^2 + \frac{\sigma^2}{2r} - \overline{x}^2}}{\overline{x} - \sqrt{\frac{\sigma^2}{2r}}} \to \underline{x_s} = \overline{x_s} - \sqrt{\frac{\sigma^2}{2r}} \log\left(\frac{\sqrt{\frac{\sigma^2}{2r}} + \sqrt{\frac{\sigma^2}{2r} + 4\omega^2}}{2\omega}\right)$. Also $(U+V)(x) \to W_s(x)$. So the total welfare approaches the socially efficient welfare. Also, both $\alpha_- = \frac{1}{2}\frac{r}{r+2\lambda}(\overline{x} + \sqrt{\frac{\sigma^2}{2r+4\lambda}})e^{-\sqrt{\frac{2r+4\lambda}{\sigma^2}}\overline{x}}$ and $\beta_- = -\frac{1}{2}\frac{r}{r+2\lambda}(\overline{x} + \sqrt{\frac{\sigma^2}{2r+4\lambda}})e^{\sqrt{\frac{2r+4\lambda}{\sigma^2}}(2\underline{x}-\overline{x})}$ approach 0 as $\lambda \to \infty$, so $(U-V)(x) \to 0$. Thus $(U+V)(x_0) = (U+V)_s(x_0)$ and $(U-V)(x_0) = 0$ in the limit, so each firm approaches half of the socially efficient total welfare.

Proof of Proposition 5

First we prove Proposition 5(1). Let $I = (\underline{x_s}, 2\omega)$, and $\tilde{\lambda} = \underline{\lambda}$. If $\lambda \leq \underline{\lambda}$, then $\underline{x} \geq \overline{x}$ by Proposition 3, so the game ends immediately. If $x_0 \in I$, then the Responder quits at time 0 and both firms get outside option of ω. By Proposition 3 and Corollary 4, as $\lambda \to \infty$, $\underline{x} \to \underline{x_s}$ and $(U+V)(x_0) \to (U+V)_s(x_0)$. So there exists a $\lambda' > \underline{\lambda}$ such that $\underline{x}(\lambda') < x_0$ and $(U+V)(x_0) > 2\omega$, because $(U+V)_s(x_0) > 2\omega$. So the total ex-ante welfare is higher under λ'. The Responder can never be worse off than ω, and the Proposer has weakly higher utility than the Responder. Thus λ' Pareto dominates λ.

For Proposition 5(2): For any λ, $\underline{x}(\lambda) > \underline{x_s}$. Take any x_0 in the interval $(\underline{x_s}, \underline{x})$. For such x_0, the ex-ante utilities for both firms under λ is ω. Then by Proposition 3, as $\lambda \to \infty$, $\underline{x} \to \underline{x_s}$ and $(U+V)(x_0) \to (U+V)_s(x_0)$. So there exists a $\lambda' > \lambda$ such that $\underline{x}(\lambda') < x_0$ and $(U+V)(x_0) > 2\omega$. This must be a Pareto improvement, similarly to the argument above.

Proof of Proposition 6

The formal proof involves three steps. First, similar to Lemma 3, an equilibrium must features a quitting threshold. Second, this threshold must be weakly above the threshold from the base model, $\underline{x}$. Otherwise, there should exist an equilibrium in the base model with a lower quitting threshold, a contradiction. Finally, solve the socially planner's problem with the additional constraint that the outside option must be taken at $\underline{x}$ and below. The solution is the equilibrium outcome from the base model.

Appendix C Online Appendix

Lemma 5

$$F(\overline{x},\lambda)=\sqrt{\frac{r}{r+2\lambda}}(\overline{x}+\sqrt{\frac{\sigma^2}{2r+4\lambda}})(\frac{2\omega+\sqrt{4\omega^2+\frac{\sigma^2}{2r}-\overline{x}^2}}{\overline{x}+\sqrt{\frac{\sigma^2}{2r}}})^{\sqrt{\frac{r+2\lambda}{r}}}-\sqrt{4\omega^2+\frac{\sigma^2}{2r}-\overline{x}^2}=0$$

has a unique solution $\overline{x}(\lambda)$ in the range of $2\omega\leq\overline{x}\leq\overline{x_s}=\sqrt{\frac{\sigma^2}{2r}+4\omega^2}$. The solution $\overline{x}(\lambda)$ is increasing in λ for $\lambda\geq\underline{\lambda}$, $\overline{x}(\underline{\lambda})=2\omega$, and $\overline{x}\to\overline{x_s}$ as $\lambda\to\infty$.

Proof. First, $\frac{2\omega+\sqrt{4\omega^2+\frac{\sigma^2}{2r}-\overline{x}^2}}{\overline{x}+\sqrt{\frac{\sigma^2}{2r}}}\leq 1$ because $\overline{x}\geq 2\omega$ by Lemma 3. Thus $\frac{\partial F}{\partial\lambda}<0$ because every term decreases in λ.

By implicit function theorem, to prove $\frac{\partial\overline{x}}{\partial\lambda}>0$, we need to prove that $\frac{\partial F}{\partial\overline{x}}>0$ for $2\omega\leq\overline{x}\leq\overline{x_s}$. We prove by the following steps. We show that $\frac{\partial F}{\partial\overline{x}}=M(\overline{x})*E(\overline{x})$ for some M and E. The term $M(\overline{x})$ is always positive. The term $E(\overline{x})$ is positive at $\overline{x}=2\omega$, and $\frac{\partial E}{\partial\overline{x}}>0$ for $\overline{x}>2\omega$, thus concluding that $\frac{\partial F}{\partial\overline{x}}>0$.

Let $M=\left(\frac{2\omega+\sqrt{4\omega^2+\frac{\sigma^2}{2r}-\overline{x}^2}}{\overline{x}+\frac{\sigma^2}{2r}}\right)^{\sqrt{\frac{r+2\lambda}{r}}}$. Then $M>0$ for $\overline{x}<\overline{x_s}=\sqrt{4\omega^2+\frac{\sigma^2}{2r}}$, then:

$$\begin{aligned}
\frac{\partial F}{\partial\overline{x}}&=\frac{\overline{x}}{\sqrt{4\omega^2+\frac{\sigma^2}{2r}-\overline{x}^2}}-\frac{\overline{x}(\overline{x}+\sqrt{\frac{\sigma^2}{2r+4\lambda}})}{2\omega+\sqrt{4\omega^2+\frac{\sigma^2}{2r}-\overline{x}^2}}\frac{1}{\sqrt{4\omega^2+\frac{\sigma^2}{2r}-\overline{x}^2}}M-\frac{\overline{x}+\sqrt{\frac{\sigma^2}{2r+4\lambda}}}{\overline{x}+\sqrt{\frac{\sigma^2}{2r}}}M+M\\
&=M*\Big[\frac{\overline{x}}{\sqrt{4\omega^2+\frac{\sigma^2}{2r}-\overline{x}^2}}(\frac{\overline{x}+\sqrt{\frac{\sigma^2}{2r+4\lambda}}}{2\omega+\sqrt{4\omega^2+\frac{\sigma^2}{2r}-\overline{x}^2}})^{\sqrt{\frac{r+2\lambda}{r}}}\\
&\quad-\frac{\overline{x}}{\sqrt{4\omega^2+\frac{\sigma^2}{2r}-\overline{x}^2}}\frac{\overline{x}+\sqrt{\frac{\sigma^2}{2r+4\lambda}}}{2\omega+\sqrt{4\omega^2+\frac{\sigma^2}{2r}-\overline{x}^2}}-\frac{\overline{x}+\sqrt{\frac{\sigma^2}{2r+4\lambda}}}{\overline{x}+\sqrt{\frac{\sigma^2}{2r}}}+1\Big]\\
&=M*E(\overline{x})
\end{aligned}\tag{25}$$

At $\overline{x}=2\omega$, $E(\overline{x})=\left(\frac{2\omega}{\sqrt{\frac{\sigma^2}{2r}+1}}\right)\left(\frac{2\omega+\sqrt{\frac{\sigma^2}{2r+4\lambda}}}{2\omega+\sqrt{\frac{\sigma^2}{2r}}}\right)>0$. We can write $E(\overline{x})$ as $E=\frac{\overline{x}}{\sqrt{4\omega^2-\overline{x}^2+\frac{\sigma^2}{2r}}}*B-$

$\frac{\overline{x}+\sqrt{\frac{\sigma^2}{2r+4\lambda}}}{\overline{x}+\sqrt{\frac{\sigma^2}{2r}}}+1$, where

$$\begin{aligned} B &= \Big(\frac{\overline{x}+\sqrt{\frac{\sigma^2}{2r}}}{2\omega+\sqrt{4\omega^2-\overline{x}^2+\frac{\sigma^2}{2r}}}\Big)^{\sqrt{\frac{r+2\lambda}{r}}} - \frac{\overline{x}+\sqrt{\frac{\sigma^2}{2r+4\lambda}}}{2\omega+\sqrt{4\omega^2-\overline{x}^2+\frac{\sigma^2}{2r}}} \\ &> \frac{\overline{x}+\sqrt{\frac{\sigma^2}{2r}}}{2\omega+\sqrt{4\omega^2-\overline{x}^2+\frac{\sigma^2}{2r}}} - \frac{\overline{x}+\sqrt{\frac{\sigma^2}{2r+4\lambda}}}{2\omega+\sqrt{4\omega^2-\overline{x}^2+\frac{\sigma^2}{2r}}} \\ &= \frac{\sqrt{\frac{\sigma^2}{2r}}-\sqrt{\frac{\sigma^2}{2r+4\lambda}}}{2\omega+\sqrt{4\omega^2-\overline{x}^2+\frac{\sigma^2}{2r}}} > 0 \end{aligned} \tag{26}$$

with $\frac{dB}{d\overline{x}} > 0$. Then taking derivative of E with respect to $\overline{x}$, we get:

$$\frac{dE}{d\overline{x}} = \frac{\sqrt{4\omega^2-\overline{x}^2+\frac{\sigma^2}{2r}}+\overline{x}^2/\sqrt{4\omega^2-\overline{x}^2+\frac{\sigma^2}{2r}}}{\sqrt{4\omega^2-\overline{x}^2+\frac{\sigma^2}{2r}}^2} * B + \frac{\overline{x}}{\sqrt{4\omega^2-\overline{x}^2+\frac{\sigma^2}{2r}}}\frac{dB}{d\overline{x}} - \frac{\sqrt{\frac{\sigma^2}{2r}}-\sqrt{\frac{\sigma^2}{2r+4\lambda}}}{(\overline{x}+\sqrt{\frac{\sigma^2}{2r}})^2} \tag{27}$$

Then $\frac{dE}{d\overline{x}}$ is positive because each of the terms is positive for $\overline{x} \in (2\omega, \overline{x_s})$. Because $E(\overline{x} = 2\omega) > 0$, this implies $E(x) > 0$ for all $\overline{x} \geq 2\omega$. Thus $\frac{dF}{d\overline{x}} = M * E > 0$ for $\overline{x} \in (2\omega, \overline{x_s})$. Combining with the fact that $\frac{dF}{d\lambda} < 0$, by implicit function theorem, there exists a function $\overline{x}(\lambda)$ in that range and $\frac{d\overline{x}}{d\lambda} > 0$.

□