Semiparametric Quantile Models for Ascending Auctions with Asymmetric Bidders

Jayeeta Bhattacharya
Queen Mary University of London, UK

Nathalie Gimenes
PUC-Rio, Rio de Janeiro, Brazil

Emmanuel Guerre
University of Kent, UK

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Abstract

The paper proposes a parsimonious and flexible semiparametric quantile regression specification for asymmetric bidders within the independent private value framework. Asymmetry is parameterized using powers of a parent private value distribution, which is generated by a quantile regression specification. As noted in Cantillon (2008), this covers and extends models used for efficient collusion, joint bidding and mergers among homogeneous bidders. The specification can be estimated for ascending auctions using the winning bids and the winner's identity. The estimation is two stage. The asymmetry parameters are estimated from the winner's identity using a simple maximum likelihood procedure. The parent quantile regression specification can be estimated using simple modifications of Gimenes (2017). A timber application reveals that weaker bidders have 30% less chances to win the auction than stronger ones. It is also found that increasing participation in an asymmetric ascending auction may not be as beneficial as using an optimal reserve price as would have been expected from a result of Bulow and Klemperer (1996) valid under symmetry.

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Keywords: Private values; asymmetry; ascending auctions; seller expected revenue; quantile regression; two stage quantile regression estimation.

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1 Introduction

Asymmetry among bidders may arise from many factors, for example, differences in taste or specialization, degree of information, productivity, costs, firm size, joint bidding or collusion among a subgroup of buyers. It is, therefore, likely that symmetric bidding is only a theoretical approximation that may not fit well many auction markets. Within the independent private value paradigm (IPV hereafter), the revenue equivalence theorem no longer holds with asymmetric bidders and first-price auction can be inefficient, see Krishna (2009) and the references therein. Cantillon (2008) supports the common belief that competition is reduced by bidders asymmetries. She shows that asymmetry decreases the seller expected revenue in first-price and second-price auctions, when compared to revenues achieved with a benchmark symmetric private value distribution. The timber auction revenue analysis of Roberts and Sweeting (2016) shows that reducing the participation of strong bidders can considerably lower the seller expected revenue.

Myerson (1981) suggests to depart from standard formats and describes an optimal auction which restores some competition by handicapping strong bidders. This mechanism critically involves the private value distribution and is difficult to implement. In an empirical study of snow removal contract sealed procurements, Flambard and Perrigne (2006) considered this optimal auction and an alternative subsidy policy. Krasnokutskaya and Seim (2011) studied a bid preference program for California highway auction, see also Marion (2007). Athey, Coey and Levin (2013) focused on set-asides and subsidies for timber auctions.

Among the aforementioned empirical works, the only papers adopting a nonparametric approach are Flambard and Perrigne (2006) and Marion (2007), who studied first-price auctions. For first-price auctions, Krasnokutskaya and Seim (2011), Athey, Levin and Seira (2011) and Athey et al. (2013) all considered parametric specifications, as did Roberts and Sweeting (2016) for ascending auctions.

There are, however, some works devoted to the nonparametric approach for ascending auctions with asymmetric bidders. A theoretical nonparametric identification result with a finite number of asymmetric types due to Komarova (2013a) shows that the asymmetric valuation distributions can be recovered from the winning bid and the identity of the winner under IPV, see also Athey and Haile (2002). Brendstrup and Paarsch (2006) have proposed a related semi-nonparametric estimation procedure. Lamy (2012) shows that nonparametric identification still holds under anonymity for second-price auctions when all the bids are observed. Set identification results are also available for affiliated models, which are not point identified as shown by Athey and Haile (2002). For affiliated values and second-price auction, Komarova (2013b) gives bounds for joint private value distribution, assuming identities are available. Coey, Larsen, Sweeney and Waisman (2017) consider a more difficult scenario, where only the winning bid is observed and anonymity is possible. They obtained bounds for the seller expected revenue and bidder surplus which extends upon the ones of Aradillas-López, Gandhi and Quint (2013) for the symmetric case.

Developing nonparametric approaches for asymmetric bidders with a discrete number of types is difficult, because a different value distribution must be estimated for each types, as in Flambard and Perrigne (2006) or Brendstrup and Paarsch (2006). Dividing the sample in subsamples defined by a given type may result in small subsamples in addition to poor nonparametric estimation rates due to the curse of dimensionality. Comparing the valuation distribution across types is not an easy task. In this paper, we tackle these two issues through a semiparametric approach allowing for a nonparametric component common to each type and using a parametric description of type heterogeneity. The common nonparametric component is a parent private value conditional distribution $F(v|x)$, where x is an auction specific good covariate. Following Gimenes (2017), we assume that $F(v|x)$ corresponds to a quantile regression model, so that this rich and flexible specification can be estimated with a standard parametric rate independently of the dimension of x. The asymmetry parameter, say λ_i, is an exponent specific to bidder i, whose private value distribution is

$$F_i(v|x) = F^{\lambda_i}(v|x).$$

The exponent λ_i can be an individual fixed effect which captures unobserved bidder characteristics. As developed in the paper, it can also be a parametric function of some observed bidder variables and fixed effect parameters. In our timber application, the buyers are either mill or logger, which are considered as weak and strong bidders respectively in all applications.

Cantillon (2008) has used a similar specification for theoretical illustration purpose, noting that it has been used to "model efficient collusion, joint bidding and mergers among homogeneous bidders", which can be relevant for many applications. Indeed, when λ_i is an integer number, $F^{\lambda_i}(v|x)$ is the distribution of the maximum value of λ_i symmetric bidders with independent valuations drawn from $F(v|x)$, as relevant, for instance, in joint bidding. This feature also shows that the asymmetry parameter λ_i is a measure of the "strength" of bidder i. A small numerical experiment in the paper parallels Cantillon (2008), adopting an econometric point of view based on the symmetric private value distribution which would be estimated ignoring asymmetry by the quantile procedure of Gimenes (2017). Such a misspecification may lead to underestimation of the optimal reserve price and seller expected revenue.

The proposed estimation is two stage, based upon the winning bid and identity of the winner. The first stage estimates the parameters appearing in the asymmetry exponent λ_i using a maximum likelihood procedure based upon the winner identity. The intuition behind this procedure is that the distribution of the winner identity only depends upon the relative buyers' strength, and hence on asymmetry parameter λ_i and not upon the common parent distribution $F(v|x)$. The second stage estimates the quantile regression specification associated with $F(v|x)$. As in Gimenes (2017), it is based on a quantile regression estimation which uses a transformation of the quantile levels. Accounting for asymmetry leads to considering a transformation which depends upon the estimated asymmetry parameter. This latter step parallels Arellano and Bonhomme (2017), who similarly estimate a quantile level transformation in a three stage quantile regression procedure.

The empirical application illustrates the methodology using USFS timber ascending auctions. Two kinds of firms are competing: firms with manufacturing capacity (mills, usually considered as strong bidders in the literature) and firms lacking manufacturing capabilities (loggers). The estimated asymmetry exponent of the loggers is 30% less than the one of the mills, suggesting that, roughly speaking, two mills should be replaced by three loggers to generate an ascending auction with similar features. The empirical application also studies the seller expected revenue as a function of the proportion of loggers and the number of buyers. It reveals economically significant variations, in the range of $9\% - 20\%$ between ascending auctions attended only by loggers or only by mills. In small auctions with two bidders, changing a logger by a mill can increase the seller optimal expected revenue by 5% in some cases, and still as high as 1% with 12 bidders. This suggests that seller expected revenue bounds that does not account for the proportion of each type can be considerably large, and that the ones averaging over types participation, as in Coey et al. (2017), can be less informative. Another finding relates to an important result of Bulow and Klemperer (1996) stating that, under symmetry, increasing participation is more beneficial than using an optimal auction. Several violations of this result are observed, especially due to the presence of weak bidders.

The paper is organized as follows. Section 2 presents the auction setup and the asymmetric quantile specification. Sections 2.2 and 2.3 give the identification strategy and discuss identification of the parameter of asymmetry under several specifications. Section 3 shows how to design the optimal reserve price policy when bidders are asymmetric and studies the consequences of a symmetric misspecification for the seller's expected revenue. The two-step estimator is proposed in Section 4 and its asymptotic distribution is obtained. A simulation of the methodology is given in Section 5 and an empirical application using USFS timber ascending auctions is studied in Section 6. The proofs of all the results given in the paper are grouped in the Appendix A. Appendix B contains tables not displayed in the application section to save space.

2 Semiparametric quantile specifications

A single and indivisible object with observed characteristics $x \in \mathbb{R}^d$ is auctioned to $N \geq 2$ bidders through an ascending auction. Each bidder has a specific characteristic Z_i, $i = 1, \ldots, N$. The auction covariates x, the number of bidders N participating in the auction and the associated bidder covariates Z_i, $i = 1, \ldots, N$ are common knowledge to buyers and sellers, and observed by the analyst. Within the IPV paradigm, each bidder $i = 1, \cdots, N$ is assumed to have a private value V_i for the auctioned good, which is not observed by other bidders. The bidder only knows his own private value, but it is common knowledge for bidders and sellers that each private value has been independently drawn from a c.d.f. $F_i(\cdot|X, Z_i)$ conditional upon (X, Z_i), $X = (1, x')'$, or equivalently, with a conditional quantile function

$$V_i(\tau|X, Z_i) := F_i^{-1}(\tau|X, Z_i), \quad \tau \text{ in } [0, 1] . \tag{2.1}$$

It will be assumed later on that the analyst observes L identically drawn auctions. For each auction ℓ, the winning bid W_ℓ and winner's identity, the number N_ℓ of bidders, the good covariate X_ℓ and the bidder characteristics $Z_\ell = [Z_{1\ell}, \ldots, Z_{N\ell}]$ are observed. As shown later, the assumption that the identity of the winner is observed can be relaxed when bidders are characterized using discrete types. In this case, it is sufficient to observe the type of the winner and the numbers of bidders within a given type.

As in the symmetric private value setting, the dominant strategy for non-winners is to bid up to their true valuation. It will, therefore, be assumed that

Assumption 1 *The winning bid is the second-highest bidder's private value.*

See Haile and Tamer (2003), Brendstrup and Paarsch (2006), Aradillas-López et al. (2013), Coey et al. (2017), and Gimenes (2017) for similar assumptions and related discussions.

2.1 Asymmetric private value quantile specification

The proposed model combines an asymmetry function

$$\lambda_i = \lambda(Z_i; \alpha_i, \beta) > 0 \tag{2.2}$$

with a parent conditional distribution $F(\cdot|X)$ which only depends upon the good covariates and is generated by a quantile regression

$$F^{-1}(\tau|X) = X'\gamma(\tau) \tag{2.3}$$

assuming that the first entry of X is a constant term. In (2.2), the α_i are bidder fixed effects parameter which can capture some unobserved bidder heterogeneity. In what follows $\alpha = [\alpha_1, \ldots, \alpha_N]$.

Assumption 2 *Suppose (2.2) and (2.3) hold. There are some α, β and a vector function $\gamma(\cdot)$ such that*

$$F_i(\cdot|X, Z_i) = [F(\cdot|X)]^{\lambda_i} \tag{2.4}$$

for all admissible X, Z_i and all $i = 1, \ldots, N$.

Cantillon (2008) refers to distributions of the type of (2.4) as a class of distributions for which a quasi-ordering of potential bidders is available. This specification accommodates asymmetries that arise from merger, joint bidding or collusion among homogeneous bidders. See e.g. Graham and Marshall (1987), Mailath and Zemsky (1991), McAfee and McMillan (1992), Brannman and Froeb (2000) and Waehrer and Perry (2003).

Assumption 2 is equivalent to the quantile specification

$$V_i(\tau|X, Z_i) = X'\gamma\left[\tau^{1/\lambda(Z_i;\alpha_i,\beta)}\right] \tag{2.5}$$

which shows that asymmetry comes from a bidder specific transformation of the quantile level τ. As detailed below, the power specification is particularly convenient to establish identification. Examples of parametric $\lambda(Z_i;\alpha_i,\beta)$ are considered later on, but nonparametric specifications are also possible although not investigated here. The slope coefficient $\gamma(\cdot)$ is the nonparametric element of the model. It can, however, be estimated with a parametric rate as expected from the quantile regression and shown later on. The asymmetric power exponent $1/\lambda(Z_i;\alpha_i,\beta)$ measures the bidder strength: if $\lambda(Z_i;\alpha_i,\beta) > \lambda(Z_j;\alpha_j,\beta)$ then bidder i dominates bidder j in a first-order stochastic dominance sense, i.e. $F_i(\cdot|X,Z_i) \leq F_j(\cdot|X,Z_j)$ with a strict inequality inside the common support of these distribution. Note that the private value distributions have the same support $[V(0|X), V(1|X)]$ of the parent distribution. When $\lambda(Z_i;\alpha_i,\beta)$ goes to infinity, $V_i(\tau|X,Z_i)$ converges to $V(1|X)$ while it goes to $V(0|X)$ when $\lambda(Z_i;\alpha_i,\beta)$ goes to 0.

Additional standard assumptions on the parent quantile slope function $\gamma(\cdot)$ and the function $\lambda(\cdot;\cdot)$ are as follows. In the last assumption, Θ is the compact set of admissible asymmetry parameters (α,β) and $\mathcal{Z}$ is the compact support of the bidder characteristic Z_i.

Assumption 3 *The vector of auction specific variables, $X = [1, x_0']'$, has a dimension of $(d+1)\times 1$. The random vector x_0 has a compact support $\mathcal{X}_0 \subset (0,+\infty)^d$. The matrix $\mathbb{E}[XX']$ has an inverse.*

Assumption 4 *$V(\cdot|X)$ is continuously differentiable over $(0,1)$ with a derivative $V^{(1)}(\cdot|X)$ which is strictly positive for all X in $\mathcal{X} = \{1\}\times\mathcal{X}_0$.*

Assumption 5 *It holds $\inf_{(z,a,b)\in\mathcal{Z}\times\Theta}\inf_{1\leq i\leq N}\lambda(z;a_i,b) > 0$. The function $\lambda(z;a_i,b)$ is twice continuously differentiable with respect to a_i and b. The true value (α,β) of the asymmetry parameter lies in the interior of Θ.*

2.2 Identification

The proposed identification procedure is in two steps, which are constructive enough to develop a simple estimation procedure. The first step aims to identify the bidder asymmetry parameters α and β from the observed winner's identity.

Lemma 1 *Suppose Assumptions 1 and 2 hold. Then for any* $i = 1, \ldots, N$

$$\mathbb{P}\left(\text{Bidder } i \text{ wins the auction} \middle| X, Z\right) = \frac{\lambda(Z_i; \alpha_i, \beta)}{\sum_{j=1}^{N} \lambda(Z_j; \alpha_j, \beta)}. \tag{2.6}$$

Proof of Lemma 1. Let

$$G(w, i|X, Z) = \mathbb{P}\left(W \leq w \text{ and } i \text{ wins the auction} \middle| X, Z\right)$$

be the joint distribution of winning bids and winner's identity. Due to private value independence, it holds, as shown in Brendstrup and Paarsch (2006),

$$\begin{aligned}
G(w, i|X, Z) &= \mathbb{P}\left(\max_{1\leq j\neq i\leq N} v_j \leq w \text{ and } \max_{1\leq j\neq i\leq N} v_j \leq v_i \middle| X, Z\right) \\
&= \mathbb{P}\left(\max_{1\leq j\neq i\leq N} v_j \leq \min(w, v_i) \middle| X, Z\right) \\
&= \int_0^w \left\{\prod_{1\leq j\neq i\leq N} F_j(u|X, Z_j)\right\} dF_i(u|X, Z_i) + \left\{\prod_{1\leq j\neq i\leq N} F_j(w|X, Z_j)\right\}(1 - F_i(w|X, Z_i)) \\
&= \int_0^w (1 - F_i(u|X, Z_i))\, d\left\{\prod_{1\leq j\neq i\leq N} F_j(u|X, Z_j)\right\}
\end{aligned}$$

where the last line is obtained by integration by parts. Then, Assumption 2 gives

$$\begin{aligned}
G(w, i|X, Z) &= \int_0^w \left(1 - [F(u|X)]^{\lambda_i}\right) d\,[F(u|X)]^{\sum_{1\leq j\neq i\leq N} \lambda_j} \\
&= [F(w|X)]^{\sum_{1\leq j\neq i\leq N} \lambda_j} - \frac{\sum_{1\leq j\neq i\leq N} \lambda_j}{\sum_{j=1}^{N} \lambda_j} [F(w|X)]^{\sum_{j=1}^{N} \lambda_j}.
\end{aligned} \tag{2.7}$$

It follows

$$\begin{aligned}
&\mathbb{P}\left(\text{Bidder } i \text{ wins the auction} \middle| X, Z\right) = G(+\infty, i|X, Z) \\
&= 1 - \frac{\sum_{1\leq j\neq i\leq N} \lambda_j}{\sum_{j=1}^{N} \lambda_j} = \frac{\lambda_i}{\sum_{j=1}^{N} \lambda_j}. \qquad \square
\end{aligned}$$

Suppose that the equation system with unknowns a and b in Θ

$$\frac{\lambda(Z_i; a_i, b)}{\sum_{j=1}^{N} \lambda(Z_j; a_j, b)} = \frac{\lambda(Z_i; \alpha_i, \beta)}{\sum_{j=1}^{N} \lambda(Z_j; \alpha_j, \beta)}, \quad i = 1, \ldots, N, \tag{2.8}$$

has a unique solution, α and β. Then, Lemma 1 shows that the winner's identity distribution identifies the asymmetry parameters α and β. Identification on a case by case basis with examples of functions $\lambda(\cdot;\cdot,\cdot)$ and parameter set Θ ensuring identification of the asymmetry parameters is given in the next section. The probability of winning is very often used to assess the presence of asymmetry among the bidders, see Laffont, Ossard and Vuong (1995), Flambard and Perrigne (2006) for first-price sealed bid auctions and Brendstrup and Paarsch (2006) for ascending auctions.

Identification of the parent quantile regression slope $\gamma(\cdot)$ follows in a second step, using the winning bid c.d.f. given that bidder i wins the auction,

$$G(w|X, Z, i) = \frac{G(w, i|X, Z)}{G(+\infty, i|X, Z)}$$

where $G(w, i|X, Z)$ is defined in (2.7). Define[1]

$$\Psi_i(\tau; Z, \alpha, \beta) = \frac{\Lambda_N(Z; \alpha, \beta)\tau^{\Lambda_{N|i}(Z;\alpha,\beta)} - \Lambda_{N|i}(Z; \alpha, \beta)\tau^{\Lambda_N(Z;\alpha,\beta)}}{\lambda(Z_i; \alpha_i, \beta)} \tag{2.9}$$

where

$$\Lambda_N(Z; \alpha, \beta) = \sum_{j=1}^{N} \lambda(Z_j; \alpha_j, \beta),$$
$$\Lambda_{N|i}(Z; \alpha, \beta) = \Lambda_N(Z; \alpha, \beta) - \lambda(Z_i; \alpha_i, \beta).$$

Then (2.7) yields that the winning bid c.d.f. given winner's identity satisfies

$$G(w|X, Z, i) = \Psi_i[F(w|X); Z, N, \alpha, \beta].$$

[1]Note that $\Psi_i(\tau; Z, \alpha, \beta)$ should be written as $\Psi_i(\tau; Z, N, \alpha, \beta)$.

Assuming that $\Psi_i(\cdot; Z, \alpha, \beta)$ is strictly increasing, then, implies that the conditional winning bids quantile function $W(\tau|X, Z, i)$ given that i wins is

$$W(\tau|X, Z, i) = F^{-1}\left[\Psi_i^{-1}(\tau; Z, \alpha, \beta)|X\right] = X'\gamma\left[\Psi_i^{-1}(\tau; Z, \alpha, \beta)\right].$$

It follows that

$$W[\Psi_i(\tau; Z, \alpha, \beta)|X, Z, i] = X'\gamma(\tau). \tag{2.10}$$

Identification of $\gamma(\cdot)$ easily follows as stated in the next Proposition.

Proposition 2 *Suppose that Assumptions 1-5 hold, and that the asymmetry parameters (α, β) are identified. Then the parent slope function $\gamma(\cdot)$ is also identified.*

Proof of Proposition 2: See proof section. □

2.3 Identified bidder asymmetry specifications

Asymmetry parameter identification is a crucial condition, which holds for the following standard choice of the function $\lambda(\cdot; \cdot, \cdot)$ under proper standardization of the asymmetry parameter. For the third and fourth example given below, it is useful to assume that the bidder covariate $Z_{i\ell}$ varies across auctions.

Example 1: Bidder fixed effects. In this example $\lambda(Z_i; \alpha_i, \beta) = \alpha_i$, and (2.8) shows that asymmetry parameter identification holds provided the system of equations with unknown $a = [a_1, \ldots, a_N]$ in Θ

$$\frac{a_i}{\sum_{j=1}^N a_j} = \frac{\alpha_i}{\sum_{j=1}^N \alpha_j}, \quad i = 1, \ldots, N,$$

has a unique solution. As well known, this is ensured when

$$\Theta = \left\{a \in \mathbb{R}_{+*}^N | a_1 = 1\right\}$$

that is, the parent private value distribution is the first bidder private value distribution.[2]

Alternatively the simplex $\Theta = \left\{ a \in \mathbb{R}^N_{+*} | \sum_{i=1}^N a_i = 1 \right\}$ is also possible. $\square$

Example 2: Linear regression. The case of the regression specification $\lambda(Z_i; \alpha_i, \beta) = Z_i'\beta$ is particularly useful when the covariate Z_i codes bidder types.[3] An example of continuous Z_i is provided by construction procurement, where Z_i can group the bidder's distance to the construction site and her capacity. When $\beta \neq 0$, (2.8) gives the system

$$\frac{Z_i'b}{\sum_{j=1}^N Z_j b} = \frac{Z_i'\beta}{\sum_{j=1}^N Z_i'\beta}, \quad i = 1, \ldots, N,$$

which is equivalent to $Z_i'bZ_j'\beta = Z_i'\beta Z_j'b$ or $b'Z_iZ_j'\beta = \beta'Z_iZ_j'b$ for all bidder pair $\{i, j\}$. If the range of Z_iZ_j' has a non-empty interior, differentiating with respect to the entries of this matrix gives $b'\beta = \beta'b$, which implies $b_{p_1}\beta_{p_2} = \beta_{p_1}b_{p_2}$, for any pair (p_1, p_2). Hence, β is identified up to a multiplicative constant and imposing that the first entry of β is 1 or that $\beta'\beta = 1$ ensures identification. $\square$

Example 3: Linear regression with bidder fixed effects. The case of $\lambda(Z_{i\ell}; \alpha_i, \beta) = \alpha_i + Z_{i\ell}'\beta$ can be dealt as in Example 2, augmenting $Z_{i\ell}$ to code bidder identities. $\square$

Example 4: Exponential linear regression with bidder fixed effects. When the $Z_{i\ell}$ entries can take negative values, a possible choice of the positive function $\lambda(\cdot; \cdot, \cdot)$ is $\lambda(Z_{i\ell}; \alpha_i, \beta) = \alpha_i \exp(Z_{i\ell}'\beta)$. For this choice (2.8) implies

$$\frac{a_j \exp\left(Z_{j\ell}'b\right)}{a_i \exp\left(Z_{i\ell}'b\right)} = \frac{\alpha_j \exp\left(Z_{j\ell}'\beta\right)}{\alpha_i \exp\left(Z_{i\ell}'\beta\right)}, \quad 1 \leq i, j \leq N.$$

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[2]It is however possible to identify α_1, strengthening Assumption 4 to ensure $V^{(1)}(\cdot)$ exists and is strictly positive over $[0, 1]$. If so, $V(\tau) - V(0)$ is equivalent to $V^{(1)}(0)\tau^{1/\alpha_1}$, showing that α_1 is identified from the lower tail of the identified parent private value distribution. Implementing this in practice may however give nonparametric consistency rates and is therefore not attempted here.

[3]Alternatively, a fixed effects specification as in Example 1 can be used provided the fixed effects α_i can only take K unknown values $\mu_1, \ldots, \mu_K$, where K is the number of types.

Taking $Z_{i\ell} = Z_{j\ell}$ shows that the fixed effects α_i are identified up to scale, so that restricting to the parameter spaces of Example 1 yields identification of α. (2.8) then becomes

$$(Z_{i\ell} - Z_{j\ell})' (b - \beta) = 0, \quad 1 \leq i, j \leq N$$

which, provided $\text{Var}(Z_{i\ell} - Z_{j\ell})$ is full rank for some (i, j), ensures identification of β. □

3 Seller revenue and asymmetry misspecification

The proposed specification is convenient to compute and analyze the seller revenue. The presence of a reserve price $R = R(X, Z, V_0)$, where V_0 is the seller private value, requires changing Assumption 1 into

Assumption 6 *There is no transaction if all private values are below the reserve price. Otherwise, the winning bid is the greater of the second-highest bidder's private values and the reserve price.*

For a reserve price in the common support $[V(0|X), V(1|X)]$, consider the quantile level $r = r(X, Z, V_0) = F(R|X)$ of R in the parent distribution. Under Assumption 4 it therefore holds that $R = V(r|X)$. It is convenient to abbreviate $\lambda(Z_i; \alpha_i, \beta)$, $\Lambda_N(Z; \alpha, \beta)$, $\Lambda_{N|i}(Z; \alpha, \beta)$ into λ_i, Λ_N and $\Lambda_{N|i}$, respectively. The seller payoff in an auction with reserve price R is

$$\pi(r) = W\mathbb{I}(W \geq R) + V_0\mathbb{I}(W < R),$$

where W is the winning bid. The corresponding expected seller revenue is

$$\Pi(r|X, Z, V_0) = \mathbb{E}[\pi(r)|X, Z, V_0].$$

3.1 Expected revenue and optimal reserve price

The next Proposition gives a quantile expression for the expected revenue and characterizes the optimal reserve prices. Let $\Lambda_N = \Lambda_N(Z; \alpha, \beta)$ and $\Lambda_{N|i} = \Lambda_{N|i}(Z; \alpha, \beta)$ be as (2.9).

Proposition 3 *Suppose Assumptions 2, 4 and 6 hold. Then*

(i) The probability of selling the auctioned good is $(1 - r^{\Lambda_N})$.

(ii) The seller expected payoff is

$$\Pi(r|X, Z, V_0) = V_0 r^{\Lambda_N} + R \sum_{i=1}^{N} r^{\Lambda_{N|i}} \left(1 - r^{\lambda_i}\right) + \int_r^1 V(t|X) \left\{ (1 - N) \Lambda_N t^{\Lambda_N - 1} + \sum_{i=1}^{N} \Lambda_{N|i} t^{\Lambda_{N|i} - 1} \right\} dt. \tag{3.1}$$

(iii) The optimal reserve price $R_* = V(r_*|X)$ *satisfies*

$$V_0 = R_* - V^{(1)}(r_*|X) \frac{r_*}{\Lambda_N} \sum_{i=1}^{N} \left(r_*^{-\lambda_i} - 1\right). \tag{3.2}$$

Proof of Proposition 3: See proof section. □

Proposition 3 allows to estimate the seller expected revenue by replacing the asymmetry coefficients λ_i and the parent private value quantile function $V(\cdot|\cdot)$ by their estimates. The optimal reserve price can be obtained solving an estimation of (3.2) but performing a numerical maximization of the estimation of $\Pi(r)$ can be preferred to avoid the estimation of the derivative $V^{(1)}(\cdot|\cdot)$.

Compared to the case of symmetric bidders, Proposition 3-(ii) shows that the optimal reserve price depends upon the number N of bidders and upon the bidder characteristics. The impact of the asymmetry coefficients λ_i on the expected seller revenue and on the optimal reserve price seems unclear. For $\Pi(r)$, the ambiguity is due to the term $-r^{\Lambda_{N|i}+\lambda_i}$ which increases with λ_i, while the other terms decrease. Observe similarly that, in (3.2), $1/\Lambda_N$ decreases with λ_i while $r^{-\lambda_i}$ increases. Cantillon (2008, Theorem 2) gives condition that allows to rank two sets of asymmetry coefficients λ_i according to seller revenue.

In many cases, the seller must decide a reserve price before observing the number N of bidders and the asymmetry parameter of the entrants. The expected revenue formula (3.1) is conditional on N and on the asymmetry parameters of the entrant, an information which is not available but can be integrated out to produce a relevant expected revenue and optimal reserve price.

3.2 The effect of a symmetric misspecification

To analyse the effect of a symmetric misspecification on the optimal reserve price and seller revenue, we perform a numerical experiment with no covariate and two asymmetric bidders with private values $F_i(v) = (v^\kappa)^{\lambda_i}$, $0 \leq v \leq 1$ and $i = 1, 2$. Higher κ gives private values closer to 1 and values of the curvature parameter κ ranging from 1 to 50 are considered. High and moderate asymmetry scenarios, with (λ_1, λ_2) set to $(0.1, 3.9)$ and $(0.1, 0.9)$ respectively are considered.

To evaluate the effect of estimating a symmetric misspecified model, we derive the limiting symmetric private value distribution by matching the distribution of winning bids with the symmetric winning bid distribution. Under Assumption 1, the winning bid is equal to the minimum between (V_1, V_2), therefore, the winning bids distribution is

$$F_{\lambda,W}(w) = \mathbb{P}(\min(V_1, V_2) \leq w) = w^{\kappa\lambda_1} + w^{\kappa\lambda_2} - w^{\kappa(\lambda_1+\lambda_2)}, \quad w \in [0,1].$$

For symmetric bidders, the function $\Psi_i(\cdot)$ does not depend upon i and is equal to

$$\Psi(\tau) = 2\tau - \tau^2 = 1 - (1-\tau)^2.$$

Therefore, the symmetric private value c.d.f. $F_{\lambda,S}(\cdot)$ which generates the winning bid distribution $F_{\lambda,W}(\cdot)$ must satisfy $\Psi[F_{\lambda,S}] = F_{\lambda,W}$ so that

$$F_{\lambda,S}(v) = 1 - \left(1 - v^{\kappa\lambda_1} - v^{\kappa\lambda_2} + v^{\kappa(\lambda_1+\lambda_2)}\right)^{1/2}, \quad v \in [0,1].$$

The c.d.f. $F_{\lambda,S}(v)$ is the limit of any nonparametric estimator obtained by matching the winning bid distribution of a misspecified symmetric bidder model with the observed one, see for instance Gimenes (2017). An optimal reserve price assuming symmetric bidders, $R_{\lambda,S} = V_{\lambda,S}(r_{\lambda,S})$ where $V_{\lambda,S}(\cdot) = F^{-1}_{\lambda,S}(\cdot)$, solves the symmetric version of (3.2)

$$R_{\lambda,S} - V^{(1)}_{\lambda,S}(r_{\lambda,S})(1 - r_{\lambda,S}) = 0$$

where the seller private value V_0 is set to 0 for the sake of simplicity. The expected seller revenue achieved with $R_{\lambda,S}$ under the true asymmetric private values distribution can then be computed using (3.1). The reserve price $R_{\lambda,S}$ and the corresponding expected seller revenue are reported in the columns labeled "Misspecified" of the next table. The optimal reserve price and seller revenue using the true private value distribution are reported under the label "Asymmetry".

Table 1: Misspecified symmetric versus true asymmetric models

			Optimal Reserve Price		Expected Seller Revenue		
λ_1	λ_2	κ	Asymmetry	Misspecified	Asymmetry	Misspecified	Percentage Loss
0.1	3.9	1	0.6630	0.5451	0.5389	0.5059	6.12%
		2	0.7550	0.5995	0.6800	0.6054	10.97%
		5	0.8558	0.6403	0.8223	0.6738	18.06%
		10	0.9092	0.6671	0.8927	0.7230	19%
		50	0.9730	0.7785	0.9707	0.7173	26.10%
0.1	0.9	1	0.4830	0.4420	0.2550	0.2535	0.59%
		2	0.5559	0.4901	0.3948	0.3887	1.55%
		5	0.6768	0.5773	0.5987	0.5767	3.67%
		10	0.7676	0.6450	0.7336	0.6930	5.53%
		50	0.8710	0.7785	0.9283	0.7148	23%

Table 1 reveals that ignoring asymmetry can lead to substantial loss in terms of seller revenue, when the curvature parameter κ is high or in the high asymmetry scenario. Note that the optimal expected revenue computed under the misspecified symmetric model is always smaller than the one achieved with the correct asymmetric model. The optimal reserve price is also substantially higher in the correct model with strong asymmetry.

The analysis presented here differs from Cantillon (2008)'s, who studies how presence of asymmetry impacts seller expected revenue. She compares auctions with asymmetric private value distributions with a symmetric benchmark, and finds that asymmetry is associated with lower expected revenue. Our comparison differs from hers in that we investigate what happens if a truly asymmetric model is *misspecified* as symmetric - in which case, we find that not accounting for asymmetry may lead to loss in revenue. In that sense, our analysis supports and extends her findings. To see this, consider different scenarios of asymmetry such that $\lambda_1 + \lambda_2 = 1$, with $\kappa = 1$, as shown in Table 2.

Table 2: Varying asymmetry levels

		Optimal Reserve Price		Expected Seller Revenue		
λ_1	λ_2	Asymmetry	Misspecified	Asymmetry	Misspecified	Percentage Loss
0.1	0.9	0.4830	0.4420	0.2550	0.2535	0.59%
0.2	0.8	0.4680	0.4433	0.2593	0.2590	0.14%
0.3	0.7	0.4550	0.4442	0.2627	0.2627	0.003%
0.4	0.6	0.4470	0.4440	0.2648	0.2648	0.0003%
0.5	0.5	0.4440	0.4449	0.2655	0.2655	0.00%

As can be seen, higher asymmetry when (λ_1, λ_2) is (0.1,0.9) has lesser revenue than the symmetric case of (0.5,0.5), supporting Cantillon's finding that "the expected revenue is lower the more asymmetric bidders are". However, given ex-ante asymmetry among bidders, a misspecified symmetric model always has smaller expected revenue and higher the asymmetry, more is the potential loss in revenue due to misspecification.

4 Estimation and asymptotic inference

Suppose that for each auction ℓ, the analyst observes the winning bid W_ℓ, the good covariate X_ℓ, the number of bidders N_ℓ, the bidder covariate Z_ℓ and the identity I_ℓ^* of the winner. The proposed estimation method is in two step and closely follows the identification procedure. As stated in Lemma 1, the probability that bidder i wins is

$$\pi\left(i|Z_\ell, N_\ell, \alpha, \beta\right) = \frac{\lambda(Z_{i\ell}; \alpha_i, \beta)}{\sum_{j=1}^{N_\ell} \lambda(Z_{j\ell}; \alpha_j, \beta)}$$

so that the asymmetry parameter (α, β) can be estimated using the maximum likelihood estimator

$$\left(\widehat{\alpha}, \widehat{\beta}\right) = \arg \max_{(\alpha,\beta)\in\Theta} \sum_{\ell=1}^{L} \ln \pi \left(I_\ell^* | Z_\ell, N_\ell, \alpha, \beta\right). \tag{4.1}$$

The second step consists in the estimation of the parent quantile slope and is based upon (2.10), which identifies $\gamma(\cdot)$ as shown in Proposition 2. Define, for $\Psi_i(\tau; Z, N, \alpha, \beta)$ as in (2.9),

$$\widehat{\Phi}_\ell(\tau) = \Phi_\ell\left(\tau; \widehat{\alpha}, \widehat{\beta}\right) = \Psi_{I_\ell^*}\left(\tau; Z_\ell, N_\ell, \widehat{\alpha}, \widehat{\beta}\right).$$

The quantile level $\widehat{\Phi}_\ell(\tau)$ is an estimation of the (random) quantile level $\Psi_{I_\ell^*}(\tau; Z_\ell, \alpha, \beta)$ which is such that the quantile function of the winning bid given X_ℓ, Z_ℓ, N_ℓ and I_ℓ^* satisfies

$$W\left[\Psi_{I_\ell^*}(\tau; Z_\ell, \alpha, \beta) | X_\ell, Z_\ell, N_\ell, I_\ell^*\right] = X_\ell' \gamma(\tau),$$

(see (2.10)). It suggests the quantile regression estimator

$$\widehat{\gamma}(\tau) = \arg \min_{\gamma} \sum_{\ell=1}^{L} \rho_{\widehat{\Phi}_\ell(\tau)}\left(W_\ell - X_\ell' \gamma\right) \tag{4.2}$$

where $\rho_\Phi(u) = u(\Phi - \mathbb{I}(u < 0))$, see e.g. Koenker (2005).

While the first step is a standard MLE estimator, the second step slope estimator involves a random quantile level, which in addition must be estimated. As well known since Murphy and Topel (1985), the first step estimation can affect the second step asymptotic property. However, this can easily be captured using the proof techniques in Pollard (1991). Useful assumptions and notations are as follows. In the sequel, (α, β) will be abbreviated in θ when convenient. Let $\pi^\theta(i|Z_\ell, N_\ell, \theta)$ be θ derivative of $\pi(i|Z_\ell, N_\ell, \theta)$. Under Assumption 5, the Fisher information matrix for the asymmetry parameters can be defined as

$$\mathcal{I}(\theta) = \text{Var}\left(\frac{\pi^\theta\left(I_\ell^* | Z_\ell, N_\ell, \theta\right)}{\pi\left(I_\ell^* | Z_\ell, N_\ell, \theta\right)}\right)$$

or by using the Bartlett identity when Z_ℓ has a compact support as assumed below.

Assumption 7 *The auction variables* $(X_\ell, N_\ell, Z_\ell, I_\ell^*, W_\ell)$ *are drawn identically and independently. The support* $\mathcal{Z}$ *of* Z_ℓ *is compact.*

Assumption 8 *The identification equations (2.6) and (2.10) hold. The asymmetry parameter are identified and the Fisher information matrix* $\mathcal{I}(\theta)$ *has an inverse.*

Assumption 7 is standard. Assumption 8 imposes that the auction model is correctly specified and that the asymmetry parameter are identified.

Consider now some additional notations for the second step estimator $\widehat{\gamma}(\tau)$. Let the τ derivative of $\Psi_{I_\ell^*}(\tau; Z_\ell, N_\ell, \theta)$ be denoted by $\Psi_{I_\ell^*}^\tau(\tau; Z_\ell, N_\ell, \theta)$, which exists and is strictly positive on $(0,1)$ as shown in the proof of Proposition 2. As the conditional quantile function of the winning bid is

$$W(\tau | X_\ell, Z_\ell, I_\ell^*, N_\ell) = X_\ell' \gamma \left[\Psi_{I_\ell^*}^{-1}(\tau; Z_\ell, N_\ell, \theta) \right]$$

the conditional p.d.f. of the winning bid is, under Assumption 4,

$$\begin{aligned} f_W(w | X_\ell, Z_\ell, I_\ell^*, N_\ell) &= \frac{1}{W^{(1)}\left[W^{-1}(w | X_\ell, Z_\ell, I_\ell^*, N_\ell) | X_\ell, Z_\ell, I_\ell^*, N_\ell\right]} \\ &= \frac{\Psi_{I_\ell^*}^\tau\left(\Psi_{I_\ell^*}^{-1}\left(W^{-1}(w | X_\ell, Z_\ell, I_\ell^*, N_\ell); Z_\ell, N_\ell, \theta\right); Z_\ell, N_\ell, \theta\right)}{X_\ell' \gamma^{(1)}\left[\Psi_{I_\ell^*}^{-1}\left(W^{-1}(w | X_\ell, Z_\ell, I_\ell^*, N_\ell); Z_\ell, N_\ell, \theta\right)\right]} \end{aligned}$$

which is continuous and bounded away from infinity over $(V(0|X_\ell), V(1|X_\ell))$. Let the θ

derivative of $\Psi_{I_\ell^*}(\tau; Z_\ell, N_\ell, \theta)$ be denoted by $\Psi^\theta_{I_\ell^*}(\tau; Z_\ell, N_\ell, \theta)$ and define

$$
\begin{aligned}
H(\tau) &= \mathbb{E}\left[X_\ell X_\ell' f_W\left(X_\ell'\gamma(\tau) \,|X_\ell, Z_\ell, I_\ell^*, N_\ell\right)\right], \\
J(\tau) &= \mathbb{E}\left[X_\ell X_\ell' \left(\mathbb{I}\left(W_\ell \le X_\ell'\gamma(\tau)\right) - \Psi_{I_\ell^*}(\tau; Z_\ell, N_\ell, \theta)\right)^2\right] \\
C(\tau) &= \mathbb{E}\left[\left(X_\ell\left(\mathbb{I}\left(W_\ell \le X_\ell'\gamma(\tau)\right) - \Psi_{I_\ell^*}(\tau; Z_\ell, N_\ell, \theta)\right)\right)\mathcal{I}^{-1}(\theta)\left(\frac{\pi^\theta\left(I_\ell^*|Z_\ell, N_\ell, \theta\right)}{\pi\left(I_\ell^*|Z_\ell, N_\ell, \theta\right)}\right)'\right], \\
D(\tau) &= -\mathbb{E}\left[\Psi^\theta_{I_\ell^*}(\tau; Z_\ell, N_\ell, \theta)\, X_\ell'\right].
\end{aligned}
$$

The matrices $H(\tau)$ and $J(\tau)$ are specific to the infeasible quantile regression estimator $\widetilde{\gamma}(\tau)$ of $\gamma(\tau)$ which uses the true asymmetry parameters (α, β) instead of their estimates,

$$
\widetilde{\gamma}(\tau) = \arg\min_{\gamma} \sum_{\ell=1}^{L} \rho_{\Psi_{I_\ell^*}(\tau; Z_\ell, N_\ell, \theta)}\left(W_\ell - X_\ell'\gamma\right).
$$

In particular, $H^{-1}(\tau) J(\tau) H^{-1}(\tau)$ is the asymptotic variance of $\widetilde{\gamma}(\tau)$, see Koenker (2005). The matrix $C(\tau)$ is the asymptotic covariance of the infeasible $\widetilde{\gamma}(\tau)$ and $\left(\widehat{\alpha}, \widehat{\beta}\right)$. Finally

$$
D(\tau) = \frac{\partial}{\partial\theta\partial\gamma'}\mathbb{E}\left[\rho_{\Psi_{I_\ell^*}(\tau; Z_\ell, N_\ell, \theta)}\left(W_\ell - X_\ell'\gamma(\tau)\right)\right]
$$

is the $\theta\gamma$ derivative of the population version of the objective function which is used for $\widetilde{\gamma}(\tau)$.

The asymptotic variance of the asymmetry parameter estimator $\left(\widehat{\alpha}, \widehat{\beta}\right)$ and of the feasible $\widehat{\gamma}(\tau)$ is given by the matrices $\mathcal{I}^{-1}(\theta)$ and

$$
\begin{aligned}
V(\tau) &= H^{-1}(\tau)\left\{J(\tau) + D(\tau)\mathcal{I}^{-1}(\theta)D(\tau)' + D(\tau)C(\tau)' + C(\tau)D(\tau)'\right\}H^{-1}(\tau) \\
C_{\gamma\theta}(\tau) &= H^{-1}(\tau)\left\{-C(\tau) - D(\tau)\mathcal{I}^{-1}(\theta)\right\}
\end{aligned}
$$

The next Theorem gives the asymptotic joint distribution of $\widehat{\gamma}(\tau)$ and $\left(\widehat{\alpha}, \widehat{\beta}\right)$.

Theorem 4 *Suppose Assumptions 2-5, 7 and 8 hold. Then, for any quantile level* τ *in* $(0, 1)$,

$\widehat{\gamma}(\tau)$ *and* $\widehat{\theta} = \left(\widehat{\alpha}', \widehat{\beta}'\right)'$ *are asymptotically normal with*

$$\sqrt{L}\left(\left(\widehat{\gamma}(\tau) - \gamma(\tau)\right)', \left(\widehat{\theta} - \theta\right)'\right)' \xrightarrow{d} \mathcal{N}\left(0, \begin{bmatrix} V(\tau) & C_{\gamma\theta}(\tau) \\ C_{\gamma\theta}(\tau)' & \mathcal{I}(\theta)^{-1} \end{bmatrix}\right).$$

Proof of Theorem 4: See proof section. □

While the asymptotic normality of the MLE $\widehat{\theta}$ is standard, the one of $\widehat{\gamma}(\tau)$ follows from modifying the approach of Pollard (1991) to account for the first step estimation. The asymptotic variance of these estimators can be estimated but it may be more suitable to rely on bootstrap, especially for the parent slope function $\gamma(\cdot)$. Indeed, bootstrap is more reliable for inference in quantile regression, see Koenker (2005) and the reference therein.

5 Simulations

In this section, we present the results of a Monte Carlo simulation designed to evaluate the performance of the two-step estimation procedure.

Data Generating Process. We simulate $L = 2000$ ascending auctions with $N = 5$ bidders assigned to $K = 2$ different classes: type 1 and type 2 with $\lambda_1 = 1$ and $\lambda_2 = \exp(2) = 7.39$. Bidders are assigned to each type with equal probability. Auction specific characteristics x_ℓ is a random draw from $\mathcal{U}_{[1,3]}$, for $\ell = 1 \ldots L$, with an expected value of $E[x_\ell] = 2$ and $X_\ell = [1, x_\ell]$. The parent private value conditional quantile function is generated as

$$V(\tau|X_\ell) = X_\ell'\gamma(\tau) = \gamma_0(\tau) + \gamma_1(\tau)x_\ell, \tag{5.1}$$

where the true quantile regression coefficients are

$$\gamma_0(\tau) = \tau^{\exp(1.5)}/2; \qquad \gamma_1(\tau) = \tau^{\exp(1.5)}/4.$$

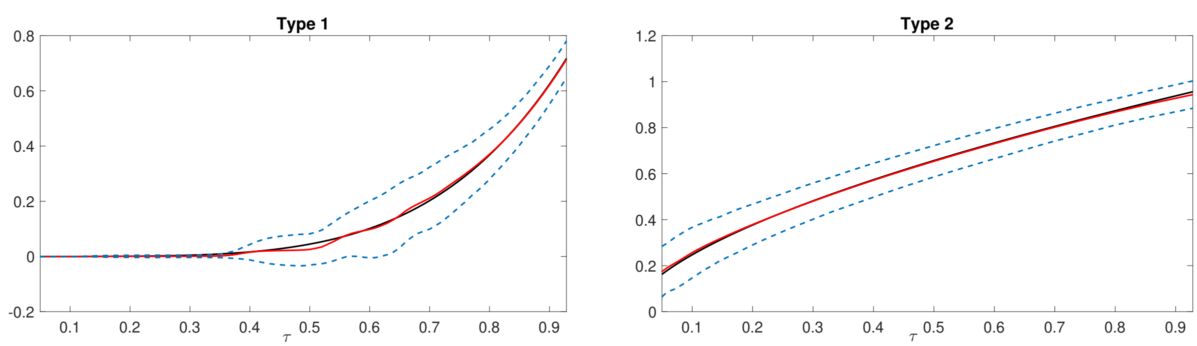
The number of simulation replications is set to 1000.

Estimation. The estimation is conducted in two steps. In the first step, the type parameters (λ_1, λ_2) are estimated using maximum likelihood estimation by maximizing (4.1) over a grid of points. The quantile regression slope $(\gamma_0(\tau), \gamma_1(\tau))$ are then estimated in a second step using (4.2). For the median x_ℓ, the estimated parent quantile function is given by $\widehat{V}(\tau|X_\ell) = \widehat{\gamma}_0(\tau) + 2\widehat{\gamma}_1(\tau)$.

Results. Figure 1 compares the true private value quantile function (in black) with the mean of the estimated private value quantile function across simulations (in red) for both type 1 and type 2, considering a median x_ℓ auction. The bias and standard error (SE) for the private value quantile function for both types is reported in Table 3. The simulation results confirm that the two step estimation procedure works well.

Figure 1: Simulation: True vs. Estimated Private Value Quantile Function

Median Auction



True in black, mean of estimation across simulations in red, 95% confidence intervals in dotted lines

6 Application

In this section, we investigate asymmetry in timber auctions from the USFS using the methodology developed in this paper. Bidders are classified as mill (with manufacturing capacity to process the timber) and logger (lacking manufacturing capabilities). For sim-

Table 3: Simulation: Bias and SE of Private Value Quantile Function

	Type 1		Type 2	
τ	Bias	SE	Bias	SE
0.1	0.0000	0.0000	0.0092	0.0560
0.2	−0.0003	0.0019	0.0020	0.0460
0.3	−0.0037	0.0022	−0.0013	0.0401
0.4	−0.0010	0.0143	−0.0023	0.0474
0.5	−0.0192	0.0288	−0.0028	0.0348
0.6	−0.0032	0.0526	−0.0030	0.0335
0.7	0.0091	0.0574	−0.0033	0.0309
0.8	0.0024	0.0460	−0.0053	0.0291
0.9	−0.0026	0.0357	−0.0103	0.0300

plicity in the exposition, mills and loggers are abbreviated by M and L, respectively, when convenient. The dataset aggregates 7,462 ascending auctions (i.e., winning bids) that occurred in the western part of the US during the period of 1982-90. The sample contains a set of variables characterizing each timber tract including the estimated volume of the timber (measured in thousand of board feet - mbf) and its estimated appraisal value (given in Dollar per unit of volume). Mills won in about 72% of the auctions. The descriptive statistics can be found in Table 4. The auctioned tract exhibits significant heterogeneity in quality and size. The contracts to extract the timber last, on average, 2 years. Bidders participation is high. On average, there are 6 bidders attending the auctions in a range of 2 to 12.

In what follows, a median auction is an auction where the appraisal value and the volume are set to their median value. With the exception of Figure 3, all the figures and tables of this section and Appendix B are for a median auction.

6.1 Private value quantile functions

We use a type fixed effect specification for the asymmetry parameter $\lambda_{i\ell}$, with $\lambda_{i\ell} = \lambda_M$ if bidder i at auction ℓ is a mill and $\lambda_{i\ell} = \lambda_L$ if it is a logger. For identification, we normalize $\lambda_M = 1$. The first step estimation gives $\widehat{\lambda}_L = 0.6988$ with a 95% confidence

Table 4: Descriptive Statistics

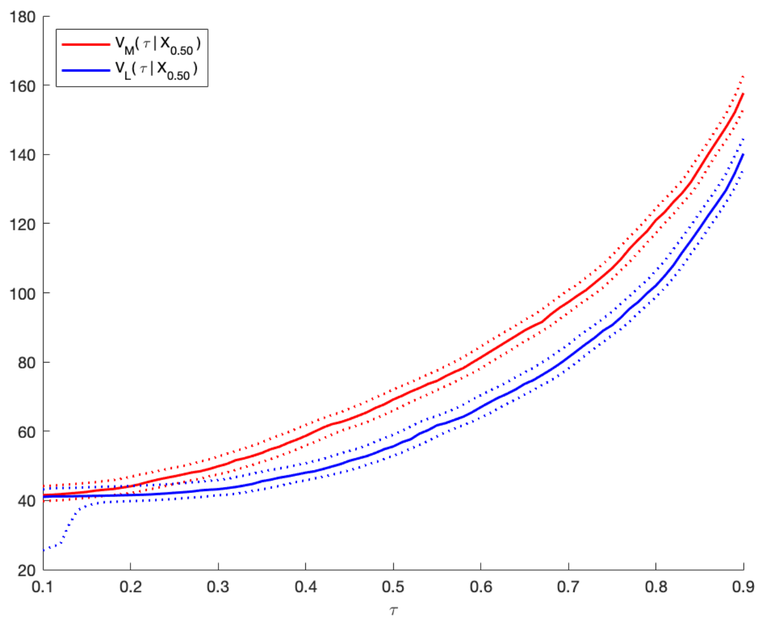
	Mean	Std. Dev.	Max	Min
Winning bids ($ per tbf)	126.43	136.22	5,145.71	0.14
Appraisal value ($ per tbf)	58.65	60.35	793.62	0.25
Volume (tbf)	4,466.89	4,418.41	39,920	8
Contract Length (years)	1.96	1.3	42	0.1
Number of bidders	5.77	3.09	12	2
Number of loggers	1.74	2.10	11	0
Number of mills	4.03	3.02	12	0
Bidders in the winner's class	4.52	2.73	12	1

interval computed by pairwise bootstrap given by $[0.6516, 0.7554]$, which shows that loggers are indeed significantly weaker than mills. In particular, the logger winning probability is 41.1% when each type are in equal proportions, 70% of the probability that a mill wins the ascending auction, which is 58.8%.

This is confirmed by Figure 2, which gives the estimated private value quantile functions of mills (red) and loggers (blue) and their 95% confidence bands computed via pairwise bootstrap method for a median auction. The private value conditional distribution of mills first-order stochastically dominates the one of loggers, especially in the upper part of the distribution.

Figure 2: Private Value Conditional Quantile Function of Loggers and Mills

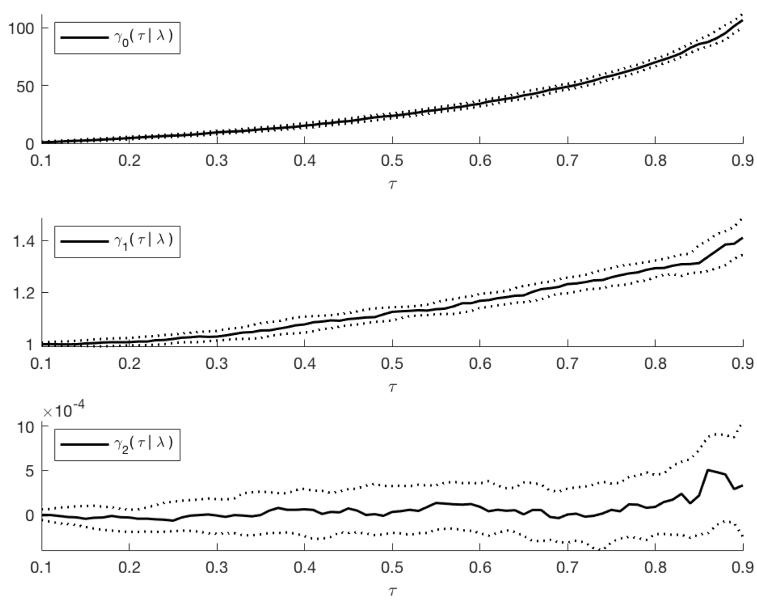
Median Auction



The 95% confidence intervals for the quantile regression estimates were computed by resampling with replacement the (X_ℓ, W_ℓ)-pair. Mills in red and loggers in blue.

The power specification of the private value quantile functions allows to highlight which variable generates asymmetry. Indeed, a constant slope function in the parent private value quantile regression means that the impact of the associated variable is identical for each type of bidders. Figure 3 gives the quantile regression coefficients of the private value parent distribution. The estimated volume slope function looks constant, and possibly not significant. As the power transformation will not make bidders to differ in terms of volume slope functions, this suggests that capacity constraint is not binding for both types. In contrast, the parent appraisal value slope function does not look constant, and this variable is much likely to generate differences across mills and loggers. Figure 3, therefore, suggests that asymmetry is driven by qualitative (e.g. ability to improve on the appraisal value of the timber) and unobserved factors (captured by the intercept), instead of capacity constraints. Interestingly, coping for asymmetry gives appraisal value slope estimated functions that vary much less across quantile levels than in Gimenes (2017).

Figure 3: Private Value Parent Quantile Regression Coefficients



The 95% confidence intervals for the quantile regression estimates were computed by resampling with replacement the (X_ℓ, W_ℓ)-pair. Top intercept, middle appraisal value and bottom volume estimated slope functions.

6.2 Expected revenue and optimal reserve price

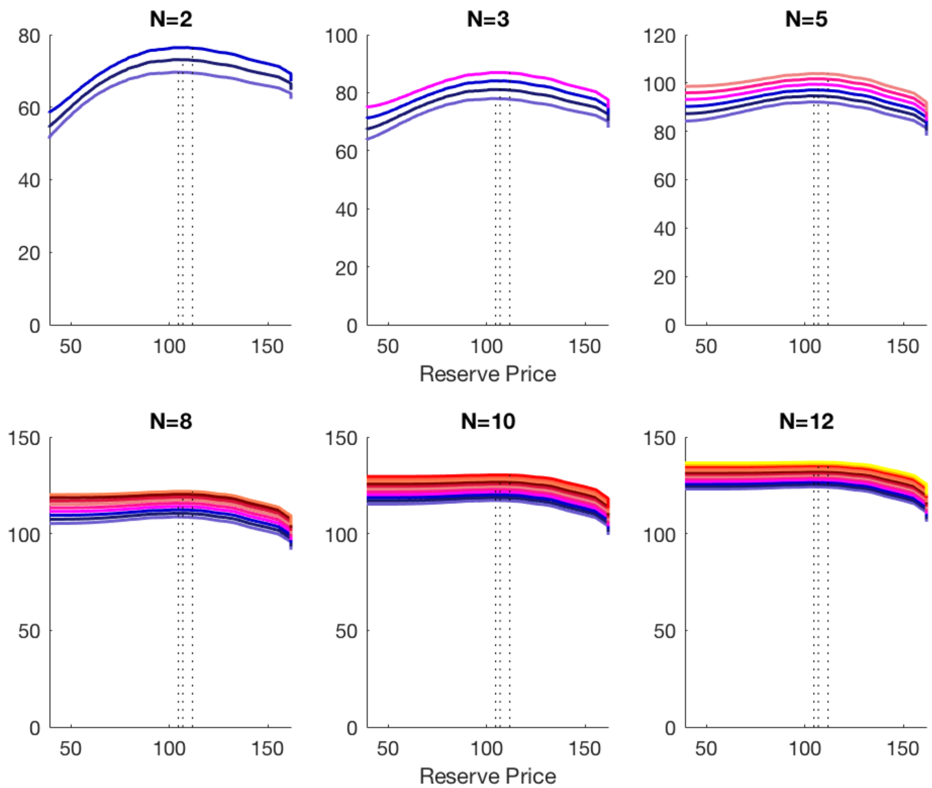
We now investigate the effect of asymmetry on the seller's expected revenue and optimal reserve price. Given that we recover all the primitives of the game, we can evaluate the seller expected revenue as the proportion of types changes. This contrasts with Coey et al. (2017) who averages over the type proportion. In sections 6.2 and 6.3, the seller's outside option value V_0 appearing in Proposition 3 is set to 0. The expected revenue can be estimated from (3.1) by plugging in the estimation $\widehat{V}(\cdot|X)$ of the parent quantile regression. Plotting $r \in [0,1] \mapsto \left(\widehat{V}(r|X), \widehat{\Pi}(r|X)\right)$ gives a graph of the estimated seller's expected revenue achieved with a reserve price $R = \widehat{V}(r|X)$.

Figure 4 shows estimates of the expected revenue as a function of the reserve price for each N and type proportion. The dotted vertical lines give the optimal reserve price for each proportion of types. As the colors of the curves become warmer (from blue to red and yellow), loggers are replaced by mills and the revenue level increases in a parallel way. The expected revenue functions have clear maximas for small numbers of bidders (typically $N = 2$ or $N = 3$), contrasting with the estimation obtained with symmetric bidders in Gimenes and Guerre (2019). For larger N, the expected revenues look flat in their central part, a fact which cannot be seen from the estimation set strategy of Coey et al. (2017).

As a consequence, implementing an optimal reserve price is mostly useful when the probability of observing a small number of bidders is high. The optimal reserve prices shown in Figure 4 and detailed in the Appendix Table B.3 depend upon N and type proportion, but exhibit a moderate 7% variation, staying in the interval $[104.7, 111.9]$ and slightly increasing with the number of mills. As the expected revenues are flat around their maxima, using a reserve price in the range $[104.7, 111.9]$ gives an expected revenue close to its maxima. This includes the optimal reserve price 107.9$ estimated from a symmetric specification, as in Gimenes (2017), given in B.3. As the expected revenue with no reserve price is mostly below 100$ when $N \leq 5$ as seen from Table 5 below, using such a reserve price may mean not selling the auctioned lot if a small number of bidders participates.[4]

[4]To see this, observe that the probability of selling is the probability that the maximum private value

Figure 4: Strategical Expected Revenue and Optimal Reserve Price



6.3 Type variation and additional bidder effects

Table 5: Non Strategical Expected Revenue

	Min ER	Max ER	Max %Δ	One logger replaced by one mill [Min %, Max %]
$N = 2$	48.02	57.65	19.61%	[8.84%, 9.89%]
	[46.37, 50.42]	[56.20, 59.70]		
$N = 3$	63.59	75.04	18.01%	[5.42%, 5.83%]
	[61.61, 66.03	[73.30, 77.47]		
$N = 5$	84.25	98.64	17.08%	[2.78%, 3.63%]
	[81.82, 87.14]	[96.51, 101.45]		
$N = 8$	105.28	120.17	14.14%	[1.35%, 2.02%]
	[102.57, 108.42]	[117.68, 123.34]		
$N = 10$	115.22	129.55	12.44%	[0.93%, 1.48%]
	[112.45, 118.58]	[126.59, 132.95]		
$N = 12$	123.05	136.52	10.95%	[0.66%, 1.12%]
	[120.23, 126.19]	[133.61, 139.96]		

The 95% confidence intervals for the quantile regression estimates were computed by resampling with replacement the (X_ℓ, W_ℓ)-pair.

In this section, we study the effects of changes in the bidder's type proportion and additional bidders on the expected revenue. For that, we set the largest N to its maximal observed value 12, see tables B.1 and B.2 in Appendix B. The 95% boostrapped confidence intervals for the expected revenue given in these tables have a length ranging from 2$ to 6$, corresponding to revenues varying between 48$ and 137$ [5]. The bootstrapped 95% confidence intervals of the strategical seller expected revenue, achieved using an optimal reserve price, and the non strategical one, obtained with a non binding reserve price, overlaps up to $N = 6$. Similarly, the revenue gain achieved when an additional bidder of any type enters

$V_{(N)}$ is above the reserve price R. The Markov inequality gives the bound $\mathbb{E}[V_{(N)}]/R$ for the latter. A proxy for $\mathbb{E}[V_{(N)}]/R$ is the non strategical revenue $\Pi(0)$ when the seller value is 0, suggesting to use the bound $\Pi(0)/R$ for the probability of selling.

[5]The bootstrap 95% confidence intervals for the optimal reserve price have a larger length, between 12$ and 14$ for an optimal reserve price between 104$ and 112$. As a matter of comparison, Coey et al. (2017)'s set identified confidence bounds for the seller revenue and optimal reserve price look huge, but they also allow for affiliated values.

Table 6: Strategical Expected Revenue

	Min ER	Max ER	Max %Δ	One logger replaced by one mill [Min %, Max %]
$N = 2$	69.65	76.44	9.75%	[4.57%, 4.95%]
	[68.64, 70.92]	[75.45, 77.77]		
$N = 3$	77.97	86.95	11.52%	[3.44%, 3.98%]
	[76.60, 79.67]	[85.62, 88.73]		
$N = 5$	92.16	103.9	12.74%	[2.14%, 2.75%]
	[90.27, 94.45]	[102.06, 106.30]		
$N = 8$	108.63	121.86	12.18%	[1.20%, 1.73%]
	[106.32, 111.36]	[119.54, 124.84]		
$N = 10$	117.16	130.37	11.28%	[0.86%, 1.33%]
	[114.70, 119.99]	[127.76, 133.61]		
$N = 12$	124.2	136.92	10.24%	[0.63%, 1.03%]
	[121.63, 127.20]	[134.11, 140.35]		

The 95% confidence intervals for the quantile regression estimates were computed by resampling with replacement the (X_ℓ, W_ℓ)-pair.

looks significant, at least for auctions with up to 7 initial bidders for additional logger and, for mill, up to some auctions with $N = 10$. Setting the largest N to 12 is therefore expected to capture all the statistically significant policy effects delivered by the sample. We now focus on each of these effects.

Revenue and types. Point estimation of bidders' private value distributions permits investigation of changes in the number of bidders of a given type. Tables 5, 6 and 7 give a summary of all universe of changes, see also Tables B.1 and B.2 in Appendix B.

Table 5 considers a non strategical expected revenue, which means that reserve price is non binding, whereas Table 6 focuses on the optimal revenue[6]. All the results are obtained for a given N. The second and third columns of both tables give the minimum and maximum

[6] As suggested in Coey, Larsen and Sweeney (2019), a comparison between a strategical and non strategical expected revenue can be fruitful to the seller due to the costs that a policy of setting an optimal reserve price may impose in practice. Recent works have highlighted the asymmetric effects on seller's revenue due to mistakes in choosing reserve prices (see e.g. Kim (2013), Ostrovsky and Schwarz (2016), Coey et al. (2019) and Gimenes (2017))

values of the seller expected revenue across type proportions. The minimum and maximum values of the revenue in both cases are obtained when only loggers and only mills are participating, respectively. The percentage change in revenue when changing all loggers into mills is given in the fourth column and is an additional measure of asymmetry. It is, on average, 15.4% in the non strategical case and 11.3% in Table 6. These order of magnitude are similar to the one found in Roberts and Sweeting (2016) who employ a parametric specification.[7] The fifth column gives the maximum and minimum percentage changes obtained when replacing one logger by one mill. All these results suggest that the seller should either incentivize mills participation or subsidize higher loggers bid as studied in Flambard and Perrigne (2006), Marion (2007) or Krasnokutskaya and Seim (2011) for the latter.

Table 7: Violations of Bulow and Klemperer (1996), $N = 2, 3, 4$

N	(Logger,Mill)	Non strat. ER	Strat. ER	Additional Logger	Additional Mill
$N = 2$	(2,0)	48.02	69.65	63.59*	67.30
	(1,1)	52.46	73.10	67.30*	71.18
	(0,2)	57.65	76.44	71.18*	75.04
$N = 3$	(3,0)	63.59	77.97	74.82	78.23
	(2,1)	67.30	81.07	78.23	81.63
	(1,2)	71.18	84.06	81.63	84.96
	(0,3)	75.04	86.95	84.96	88.17
$N = 4$	(4,0)	74.82	85.44	84.25	87.31
	(3,1)	78.23	88.24	87.31	90.30
	(2,2)	81.63	90.93	90.30	93.19
	(1,3)	84.96	93.53	93.19	95.97
	(0,4)	88.17	96.03	95.97	98.64

An underlined revenue indicates a violation of Bulow and Klemperer (1996), ie the considered non strategical revenue obtained by adding a bidder of a given type is below the strategical one. A "*" indicates that the 95% bootstrapped confidence interval of the strategical revenue and the non strategical one with an additional bidder of the considered type do not overlap.

Revenue and additional bidders. An important result by Bulow and Klemperer (1996) states that the seller's expected revenue achieved in an ascending auction with no reserve

[7] These authors also allow for entry decision but their estimate "indicate a moderate effect of selection".

price but an additional bidder is higher than the one of any allocation mechanism, that includes the case of an ascending auction with an optimal reserve price, under symmetry and a downward sloping marginal revenue condition.[8] Table 7 reports several violations of Bulow and Klemperer (1996) arising in our asymmetric framework. The "Strat. ER" column of Table 7 indicates the estimated optimal expected revenue achieved with $N = 2, 3$ and 4 bidders, with number of loggers or mills as indicated in the second column. The last two columns give the estimated non strategical expected revenue obtained when adding a logger or a mill.

Table 7 shows that using an optimal reserve price is always more profitable than adding a weak logger bidder. Adding a mill bidder is also less profitable than using the optimal auction but only when $N = 2$ and in a much less significant way than adding a logger. Table 7 shows that the difference of revenue using the optimal auction and adding a logger decreases with N, in average across type proportion. By contrast the revenue difference using the optimal auction and adding a mill increase with N.[9] The systematic violations of Bulow and Klemperer (1996) when adding a logger suggests that the logger private value distribution does not satisfy the downward sloping marginal revenue condition.[10] When $N \geq 4$, using the optimal reserve price is less profitable than participation of an additional bidder of any type, up to few minor exceptions. However the differences of expected revenue between an optimal reserve price and an additional bidder are at best in the range of 3\$, which is close to the half length of the boostrapped 95% confidence interval for the strategical

[8] See Coey et al. (2019) for a recent econometric application to entry exogeneity.

[9] Tables B.1 and B.2 in Appendix B also report the revenues obtained for an estimation of a symmetric private value model as in Gimenes (2017). Interestingly violations of Bulow and Klemperer (1996) occur for $N = 2, 3$ but not for larger N.

[10] The downwards sloping marginal revenue condition of Bulow and Klemperer (1996) requires that

$$-\frac{d}{dt}\left[V_i(t)(1-t)\right] = V\left(t^{1/\lambda_i}\right) - (1-t)\frac{t^{1/\lambda_i - 1}}{\lambda_i}V^{(1)}\left(t^{1/\lambda_i}\right)$$

increases with t. If $V^{(1)}(0) > 0$ and $1/2 < \lambda_i < 1$, the leading term when t goes to 0 of the derivative of this function is $-(1/\lambda_i - 1)\frac{t^{1/\lambda_i - 2}}{\lambda_i}V^{(1)}(0)$ which is negative, so that the considered condition is not compatible with our estimation of λ_L.

and non strategical seller's expected revenues.

7 Conclusion

The paper considers a semiparametric specification for asymmetric private value distribution under the independent private value distribution setup. The bidders share a common parent distribution which is generated by a quantile regression model. Asymmetry is driven by powers applied to the parent distribution. These powers can depend upon individual and/or group fixed effects, bidder and/or auction specific variables. The specification can be estimated by a two stage procedure from the winning bid and winner's identity. This quantile regression specification is not affected by the curse of dimensionality and can cope with data-rich environment. Unlike common parametric specifications, it is expected to be less affected by misspecification due to its nonparametric nature. Usual parametric rates nevertheless apply and estimation techniques remain standard. The parametric power component of the model allows for a simple evaluation of bidder's asymmetry and of its economic implications.

A timber auction application has been used to illustrate the implication of asymmetry. The estimated asymmetry parameter means that weaker bidders have 30% less chances to win the auction than stronger ones. The quantile regression specification allows to detect the variables that affect the bidders in a symmetric way, here volume, suggesting that bidders face similar capacity constraints, and the other variables that represent characteristics of asymmetry. The shape of the expected revenue varies a lot with the number N of bidders, being mostly flat for $N > 5$, with an optimal revenue close to the one achieved in the absence of a reserve price. For small N, the choice of a reserve price does matter, but the estimated optimal one does not vary too much with N and type proportion. The effect of asymmetry is mild here, and using the one estimated from a misspecified symmetric model should protect the seller against revenue loss occurring for small N. On the other hand, and as expected, the proportion of small bidders may importantly affect the seller' expected revenue. This suggests that the seller can benefit from preference policies which would strengthen the weak bidders. A striking finding is that, in small auctions with less than four bidders, increasing participation, as recommended by Bulow and Klemperer (1996) in a symmetric environment, may give a smaller revenue than using an optimal reserve price, due to the presence of weak bidders. As a consequence, the choice of a proper reserve price may be a more important tool under asymmetry than when the bidders are symmetric.

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Appendix A - Proof section

A.1 Proof of Proposition 2

Assumption 5 yields that the function $\Psi_i(\tau; Z, \alpha, \beta) = \Psi_i(\tau)$ is well-defined as $\lambda(Z_i; \alpha_i, \beta) = \lambda_i > 0$. Set $\Lambda_N = \Lambda_N(Z; \alpha, \beta)$, $\Lambda_{N|i} = \Lambda_{N|i}(Z; \alpha, \beta) = \Lambda_N - \lambda_i$ so that

$$\Psi_i(\tau) = \frac{\Lambda_N \tau^{\Lambda_{N|i}} - \Lambda_{N|i}\tau^{\Lambda_N}}{\lambda_i},$$
$$\frac{\partial \Psi_i(\tau)}{\partial \tau} = \frac{\Lambda_N \Lambda_{N|i} \tau^{\Lambda_{N|i}-1}}{\lambda_i}\left(1 - \tau^{\lambda_i}\right).$$

Hence $\Psi_i(\cdot)$ is continuous and strictly increasing. (2.10) and Assumption 3 then yield

$$\gamma(\tau) = \mathbb{E}^{-1}[XX'] \times \mathbb{E}[XW(\Psi_i(\tau)|X, Z, i)]$$

for all τ in $[0,1]$. □

A.2 Proof of Proposition 3

Ignore, for the sake of brevity, the conditioning variables. Under assumption 6, the seller possible payoffs are

$$\pi(r) = \begin{cases} V_0 & \text{if } V_{N:N} < R \\ R & \text{if } V_{N-1:N} < R \leq V_{N:N}, \\ V_{N-1:N} & \text{if } R \leq V_{N-1:N}, \end{cases}$$

where $V_{i:N}$ is the ith-lowest order statistics of private values, i.e. $V_{N:N}$ is the first highest order statistic and $V_{N-1:N}$ the second largest one. Recall that $r = F(R)$, or equivalently $R = V(r)$. The next three points evaluate the contribution of each of the three events above to the seller revenue.

1. $\mathbb{P}(V_{N:N} < R) = \mathbb{P}(V_i < R, \forall i = 1, \ldots, N) = \Pi_{i=1}^N F_i(R) = \Pi_{i=1}^N [F(V(r))]^{\lambda_i} = r^{\Lambda_N}$. It follows that the probability of selling is $1 - r^{\Lambda_N}$, hence, Proposition 3-(i) is proven. The contribution of this event to the seller revenue is $\pi_1(r) = V_0 r^{\Lambda_N}$;

2. $\mathbb{P}(V_{N-1:N} < R \leq V_{N:N}) = \sum_{i=1}^N \Pi_{j \neq i} F_j(R)(1 - F_i(R)) = \sum_{i=1}^N r^{\Lambda_{N|i}}(1 - r^{\lambda_i})$. The contribution of this second event to the seller revenue is $\pi_2(r) = V(r) \sum_{i=1}^N r^{\Lambda_{N|i}}(1 - r^{\lambda_i})$;

3. Let $F_{N-1:N}(v)$ denote the c.d.f. of the second-highest order statistic $V_{N-1:N}$, which is

$$F_{N-1:N}(v) = \prod_{i=1}^N F_i(v) + \sum_{i=1}^N \prod_{j \neq i} F_j(v)(1 - F_i(v)).$$

   Under Assumption 2

$$\begin{aligned} F_{N-1:N}(v) &= [F(v)]^{\Lambda_N} + \sum_{i=1}^N \left[(1 - (F(v))^{\lambda_i}).(F(v))^{\Lambda_{N|i}}\right] \\ &= [F(v)]^{\Lambda_N} + \sum_{i=1}^N [(F(v))^{\Lambda_{N|i}} - (F(v))^{\Lambda_N}] \\ &= (1 - N)(F(v))^{\Lambda_N} + \sum_{i=1}^N (F(v))^{\Lambda_{N|i}}. \end{aligned}$$

   The change of variable $v = V(t)$ then gives that the contribution of the third event to

$\pi(r)$ as

$$\pi_3(r) = \int_r^1 v dF_{N-1:N}(v) = \int_r^1 V(t)\, d\left[(1-N)t^{\Lambda_N} + \sum_{i=1}^N t^{\Lambda_{N|i}}\right]$$
$$= \int_r^1 V(t)\left\{(1-N)\Lambda_N t^{\Lambda_N - 1} + \sum_{i=1}^N \Lambda_{N|i} t^{\Lambda_{N|i}-1}\right\} dt.$$

As $\Pi(r) = \pi_1(r) + \pi_2(r) + \pi_3(r)$, Proposition 3-(ii) is proved. It also follows that

$$\frac{\partial \Pi(r)}{\partial r} = V_0 \Lambda_N r^{\Lambda_N - 1} + V^{(1)}(r) \sum_{i=1}^N r^{\Lambda_{N|i}}(1 - r^{\lambda_i}) + R \sum_{i=1}^N \left(\Lambda_{N|i} r^{\Lambda_{N|i}-1} - \Lambda_N r^{\Lambda_N - 1}\right)$$
$$- R\left\{\Lambda_N r^{\Lambda_N - 1} + \sum_{i=1}^N \left(\Lambda_{N|i} r^{\Lambda_{N|i}-1} - \Lambda_N r^{\Lambda_N - 1}\right)\right\}$$
$$= (V_0 - R)\Lambda_N r^{\Lambda_N - 1} + V^{(1)}(r) \sum_{i=1}^N r^{\Lambda_{N|i}}(1 - r^{\lambda_i})$$
$$= \Lambda_N r^{\Lambda_N - 1}\left\{V_0 - R - \frac{V^{(1)}(r) r}{\Lambda_N} \sum_{i=1}^N (1 - r^{-\lambda_i})\right\}.$$

Note that the optimal r_* must belong to the open set $(0,1)$. Hence the FOC $\frac{\partial \Pi(r_*)}{\partial r} = 0$ gives that Proposition 3-(iii) holds. □

A.3 Proof of Theorem 4

By Theorems 2.5 and 3.3, the proof of Theorem 3.2 in Newey and McFadden (1994), it holds under Assumptions 5, 7 and 8

$$\sqrt{L}\left(\widehat{\theta} - \theta\right) = \widehat{\Sigma} + o_{\mathbb{P}}(1), \quad \widehat{\Sigma} = \mathcal{I}(\theta)^{-1} \frac{1}{\sqrt{L}} \sum_{\ell=1}^L \frac{\pi^\theta(I_\ell^* | Z_\ell, N_\ell, \theta)}{\pi(I_\ell^* | Z_\ell, N_\ell, \theta)} + o_{\mathbb{P}}(1). \tag{A.1}$$

For $\widehat{\gamma}(\tau)$, define

$$\widehat{Q}(\gamma; \vartheta) = \frac{1}{L} \sum_{\ell=1}^L \rho_{\Psi_{I_\ell^*}(\tau; Z_\ell, N_\ell, \vartheta)}\left(W_\ell - X_\ell' \gamma\right)$$

which is such that $\widehat{\gamma}(\tau) = \arg\min_{\gamma} \widehat{Q}\left(\gamma; \widehat{\theta}\right)$ and $\widetilde{\gamma}(\tau) = \arg\min_{\gamma} \widehat{Q}(\gamma; \theta)$. The proof makes use of the following partial derivatives

$$\widehat{Q}_{\vartheta} = \left.\frac{\partial \widehat{Q}(\gamma; \vartheta)}{\partial \vartheta}\right|_{(\gamma,\vartheta)=(\gamma(\tau),\theta)} = \frac{1}{L}\sum_{\ell=1}^{L}\left(W_{\ell} - X_{\ell}'\gamma(\tau)\right)\Psi_{I_{\ell}^{*}}^{\theta}\left(\tau; Z_{\ell}, N_{\ell}, \theta\right), \quad Q_{\vartheta} = \mathbb{E}\left[\widehat{Q}_{\vartheta}\right],$$

$$\widehat{Q}_{\vartheta\vartheta} = \frac{1}{L}\sum_{\ell=1}^{L}\left(W_{\ell} - X_{\ell}'\gamma(\tau)\right)\Psi_{I_{\ell}^{*}}^{\theta\theta}\left(\tau; Z_{\ell}, N_{\ell}, \theta\right), \quad Q_{\vartheta\vartheta} = \mathbb{E}\left[\widehat{Q}_{\vartheta\vartheta}\right]$$

$$\widehat{Q}_{\vartheta\gamma} = -\frac{1}{L}\sum_{\ell=1}^{L}\Psi_{I_{\ell}^{*}}^{\theta}\left(\tau; Z_{\ell}, N_{\ell}, \theta\right) X_{\ell}', \quad D(\tau) = \mathbb{E}\left[\widehat{Q}_{\vartheta\gamma}\right].$$

Let $\widehat{S}/\sqrt{L} = \widehat{S}(\tau)/\sqrt{L}$ be the γ-derivative of $\widehat{Q}(\gamma; \theta)$ taken at $\gamma(\tau)$

$$\widehat{S} = \frac{1}{\sqrt{L}}\sum_{\ell=1}^{L} X_{\ell}\left[\mathbb{I}\left(W_{\ell} \leq X_{\ell}'\gamma(\tau)\right) - \Psi_{I_{\ell}^{*}}\left(\tau; Z_{\ell}, N_{\ell}, \theta\right)\right].$$

Define the objective function

$$\widehat{\mathcal{Q}}(\xi) = L\left\{\widehat{Q}\left(\gamma(\tau) + \frac{\xi}{\sqrt{L}}; \widehat{\theta}\right) - \widehat{Q}(\gamma(\tau); \theta) - \widehat{Q}_{\vartheta}'\left(\widehat{\theta} - \theta\right) - \frac{\left(\widehat{\theta} - \theta\right)'\widehat{Q}_{\vartheta\vartheta}\left(\widehat{\theta} - \theta\right)}{2}\right\}$$

which is such that

$$\sqrt{L}\left(\widehat{\gamma}(\tau) - \gamma(\tau)\right) = \arg\min_{\xi} \widehat{\mathcal{Q}}(\xi).$$

For simplicity of notation, denote $\Psi_{I_{\ell}^{*}}(\tau; Z_{\ell}, N_{\ell}, \theta) = \Psi_{I_{\ell}^{*}}$. Arguing as in Pollard (1991, p.192) yields, for each fixed ξ,

$$L\left\{\widehat{Q}\left(\gamma(\tau) + \frac{\xi}{\sqrt{L}}; \theta\right) - \widehat{Q}(\gamma(\tau); \theta)\right\} = \widehat{S}'\xi + \frac{1}{2}\xi' H(\tau)\xi + \sum_{\ell=1}^{L}\left(\widetilde{R}_{\ell}(\xi) - \mathbb{E}\left[\widetilde{R}_{\ell}(\xi)\right]\right) \tag{A.2}$$

where

$$\widetilde{R}_\ell(\xi) = \left\{ \rho_{\Psi_{I_\ell^*}} \left(W_\ell - X_\ell' \left(\gamma(\tau) + \frac{\xi}{\sqrt{L}} \right) \right) - \rho_{\Psi_{I_\ell^*}} \left(W_\ell - X_\ell' \gamma(\tau) \right) \right\} - \left(\frac{1}{\sqrt{L}} X_\ell \left[\mathbb{I}\left(W_\ell \leq X_\ell' \gamma(\tau) \right) - \Psi_{I_\ell^*} \right] \right)' \xi$$

$\sum_{\ell=1}^{L} \left(\widetilde{R}_\ell(\xi) - \mathbb{E}\left[\widetilde{R}_\ell(\xi) \right] \right)$ contributes only $o_{\mathbb{P}}(1)$ to (A.2). To see this note that $\rho_a(b) = (a - \mathbb{I}(b < 0))b = \int_0^b (a - \mathbb{I}(t < 0))dt$ and denote $\delta_\ell(\xi) = X_\ell'\xi/\sqrt{L}$ and $\widetilde{D}_\ell(\tau) = W_\ell - X_\ell'\gamma(\tau)$

$$\begin{aligned} \widetilde{R}_\ell(\xi) &= \rho_{\Psi_{I_\ell^*}} \left(\widetilde{D}_\ell(\tau) - \delta_\ell(\xi) \right) - \rho_{\Psi_{I_\ell^*}} \left(\widetilde{D}_\ell(\tau) \right) - \delta_\ell(\xi) \left[\mathbb{I}\left(\widetilde{D}_\ell(\tau) \leq 0 \right) - \Psi_{I_\ell^*}(\tau; \theta) \right] \\ &= \int_0^{\delta_\ell(\xi)} \left[\mathbb{I}\left(\widetilde{D}_\ell(\tau) \leq t \right) - \mathbb{I}\left(\widetilde{D}_\ell(\tau) \leq 0 \right) \right] dt. \end{aligned}$$

Using Cauchy-Schwarz inequality

$$\widetilde{R}_\ell(\xi)^2 \leq |\delta_\ell(\xi)| \left| \int_0^{\delta_\ell(\xi)} \mathbb{I}\left(\left| \widetilde{D}_\ell(\tau) \right| \leq |t| \right) dt \right| \leq |\delta_\ell(\xi)| \int_0^{|\delta_\ell(\xi)|} \mathbb{I}\left(\left| \widetilde{D}_\ell(\tau;) \right| \leq |t| \right) dt.$$

Denote $||f_W(\cdot|\cdot)||_\infty = \sup_{w,x,z} |f_W(w|X, Z, I^*, N)|$.

$$\begin{aligned} \mathbb{E}\left[\widetilde{R}^2(\xi) | X, Z, I^*, N \right] &\leq |\delta(\xi)| \int_0^{|\delta(\xi)|} \left\{ \int \mathbb{I}\left(|w - X'\gamma(\tau)| \leq |t| \right) f_W(w|X, Z, I^*, N) dw \right\} dt, \\ &\leq ||f_W(\cdot|\cdot)||_\infty |\delta(\xi)| \int_0^{|\delta(\xi)|} \left\{ \int \mathbb{I}\left(|w - X'\gamma(\tau)| \leq |t| \right) dw \right\} dt, \\ &\leq ||f_W(\cdot|\cdot)||_\infty |\delta(\xi)| \int_0^{|\delta(\xi)|} 2|t| dt = ||f_W(\cdot|\cdot)||_\infty |\delta(\xi)|^3 \leq \frac{C||X||^3||\xi||^3}{L^{3/2}}. \\ \mathbb{E}\left[\widetilde{R}^2(\xi) \right] = \mathbb{E}\left[\mathbb{E}\left[\widetilde{R}^2(\xi) | X, Z, I^*, N \right] \right] &\leq \mathbb{E}\left[\frac{C||X||^3||\xi||^3}{L^{3/2}} \right] \leq \frac{C||\xi||^3}{L^{3/2}}. \end{aligned}$$

Due to cancellation of cross-product terms

$$\mathbb{E}\left[\left| \sum_{\ell=1}^{L} \left(\widetilde{R}_\ell(\xi) - \mathbb{E}\left[\widetilde{R}_\ell(\xi) \right] \right) \right|^2 \right] \leq \sum_{\ell=1}^{L} \mathbb{E}\left[\widetilde{R}_\ell^2(\xi) \right] \leq \frac{C||\xi||^3}{\sqrt{L}} = o(1).$$

Lemma 2.4 in Newey and McFadden (1994) also gives

$$\begin{aligned}
&\widehat{Q}\left(\gamma(\tau)+\frac{\xi}{\sqrt{L}};\widehat{\theta}\right)-\widehat{Q}\left(\gamma(\tau)+\frac{\xi}{\sqrt{L}};\theta\right)-\widehat{Q}'_{\vartheta}\left(\widehat{\theta}-\theta\right)\\
&=\left[\int_0^1\left\{\widehat{Q}'_{\vartheta}\left(\gamma(\tau)+\frac{\xi}{\sqrt{L}};\theta+t\left(\widehat{\theta}-\theta\right)\right)-\widehat{Q}'_{\vartheta}\left(\gamma(\tau);\theta\right)\right\}dt\right]\left(\widehat{\theta}-\theta\right)\\
&=\frac{\left(\widehat{\theta}-\theta\right)'\widehat{Q}_{\vartheta\vartheta}\left(\widehat{\theta}-\theta\right)}{2}+\left(\widehat{\theta}-\theta\right)'\widehat{Q}_{\vartheta\gamma}\frac{\xi}{\sqrt{L}}+o_{\mathbb{P}}\left(\frac{1}{L}\right)\\
&=\frac{\left(\widehat{\theta}-\theta\right)'\widehat{Q}_{\vartheta\vartheta}\left(\widehat{\theta}-\theta\right)}{2}+\left(\widehat{\theta}-\theta\right)'D(\tau)\frac{\xi}{\sqrt{L}}+o_{\mathbb{P}}\left(\frac{1}{L}\right).
\end{aligned}$$

Hence, for each fixed ξ,

$$\begin{aligned}
\widehat{\mathcal{Q}}(\xi)&=L\left\{\widehat{Q}\left(\gamma(\tau)+\frac{\xi}{\sqrt{L}};\widehat{\theta}\right)-\widehat{Q}\left(\gamma(\tau)+\frac{\xi}{\sqrt{L}};\theta\right)-\widehat{Q}'_{\vartheta}\left(\widehat{\theta}-\theta\right)-\frac{\left(\widehat{\theta}-\theta\right)'\widehat{Q}_{\vartheta\vartheta}\left(\widehat{\theta}-\theta\right)}{2}\right\}\\
&\quad+L\left\{\widehat{Q}\left(\gamma(\tau)+\frac{\xi}{\sqrt{L}};\theta\right)-\widehat{Q}\left(\gamma(\tau);\theta\right)\right\}\\
&=\left(\widehat{S}+D(\tau)\sqrt{L}\left(\widehat{\theta}-\theta\right)\right)'\xi+\frac{1}{2}\xi' H(\tau)\xi+o_{\mathbb{P}}(1)\\
&=\left(\widehat{S}+D(\tau)\widehat{\Sigma}\right)'\xi+\frac{1}{2}\xi' H(\tau)\xi+o_{\mathbb{P}}(1)
\end{aligned}$$

where the last line is from (A.1). Applying the convexity arguments in Pollard (1991) then gives, since $\sqrt{L}\left(\widehat{\gamma}(\tau)-\gamma(\tau)\right)=\arg\min_{\xi}\widehat{\mathcal{Q}}(\xi)$,

$$\sqrt{L}\left(\widehat{\gamma}(\tau)-\gamma(\tau)\right)=-H(\tau)^{-1}\left(\widehat{S}+D(\tau)\widehat{\Sigma}\right)+o_{\mathbb{P}}(1).$$

Then the joint asymptotic distribution of $\sqrt{L}\left(\widehat{\gamma}(\tau)-\gamma(\tau)\right)$ and $\sqrt{L}\left(\widehat{\theta}-\theta\right)$ is the one of $-H(\tau)^{-1}\left(\widehat{S}+D(\tau)\widehat{\Sigma}\right)$ and $\widehat{\Sigma}$ by (A.1), which by the CLT is a centered normal with covariance matrix

$$\left[\begin{array}{cc} \mathrm{Var}\left(H\left(\tau\right)^{-1}\left(\widehat{S}+D\left(\tau\right)\widehat{\Sigma}\right)\right) & -\mathrm{Cov}\left(H\left(\tau\right)^{-1}\left(\widehat{S}+D\left(\tau\right)\widehat{\Sigma}\right),\widehat{\Sigma}\right) \\ -\mathrm{Cov}\left(\widehat{\Sigma},H\left(\tau\right)^{-1}\left(\widehat{S}+D\left(\tau\right)\widehat{\Sigma}\right)\right) & \mathcal{I}\left(\theta\right)^{-1} \end{array} \right],$$

which can be written as in the Theorem. □

Appendix B - Tables

This appendix displays tables that were commented on but not included in the empirical application. All tables are for median ascending auctions. The second column in all the three tables gives the corresponding estimates considering the methodology proposed in Gimenes (2017) with symmetric bidders for $N = 2, 3, \ldots, 12$. If the econometrician does not take asymmetry into account, the seller's expected revenue, $\Pi(r|X, N, V_0)$ is

$$\begin{aligned}\Pi(r|X, N, V_0) = {} & V_0 r^N + RNr^{N-1}(1-r) \\ & + N(N-1)\int_r^1 V(t|X)\, t^{N-2}(1-t)\, dt\end{aligned}$$

and the optimal reserve price is $R_* = V(r_*|X)$ with $r_* = \arg\max_r \Pi(r|X, N, V_0)$. The seller value V_0 is set to 0 and the private value quantile function is estimated as in Gimenes (2017). Note that this differs from Section 3.2, where the true asymmetric distribution was used to compute the revenue achieved using a optimal reserve price from the misspecified symmetric model. Estimates taking into account asymmetry among the bidders, as discussed in this paper, are given on columns three to eight. They were computed using the expressions 3.1 and 3.2. The proportion of bidder's types are given in parentheses following the rule $(\#Loggers, \#Mills)$.

Table B.1: Seller Non Strategical Expected Revenue as a Function of the Proportion of Types

	Gimenes (2017)'s Approach	Asymmetric Approach					
		(2,0)	(1,1)	(0,2)			
$N = 2$	54.76	48.02	52.46	57.65			
	[53.42, 56.68]	[46.37, 50.42]	[51.07, 54.40]	[56.20, 59.70]			
		(3,0)	(2,1)	(1,2)	(0,3)		
$N = 3$	70.69	63.59	67.30	71.18	75.04		
	[69.12, 72.83]	[61.61, 66.03]	[65.72, 69.44]	[69.64, 73.31]	[73.30, 77.47]		
		(4,0)	(3,1)	(2,2)	(1,3)	(0,4)	
$N = 4$	82.99	74.82	78.23	81.63	84.96	88.17	
	[81.36, 85.29]	[72.59, 77.49]	[76.40, 80.58]	[80.03, 83.85]	[83.23, 87.32]	[86.19, 90.83]	
		(5,0)	(4,1)	(3,2)	(2,3)	(1,4)	(0,5)
$N = 5$	92.95	84.25	87.31	90.30	93.19	95.97	98.64
	[91.05, 95.41]	[81.82, 87.14]	[85.27, 89.86]	[88.48, 92.62]	[91.42, 95.58]	[94.07, 98.54]	[96.51, 101.45]
		(6,0)	(5,1)	(4,2)		(1,5)	(0,6)
$N = 6$	101.14	92.31	95.02	97.65	(...)	104.93	107.16
	[99.23, 103.68]	[89.76, 95.32]	[92.84, 97.71]	[95.68, 100.15]		[102.90, 107.64]	[104.87, 110.08]
		(7,0)	(6,1)	(5,2)		(1,6)	(0,7)
$N = 7$	107.96	99.25	101.65	103.97	(...)	112.35	114.23
	[105.97, 110.64]	[96.62, 102.31]	[99.32, 104.41]	[101.88, 106.60]		[110.18, 115.17]	[111.84, 117.24]
		(8,0)	(7,1)	(6,2)		(1,7)	(0,8)
$N = 8$	113.72	105.28	107.41	109.47	(...)	118.57	120.17
	[111.49, 116.55]	[102.57, 108.42]	[104.99, 110.26]	[107.26, 112.19]		[116.26, 121.51]	[117.68, 123.34]
		(9,0)	(8,1)	(7,2)		(1,8)	(0,9)
$N = 9$	118.62	110.56	112.46	114.30	(...)	123.85	125.22
	[116.45, 121.56]	[107.80, 113.69]	[110.01, 115.36]	[112.01, 117.07]		[121.42, 126.98]	[122.64, 128.47]
		(10,0)	(9,1)	(8,2)		(1,9)	(0,10)
$N = 10$	122.83	115.22	116.93	118.56	(...)	128.36	129.55
	[120.30, 125.84]	[112.45, 118.58]	[114.23, 120.07]	[115.96, 121.55]		[125.56, 131.63]	[126.59, 132.95]
		(10,0)	(9,1)	(8,2)		(1,9)	(0,10)
$N = 11$	126.47	119.36	120.89	122.36	(...)	132.25	133.28
	[124.11, 129.56]	[116.54, 122.48]	[118.31, 123.86]	[120.01, 125.19]		[129.56, 135.51]	[130.48, 136.65]
		(12,0)	(11,1)	(10,2)		(1,11)	(0,12)
$N = 12$	129.62	123.05	124.43	125.76	(...)	135.62	136.52
	[126.93, 132.78]	[120.23, 126.19]	[121.91, 127.43]	[123.32, 128.65]		[132.83, 138.94]	[133.61, 139.96]

The 95% confidence intervals for the quantile regression estimates were computed by resampling with replacement the (X_ℓ, W_ℓ)-pair. As one goes from left to right, loggers are replaced by mills and the proportion of mills increases keeping the number of bidders fixed. In the vertical direction, as one goes from the top of the table to the bottom, the number of mills is kept fixed and loggers are included increasing the number of bidders.

Table B.2: Seller Strategical Expected Revenue as a Function of the Proportion of Types

	Gimenes (2017)'s Approach	Asymmetric Approach					
		(2,0)	(1,1)	(0,2)			
$N = 2$	73.77	69.65	73.1	76.44			
	[72.93, 74.97]	[68.64, 70.92]	[72.31, 74.20]	[75.45, 77.77]			
		(3,0)	(2,1)	(1,2)	(0,3)		
$N = 3$	83.37	77.97	81.07	84.06	86.95		
	[82.20, 84.97]	[76.60, 79.67]	[79.97, 82.54]	[82.97, 85.65]	[85.62, 88.73]		
		(4,0)	(3,1)	(2,2)	(1,3)	(0,4)	
$N = 4$	91.73	85.44	88.24	90.93	93.53	96.03	
	[90.46, 93.64]	[83.78, 87.48]	[86.88, 90.02]	[89.62, 92.71]	[92.15, 95.45]	[94.41, 98.17]	
		(5,0)	(4,1)	(3,2)	(2,3)	(1,4)	(0,5)
$N = 5$	99.03	92.16	94.69	97.12	99.46	101.72	103.9
	[97.37, 101.24]	[90.27, 94.45]	[93.13, 96.71]	[95.59, 99.09]	[97.99, 101.56]	[100.08, 103.95]	[102.06, 106.30]
		(6,0)	(5,1)	(4,2)		(1,5)	(0,6)
$N = 6$	105.4	98.22	100.51	102.71	(...)	108.83	110.73
	[103.80, 107.78]	[96.15, 100.73]	[98.74, 102.75]	[101.03, 104.89]		[106.98, 111.29]	[108, 68, 113.36]
		(7,0)	(6,1)	(5,2)		(1,6)	(0,7)
$N = 7$	110.98	103.69	105.76	107.75	(...)	115.02	116.67
	[109.23, 113.54]	[101.50, 106.36]	[103.83, 108.15]	[105.91, 110.11]		[112.96, 117.68]	[114.48, 119.45]
		(8,0)	(7,1)	(6,2)		(1,7)	(0,8)
$N = 8$	115.88	108.63	110.51	112.32	(...)	120.41	121.86
	[113.72, 118.60]	[106.32, 111.36]	[108.47, 113.02]	[110.35, 114.83]		[118.21, 123.23]	[119.54, 124.84]
		(9,0)	(8,1)	(7,2)		(1,8)	(0,9)
$N = 9$	120.17	113.11	114.81	116.46	(...)	125.13	126.39
	[118.16, 122.99]	[110.68, 115.90]	[112.66, 117.45]	[114.40, 119.09]		[122.79, 128.11]	[123.94, 129.52]
		(10,0)	(9,1)	(8,2)		(1,9)	(0,10)
$N = 10$	123.95	117.16	118.72	120.21	(...)	129.26	130.37
	[121.56, 126.89]	[114.70, 119.99]	[116.47, 121.48]	[118.09, 122.94]		[126.79, 132.35]	[127.76, 133.61]
		(11,0)	(10,1)	(9,2)		(1,10)	(0,11)
$N = 11$	127.27	120.85	122.26	123.62	(...)	132.88	133.86
	[125.06, 130.29]	[118.33, 123.75]	[119.93, 125.11]	[121.42, 126.43]		[130.29, 136.05]	[131.14, 137.17]
		(12,0)	(11,1)	(10,2)		(1,11)	(0,12)
$N = 12$	130.2	124.2	125.48	126.72	(...)	136.06	136.92
	[127.63, 133.36]	[121.63, 127.20]	[123.11, 128.40]	[124.40, 129.60]		[133.35, 139.36]	[134.11, 140.35]

The 95% confidence intervals for the quantile regression estimates were computed by resampling with replacement the (X_ℓ, W_ℓ)-pair. As one goes from left to right, loggers are replaced by mills and the proportion of mills increases keeping the number of bidders fixed. In the vertical direction, as one goes from the top of the table to the bottom, the number of mills is kept fixed and loggers are included increasing the number of bidders.

Table B.3: Optimal Reserve Price as a Function of the Proportion of Types

	Gimenes (2017)'s Approach	Asymmetric Approach					
		(2,0)	(1,1)	(0,2)			
$N = 2$	107.85	104.65	106.88	111.85			
	[102.73, 116.36]	[102.17, 114.11]	[103.33, 116.24]	[103.82, 117.47]			
		(3,0)	(2,1)	(1,2)	(0,3)		
$N = 3$	107.85	104.65	104.65	106.88	111.85		
	[102.80, 116.36]	[102.16, 114.11]	[103.0, 115.71]	[103.58, 116.70]	[103.90, 117.47]		
		(4,0)	(3,1)	(2,2)	(1,3)	(0,4)	
$N = 4$	107.85	104.65	104.65	106.88	106.88	111.85	
	[102.80, 116.36]	[102.17, 114.11]	[102.87, 115.57]	[103.33, 116.31]	[103.94, 117.76]		
		(5,0)	(4,1)	(3,2)	(2,3)	(1,4)	(0,5)
$N = 5$	107.85	104.65	104.65	104.65	106.88	111.85	111.85
	[102.80, 116.38]	[102.17, 114.11]	[102.81, 115.57]	[103.12, 115.99]	[103.54, 116.70]	[103.66, 117.13]	[103.97, 117.75]
		(6,0)	(5,1)	(4,2)		(1,5)	(0,6)
$N = 6$	107.85	104.65	104.65	104.65	(...)	111.85	111.85
	[102.88, 116.38]	[102.17, 114.11]	[102.79, 115.57]	[103.0, 115.92]		[103.72, 117.41]	[103.97, 117.76]
		(7,0)	(6,1)	(5,2)		(1,6)	(0,7)
$N = 7$	107.85	104.65	104.65	104.65	(...)	111.85	111.85
	[102.88, 116.38]	[102.19, 114.15]	[102.59, 115.41]	[102.98, 115.71]		[103.77, 117.62]	[103.97, 117.77]
		(8,0)	(7,1)	(6,2)		(1,7)	(0,8)
$N = 8$	107.85	104.65	104.65	104.65	(...)	111.85	111.85
	[102.88, 116.41]	[102.19, 114.26]	[102.51, 115.41]	[102.95, 115.71]		[103.77, 117.61]	[103.97, 117.77]
		(9,0)	(8,1)	(7,2)		(1,8)	(0,9)
$N = 9$	107.85	104.65	104.65	104.65	(...)	111.85	111.85
	[102.85, 116.41]	[102.19, 114.64]	[102.51, 115.41]	[102.93, 115.71]		[103.77, 117.62]	[103.90, 117.77]
		(10,0)	(9,1)	(8,2)		(1,9)	(0,10)
$N = 10$	107.85	104.65	104.65	104.65	(...)	111.85	111.85
	[102.85, 116.41]	[102.19, 114.64]	[102.50, 115.41]	[102.81, 115.68]		[103.77, 117.61]	[103.90, 117.75]
		(11,0)	(10,1)	(9,2)		(1,10)	(0,11)
$N = 11$	107.85	104.65	104.65	104.65	(...)	111.85	111.85
	[102.80, 116.41]	[102.19, 114.73]	[102.48, 115.41]	[102.80, 115.68]		[103.77, 117.62]	[103.82, 117.75]
		(12,0)	(11,1)	(10,2)		(1,11)	(0,12)
$N = 12$	107.85	104.65	104.65	104.65	(...)	111.85	111.85
	[102.73, 116.41]	[102.19, 114.73]	[102.45, 115.41]	[102.79, 115.49]		[103.77, 117.61]	[103.77, 117.75]

The 95% confidence intervals for the quantile regression estimates were computed by resampling with replacement the (X_ℓ, W_ℓ)-pair. As one goes from left to right, loggers are replaced by mills and the proportion of mills increases keeping the number of bidders fixed. In the vertical direction, as one goes from the top of the table to the bottom, the number of mills is kept fixed and loggers are included increasing the number of bidders.