Nested Pseudo Likelihood Estimation of Continuous-Time Dynamic Discrete Games

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Abstract

This paper introduces a sequential estimation algorithm to estimate dynamic discrete choice models in continuous time. Dynamic discrete choice models have been used in various areas including dynamic demand models and firm entry-exit models which are mostly developed in discrete time setting. We extend the nested pseudo likelihood (NPL) estimator introduced by Aguirregabiria and Mira (2007) to continuous time models to better approximate the reality in certain economic contexts and to lessen the computational burden. We find that the NPL estimator in continuous time models has satisfying large sample properties as in discrete time models. Moreover, we present the local convergence condition in the iterative NPL algorithm and the zero Jacobian property assuring the local convergence in single agent models. Monte Carlo experiments using a five-player dynamic discrete game are executed to show the relative efficiency of the NPL estimator compared to two-step estimators.

1 Introduction

This paper introduces a sequential estimation algorithm to estimate dynamic discrete choice models in continuous time. In *dynamic* models, compared to static models, forward-looking agents make a decision based on expected future payoffs each period. *Discrete choice* models focus on agents' discrete choices such as a firm's entry-exit decision or a worker's retirement. Dynamic discrete choice models have been rigorously

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studied in empirical industrial organization field since they were pioneered by Wolpin (1984), Pakes (1986) and Rust (1987). They include literature on consumer inventory (Song and Chintagunta, 2003; Hendel and Nevo, 2006), dynamic demand of durable goods (Schiraldi, 2011; Gowrisankaran and Rysman, 2012), and firm entry and exit (Benkard, 2004; Jia, 2008; Holmes, 2011)[1]. These examples are all based on discrete time settings.

In contrast, *continuous time* models have been introduced only recently in this field despite their flexibility and computational advantage. Continuous time models refer to models where state variables change sequentially in continuous time. For example, suppose a supermarket chain is planning to open a new store in a market and market demand is low at the beginning of a year. Then, at some point during the year market demand increases and observing this change, the chain opens a new store later in the year. In continuous time models, they allow state variables, market demand and the number of stores in this case, to change sequentially during a period of time. However, in discrete time models, they assume that all variables change simultaneously at the beginning of the year. So, discrete time models can mispresent the reality by setting a store opening and market demand increase happen at the same time. As seen in the example, there exist economic contexts that continuous time models can better approximate.

Continuous time models also help to reduce computational burden especially in dynamic discrete games by avoiding the "curse of dimensionality" which arises from the cost of computing players' expectations over all possible future states (Doraszelski and Judd, 2012). Moreover, Doraszelski and Judd (2012) show that continuous time models can be simpler and computationally more tractable by restricting the number of possible state changes. In discrete time models, every state variable must move at the same time so if there are n state variables and each state variables can move to κ states, the number of possible next states is κ^n. In contrast, continuous time models only allow one state variable to move at an instant, so there are only κn possible states for next period. This computational advantage makes continuous time models more attractive in large scale models.

The goal of this paper is to introduce the nested pseudo likelihood (NPL) estimator and present its properties in dynamic discrete choice models in continuous time. The NPL estimator has been introduced by Aguirregabiria and Mira (2002, 2007) in discrete time. Aguirregabiria and Mira (2002) develops the NPL algorithm in a single agent model and their subsequent paper in 2007 expands the algorithm to a dynamic discrete

[1] Doraszelski and Pakes (2007) and Aguirregabiria and Mira (2010) provide the survey of applied works in dynamic discrete choice models in discrete time.

game. Compared to the two-step pseudo maximum likelihood (PML) estimator, the NPL estimator has a number of advantages. First, it does not require consistent estimates for the initial conditional choice probabilities. Second, the NPL estimator performs better in finite sample as the nonparametric estimates for choice probabilities can be imprecise in two-step estimators. This paper presents that these advantages of the NPL estimator are also preserved in continuous time setting.

Despite its advantages, researchers have expressed concerns involving the convergence properties of the NPL algorithm (Pesendorfer and Schmidt-Dengler, 2010; Egesdal et al., 2015). Following literature has shown counterexamples where NPL estimator fails to converge or converges to an inconsistent estimate (Pesendorfer and Schmidt-Dengler, 2010; Egesdal et al., 2015). Kasahara and Shimotsu (2012) present a local convergence condition for the NPL estimator in discrete time models which we extend to continuous time models.

This paper is based on the literature on the NPL estimator and dynamic discrete choice models in continuous time. We examine the properties of the NPL estimator in continuous time and develop a continuous time version of the five-player game example in Aguirregabiria and Mira (2007). Bugni and Bunting (2018) introduce K-ML estimator which nests the NPL estimator as it becomes equivalent to the NPL estimator by letting the number of iterations $K \to \infty$. We rely on Bugni and Bunting (2018) to present asymptotic properties for the policy iteration estimator where the number of iterations for the NPL algorithm is finite.

A dynamic discrete choice model in continuous time has been introduced by Doraszelski and Judd (2012) to reduce the computational burden and to adapt the economic problem to more probable setting. This has been developed into an empirical model with a conditional choice probabilities (CCP) representation and two-step estimator by Arcidiacono et al. (2016) – henceforth, ABBE (2016). Blevins (2016) presents theoretical, computational and econometric properties of dynamic discrete choice games and extends results from ABBE (2016). Since then, CCP estimation in dynamic discrete choice models has been applied in various areas, including transportation industry (Mazur, 2017; Qin, 2017), supermarket industry (Schiraldi, Smith, and Takahashi, 2012), TV advertising (Deng and Mela, 2018), nightlife industry (Cosman, 2017), online games (Nevskaya and Albuquerque, 2019), and job search (Arcidiacono et al., 2019). We step further and first introduce the NPL estimator in continuous time which is an iterative version of two-step estimators designed to improve efficiency.

The rest of the paper is organized as follows. Section 2 develops a dynamic discrete game in continuous time with an example. Section 3 introduces the NPL algorithm to estimate the parameters in the model. In Section 4, we simulate a five-player game in retail industry and present the performance of the NPL estimator. Section 5 concludes.

2 Model

In this section, we introduce a dynamic discrete game in continuous time in ABBE (2016). We include a five player game example which is a continuous time version of the game introduced in Section 4 in Aguirregabiria and Mira (2007).

2.1 Setting

In the model, agents, denoted by $i = 1, \ldots, N$, choose their action j from a choice set $\mathcal{A} = \{0, 1, \ldots, J-1\}$ in continuous time $t \in [0, \infty)$. This results in changes in endogenous state variables. We also introduce exogenous state variables which change according to Poisson process.

State space. The states are discrete and finite, so every state at time t can be represented by an state vector x in a finite state space $\mathcal{X} = \{x_1, \ldots, x_K\}$ where each state is indexed by $k = 1, 2, \ldots, K$.

Example 1. Suppose that there are five firms competing in a local retail market. They can either choose to open/exit ($j = 1$), or do nothing ($j = 0$). Since this is a small market, firms can have at most one store, so at each state k, the number of stores that firm i can own can be denoted as $a_{ik} \in \{0, 1\}$ for $i = 1, \ldots, 5$. We introduce another variable s_k which measures the market size discretized to $s_k = \{1, 2, \ldots, 5\}$. Then, we can write every state by a 6×1 vector x_k for $k = 1, \ldots, K$ where $K = 5 \times 2^5 = 160$:

$$x_k = (s_k, a_{1k}, a_{2k}, a_{3k}, a_{4k}, a_{5k}). \qquad \blacksquare$$

Poisson processes[2]. Consider a situation where there is a state jump at some time T_n and the next jump is at T_{n+1}. The state jump can be a firm's entry to a new market or any changes in other state variables such as an increase in population. We call T_n as the n-th event time and the difference between two absolute times as inter-event time denoted as τ_n. Specifically, we consider a following process:

$$T_{n+1} = T_n + \tau_n$$

[2]Definitions and notations in this section are based on Schuette and Metzner (2009).

where $n \geq 1$ and $T_0 = 0$.

We assume that the sequence of inter-event times $\{\tau_n\}_{n \in \mathbb{N}}$ is an independent and identically distributed sequence of exponential random variables with parameter $\lambda > 0$. Then, the number of events up to some time t, denoted as $N(t)$, follows Poisson distribution:

$$P[N(t) = s] = \frac{(\lambda t)^s}{s!} e^{-\lambda t}.$$

In the model, we introduce two types of Poisson processes. First, agents receive a chance to move to another state according a Poisson process with a rate parameter λ_{ik} in state k. For example, by assuming $\lambda_{ik} = \lambda = 1$, for $i = 1, \ldots, N$ and $k = 1, \ldots, K$, we are assuming that every agent receives a chance to move once a year on average if we set a year as unit of time. Since Poisson process has the property of $E[N(t)] = \lambda t$, the expected number of events during a fixed period of time equals $1/\lambda$. Second, exogenous state variables change by another Poisson process which can be characterized by $K \times K$ intensity matrix. In a N-player game, we have $(N + 1)$ Poisson processes. We present the details on intensity matrix below.

State changes. There are two types of state changes – endogenous state changes which are relying on each agent's decision and exogenous state changes driven by the nature. As endogenous state changes only when an agent makes a choice, we can define a state continuation function $l(i, j, k)$ which maps an agent i's choice j and current state k to next state l. We assume that choice $j = 0$, where $l(i, 0, k) = k$, is costless and that all choices j are meaningfully distinct $l(i, j, k) \neq l(i, j', k)$ for all states $k = 1, \ldots, K$.

On the other hand, nature chooses to change exogenous state variables which are not affected by agents' choice, such as market size, income, or population. In the continuous time setting, we implicitly assume that only one state change occurs at the same instant, so simultaneous change in multiple state variables is a zero-measure event.

Example 1 (continued). When firm 2 decides to open a new store at state k, the continuation state index will be $l(i, j, k) = l(2, 1, k)$. Then, the new state $x_{l(i,j,k)}$ is $x_{l(i,j,k)} = (s_k, a_{1k}, a_{2k} + 1, \ldots, a_{5k})$. For exogenous state change, when the market size increases by 1 at state k, the state k moves to l and the next state x' can be written as $x' = (s_k + 1, a_{1k}, a_{2k}, \ldots, a_{5k})$. Note that only one state variable moves at a time and other state variables remain the same as previous state. ■

Intensity matrices. Since Poisson process is a type of continuous time Markov process, it also shares properties of Markov jump processes. It is known that a finite-state Markov jump process can be characterized by an intensity matrix which is also called transition rate matrix. This is the counterpart of one-step transition probability matrix in a discrete time model in that each component of intensity matrix q_{kl} represents the rate departing from k and arriving in state l. The difference is that the intensity rate shows the transition rate for an instant instead of one period of time. So, when we define transition probability matrix over some small time interval h as $P(h)$, we can write $P(h) = I + Qh$. Note that as $h \to 0$, $P(h)$ approaches the identity matrix.

Then, for states k, l with $k \neq l$, we have

$$\begin{cases} P(x_{t+h} = l | x_t = k) = q_{kl} h \\ P(x_{t+h} = k | x_t = k) = 1 - \sum_{l \neq k} q_{kl} h \end{cases}$$

We can formally define the intensity matrix Q:

$$Q = \begin{bmatrix} q_{11} & q_{12} & q_{13} & \cdots & q_{1K} \\ q_{21} & q_{22} & q_{23} & \cdots & q_{2K} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ q_{K1} & q_{K2} & q_{K3} & \cdots & q_{KK} \end{bmatrix}$$

where

$$q_{kl} = \begin{cases} \lim_{h \to 0} \frac{Pr(x_{t+h} = l | x_t = k)}{h} & \text{for } l \neq k \\ -\sum_{l \neq k} q_{kl} & \text{otherwise.} \end{cases}$$

For endogenous state changes, we can write q_{kl} as a multiplication of move arrival rate and choice probability. First, each agent receives an opportunity to move according to Poisson process at rate λ_{ik}. Then, we also assume that agents are forward looking and discount the future payoff with rate $\rho \in (0, \infty)$. They observe other agents' moves and optimally choose their action $j \in \mathcal{A}$ on their beliefs on other agents' actions and natures' moves. We denote the conditional choice probability of agent i choosing action j in state k with σ_{ijk}. So, at some state k, an agent receives a move opportunity by rate λ_i and chooses action j by probability σ_{ijk}, which means the possibility of state change at state k is $\lambda \sigma_{ijk}$. Each element q_{ikl} of Q_i can be written as follows:

$$q_{ikl} = \begin{cases} \lambda_{ik} \sigma_{ijk} & \text{if } l \neq k \text{ and } l(i, j, k) = l \\ -\sum_{l \neq k} q_{ikl} & \text{otherwise.} \end{cases}$$

We denote an intensity matrix for exogenous state changes as Q_0 and elements of Q_0 as q_{kl}. If we assume that Poisson processes are independent, we can describe the state dynamics with an aggregate intensity matrix $Q = Q_0 + Q_1 + Q_2 + \ldots + Q_N$.

Example 1 (continued). In the previous example, nature changes the market size according to Poisson process characterized by Q_0. We impose a restriction that market size only changes by 1 at an instant. Firms receive a move opportunity by rate λ and will choose to move by choice probability σ_{ijk}. So, q_{ikl} for firm i's intensity matrix can be written as:

$$q_{ikl} = \begin{cases} \lambda\sigma_{i1k} & \text{if firm } i \text{ enters or exits and } l(i,1,k) = l \\ -\lambda\sigma_{i1k} & \text{if firm } i \text{ does nothing} \\ 0 & \text{otherwise.} \end{cases} \qquad \blacksquare$$

Payoffs. In continuous time models, we can distinguish agents' flow payoffs from instantaneous choice-specific payoffs. First, agents receive flow payoffs at each state k regardless of their choices. We denote flow payoff by u_{ik} and assume $|u_{ik}| < \infty$ for $i = 1, \ldots, N$ and $k = 1, \ldots, K$. On the other hand, receiving a chance to move at rate λ_{ik} and choosing action j, the agent will receive some instantaneous choice-specific payoff which can be again divided into a deterministic component ψ_{ijk} where $|\psi_{ijk}| < \infty$ and a stochastic component $\varepsilon_{ijk} \in \mathbb{R}$. The first part ψ_{ijk} is a common knowledge and it can be observed both by the agent and the econometrician but the second part ε_{ijk} is only observed by agent i.

Example 1 (continued). We consider a log-linear profit function for flow payoff including competition effects and market size.

$$u_{ik} = \theta_{RS} \ln(s_k) - \theta_{RN} \ln\big(1 + \sum_{m \neq i} a_{mk}\big) - \theta_{FC,i} \tag{1}$$

We estimate the fixed costs $\theta_{FC,i}$ for $i = 1, \ldots, 5$ and other profit function parameters θ_{RS} and θ_{RN}. Note that there is no entry cost in flow payoffs since entry cost is only paid when a firm chooses to open a new store, which means it is choice-specific, compared to Equation (48) in Aguirregabiria and Mira (2007). When a firm decides an action j, they receive an instantaneous payoff $\psi_{ijk} + \varepsilon_{ijk}$ where

$$\psi_{ijk} = \begin{cases} -\theta_{EC} & \text{if } j = 1 \text{ and } a_{ik} = 0 \\ 0 & \text{otherwise.} \end{cases} \qquad \blacksquare$$

2.2 The Bellman optimality and CCP representation

In a Markov structure game, every agent plays a stationary Markov strategy based on their beliefs regarding other agents' choice probabilities denoted by ς_i. Each agent i

forms their belief on $(N-1)$ agents' J choices in K states so, ς_i is a collection of $(N-1)\times J\times K$ probabilities, ς_{imjk} for player $m\neq i$, choice j and state k.

We now introduce the value function and equations for conditional choice probability (CCP). Agents establish the value function based on the expectations on other agents' and nature's moves, and their own move opportunities. We follow the derivation of instantaneous Bellman equation in Blevins (2016). A probability of agent i getting a move opportunity for a small increment of time at state k is $\lambda_{ik}h$ under Poisson process. Denoting each agent i's discount rate as ρ_i, the discount factor for time increment h is $1/(1+\rho_i h)$. Given agent i's beliefs, ς_i, we can write the Bellman equation as below:

$$V_{ik}=\frac{1}{1+\rho_i h}\Big[u_{ik}h+\sum_{l\neq k}q_{0kl}hV_{il}+\sum_{m\neq i}\lambda_{mk}h\sum_{j=0}^{J}\varsigma_{imjk}V_{i,l(m,j,k)}+\lambda_{ik}h\mathbb{E}\max_j\{\psi_{ijk}+\varepsilon_{ijk}+V_{l(i,j,k)}\}$$
$$+\Big(1-\sum_{i=1}^{N}\lambda_{ik}h-\sum_{l\neq k}q_{0kl}h\Big)V_{ik}+o(h)\Big].$$

For a small amount of time h, agent i receives a flow payoff of $u_{ik}h$, a value of V_l if the state changes by nature, and $\psi_{ijk}+\varepsilon_j+V_{l(i,j,k)}$ if agent i chooses j after receiving a move opportunity. The last line accounts for the situation where the agent does not receive a move opportunity or nature does not move for time h. By rearranging and letting $h\to 0$, we get a simpler form of the Bellman equation:

$$V_{ik}=\frac{u_{ik}+\sum_{l\neq k}q_{kl}V_{il}+\sum_{m\neq i}\lambda_{mk}\sum_j\varsigma_{imjk}V_{i,l(m,j,k)}+\lambda_{ik}E\max_j\{\psi_{ijk}+\epsilon_{ijk}+V_{i,l(i,j,k)}\}}{\rho+\sum_{l\neq k}q_{kl}+\sum_{i=1}^{N}\lambda_{ik}}.$$

Now we define a best response function for each agent. A Markov strategy for agent i is a mapping σ_i from each state $(k,\epsilon_{ijk})\in\mathcal{X}\times\mathbb{R}^J$ to an action $j\in\mathcal{A}_{ik}$. Then, the best response for an agent i is defined as:

$$\sigma_i(k,\epsilon;\varsigma_i)=j\iff\psi_{ijk}+\epsilon_{ijk}+V_{i,l(i,j,k)}\geq\psi_{ij'k}+\epsilon_{ij'k}+V_{i,l(i,j',k)}\quad\forall j'\in\mathcal{A}_{ik}.$$

Then, the choice probability for each agent becomes:

$$\sigma_{ijk}=\Pr[\delta_i(k,\epsilon_i;\varsigma_i)=j|k].$$

We now collect assumptions needed for the existence of equilibrium.

Assumption 1. *(Discrete states) The state space is finite:* $K=|\mathcal{X}|<\infty$.

Assumption 2. *(Bounded rates and payoffs) The discount rate ρ_i, move arrival rate, rates of state changes due to nature, and payoffs are all bounded for all $i = 1, \ldots, N$, $j = 0, \ldots, J-1$, $k = 1, \ldots, K$, $l = 1, \ldots, K$ with $l \neq k$: (a) $0 < \rho_i < \infty$, (b) $0 < \lambda_{ik} < \infty$, (c) $0 \leq q_{kl} < \infty$, (d) $|u_{ik}| < \infty$, and (e) $|\psi_{ijk}| < \infty$.*

Assumption 1 and 2 are needed to define a continuous time Markov chain. In Assumption 2, we rule out infinite rates and payoffs to make it possible to estimate by restricting the value to a finite number.

Assumption 3. *(Additive separability) In each state k, the instantaneous payoff associated with choice j is additively separable as $\psi_{ijk} + \varepsilon_{ijk}$.*

Assumption 4. *(Distinct actions) For all $i = 1, \ldots, N$ and $k = 1, \ldots, K$, the continuation state function $l(i, j, k)$ and the choice-specific payoffs ψ_{ijk} satisfy the following two properties: (a) choice $j = 0$ is a costless continuation choice with $l(i, j, k) = k$ and $\psi_{ijk} = 0$, and (b) all choices j are meaningfully distinct in the sense that the continuation states differ: $l(i, j, k) \neq l(i, j', k)$ for all $j = 0, \ldots, J-1$ and $j' \neq j$.*

Assumption 5. *(Private information) The errors ε_{ik} are i.i.d. over time with joint distribution F which is absolutely continuous with respect to Lebesgue measure (with joint density f), has finite first moments, and has support equal to $\mathbb{R}^J$.*

Assumption 3 is common in the literature as in Assumption 1 in Aguirregabiria and Mira (2002) and Aguirregabiria and Mira (2007). Assumption 4 assumes that all choices should result in different states. For example, choice $j = 1$ and $j = 2$ should not be defined separately if they both makes the state move from $k = 3$ to $k = 4$. Distribution assumption on ε is also used widely in previous literature as Assumption 2 in Aguirregabiria and Mira (2002) and Aguirregabiria and Mira (2007).

We define Markov perfect equilibrium as in Aguirregabiria and Mira (2007) and ABBE (2016).

Definition 1 (Markov perfect equilibrium). *A collection of Markov strategies $\{\delta_1, \ldots, \delta_N\}$ is a Markov perfect equilibrium if for all i and for any $(k, \epsilon_{ik}) \in \mathcal{X} \times \mathbb{R}^J$, $\delta_i(k, \epsilon_{ik})$ is a best response.*

By Proposition 5 in ABBE (2016), under Assumptions 1-5, a Markov perfect equilibrium exists. Moreover, under Assumptions 1-5 and assuming that λ_{ik} is constant across all agents and states denoted by λ, ABBE (2016) show that we can express the value function as below.

$$V_i(\theta,\sigma) = \Big[(\rho + N\lambda)I - \lambda \sum_{m=1}^{N} \Sigma_m(\sigma_m) - Q_0\Big]^{-1} [u_i(\theta) + \lambda_i E_i(\theta,\sigma)] \tag{2}$$

where $\Sigma_m(\sigma_m)$ is the $K \times K$ state transition matrix induced by the actions of player m given the choice probabilities σ_m and where $E_i(\theta,\sigma)$ is a $K \times 1$ vector where each element k is the ex-ante expected value of the choice-specific payoff in state k, $\sum_j \sigma_{ijk}[\psi_{ijk} + e_{ijk}(\theta,\sigma)]$ where $e_{ijk}(\theta,\sigma)$ is the expected value of ε_{ijk} given that choice j is optimal,

$$\frac{1}{\sigma_{ijk}} \int \varepsilon_{ijk} \cdot 1\{\varepsilon_{ij'k} - \varepsilon_{ijk} \leq \psi_{ijk} - \psi_{ij'k} + V_{l(i,j,k)}(\theta,\sigma) - V_{l(i,j',k)}(\theta,\sigma)\ \forall j'\} f(\varepsilon_{ik}) d\varepsilon_{ik}.$$

We now have two important equations that forms a players' decision problem.

1. Bellman optimality: We stack Equation (2) and define the value function vector as $\Upsilon(\theta,\sigma)$.

$$\Upsilon(\theta,\sigma) \equiv V(\theta,\sigma) = \Big[(\rho + N\lambda)I - \lambda \sum_{m=1}^{N} \Sigma_m(\theta,\sigma_m) - Q_0\Big]^{-1} [u + \lambda E(\theta,\sigma)] \tag{3}$$

2. Conditional choice probability

$$\Gamma(v) \equiv \sigma$$

where σ is a $NJK \times 1$ vector with $\sigma_{ijk} = \Pr[\delta_i(k,\varepsilon) = j|k]$. Assuming ε follows i.i.d. $TIEV(0,1)$,

$$\sigma_{ijk} = \frac{\exp(\psi_{ijk} + V_{l(i,j,k)})}{\sum_{j'} \exp(\psi_{ij'k} + V_{l(i,j',k)})} \tag{4}$$

We now introduce a policy iteration operator Ψ in the space of conditional choice probabilities. Substituting (1) into (2),

$$\sigma = \Psi(\theta,\sigma) \equiv \Gamma(\Upsilon(\theta,\sigma)) \tag{5}$$

where Υ is a policy valuation operator that maps an $NJK \times 1$ vector of conditional choice probabilities into an $NK \times 1$ vector in value function space. Γ is a policy improvement operator that maps an $NK \times 1$ vector in value function space into $NJK \times 1$ vector of conditional choice probabilities.

3 The NPL estimator in continuous time

3.1 NPL algorithm

For identification, we first introduce two more assumptions for identification and estimation with continuous time data observed only at fixed time intervals (Assumption 6, 7 in ABBE, 2016; Assumption 8, 9 in Blevins, 2016).

Assumption 6. *The mapping $Q \to \{Q_0, Q_1, \ldots, Q_N\}$ is known.*

Assumption 7 (Multiple equilibria)**.** *(i) In every market $m = 1, \ldots, M$, every agent expects a single equilibrium to be played which results an intensity matrix Q. (ii) The distribution of state transition at any time $t \in [0, \infty)$ is consistent with the intensity matrix $P(\Delta) = exp(\Delta Q)$.*

In continuous time models, there can exist two types of data. First type is continuous time data which is observed every instant. Stock prices can be an example close to this type. Second, some data generated by a continuous time model can be only observed at discrete times. For example, population can change any time during the year but annual population data is only observed once a year. We can construct two different likelihood functions for each type of data.

Fully observed continuous time data. Consider a dataset of observations in market m and n- th move, $\{k_{mn}, t_{mn} : m = 1, \ldots, M, n = 1, \ldots, T_m\}$ sampled in interval $[0, \bar{T}]$. State k_{mn} denotes the state immediately before state change at time t_{mn} and time t_{mn} is the time of n-th state change in market m. We assume that the observations are independently and identically distributed.

We follow the derivation in Section 6.1. in ABBE (2016) to construct the likelihood function. Let h denote a $K(K-1) + NJK$ vector of hazard rates for state change:

$$h = (q_{12}, q_{13}, \ldots, q_{K-1,K}, \lambda_{11}\sigma_{111}, \ldots, \lambda_{11}\sigma_{1Jk}, \ldots, \lambda_{N1}\sigma_{N11}, \ldots, \lambda_{NK}\sigma_{NJK}).$$

Since nature moves according to Poisson process with rate parameter q_{kl} and agents chooses action j after receiving a move opportunity according to Poisson process with rate parameter λ_{ik}, the probability that there is a state change within τ units of time is:

$$\Big(\sum_{l \neq k} q_{kl} + \sum_i \lambda_{ik} \sum_{j \neq 0} \sigma_{ijk} \Big) \exp\Big(-\tau \big(\sum_{l \neq k} q_{kl} + \sum_i \lambda_{ik} \sum_{j \neq 0} \sigma_{ijk} \big) \Big).$$

When there is a state change from state k, conditional probability that the change is due to agent i's action j is

$$\frac{\lambda_{ik}\sigma_{ijk}}{\sum_{l \neq k} q_{kl} + \sum_i \lambda_{ik} \sum_{j \neq 0} \sigma_{ijk}}$$

and conditional probability that it is due to nature is

$$\frac{q_{kl}}{\sum_{l\neq k} q_{kl} + \sum_i \lambda_{ik} \sum_{j\neq 0} \sigma_{ijk}}.$$

Denoting $g(\tau, k; h) = \exp\big(- \tau(\sum_{l\neq k} q_{kl} + \sum_i \lambda_{ik} \sum_{j\neq 0} \sigma_{ijk})\big)$, the likelihood of state change is

$$\begin{cases} \lambda_{ik}\sigma_{ijk} g(\tau, k; h) & \text{if agent } i \text{ chooses action } j \\ q_{kl} g(\tau, k; h) & \text{if nature moves.} \end{cases} \tag{6}$$

Let $\tau_{m,T+1}$ be the time interval between the last state change at time T and $\bar{T}$, and $k_{m,T+1} \equiv k_{m,T}$. We introduce an indicator function $I_{mn}(i,j)$ which is 1 when n-th state change at market m is induced by agent i's action j and 0 otherwise. We also define $I_{mn}(0,l)$ as 1 when n-th state change at market m is by nature's move to state l and 0 otherwise.

Combining the results above, the likelihood function becomes

$$\begin{aligned} L(\theta, h) = \frac{1}{M} \sum_{m=1}^{M} \Big[\sum_{n=1}^{T} \Big\{ \ln g(\tau, k; h) + \sum_{l\neq k_{mn}} I_{mn}(0,l) \ln q_{k_{mn},l} + \sum_i \sum_{j\neq 0} I_{mn}(i,j) \ln(\lambda_{imk}\sigma_{ijk}) \Big\} \\ + \ln g(\tau_{m,T+1}, k_{m,T+1}; h) \Big]. \end{aligned} \tag{7}$$

The first term inside the summation is the common term in Equation (6) and the second term is from the likelihood of state change due to nature. The third term is considering agents' moves and the last term is the likelihood of no state change after time $t_{m,T}$.

Since state changes due to nature is exogenous, the log-likelihood function of the model can be decomposed into conditional choice probability and nature's transition probability terms. Consistent estimates for elements of Q_0, $q = (q_{12}, \ldots, q_{K-1,K})$, can be obtained from transition data without having to solve the Markov decision model. Therefore, we assume that q is known and focus on the estimation of θ and $\sigma = (\sigma_{111}, \ldots, \sigma_{NJK})$. Then, the likelihood function becomes

$$L_M(\theta, \sigma) = \frac{1}{M} \sum_{m=1}^{M} \Big[\sum_{n=1}^{T} \Big\{ \ln g(\tau, k; \sigma) + \sum_i \sum_{j\neq 0} I_{mn}(i,j) \ln(\lambda_{imk}\sigma_{ijk}) \Big\} + \ln g(\tau_{m,T+1}, k_{m,T+1}; \sigma) \Big].$$

Discretely observed continuous time data. Consider a dataset of $\{k_{mn} : m = 1, \ldots, M, n = 0, \ldots, T\}$ which are sampled at times on the lattice $\{n\Delta : n = 0, \ldots, T\}$. We also assume that q is known as above. Given q and σ, we can construct a transition matrix. Define the pseudo likelihood function

$$L_M(\theta, \sigma) = \frac{1}{M} \sum_{m=1}^{M} \sum_{n=1}^{T} \ln P_{k_{m,n-1}, k_{mn}}(\Delta; \Psi(\theta, \sigma)) \tag{8}$$

where $P_{k,l}(\Delta; \Psi(\theta, \sigma))$ denotes the (k, l) element of the transition matrix induced by $\sigma = \Psi(\theta, \sigma)$.

We calculate $P(\Delta)$ using matrix exponential as suggested in ABBE (2016). First, assume that we have a continuous time data observed only at discrete times with unit interval, $\Delta = 1$. Let $Z(\theta, \sigma)^r$ denote the transition matrix from some state k to another state k' after an unit interval of time $\Delta = 1$ in r steps. Then, let $a_{m,n}$ denote an indicator vector for an observation n in market m which is one in position $k_{m,n}$ and zero elsewhere. We can write the likelihood function as below:

$$L_M(\theta, \sigma) = \frac{1}{M} \sum_{m=1}^{M} \sum_{n=1}^{T} \ln \Big[\sum_{r=0}^{\infty} \frac{\lambda^r \exp(-\lambda)}{r!} a'_{m,n} Z(\theta, \sigma)^r a_{m,n+1} \Big]. \tag{9}$$

Then, using the definition of matrix exponential, we can rewrite the equation above.

$$L_M(\theta, \sigma) = \frac{1}{M} \sum_{m=1}^{M} \sum_{n=1}^{T} \ln[a'_{m,n} \exp(Q(\theta, \sigma)) a_{m,n+1}]. \tag{10}$$

where Q is the aggregate intensity matrix for the model.

NPL algorithm. We propose the nested pseudo likelihood algorithm to obtain the NPL estimator in continuous time setting. Let $\hat{\sigma}^0$ be an initial guess of the vector of players' choice probabilities. Given $\hat{\sigma}^0$, for $l \geq 1$,

1. Given $\hat{\sigma}^{l-1}$, update $\hat{\theta}$ by
$$\hat{\theta}^l = \operatorname*{argmax}_{\theta \in \Theta} L_M(\theta, \hat{\sigma}^{l-1}) \tag{11}$$

2. Update $\hat{\sigma}$ using the equilibrium condition, i.e.
$$\hat{\sigma}^l = \Psi(\hat{\theta}^l, \hat{\sigma}^{l-1}) \tag{12}$$

Iterate in l until convergence in σ and θ is reached. Since we usually have observe continuous time data at discrete times in practice, we focus on properties of the NPL estimator using the likelihood function in Equation (7) in next section.

Remark on the definition of the NPL estimator. Aguirregabiria and Mira (2002) define $\hat{\theta}^l$ as policy iteration (PI) estimator and the estimation procedure as the NPL algorithm. In Aguirregabiria and Mira (2007), they define the same estimator as K-stage estimator[3] and define the NPL estimator as the limit of the sequence $\{\hat{\theta}^l, \hat{\sigma}^l\}_{l=1}^L$ with certain conditions. Specifically, if the sequence $\{\hat{\theta}^l, \hat{\sigma}^l\}_{l=1}^L$ converges, its limit $(\tilde{\theta}, \tilde{\sigma})$, which is called NPL fixed point, satisfies two conditions:

$$\tilde{\theta} = \underset{\theta \in \Theta}{\operatorname{argmax}}\, L(\theta, \tilde{\sigma}) \quad \text{and} \quad \tilde{\sigma} = \Psi(\tilde{\theta}, \tilde{\sigma}).$$

It is possible that there exist multiple NPL fixed points, so they choose the one with highest maximum likelihood and define it as the NPL estimator, $(\hat{\theta}_{NPL}, \hat{\sigma}_{NPL})$. Subsequent studies use the definition of NPL estimator in Aguirregabiria and Mira (2007), so we follow the same definition (Pesendorfer and Schmidt-Dengler, 2010; Kasahara and Shimotsu, 2012). Henceforth, we define $(\hat{\theta}^l, \hat{\sigma}^l)$ as the PI estimator and $(\hat{\theta}, \hat{\sigma}) = (\hat{\theta}_{NPL}, \hat{\sigma}_{NPL})$ as the NPL estimator dropping the subscript for simplicity.

3.2 Large sample properties

In this section, we present large sample properties of NPL estimator in continuous time. We start from the PI estimator in single agent discrete choice models and expand to the PI estimator and the npl estimator in dynamic discrete games.

3.2.1 Single agent discrete choice models

Proposition 1 shows that under certain regularity conditions, the NPL estimator in a continuous time single agent discrete choice model has consistency and asymptotic normality. All proofs are provided in Appendix.

Proposition 1. *Let σ^* be the true set of conditional choice probabilities and let Θ and $\Sigma \equiv [0,1]^{JK}$ be the set of possible values of θ and σ, respectively. Consider the following regularity conditions:*

(a) Θ is a compact set.

(b) $\Psi(\theta, \sigma)$ is continuous and twice continuously differentiable in θ and σ.

(c) Let $\Psi_{jk}(\theta, \sigma)$ be the corresponding element of $\Psi(\theta, \sigma)$ for some choice j and state k. Then, $\Psi_{jk}(\theta, \sigma) > 0$ for any $j = 0, 1, \ldots, J-1$, $k = 1, \ldots, K$, and any $\{\theta, \sigma\} \in \Theta \times \Sigma$.

(d) There is a unique $\theta^* \in \text{int}(\Theta)$ such that $\sigma^* = \Psi(\theta^*, \sigma^*)$.

(e) $\hat{\sigma}^0$ is a strongly consistent estimator of σ^* such that

[3] They call it K-stage estimator as they use K as a notation for the number of stages. They also define $(\hat{\theta}^1, \hat{\sigma}^1)$ as the Pseudo Maximum Likelihood (PML) estimator.

$$\left[\sqrt{M}\nabla_\theta L_M(\theta^*,\sigma^*); \sqrt{M}(\hat{\sigma}^0-\sigma^*)'\right]' \xrightarrow{d} N(0,\Omega)$$

Then,

$$\sqrt{M}(\hat{\theta}^l-\theta^*) \xrightarrow{d} N(0,\Omega_{\theta\theta'}^{-1}) \tag{13}$$

where $\Omega_{\theta\theta'} = E[\nabla_\theta s_m \nabla_{\theta'} s_m]$ *and* s_m *is defined as*

$$s_m \equiv \sum_{n=1}^{T} \ln P_{k_{m,n-1}k_{mn}}\big(\Delta;\Psi(\theta^*,\sigma^*)\big).$$

Assumption (a), (b), and (c) are usual regularity conditions, and (d) is needed for identification. Assumption (e) requires that the initial value for estimation is consistent. As seen from the asymptotic distribution in Equation (13), the asymptotic variance is not affected by estimates for CCPs. This means that there is no inefficiency arising from CCP estimation in a single agent model.

3.2.2 Dynamic discrete games

We now extend the model to a dynamic discrete game with N players and present some features for the PI estimator in games.

Proposition 2. *Let* σ^* *be the true set of conditional choice probabilities and let* Θ *and* $\Sigma \equiv [0,1]^{NJK}$ *be the set of possible values of* θ *and* σ*, respectively. Consider the following regularity conditions:*

(a) Θ *is a compact set.*

(b) $\Psi(\theta,\sigma)$ *is continuous and twice continuously differentiable in* θ *and* σ*.*

(c) $\Psi_{ijk}(\theta,\sigma)$ *be the corresponding element of* $\Psi(\theta,\sigma)$ *for player i's choice j at state k. Then,* $\Psi_{ijk}(\theta,\sigma) > 0$ *for any* $i = 1,\ldots,N$, $j = 0,1,\ldots,J-1$, $k = 1,\ldots,K$, *and any* $\{\theta,\sigma\} \in \Theta\times\Sigma$.

(d) There is a unique $\theta^* \in \text{int}(\Theta)$ *such that* $\sigma^* = \Psi(\theta^*,\sigma^*)$.

(e) $\hat{\sigma}^0 = (1/M)\sum_{m=1}^{M} r_m$ *is a strongly consistent estimator of* σ^* *such that* $\sqrt{M}(\hat{\sigma}^0-\sigma^*) \xrightarrow{d} N(0,\Sigma^0)$.

Then, for all $l \leq L$,

$$\sqrt{M}(\hat{\theta}^l-\theta^*) \xrightarrow{d} N\big(0,\Omega_{\theta\theta'}^{-1}+\Omega_{\theta\theta'}^{-1}\Omega_{\theta\sigma'}\Sigma^{l-1}\Omega'_{\theta\sigma'}\Omega_{\theta\theta'}^{-1}\big)$$

where $\Omega_{\theta\theta'} = E[\nabla_\theta s_m \nabla_{\theta'} s_m]$, $\Omega_{\theta\sigma'} = E[\nabla_\theta s_m \nabla_{\sigma'} s_m]$, Σ^l *is the asymptotic variance of* $\sqrt{M}(\hat{\sigma}^l - \sigma^*)$*, and* $s_m = \sum_{n=1}^{T} P_{k_{m,n-1},k_{m,n}}(\Delta;\Psi(\theta,\sigma))$*. Moreover, for* $l \leq L$, $V^l - V^{l+1}$ *is positive definite* *where* $V^l = \Omega_{\theta\theta'}^{-1} + \Omega_{\theta\theta'}^{-1}\Omega_{\theta\sigma'}\Sigma^l\Omega'_{\theta\sigma'}\Omega_{\theta\theta'}^{-1}$.

The results above are for the PI estimator $\hat{\theta}^l$, $l = 1, \ldots, L$. In Aguirregabiria and Mira (2007), they provide large sample properties for the NPL estimator. We also present similar properties for $\hat{\theta}$ in our continuous time model. A local convergence condition for $\hat{\theta}^l$ will be provided in the next section.

We first introduce definitions and notations in Aguirregabiria and Mira (2007). The NPL operator $\phi_M(\cdot)$ is used to describe NPL algorithm by a single operator:

$$\phi_M(\sigma) = \Psi(\tilde{\theta}_M(\sigma), \sigma) \quad \text{where} \quad \tilde{\theta}_M(\sigma) \equiv \underset{\theta \in \Theta}{\operatorname{argmax}}\, L_M(\theta, \sigma).$$

Let Y be the set of NPL fixed points, $Y \equiv \{(\theta, \sigma) \in \Theta \times [0,1]^{NJK} : \theta = \tilde{\theta}(\sigma)$ and $\sigma = \phi(\sigma)\}$. Then, choosing the NPL fixed point that has the maximum value of the likelihood,

$$(\hat{\theta}, \hat{\sigma}) \equiv \underset{(\theta,\sigma) \in Y}{\operatorname{argmax}}\, L_M(\theta, \sigma).$$

Proposition 3 shows large sample properties for $(\hat{\theta}, \hat{\sigma})$. We denote the counterparts of $\tilde{\theta}_M(\sigma)$ and $\tilde{\phi}_M(\sigma)$ as $\tilde{\theta}^*(\sigma) \equiv \operatorname{argmax}_{\theta \in \Theta} L(\theta, \sigma)$ and $\phi^*(\sigma) = \Psi(\tilde{\theta}^*(\sigma), \sigma)$

Proposition 3. *Let σ^* be the true set of conditional choice probabilities and let Θ and $\Sigma \equiv [0,1]^{NJK}$ be the set of possible values of θ and σ, respectively. Consider the following regularity conditions:*

(a) Θ is a compact set.

(b) $\Psi(\theta, \sigma)$ is continuous and twice continuously differentiable in θ and σ.

(c) $\Psi_{ijk}(\theta, \sigma)$ be the corresponding element of $\Psi(\theta, \sigma)$ for player i's choice j at state k. Then, $\Psi_{ijk}(\theta, \sigma) > 0$ for any $i = 1, \ldots, N$, $j = 1, \ldots, J-1$, $k = 1, \ldots, K$, and any $\{\theta, \sigma\} \in \Theta \times \Sigma$.

(d) There is a unique $\theta^* \in \operatorname{int}(\Theta)$ such that $\sigma^* = \Psi(\theta^*, \sigma^*)$.

(e) (θ^*, σ^*) is an isolated population NPL fixed point.

(f) There exists a closed neighborhood of σ^*, $\mathcal{N}(\sigma)$, such that, for all $\sigma \in \mathcal{N}(\sigma^*)$, $L(\theta, \sigma)$ is globally concave in θ and $\partial^2 L(\theta, \sigma^*)/\partial\theta\partial\theta'$ is a nonsingular matrix.

(g) The operator $\phi^*(\sigma) - \sigma$ has a nonsingular Jacobian matrix.

Then,

$$\sqrt{M}(\hat{\theta} - \theta^*) \xrightarrow{d} N\big(0, [\Omega_{\theta\theta'} + \Omega_{\theta\sigma'}(I - \Psi_\sigma)^{-1}\Psi_\theta]^{-1}\Omega_{\theta\theta'}[\Omega_{\theta\theta'} + \Psi'_\theta(I - \Psi'_\sigma)^{-1}\Omega'_{\theta\sigma'}]^{-1}\big)$$

where $\Psi_\theta = \nabla_\theta \Psi(\theta, \sigma)$ *and* $\Psi_\sigma = \nabla_\sigma \Psi(\theta, \sigma)$.

Assumptions from (a) to (d) is same as Assumption (a)-(d) in Proposition 2 and

Assumption (e), (f), and (g) corresponds to Assumption (v), (vi), and (vii) in Proposition 2 of Aguirregabiria and Mira (2007). Note that it does not require the consistency of $\hat{\sigma}^0$.

3.3 Convergence of the NPL algorithm

The NPL algorithm is a sequential method to find the NPL estimator. This is why there is a probability for the NPL algorithm to fail to converge to a consistent estimator even if the NPL estimator is consistent under regularity conditions. Ideally, the NPL algorithm could converge to the consistent estimator if it can be evaluated at every fixed point which is not often possible in practice. Starting from Psendorfer and Shmidt-Dengler (2010), there has been papers that covered the convergence issue of the NPL algorithm.

Kasahara and Shimotsu (2012) show that a key determinant of the convergence of the NPL algorithm is the contraction property of the mapping Ψ and propose a local convergence condition for NPL estimator in discrete time. We extend their argument to continuous time and provide an analogous condition in continuous time.

For simplicity, we define $\Psi_\theta^* \equiv \nabla_{\theta'}\Psi(\theta^*, \sigma^*)$ and $\Psi_\sigma^* \equiv \nabla_{\sigma'}\Psi(\theta^*, \sigma^*)$. Let P be an aggregate transition probability matrix between states. Then, each element P_{kl} of P becomes

$$P_{kl} = \begin{cases} \lambda_{ik}\sigma_{ijk} \quad \text{where } l(i,j,k) = l & \text{if state change from } k \text{ to } l \text{ is endogenous.} \\ [\exp(Q_0)]_{kl} & \text{if state change from } k \text{ to } l \text{ is exogenous.} \end{cases}.$$

Let P^* denote a $K^2 \times 1$ vector which is a vectorized version of $P(\Delta; \Psi(\theta^*, \sigma^*))$, so, $P^* = (P_{11}, P_{12}, \ldots, P_{KK})$. Then, define a scalar $a_{ij,kl}$ as:

$$a_{ij,kl} = \begin{cases} 1 & \text{if state change from } k \text{ to } l \text{ is induced by agent } i\text{'s choice } j \\ 0 & \text{otherwise.} \end{cases}$$

We construct a $NJK \times K^2$ matrix, A, by stacking $a_{ij,kl}$ accordingly and denote $\Delta_\sigma = A\,\text{diag}(P^*)^{-1}A'$.Then, we can rewrite $\Omega_{\theta\theta'} = \Psi_\theta^{*\prime}\Delta_\sigma\Psi_\theta^*$ and $\Omega_{\theta\sigma'} = \Psi_\theta^{*\prime}\Delta_\sigma\Psi_\sigma^*$. We also define $M_{\Psi_\theta} \equiv I - \Psi_\theta^*(\Psi_{\theta'}^*\Delta_\sigma\Psi_\theta^*)^{-1}\Psi_{\theta'}^*\Delta_\sigma$.

Assumption 8. *(a) Assumptions in Proposition 3 hold. (b) $\Psi(\theta, \sigma)$ is three times continuously differentiable in a closed neighborhood $\mathcal{N}$ of (θ^*, σ^*). (c) $\Omega_{\theta\theta'}$ is nonsingular.*

We now present a local convergence condition for NPL estimator $\hat{\sigma}^l$. If $\hat{\sigma}^l$ converges to $\hat{\sigma}$, $\hat{\theta}^l$ converges to $\hat{\theta}$ by the continuity of $L(\theta, \sigma)$ and applying the Theorem of Maximum in Equation (11). So, it is sufficient to show the convergence for $\hat{\sigma}^l$. The convergence condition below is analogous to Proposition 2 in Kasahara and Shimotsu (2012).

Proposition 4. *Suppose that Assumption 8 holds and that $\rho(M_{\Psi_\theta}\Psi_\sigma^*) < 1$. Then, there exists a neighborhood $\mathcal{N}_1$ of σ^* such that, for any initial value $\hat{\sigma}^0 \in \mathcal{N}_1$, we have $\lim_{L\to\infty} \hat{\sigma}^l = \hat{\sigma}$ almost surely.*

We now present the zero Jacobian property for $\Psi(\theta, \sigma)$ to show that $\Psi_\sigma^* = 0$ in a single agent model which makes $\rho(M_{\Psi_\theta}\Psi_\sigma^*) < 1$ satisfied.

Proposition 5. *In a single agent model, the Jacobian matrix Ψ_σ of $\Psi(\theta, \sigma)$ is zero at the fixed point σ.*

Since the Jacobian matrix Ψ_σ is zero at the fixed point and $\sigma^* = \Psi(\theta^*, \sigma^*)$, $\rho(M_\Psi \Psi_\sigma^*)$ is zero. Then, by Proposition 4, the NPL estimator always converges to $(\hat{\theta}, \hat{\sigma})$ in a single agent dynamic choice model in continuous time.

4 Monte Carlo experiments

In this section, we present Monte Carlo experiments using the dynamic discrete game described in Section 4 of Aguirregabiria and Mira (2007) which is presented as Example 1 in Section 2.

4.1 Data generating process

Throughout the experiments, we set the move arrival rate $\lambda_{ik} = 1$ and the discount rate $\rho_{ik} = 0.05$ for $i = 1, \ldots, N$ and $k = 1, \ldots, K$. We also assume that the market size moves by 1 at an instant and is constant for every state k. So, q_1 and q_2, which are the rate for increase in market size and the rate for decrease, respectively, fully characterize the nature's intensity matrix Q_0. As mentioned above, we can obtain consistent estimates for q_1 and q_2, so, we assume that we know Q_0 and focus on estimating parameters in firms' payoffs. We use the same transition probability matrix for market size as the discrete time model in Aguirregabiria and Mira (2007)[4].

[4] The transition probability matrix used for market size s_k is

$$\begin{bmatrix} 0.8 & 0.2 & 0.0 & 0.0 & 0.0 \\ 0.2 & 0.6 & 0.2 & 0.0 & 0.0 \\ 0.0 & 0.2 & 0.6 & 0.2 & 0.0 \\ 0.0 & 0.0 & 0.2 & 0.6 & 0.2 \\ 0.0 & 0.0 & 0.0 & 0.2 & 0.8 \end{bmatrix}.$$

The profit function in Equation (1) is used for estimation:

$$u_{ik} = \theta_{RS} \ln(s_k) - \theta_{RN} \ln \big(1 + \sum_{m \neq i} a_{mk}\big) - \theta_{FC,i}.$$

The parameters we estimate are fixed costs $\theta_{FC,i}$, entry costs θ_{EC}, competition effect θ_{RN}, and market effect θ_{RS}. Fixed costs $\theta_{FC,i}$ are set as $(\theta_{FC,1}, \theta_{FC,2}, \theta_{FC,3}, \theta_{FC,4}, \theta_{FC,5}) = (-1.9, -1.8, -1.7, -1.6, -1.5)$. The parameter θ_{RS} equals to 1. We run six experiments varying by entry costs $\theta_{EC} = (1.0, 2.0, 4.0)$ and $\theta_{RN} = (0.0, 1.0, 2.0)$. The stochastic component in instantaneous payoff ϵ_{ijk} are independently and identically distributed and follow Type 1 extreme value distribution. The support of the logarithm of market size is $\{1, 2, 3, 4, 5\}$ and each player chooses the number of stores from $a_{ik} = \{0, 1\}$. So, the size of state space is $K = 5 \times 2^5 = 160$.

Two different types of data are used for estimation. First, we use continuous time data observed every instant. To generate the continuous time data, we calculate the MPE of this game and obtain the steady-state distribution of each state. Then, we draw 50,000 initial states $k_{m,0}$ using the steady-state distribution. Then, from the initial states, we can draw the next state, $(a_{k_{m,n}} | x_{k_{m,n-1}})$ using the equilibrium conditional choice probabilities. Second, continuous time data observed in fixed time intervals is used. In this case, initial states are also drawn using the steady-state distribution. Then, we obtain a transition probability matrix P over a unit interval using $Q(\Delta)$. We use P to obtain the next states.

For each experiment, we draw a sample of 400 markets for one period of time. Table 1 presents the MPE in the data generating process when we observe the full continuous time data. First three experiments differ only by strategic interactions, θ_{RN}, which is increasing from Experiment 1 to 3. Increase in θ_{RN} decreases average number of active firms. Its effect on average number of entrants and exits is uncertain because the probability of entry (exit) is increasing (decreasing) but average number of inactive (active) firms is decreasing. In this setting, both average number of entrants and exits are increasing when θ_{RN} increases. As excess turnover increases, there will be a larger change in the number of active firms from previous to current period. As a result, AR(1) coefficient for current active firms decreases. As θ_{RN} decreases the flow payoff for all firms, firms are less likely to be active regardless of fixed costs.

Table 1: Description of the steady state in the DGP (Fully observed continuous data)

Descriptive Statistics	Exp. 1 $\theta_{EC} = 1.0$ $\theta_{RN} = 0.0$	Exp. 2 $\theta_{EC} = 1.0$ $\theta_{RN} = 1.0$	Exp. 3 $\theta_{EC} = 1.0$ $\theta_{RN} = 2.0$	Exp. 4 $\theta_{EC} = 0.0$ $\theta_{RN} = 1.0$	Exp. 5 $\theta_{EC} = 2.0$ $\theta_{RN} = 1.0$	Exp. 6 $\theta_{EC} = 4.0$ $\theta_{RN} = 1.0$
Average #active firms[1]	3.7160	2.7702	2.0453	2.7342	2.8052	2.8250
	(1.4647)	(1.5940)	(1.3713)	(1.4631)	(1.7091)	(1.8405)
AR(1) for #active[2]	0.7333	0.6893	0.4969	0.4599	0.8255	0.9342
Average #entrants	0.5468	0.7212	0.7755	1.0287	0.4793	0.2140
Average #exits	0.5412	0.7246	0.7779	1.0253	0.4785	0.2118
Excess turnover[3]	0.3778	0.4948	0.5140	0.8847	0.2294	0.0350
Correlation between entries and exits	0.0085	-0.1991	-0.2738	-0.2850	-0.1473	-0.1000
Prob. of being active						
Firm 1	0.7077	0.4983	0.3344	0.5049	0.4892	0.4582
Firm 2	0.7235	0.5256	0.3700	0.5233	0.5233	0.5044
Firm 3	0.7455	0.5574	0.4094	0.5495	0.5622	0.5565
Firm 4	0.7620	0.5792	0.4445	0.5681	0.5959	0.6172
Firm 5	0.7773	0.6098	0.4870	0.5883	0.6346	0.6886

[1] Values in parentheses are standard deviations.
[2] AR(1) for #active is the autoregressive coefficient regressing the number of current active firms on the number of active firms in previous period.
[3] Excess turnover is defined as (#entrants + #exits) - abs(#entrants - #exits).

In Experiments 4 to 6, θ_{EC} varies by 2. The number of active firms increases as θ_{EC} increases, which might be surprising, but this is due to the fact that probability of being active differs across firms. Specifically, more efficient firms tend to stay active as the entry costs increase as seen in Firm 4 and 5 with lower fixed costs.

Table 2 shows the MPE when we use the data only observed discretely. This is similar to the steady state using fully observed continuous time data since they share the same step of drawing the initial states but only differs in the way of drawing states after the transition over a unit interval as mentioned above.

Table 2: Description of the steady state in the DGP (Discretely observed continuous data)

Descriptive Statistics	Exp. 1 $\theta_{EC} = 1.0$ $\theta_{RN} = 0.0$	Exp. 2 $\theta_{EC} = 1.0$ $\theta_{RN} = 1.0$	Exp. 3 $\theta_{EC} = 1.0$ $\theta_{RN} = 2.0$	Exp. 4 $\theta_{EC} = 0.0$ $\theta_{RN} = 1.0$	Exp. 5 $\theta_{EC} = 2.0$ $\theta_{RN} = 1.0$	Exp. 6 $\theta_{EC} = 4.0$ $\theta_{RN} = 1.0$
Average #active firms[1]	3.7107	2.7744	2.0468	2.7351	2.8027	2.8214
	(1.4427)	(1.5338)	(1.2510)	(1.3921)	(1.6612)	(1.8139)
AR(1) for #active firms[2]	0.8012	0.7879	0.6909	0.6720	0.8648	0.9381
Average #entrants	0.3783	0.5024	0.5388	0.6514	0.3653	0.1861
Average #exits	0.3779	0.5008	0.5385	0.6464	0.3667	0.1870
Excess turnover[3]	0.2025	0.3096	0.3798	0.4770	0.1768	0.0413
Correlation between entries and exits	-0.0030	-0.0859	-0.0669	-0.1240	-0.0607	-0.0545
Prob. of being active						
Firm 1	0.7030	0.4980	0.3352	0.5032	0.4878	0.4567
Firm 2	0.7237	0.5286	0.3694	0.5263	0.5244	0.5045
Firm 3	0.7449	0.5530	0.4115	0.5468	0.5611	0.5549
Firm 4	0.7602	0.5806	0.4443	0.5687	0.5953	0.6179
Firm 5	0.7790	0.6141	0.4863	0.5902	0.6341	0.6875

[1] Values in parentheses are standard deviations.
[2] AR(1) for #active is the autoregressive coefficient regressing the number of current active firms on the number of active firms in previous period.
[3] Excess turnover is defined as (#entrants + #exits) - abs(#entrants - #exits).

4.2 Estimation results

We use different initial values for initial choice probabilities: (i) true conditional choice probabilities, (ii) frequency estimates, (iii) semi-parametric estimates from a logit model using market size and the number of rival firms as independent variables, and (iv) random draws from uniform distribution $(0, 1)$. For (ii), (iii), and (iv), we both estimate the two-step estimator by setting the number of iterations $l = 1$ and the NPL estimator which iterates until it converges.

Table 3 shows the results for continuous time data. First, NPL estimators initialized from different estimates, frequency estimator, logit estimator, and random draws, converged to same estimates within 20 iterations. So, we only report one result for the NPL estimates, denoted as "NPL" in the table. This result suggests that NPL estimators converge regardless of consistency of the initial estimates we used in this setting.

Another point to notice is that the NPL estimator shows similar performance as two-step PML estimator using true choice probabilities. Table 4 presents the square-root mean square error (MSE) relative to the two-step PML estimates using true CCPs. All

MSEs for the NPL estimator are close to one and they are lower than MSEs from other estimators in most cases. Since two-step PML estimators are infeasible in practice, the NPL estimator can be the best alternative for similar performance.

Compared to two-step estimator initialized from logit estimates, the NPL estimates perform better in terms of standard deviations. For all experiments, biases in θ_{RN} are lower in the NPL estimates than in the two-step estimates. This is because the NPL estimator repeatedly impose the MPE condition leading to relative efficiency and more precise estimates for strategic interaction.

Even though the two-step PML using frequency estimator uses consistent estimates for the initial guess, it shows a large bias across all parameters. The asymptotic variance is not high, and sometimes lower than two-step estimators using random estimates but the bias is consistently larger causing relatively larger MSEs in Table 4. This finite sample bias can be lowered by iterating the estimation which can be seen in results for the NPL estimator.

Table 3: Monte Carlo experiment results (Fully observed continuous data)

Exp.	Estimator	Parameters			
		$\theta_{FC,1}$	θ_{RS}	θ_{EC}	θ_{RN}
1	**True values**	**-1.9000**	**1.0000**	**1.0000**	**0.0000**
	2S-True	-1.9112 (0.2933)	1.0052 (0.0954)	1.0274 (0.1342)	0.0169 (0.2802)
	2S-Freq	-0.3623 (0.2410)	0.3700 (0.1126)	1.7908 (0.1435)	0.3567 (0.3015)
	2S-Logit	-1.9136 (0.2948)	1.0071 (0.0952)	1.0272 (0.1346)	0.0194 (0.2805)
	2S-Random	-2.1992 (0.3977)	1.1214 (0.1031)	1.0314 (0.1352)	-0.0427 (0.3415)
	NPL	-1.9144 (0.2950)	1.0072 (0.0953)	1.0273 (0.1345)	0.0190 (0.2819)
2	**True values**	**-1.9000**	**1.0000**	**1.0000**	**1.0000**
	2S-True	-1.9099 (0.2009)	1.0016 (0.0953)	1.0183 (0.1306)	1.0137 (0.2643)
	2S-Freq	-0.8255 (0.1896)	0.4671 (0.1117)	1.4694 (0.1211)	0.5877 (0.2860)
	2S-Logit	-1.8932 (0.2113)	1.0005 (0.1023)	1.0185 (0.1304)	1.0251 (0.2807)
	2S-Random	-1.7843 (0.2889)	0.9587 (0.0724)	1.0224 (0.1309)	1.0140 (0.3030)
	NPL	-1.9163 (0.2078)	1.0032 (0.1030)	1.0184 (0.1304)	1.0145 (0.2761)
3	**True values**	**-1.9000**	**1.0000**	**1.0000**	**2.0000**
	2S-True	-1.9401 (0.1770)	1.0117 (0.0808)	1.0070 (0.0960)	2.0182 (0.2665)
	2S-Freq	-1.1756 (0.2523)	0.5865 (0.1211)	1.3317 (0.1055)	1.1303 (0.3315)
	2S-Logit	-1.8700 (0.2034)	1.0077 (0.0916)	1.0070 (0.0962)	2.0714 (0.2999)
	2S-Random	-1.2428 (0.2885)	0.8001 (0.0624)	1.0215 (0.0928)	1.8789 (0.3228)
	NPL	-1.9488 (0.2013)	1.0147 (0.0930)	1.0070 (0.0961)	2.0196 (0.2867)
4	**True values**	**-1.9000**	**1.0000**	**0.0000**	**1.0000**
	2S-True	-1.9208 (0.2382)	1.0219 (0.0950)	-0.0071 (0.1149)	1.0665 (0.3163)
	2S-Freq	-0.9979 (0.2103)	0.5025 (0.1096)	0.4399 (0.1097)	0.6593 (0.3496)
	2S-Logit	-1.9081 (0.2458)	1.0233 (0.1015)	-0.0073 (0.1146)	1.0835 (0.3347)
	2S-Random	-1.5546 (0.2965)	0.8156 (0.0697)	0.0249 (0.1138)	0.8344 (0.2922)
	NPL	-1.9260 (0.2410)	1.0247 (0.1021)	-0.0071 (0.1149)	1.0727 (0.3297)
5	**True values**	**-1.9000**	**1.0000**	**2.0000**	**1.0000**
	2S-True	-1.9320 (0.1902)	1.0175 (0.1033)	2.0155 (0.1334)	1.0336 (0.2189)
	2S-Freq	-0.7240 (0.2086)	0.4921 (0.1171)	2.4653 (0.1393)	0.6563 (0.2428)
	2S-Logit	-1.9079 (0.2029)	1.0165 (0.1118)	2.0152 (0.1337)	1.0493 (0.2381)
	2S-Random	-2.1011 (0.3297)	1.1627 (0.0987)	1.9982 (0.1292)	1.3001 (0.3330)
	NPL	-1.9416 (0.2045)	1.0219 (0.1132)	2.0155 (0.1336)	1.0390 (0.2346)
6	**True values**	**-1.9000**	**1.0000**	**4.0000**	**1.0000**
	2S-True	-1.9277 (0.1966)	1.0167 (0.1094)	4.0347 (0.2232)	1.0180 (0.1934)
	2S-Freq	-0.5088 (0.1907)	0.4460 (0.1182)	4.4598 (0.2255)	0.5811 (0.2107)
	2S-Logit	-1.8957 (0.2064)	1.0113 (0.1205)	4.0319 (0.2237)	1.0291 (0.2098)
	2S-Random	-2.8593 (0.4994)	1.6734 (0.1611)	3.9831 (0.2210)	2.0784 (0.5526)
	NPL	-1.9448 (0.2087)	1.0214 (0.1206)	4.0332 (0.2232)	1.0180 (0.2040)

Note: Values in parentheses are standard deviations.

Table 4: Square-root MSEs relative to the two-step PML with true CCPs
(Fully observed continuous data)

Exp.	Estimator	Parameters			
		$\theta_{FC,1}$	θ_{RS}	θ_{EC}	θ_{RN}
1	2S-Freq	5.3031	6.6989	5.8694	1.6638
	2S-Logit	1.0056	0.9992	1.0027	1.0015
	2S-Random	1.6955	1.6671	1.0137	1.2260
	NPL	1.0064	1.0004	1.0022	1.0063
2	2S-Freq	5.4232	5.7112	3.6768	1.8962
	2S-Logit	1.0506	1.0727	0.9987	1.0650
	2S-Random	1.5469	0.8746	1.0070	1.1461
	NPL	1.0362	1.0806	0.9991	1.0450
3	2S-Freq	4.2254	5.2762	3.6157	3.4849
	2S-Logit	1.1325	1.1259	1.0016	1.1543
	2S-Random	3.9534	2.5644	0.9896	1.2909
	NPL	1.1408	1.1534	1.0012	1.0759
4	2S-Freq	3.8739	5.2251	3.9373	1.5103
	2S-Logit	1.0287	1.0685	0.9973	1.0674
	2S-Random	1.9037	2.0219	1.0114	1.0392
	NPL	1.0140	1.0778	0.9993	1.0445
5	2S-Freq	6.1921	4.9729	3.6155	1.9001
	2S-Logit	1.0525	1.0779	1.0018	1.0978
	2S-Random	2.0023	1.8152	0.9619	2.0241
	NPL	1.0818	1.0998	1.0015	1.0738
6	2S-Freq	7.0717	5.1198	2.2676	2.4137
	2S-Logit	1.0397	1.0937	1.0005	1.0901
	2S-Random	5.4465	6.2583	0.9815	6.2374
	NPL	1.0752	1.1074	0.9993	1.0540

Table 5 and 6 report the similar results for estimation using discretely observed continuous time data. The NPL estimates converge to the same estimates as before and they show similar results as the two-step PML using true CCPs as initial guess.

Table 5: Monte Carlo experiment results (Discretely observed continuous data)

Exp.	Estimator	Parameters			
		$\theta_{FC,1}$	θ_{RS}	θ_{EC}	θ_{RN}
1	**True values**	**-1.9000**	**1.0000**	**1.0000**	**0.0000**
	2S-True	-1.8696 (0.4613)	1.0321 (0.1864)	1.0126 (0.3056)	0.0710 (0.4025)
	2S-Freq	-0.2995 (0.2997)	0.3971 (0.1259)	1.9794 (0.3101)	0.3994 (0.2739)
	2S-Logit	-1.7440 (0.4069)	0.9755 (0.1655)	1.0055 (0.3129)	0.0860 (0.3060)
	2S-Random	-2.1636 (0.5416)	1.1508 (0.1539)	1.0134 (0.2922)	-0.0045 (0.4948)
	NPL	-1.8830 (0.4711)	1.0383 (0.1952)	1.0146 (0.3040)	0.0747 (0.4094)
2	**True values**	**-1.9000**	**1.0000**	**1.0000**	**1.0000**
	2S-True	-1.9433 (0.3396)	1.0385 (0.1613)	0.9871 (0.2539)	1.0938 (0.3765)
	2S-Freq	-0.6917 (0.2469)	0.4395 (0.1124)	1.6766 (0.2417)	0.5812 (0.2729)
	2S-Logit	-1.7812 (0.3104)	0.8945 (0.1339)	0.9858 (0.2555)	0.8286 (0.2973)
	2S-Random	-1.7902 (0.4500)	0.9946 (0.1114)	0.9748 (0.2538)	1.1124 (0.4437)
	NPL	-1.9558 (0.3573)	1.0461 (0.1775)	0.9880 (0.2527)	1.1051 (0.4033)
3	**True values**	**-1.9000**	**1.0000**	**1.0000**	**2.0000**
	2S-True	-1.9298 (0.2940)	1.0302 (0.1646)	0.9909 (0.2158)	2.0662 (0.4787)
	2S-Freq	-0.8447 (0.2504)	0.4236 (0.1133)	1.5584 (0.2076)	0.7669 (0.3106)
	2S-Logit	-1.6782 (0.2705)	0.8325 (0.1329)	0.9908 (0.2160)	1.5422 (0.3803)
	2S-Random	-1.2187 (0.3895)	0.8255 (0.1008)	0.9921 (0.2087)	1.9675 (0.5099)
	NPL	-1.9521 (0.3397)	1.0416 (0.1953)	0.9912 (0.2159)	2.0869 (0.5427)
4	**True values**	**-1.9000**	**1.0000**	**0.0000**	**1.0000**
	2S-True	-1.9361 (0.4290)	0.9941 (0.1813)	0.0073 (0.3143)	0.9462 (0.4605)
	2S-Freq	-0.6702 (0.2656)	0.3328 (0.0856)	0.8494 (0.2739)	0.3373 (0.2498)
	2S-Logit	-1.7521 (0.3658)	0.8327 (0.1387)	0.0044 (0.3153)	0.6578 (0.3278)
	2S-Random	-1.6005 (0.4331)	0.8264 (0.1074)	0.0254 (0.2973)	0.7897 (0.4397)
	NPL	-1.9443 (0.4427)	0.9993 (0.1939)	0.0068 (0.3146)	0.9524 (0.4843)
5	**True values**	**-1.9000**	**1.0000**	**2.0000**	**1.0000**
	2S-True	-1.9124 (0.2865)	1.0140 (0.1627)	1.9971 (0.2054)	1.0125 (0.3300)
	2S-Freq	-0.6527 (0.2805)	0.4642 (0.1309)	2.5805 (0.2464)	0.5930 (0.2421)
	2S-Logit	-1.7567 (0.2654)	0.8959 (0.1376)	1.9996 (0.2050)	0.8150 (0.2730)
	2S-Random	-2.0970 (0.3999)	1.1894 (0.1401)	1.9635 (0.2184)	1.3346 (0.4834)
	NPL	-1.9285 (0.3105)	1.0209 (0.1811)	1.9961 (0.2065)	1.0190 (0.3562)
6	**True values**	**-1.9000**	**1.0000**	**4.0000**	**1.0000**
	2S-True	-1.9634 (0.2460)	1.0438 (0.1470)	4.0477 (0.2482)	1.0649 (0.2560)
	2S-Freq	-0.7220 (0.3098)	0.5674 (0.1777)	4.4336 (0.2896)	0.7483 (0.2930)
	2S-Logit	-1.8252 (0.2365)	0.9631 (0.1388)	4.0460 (0.2452)	0.9616 (0.2545)
	2S-Random	-2.9300 (0.5703)	1.8220 (0.2097)	3.9921 (0.2574)	2.4322 (0.7285)
	NPL	-1.9921 (0.2694)	1.0570 (0.1675)	4.0473 (0.2497)	1.0799 (0.2934)

Note: Values in parentheses are standard deviations.

Table 6: Square-root MSEs relative to the two-step PML with true CCPs (Discretely observed continuous data)

		Parameters			
Exp.	Estimator	$\theta_{FC,1}$	θ_{RS}	θ_{EC}	θ_{RN}
1	2S-Freq	3.5224	3.2572	3.3591	1.1850
	2S-Logit	0.9426	0.8844	1.0232	0.7777
	2S-Random	1.3030	1.1393	0.9565	1.2108
	NPL	1.0197	1.0519	0.9951	1.0182
2	2S-Freq	3.6028	3.4469	2.8259	1.2885
	2S-Logit	0.9709	1.0280	1.0065	0.8844
	2S-Random	1.3533	0.6723	1.0030	1.1797
	NPL	1.0566	1.1059	0.9949	1.0742
3	2S-Freq	3.6707	3.5096	2.7585	2.6315
	2S-Logit	1.1837	1.2774	1.0009	1.2316
	2S-Random	2.6560	1.2038	0.9673	1.0574
	NPL	1.1632	1.1926	1.0006	1.1373
4	2S-Freq	2.9227	3.7093	2.8391	1.5275
	2S-Logit	0.9166	1.1981	1.0030	1.0222
	2S-Random	1.2232	1.1255	0.9492	1.0514
	NPL	1.0335	1.0692	1.0012	1.0497
5	2S-Freq	4.4578	3.3784	3.0697	1.4344
	2S-Logit	1.0517	1.0566	0.9978	0.9989
	2S-Random	1.5544	1.4431	1.0777	1.7804
	NPL	1.0871	1.1165	1.0055	1.0802
6	2S-Freq	4.7959	3.0490	2.0634	1.4625
	2S-Logit	0.9768	0.9366	0.9873	0.9747
	2S-Random	4.6357	5.5309	1.0191	6.0838
	NPL	1.1210	1.1537	1.0055	1.1515

4.3 Misspecified case

In this section, we simulate another experiment where the generated data is continuous but a discrete time model is estimated. We report the NPL estimator with random draws from uniform distribution used for initial choice probabilities in Table 7. We also present correct estimates for comparison from Table 5. As expected, bias increases in all estimates.

It is also noticeable that biases tend to be larger when entry costs decrease. Continuous time models allow moving sequentially between periods and there will be more moves when entry costs decrease. Since discrete time models restrict the moves to be simultaneous only once in the unit of time, biases will be larger when there are more entries between periods due to lower entry costs.

In the example, the NPL estimator yields larger bias in entry costs when the model is misspecified as a discrete time model. Continuous time models distinguish between flow payoffs and instantaneous payoffs where as discrete time models assume that agents receive all payoffs at once. Since entry costs are instantaneous payoffs received only when an agent makes a choice, they can cause larger bias when we do not distinguish two different types of payoffs.

Table 7: Misspecified case (Continuous time data and discrete time model estimation)

Exp.	Values	Parameters			
		$\theta_{FC,1}$	θ_{RS}	θ_{EC}	θ_{RN}
1	**True values**	**-1.9000**	**1.0000**	**1.0000**	**0.0000**
	Correct	-1.8830 (0.4711)	1.0383 (0.1952)	1.0146 (0.3040)	0.0747 (0.4094)
	Misspecified	-0.9369 (0.1928)	0.4604 (0.0762)	2.3993 (0.1555)	-0.1135 (0.2300)
	Bias	0.9631	-0.5396	1.3993	-0.1135
2	**True values**	**-1.9000**	**1.0000**	**1.0000**	**1.0000**
	Correct	-1.9558 (0.3573)	1.0461 (0.1775)	0.9880 (0.2527)	1.1051 (0.4033)
	Misspecified	-0.9221 (0.1444)	0.4176 (0.0889)	2.2785 (0.1083)	0.2574 (0.3024)
	Bias	0.9779	-0.5824	1.2785	-0.7426
3	**True values**	**-1.9000**	**1.0000**	**1.0000**	**2.0000**
	Correct	-1.9521 (0.3397)	1.0416 (0.1953)	0.9912 (0.2159)	2.0869 (0.5427)
	Misspecified	-0.8631 (0.1433)	0.4615 (0.0999)	2.1725 (0.1188)	0.9247 (0.3744)
	Bias	1.0369	-0.5385	1.1725	-1.0753
4	**True values**	**-1.9000**	**1.0000**	**0.0000**	**1.0000**
	Correct	-1.9443 (0.4427)	0.9993 (0.1939)	0.0068 (0.3146)	0.9524 (0.4843)
	Misspecified	-0.7994 (0.1909)	0.3607 (0.0887)	1.7383 (0.1084)	0.2212 (0.3696)
	Bias	1.1006	-0.6393	1.7383	-0.7788
5	**True values**	**-1.9000**	**1.0000**	**2.0000**	**1.0000**
	Correct	-1.9285 (0.3105)	1.0209 (0.1811)	1.9961 (0.2065)	1.0190 (0.3562)
	Misspecified	-1.0088 (0.1240)	0.4522 (0.0797)	2.9451 (0.1373)	0.2705 (0.2210)
	Bias	0.8912	-0.5478	0.9451	-0.7295
6	**True values**	**-1.9000**	**1.0000**	**4.0000**	**1.0000**
	Correct	-1.9921 (0.2694)	1.0570 (0.1675)	4.0473 (0.2497)	1.0799 (0.2934)
	Misspecified	-1.1024 (0.1418)	0.5281 (0.0937)	4.5233 (0.2036)	0.3995 (0.1889)
	Bias	0.7976	-0.4719	0.5233	-0.6005

Note: Values in parentheses are standard deviations. Biases are calculated as (Estimate from misspecified model - True value).

Estimates for θ_{RS} show small standard deviations but biases are larger than those of estimates from correctly specified models in all experiments. This result suggests that when a researcher wrongly chooses a discrete time model to estimate parameters from continuous data, it is possible to acquire wrong estimates with large finite sample bias although it seems precise with small standard errors. This suggests that a researcher should carefully specify the model by looking at the data and understanding how state variables change.

4.4 Strategic interaction and the convergence issue

In the specification of Experiment 1, the strategic interaction between agents, denoted by θ_{RN}, is zero which means this corresponds to a single agent model. As shown in the zero Jacobian property, this means that the mapping Ψ is stable, so no convergence issue will arise. However, as θ_{RN} increases, it is more likely that the mapping Ψ becomes unstable. Kasahara and Shimotsu (2012) shows that a Markov perfect equilibrium for which the local convergence holds exists if the contemporaneous and dynamic interaction between firms is small with a two-firm example.

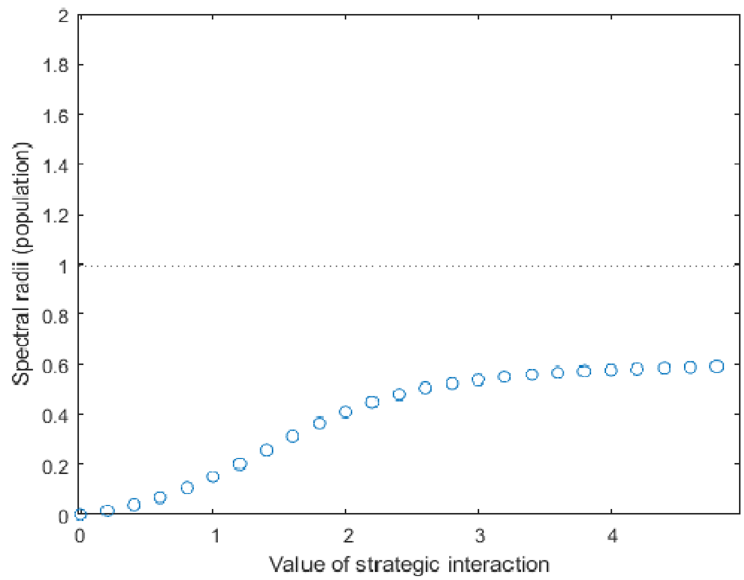
In Section 2.3 in Kasahara and Shimotsu (2012), they show that $\rho(M_{\Psi}\Psi_p^0)$ is close to $\rho(\Psi_p^0)$ in a typical setting where the number of states is larger than the number of parameters to estimate. It is also true in continuous-time games, so we focus on $\rho(\Psi_\sigma)$ since Ψ_σ is often related to the characteristics of the economic model ((Kasahara and Shimotsu, 2012)).

Aguirregabiria and Marcoux (2019) looks into the relationship between θ_{RN} and the spectral radius $\rho(\Psi_p)$ using the five player game example which is a discrete time version of the example used here. We use the same setting of Experiment 3 in continuous time and examine how $\rho(\Psi_\sigma)$ changes with regard to θ_{RN}. Our conjecture is that $\rho(\Psi_\sigma)$ is smaller than the spectral radius in a discrete time setting since continuous time games do not allow simultaneous moves between agents. This makes agents observe their competitor's actions before they receive a chance to move rather than every agent making a choice at a same time based on expectations. Thus, it is more likely that the best response mapping in a continuous time game is more stable.

Figure 1 shows that our conjecture is correct. It is analogous to Figure 1 in Aguirregabiria and Marcoux (2019). The Jacobian Ψ_σ is calculated using the numerical gradient based on 10,000 simulted observations. The spectral radius when there is no strategic interaction ($\theta_{RN} = 0$) is zero as we proved in Proposition 5. As θ_{RN} increases to five, it still does not reach 1 which means there will be no convergence issue. In contrast, in Figure 1 in Aguirregabiria and Marcoux (2019), ρ reaches the line of 1 when θ_{RN} is

2.4. This implies that in this example, it is possible to allow more strategic interactions in a continuous time setting without worrying about the convergence to inconsistent estimator.

Figure 1: Spectral radii for different θ_{RN}



5 Conclusion

This paper introduces the NPL estimator for dynamic discrete choice models in continuous time and presents features that parallel discrete time models. First, under regularity conditions, the NPL estimator in continuous time models is consistent and asymptotically normal without requiring the consistency of the initial guess for conditional choice probabilities.

Second, we presented the local convergence condition in the iterative NPL algorithm since researchers have expressed concerns involving the convergence properties of the NPL algorithm. We also showed that zero Jacobian property assuring the local convergence is always satisfied in single agent models.

Third, Monte Carlo experiments with a simple example which is analogous to an example in Aguirregabiria and Mira (2007) showed that the NPL estimator performs better than other two-step estimators initialized from estimates or random draws for conditional choice probabilities. Estimating a continuous time model with discrete time estimator results a large bias yet with low standard deviations which may mislead to wrong conclusions. Spectral radii for different strategic interactions imply that continuous-time game may have less issues with convergence.

This paper broadens the choice of estimators which can be used for dynamic discrete choice models in continuous time models. In the future, we expect to extend and develop estimators which can be used in richer settings in continuous time models including unobserved heterogeneity and continuous choices.

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A Large sample properties of the NPL estimator

A.1 Single agent models

We provide the proof for Proposition 1 following the proof of Proposition 4 in Aguirregabiria and Mira (2002).

Consistency. First, we prove the consistency of $\hat{\theta}^l$.

Step 1. If $\hat{\sigma}^{l-1}$ is consistent, then $L_M(\theta, \hat{\sigma}^{l-1})$ converges a.s. and uniformly in θ to a deterministic function $L(\theta, \sigma^*)$.

By Lemma 24.1 in Gourieroux and Monfort (1995), we have that if (i) $L_M(\theta, \sigma)$ converges a.s. and uniformly in (θ, σ) to $L(\theta, \sigma)$; (ii) $L(\theta, \sigma)$ is uniformly continuous in (θ, σ); and (iii) a sequence $\{\hat{\sigma}^{l-1}\}$ converges a.s. to σ^*; then $L_M(\theta, \hat{\sigma}^{l-1})$ converges a.s. and uniformly in θ to $L(\theta, \sigma^*)$.

For condition (i), we apply the uniform law of large numbers of Newey and McFadden (1994, Lemma 2.4) from conditions: 1) The data are i.i.d.; 2) Θ is compact.; 3) The observation likelihood $L_M(\theta, \sigma)$ is continuous at each (θ, σ) with probability one; 4) The observation likelihood is strictly bounded between 0 and 1. 1), 2), and 3) are true from assumptions. For 4), we need to show that a matrix exponential of intensity matrix Q does not have zero elements. The additional assumption that the Markov chain processes are reducible assures that the likelihood for each observation is bounded between 0 and 1.

By condition (b), $\Psi(\theta, \sigma)$ is continuous in θ and matrix exponential operation preserves continuity, so (ii) holds. (iii) holds by assumption.

Step 2. If $\hat{\theta}^{l-1}$ is consistent, then $\hat{\theta}^l \equiv \text{argmax}_{\theta \in \Theta} L_M(\theta, \hat{\sigma}^{l-1})$ converges a.s. to θ^*.

Let $\hat{\sigma}^l = \Psi(\text{argmax}_{\theta \in \Theta} L_M(\theta, \hat{\sigma}^{l-1}), \hat{\sigma}^{l-1})$. By Property 24.2 in Gourieroux and Monfort (1995), if: (i) $L_M(\theta, \sigma)$ converges a.s. and uniformly in θ to $L(\theta, \sigma)$; (ii) $L(\theta, \sigma)$ has a unique maximum in Θ at θ^*, then $\hat{\theta}^l \equiv \text{argmax}_{\theta \in \Theta} L_M(\theta, \hat{\sigma}^{l-1})$ converges a.s. to θ^*. Condition (i) is proven in Step 1. For condition (ii), condition (d) implies that θ^* is the only element of Θ that maximizes $L_M(\theta, \hat{\sigma}^{l-1})$.

Step 3. For $l \geq 1$, if $\hat{\sigma}^{l-1} \xrightarrow{a.s.} \sigma^*$, then $\hat{\sigma}^l \xrightarrow{a.s.} \sigma^*$.

By definition, $\hat{\sigma}^l = \Psi(\hat{\theta}^l, \hat{\sigma}^{l-1})$. By Step 2, $\hat{\theta}^l \xrightarrow{\text{a.s.}} \theta^*$. Since Ψ is continuous in (θ, σ), by the Slutsky theorem, $\hat{\sigma}^l \xrightarrow{\text{a.s.}} \sigma^*$.

By condition (e), $\hat{\sigma}^0 \xrightarrow{\text{a.s.}} \sigma^*$, then by simple induction on step 2 and 3, we have the result.

Asymptotic normality. Now, we prove the asymptotic normality of $\hat{\theta}$. To apply an induction argument, we show that if $(\hat{\theta}^l, \hat{\sigma}^{l-1})$ is $\sqrt{M}$-consistent and asymptotic normal, then $\sqrt{M}(\hat{\theta}^l - \theta^*) \xrightarrow{\text{d}} N(0, \Omega_{\theta\theta'}^{-1})$, and $\hat{\sigma}^l \equiv \Psi(\hat{\theta}^l, \hat{\sigma}^{l-1})$ is also $\sqrt{M}$-consistent and asymptotic normal.

Step 1. $\sqrt{M}(\hat{\theta}^l - \theta^*) \xrightarrow{d} N(0, V^*)$ *and* V^* *only depends on the upper left* $r \times r$ *submatrix of* Ω *where* r *is the dimension of the parameter vector* θ*.*

First, assume that $\hat{\sigma}^{l-1}$ is a consistent estimator of σ^* such that $[\sqrt{M}\nabla_\theta L_M(\theta^*, \sigma^*); \sqrt{M}(\hat{\sigma}^{l-1} - \sigma^*)']' \xrightarrow{d} N(0, \Omega)$. Given conditions (b), (d) and the definition of $\hat{\theta}^l$, the first order conditions of optimality imply that $\nabla_\theta L_M(\hat{\theta}^l, \hat{\sigma}^{l-1}) = 0$. Since $L_M(\theta, \sigma)$ is twice continuously differentiable, we can apply the mean value theorem:

$$0 = \nabla_\theta L_M(\theta^*, \sigma^*) + \nabla_{\theta\theta'} L_M(\theta^*, \sigma^*)(\hat{\theta}^l - \theta^*) + \nabla_{\theta\sigma'} L_M(\theta^*, \sigma^*)(\hat{\sigma}^{l-1} - \sigma^*) + o_p(1)$$

Note that $\nabla_{\theta\theta'} L_M(\theta^*, \sigma^*) \xrightarrow{\text{p}} -\Omega_{\theta\theta'}$ and $\nabla_{\theta\sigma'} L_M(\theta^*, \sigma^*) \xrightarrow{\text{p}} -\Omega_{\theta\sigma'}$, where $\Omega_{\theta\sigma'} \equiv E[\nabla_\theta s_m \nabla_{\sigma'} s_m]$ by Theorems 4.2.1 and 4.1.5 of Amemiya (1985). Rearranging the equation above,

$$\sqrt{M}(\hat{\theta}^l - \theta^*) = \Omega_{\theta\theta'}^{-1}\big[\sqrt{M}\nabla_\theta L_M(\theta^*, \sigma^*) - \Omega_{\theta\sigma'}\sqrt{M}(\hat{\sigma}^{l-1} - \sigma^*)\big] + o_p(1)$$

which leads to

$$\sqrt{M}(\hat{\theta}^l - \theta^*) \xrightarrow{\text{d}} N(0, V^*)$$

where

$$V^* = \Omega_{\theta\theta'}^{-1}(I; \Omega_{\theta\sigma'})\Omega(I; \Omega_{\theta\sigma'})'\Omega_{\theta\theta'}^{-1}$$

Note that by zero Jacobian property of Ψ_σ in Proposition 5,

$$\begin{aligned} \nabla_\sigma L_M(\theta^*, \Psi(\theta^*, \sigma^*)) &= \frac{\partial L_M(\cdot)}{\partial \Psi(\cdot)} \frac{\partial \Psi(\theta^*, \sigma^*)}{\partial \sigma} \\ &= 0 \end{aligned}$$

By information matrix equality,

$$\nabla_{\theta\sigma'} L(\theta^*, \sigma^*) = E\Big(\frac{\partial L_M(\theta^*, \sigma^*)}{\partial \theta} \frac{\partial L_M(\theta^*, \sigma^*)}{\partial \sigma'}\Big) = 0$$

and this leads to $\Omega_{\theta\sigma} = 0$. Then,

$$\begin{aligned} V^* &= \Omega_{\theta\theta'}^{-1}\Omega_{r\times r}\Omega_{\theta\theta'}^{-1} \\ &= \Omega_{\theta\theta'}^{-1} \end{aligned}$$

where $\Omega_{r\times r}$ is the upper left $r \times r$ submatrix of Ω and the second equality comes from the fact that

$$
\begin{aligned}
\Omega_{r\times r} &= var(\sqrt{M}\nabla_\theta L_M(\theta^*,\sigma^*)) \\
&= Mvar(\nabla_\theta L_M(\theta^*,\sigma^*)) \\
&= Mvar(\frac{1}{M}\sum_{m=1}^{M}\nabla_\theta s_m) \\
&= \frac{1}{M}\sum_{m=1}^{M} var(\nabla_\theta s_m) \\
&= \frac{1}{M}\sum_{m=1}^{M} E(\nabla_\theta s_m \nabla_{\theta'} s_m) \\
&= \Omega_{\theta\theta'}
\end{aligned}
$$

Step 2. $[\sqrt{M}\nabla_\theta L_M(\theta^*,\sigma^*);\sqrt{M}(\hat{\sigma}^l-\sigma^*)']' \xrightarrow{d} N(0,\Omega^*)$, and the upper left $r\times r$ submatrices of Ω and Ω^* are identical.

Define

$$
\omega_M^l \equiv [\sqrt{M}\nabla_\theta L_M(\theta^*,\sigma^*);\sqrt{M}(\hat{\sigma}^l-\sigma^*)'\Big]'
$$

We know that $\hat{\sigma}^{l+1} = \Psi(\hat{\theta}^{l+1},\hat{\sigma}^l)$ and $\sigma^* = \Psi(\theta^*,\sigma^*)$. We apply mean value expansion to $(\hat{\sigma}^{l+1}-\sigma^*)$:

$$
\begin{aligned}
\hat{\sigma}^{l+1}-\sigma^* &= \Psi(\hat{\theta}^l,\hat{\sigma}^l) - \Psi(\theta^*,\sigma^*) \\
&= \Psi_\theta(\theta^*,\sigma^*)(\hat{\theta}^{l+1}-\theta^*) + \Psi_\sigma(\theta^*,\sigma^*)(\hat{\sigma}^l-\sigma^*) + o_p(1)
\end{aligned}
$$

Then, from the same mean expansion in step 1, we have $\sqrt{M}(\hat{\theta}^l-\theta^*) = \Omega_{\theta\theta'}^{-1}\sqrt{M}\nabla_\theta L_M(\theta^*,\sigma^*) + o_p(1)$, so we substitute into the equation above,

$$
\sqrt{M}(\hat{\sigma}^{l+1}-\sigma^*) = \Psi_\theta(\theta^*,\sigma^*)\Omega_{\theta\theta'}^{-1}\sqrt{M}\nabla_\theta L_M(\theta^*,\sigma^*) + \Psi_\sigma(\theta^*,\sigma^*)\sqrt{M}(\hat{\sigma}^l-\sigma^*) + o_p(1).
$$

We rewrite in matrix form:

$$
\begin{bmatrix} \sqrt{M}\nabla_\theta L_M(\theta^*,\sigma^*) \\ \sqrt{M}(\hat{\sigma}^{l+1}-\sigma^*) \end{bmatrix} = \begin{bmatrix} I & 0 \\ \Psi_\theta(\theta^*,\sigma^*)\Omega_{\theta\theta'}^{-1} & \Psi_\sigma(\theta^*,\sigma^*) \end{bmatrix} \begin{bmatrix} \sqrt{M}\nabla_\theta L_M(\theta^*,\sigma^*) \\ \sqrt{M}(\hat{\sigma}^l-\sigma^*) \end{bmatrix}. \qquad (14)
$$

So, we have $\omega_M^{l+1} = A\omega_M^{l+1}$ where A is the first matrix in the RHS of Equation (14). It follows that if ω_M^l is asymptotically normal, ω_M^{l+1} is also asymptotically normal. From Step 1, we also know that V^* depends on upper $r\times r$ submatrix of Ω and The upper-left $r\times r$ matrix of A is the identity matrix from Equation (14). Therefore, $r\times r$ submatrices of

Ω and Ω^* are equal. Since $\sqrt{M}(\hat{\theta}^l - \theta^*) = \Omega_{\theta\theta'}^{-1}\sqrt{M}\nabla_\theta L_M(\theta^*, \sigma^*) + o_p(1)$, and denoting $\sqrt{M}(\hat{\theta}^l - \theta^*) \xrightarrow{d} N(0, V^{**})$, we have

$$\begin{aligned} V^{**} &= \Omega_{\theta\theta'}^{-1}\Omega_{r\times r}^{*}\Omega_{\theta\theta'}^{-1} \\ &= \Omega_{\theta\theta'}^{-1}\Omega_{r\times r}\Omega_{\theta\theta'}^{-1} \\ &= \Omega_{\theta\theta'}^{-1}. \end{aligned}$$

From step 1, we showed that when $\hat{\sigma}^{l-1}$ is a consistent estimator of σ^* s.t. $\omega^l \xrightarrow{d} N(0, \Omega)$, it follows $\sqrt{M}(\hat{\theta}^l - \theta^*) \xrightarrow{d} N(0, \Omega_{\theta\theta'}^{-1})$. Step 2 proves that $\omega^{l+1} \xrightarrow{d} N(0, \Omega^*)$ which implies $\sqrt{M}(\hat{\theta}^{l+1} - \theta^*) \xrightarrow{d} N(0, \Omega_{\theta\theta'}^{-1})$. Since we start from asymptotic normal estimator $\hat{\sigma}^0$ such that $\omega^0 \xrightarrow{d} N(0, \Omega)$, this is true for all $l \leq L-1$ by induction.

A.2 Dynamic discrete games

We now expand a single agent model to a game and prove large sample properties of the PI estimator in continuous time presented in Proposition 2.

Proof. **Consistency.** We prove $(\hat{\theta}^l, \hat{\sigma}^{l-1}) = (\theta^*, \sigma^*) + o_p(1)$ by induction. First, we show that it holds for $l = 1$. $\hat{\sigma}^0 - \sigma^* = o_p(1)$ follows from condition (e) that we start the iteration from a consistent estimator $\hat{\sigma}^0$ for σ^*. To show that $\hat{\theta}^1 - \theta^* = o_p(1)$, we first prove that $\sup_{\theta\in\Theta} |L_M(\theta, \tilde{\sigma}) - L(\theta, \tilde{\sigma})| = o_p(1)$ for some $\sqrt{M}$-consistent estimator $\tilde{\sigma}$ for σ^*.

$$\sup_{\theta\in\Theta} |L_M(\theta, \tilde{\sigma}) - L(\theta, \tilde{\sigma})| \leq \sup_{\theta\in\Theta} |L_M(\theta, \tilde{\sigma}) - L_M(\theta, \sigma^*)| + \sup_{\theta\in\Theta} |L_M(\theta, \tilde{\sigma}) - L(\theta, \sigma^*)| = o_p(1) \tag{15}$$

since $\sup_{\theta\in\Theta} |L_M(\theta, \tilde{\sigma}) - L_M(\theta, \sigma^*)| = o_p(1)$ by consistency of $\tilde{\sigma}$ and $\sup_{\theta\in\Theta} |L_M(\theta, \tilde{\sigma}) - L(\theta, \sigma^*)|$ by uniform convergence of $L_M(\theta, \sigma)$ in θ to $L(\theta, \sigma^*)$ as proved in step 1 in Proposition 1.

By substituting $\hat{\sigma}^0$ to $\tilde{\sigma}$ in Equation (15), we have $\sup_{\theta\in\Theta} |L_M(\theta, \hat{\sigma}^0) - L(\theta, \hat{\sigma}^0)| = o_p(1)$. We also know that $L_M(\theta, \sigma)$ is continuous in θ and $L(\theta, \sigma)$ is uniquely maximized at θ^* from the assumption. By Newey and McFadden (1994), $\hat{\theta}^1 - \theta^* = o_p(1)$.

Now we assume that $(\hat{\theta}^l, \hat{\sigma}^{l-1}) = (\theta^*, \sigma^*) + o_p(1)$ for $l \leq L-1$ and show that it holds for $l+1$. Using the intermediate value theorem, we expand $\hat{\sigma}^l - \sigma^*$,

$$\begin{aligned} \hat{\sigma}^l - \sigma^* &= \Psi(\hat{\sigma}^l, \hat{\sigma}^{l-1}) - \Psi(\theta^*, \sigma^*) \\ &= \Psi_\theta(\theta^*, \sigma^*)(\hat{\theta}^l - \theta^*) + \Psi_\sigma(\theta^*, \sigma^*)(\hat{\sigma}^{l-1} - \sigma^*) + o_p(1) = o_p(1) \end{aligned}$$

where the second equality comes from the inductive hypothesis. Substituting $\hat{\sigma}^l$ in Equation (15), we have $\sup_{\theta\in\Theta}|L_M(\theta,\hat{\sigma}^l) - L(\theta,\hat{\sigma}^l)| = o_p(1)$. From condition (b), We also know that $L_M(\theta,\sigma)$ is continuous in θ and $L(\theta,\sigma)$ is uniquely maximized at θ^* from condition (d). By Newey and McFadden (1994), $\hat{\theta}^{l+1} - \theta^* = o_p(1)$. Since we start from a consistent estimator $\hat{\sigma}^0$, we have the consistency for $\hat{\theta}^l$ for all $l \leq L$ by induction.

Asymptotic normality. We prove that $(\hat{\theta}^l, \hat{\sigma}^{l-1}) \xrightarrow{d} N(0, \Omega_{\theta\theta'} + \Omega_{\theta\theta'}\Omega_{\theta\sigma}\Sigma^{l-1}\Omega_{\theta\sigma'}\Omega^{-1}_{\theta\theta'}$ by induction. We first show that the result above holds for $l=1$. Let $R_M = \frac{1}{M}\sum_{m=1}^{M} r_m$ denote the objective function for some $\sqrt{M}$-consistent estimator $\hat{\sigma}^0$ for σ^* and $\nabla_\theta s^l_m = \sum_{m=1}^{M}\nabla_\theta \ln P_{k_{m,n-1},k_m,n}(\Delta;\Psi(\theta,\hat{\sigma}^{l-1}))$ denote the pseudo score. We expand the first order condition $\nabla_\theta L_M(\hat{\theta}^1,\hat{\sigma}^0) = 0$ using the mean intermediate value theorem.

$$0 = \nabla_\theta L_M(\theta^*,\sigma^*) + \nabla_{\theta\theta'} L_M(\theta^*,\sigma^*)(\hat{\theta}^1 - \theta^*) + \nabla_{\theta\sigma'} L_M(\theta^*,\sigma^*)(\hat{\sigma}^0 - \sigma^*) + o_p(1) \quad (16)$$

By the central limit theorem and information matrix inequality, $\nabla_{\theta\theta'} L_M(\theta^*,\sigma^*) \xrightarrow{p} -\Omega_{\theta\theta'}$ and $\nabla_{\theta\sigma'} L_M(\theta^*,\sigma^*) \xrightarrow{p} -\Omega_{\theta\sigma'}$. Then,

$$\sqrt{M}(\hat{\theta}^1 - \theta^*) = \Omega^{-1}_{\theta\theta'}\Big\{ -\Omega_{\theta\sigma'}\Big(\frac{1}{\sqrt{M}}\sum_{m=1}^{M}\nabla_\sigma r_m\Big) + \Big(\frac{1}{\sqrt{M}}\sum_{m=1}^{M}\nabla_\theta s^0_m\Big)\Big\} + o_p(1). \quad (17)$$

From condition (b), $\Psi(\theta,\sigma)$ is continuously differentiable and all elements of $\exp(Q(\theta,\sigma))$ is bounded between 0 and 1 if we assume that the Markov chain processes are reducible. Then, by the general information matrix equality, $E[\nabla_\sigma r_m \nabla_{\theta'} s^0_m] = 0$, and $E[\nabla_\sigma r_m \nabla_\sigma s^0_m] = I$. Then,

$$\frac{1}{\sqrt{M}}\Big(\sum_{m=1}^{M}\nabla_\theta s^0_m\Big) - \Omega_{\theta\sigma'}\Big(\frac{1}{\sqrt{M}}\sum_{m=1}^{M}\nabla_\sigma r_m\Big) \xrightarrow{d} N(0, \Omega_{\theta\theta'} + \Omega_{\theta\sigma'}\Sigma^0\Omega'_{\theta\sigma'}).$$

Substituting into equation (17), we have asymptotic normality for $\hat{\theta}^1$.

$$\sqrt{M}(\hat{\theta}^1 - \theta^*) \xrightarrow{d} N(0, \Omega^{-1}_{\theta\theta'} + \Omega^{-1}_{\theta\theta'}\Omega_{\theta\sigma'}\Sigma^0\Omega'_{\theta\sigma'}\Omega^{-1}_{\theta\theta'}).$$

Now we assume that $\sqrt{M}(\hat{\sigma}^{l-1} - \sigma^*) \xrightarrow{d} N(0, \Sigma^{l-1})$ and $\sqrt{M}(\hat{\theta}^l - \theta^*) \xrightarrow{d} N(0, \Omega^{-1}_{\theta\theta'} + \Omega^{-1}_{\theta\theta'}\Omega_{\theta\sigma'}\Sigma^{l-1}\Omega'_{\theta\sigma'}\Omega^{-1}_{\theta\theta'})$ for some $l \leq L-1$. We can consider $\hat{\theta}^{l+1}$ as a two step estimator based on the moment equation $E[\nabla_\theta L_M(\theta,\hat{\sigma}^l)] = 0$. We use the preliminary estimate $\hat{\sigma}^l$ of σ^* from the previous iteration based on the moment equation, $E[\Psi(\hat{\theta}^l,\hat{\sigma}^l) - \sigma^*] = 0$.

We can write the moment conditions as $\tilde{g}(\theta, \sigma)$.

$$\tilde{g}(\hat{\theta}, \sigma) = \begin{bmatrix} g(\theta, \sigma) \\ h(\sigma) \end{bmatrix} = \begin{bmatrix} \nabla_\theta L_M(\theta, \hat{\sigma}^l) \\ \Psi(\hat{\theta}^l, \sigma) - \sigma^* \end{bmatrix}.$$

Applying the general information matrix inequality,

$$E(\nabla_\theta \tilde{g}(\theta, \sigma)) = -E[\tilde{g}(\theta, \sigma) \nabla_\theta s_m^{l+1\prime}]$$

This results $E[\nabla_\theta h(\sigma)] = -E[\nabla_\theta (\Psi_m(\hat{\theta}^l, \sigma) - \sigma^*) \nabla_\theta s_m^{l+1\prime}] = 0$. As before, we expand the first moment condition $\nabla_\theta L_M(\hat{\theta}^{l+1}, \hat{\sigma}^l) = 0$ using the intermediate value theorem.

$$0 = \nabla_\theta L_M(\theta^*, \sigma^*) + \nabla_{\theta\theta'} L_M(\theta^*, \sigma^*)(\hat{\theta}^{l+1} - \theta^*) + \nabla_{\theta\sigma'} L_M(\theta^*, \sigma^*)(\hat{\sigma}^l - \sigma^*) + o_p(1). \quad (18)$$

Then, $\sqrt{M}(\hat{\theta}^{l+1} - \theta^*)$ can be written as

$$\sqrt{M}(\hat{\theta}^{l+1} - \theta^*) = \Omega_{\theta\theta'}^{-1} \Big\{ -\Omega_{\theta\sigma'} \sqrt{M}\big(\Psi(\hat{\theta}^l, \sigma) - \sigma^*\big) + \Big(\frac{1}{\sqrt{M}} \sum_{m=1}^{M} \nabla_\theta s_m^{l+1}\Big) \Big\} + o_p(1) \quad (19)$$

where $\sqrt{M}(\Psi(\hat{\theta}^l, \sigma) - \sigma^*) = \sqrt{M}(\hat{\sigma}^l - \sigma^*) \xrightarrow{d} N(0, \Sigma^l)$. Then, from equation (19),

$$\sqrt{M}(\hat{\theta}^{l+1} - \theta^*) \xrightarrow{d} N(0, \Omega_{\theta\theta'}^{-1} + \Omega_{\theta\theta'}^{-1} \Omega_{\theta\sigma'} \Sigma^l \Omega'_{\theta\sigma'} \Omega_{\theta\theta'}^{-1}).$$

We briefly show the efficiency of L-stage estimator relative to a two-step estimator. It is sufficient to show that $\Sigma^l - \Sigma^{l-1}$ is positive definite. Changing l to $l-1$ in Equation (18) and using the asymptotic variance of $\sqrt{M}(\hat{\theta}^l - \theta^*)$,

$$\sqrt{M}(\hat{\sigma}^{l-1} - \sigma^* \xrightarrow{d} N(0, \Sigma^l + \Omega_{\theta\sigma'}^{-1} \Omega_{\theta\theta'} \Omega'^{-1}_{\theta\sigma'}).$$

Then, $\Sigma^{l-1} - \Sigma^l = \Omega_{\theta\sigma'}^{-1} \Omega_{\theta\theta'} \Omega'_{\theta\sigma'}{}^{-1}$ which is positive definite. This means that the asymptotic variance of $\sqrt{M}(\hat{\theta}^l - \theta^*)$ decreases as the number of iterations increases. □

We omit the detailed proof for Proposition 3 as it is same as proof in Appendix of Aguirregabiria and Mira (2007) since the NPL operator has same properties in continuous time models. Changing $Q_0(\theta, P)$ to $L(\theta, \sigma)$, and P to σ will be sufficient to prove an analogous proposition in continuous time models.

As Kasahara and Shimotsu (2012) mention in Proposition 6, it is also possible to change weak consistency to strong consistency by changing "with probability approaching 1" to "a.s." in Step 2-5.

B Convergence of NPL estimator in continuous time

B.1 Proof for Proposition 4

We first prove an auxiliary proposition needed for convergence condition. This is analogous to Proposition 7 in Kasahara and Shimotsu (2012).

Proposition 6. *Suppose that Assumption 8 holds. Then, there exists a neighborhood $\mathcal{N}_1$ of σ^* such that $\hat{\theta}^l - \hat{\theta} = O(||\hat{\theta}^{l-1} - \hat{\theta}||)$ a.s. and $\hat{\sigma}^l - \hat{\sigma} = M_{\Psi_\theta}\Psi^*_\sigma(\hat{\sigma}^{l-1} - \hat{\sigma}) + O(M^{-1/2}||\hat{\sigma}^{l-1} - \hat{\sigma}|| + ||\hat{\sigma}^{l-1} - \hat{\sigma}||^2)$ a.s. uniformly in $\hat{\sigma}^{l-1} \in \mathcal{N}_1$.*

Proof. For $\varepsilon > 0$, define a neighborhood $\mathcal{N}(\varepsilon) = \{(\theta, \sigma) : ||\theta - \theta^*|| + ||\sigma - \sigma^*|| < \varepsilon\}$. Then, there exists $\varepsilon_1 > 0$ such that $\mathcal{N}(\varepsilon_1) \subset \mathcal{N}$ and $\sup_{(\theta,\sigma)\in\mathcal{N}(\varepsilon_1)} ||\nabla_{\theta\theta'} L(\theta, \sigma)^{-1}|| < \infty$ since $\nabla_{\theta\theta'} L(\theta, \sigma)$ is continuous and $\nabla_{\theta\theta'} L(\theta^*, \sigma^*)$ is nonsingular.

We assume that $(\hat{\theta}^l, \hat{\sigma}^{l-1}) \in \mathcal{N}(\varepsilon_1)$ which we will show in the end of this proof.

Step 1. We first prove the first statement: $\hat{\theta}^l - \hat{\theta} = O(||\hat{\theta}^{l-1} - \hat{\theta}||)$ a.s.

From the first step in NPL iteration, we know that $\nabla_\theta L_M(\hat{\theta}^l, \hat{\sigma}^{l-1}) = 0$. Applying the mean value theorem around $(\hat{\theta}, \hat{\sigma})$, we have

$$0 = \nabla_{\theta\theta'} L_M(\bar{\theta}, \bar{\sigma})(\hat{\theta}^l - \hat{\theta}) + \nabla_{\theta\sigma'} L_M(\bar{\theta}, \bar{\sigma})(\hat{\sigma}^{l-1} - \hat{\sigma}) + o_p(1). \tag{20}$$

where $(\bar{\theta}, \bar{\sigma})$ lie between $(\hat{\theta}^l, \hat{\sigma}^{l-1})$ and $(\hat{\theta}, \hat{\sigma})$.

Then, we can rewrite

$$\hat{\theta}^l - \hat{\theta} = -\nabla_{\theta\theta'} L_M(\bar{\theta}, \bar{\sigma})^{-1} \nabla_{\theta\sigma'} L_M(\bar{\theta}, \bar{\sigma})(\hat{\sigma}^{l-1} - \hat{\sigma}) + o_p(1) \tag{21}$$

Note that (i) $(\bar{\theta}, \bar{\sigma}) \in \mathcal{N}(\varepsilon_1)$ from assumption and $(\bar{\theta}, \bar{\sigma}) \in \mathcal{N}$ since $\mathcal{N}(\varepsilon_1) \subset \mathcal{N}$, and (ii) $\sup_{(\theta,\sigma)\in\mathcal{N}(\varepsilon_1)} ||\nabla_{\theta\theta'} L_M(\theta, \sigma)^{-1} \nabla_{\theta\sigma'} L_M(\theta, \sigma)|| = O(1)$ a.s. since $\sup_{(\theta,\sigma)\in\mathcal{N}(\varepsilon_1)} ||\nabla_{\theta\theta'} L(\theta, \sigma)^{-1}|| < \infty$ and $\sup_{(\theta,\sigma)\in N} ||\nabla^2 L_M(\theta, \sigma) - \nabla^2 L(\theta, \sigma)|| = o(1)$ by general uniform convergence[5].

Then, we have the first result: $\hat{\theta}^l - \hat{\theta} = O(||\hat{\sigma}^{l-1} - \hat{\sigma}||)$ a.s.

Step 2. We now prove $\hat{\sigma}^l - \hat{\sigma} = M_{\Psi_\theta}\Psi^*_\sigma(\hat{\sigma}^{l-1} - \hat{\sigma}) + O(M^{-1/2}||\hat{\sigma}^{l-1} - \hat{\sigma}|| - ||\hat{\sigma}^{l-1} + \hat{\sigma}||^2)$ a.s. uniformly in $\hat{\sigma}^{l-1} \in \mathcal{N}_1$.

We use the Taylor expansion, root-M consistency of $(\hat{\theta}, \hat{\sigma})$, and the information matrix equality for the results below:

[5](i) Θ is compact implying total boundedness. (ii) $\nabla^2 L_M(\theta, \sigma) \rightarrow \nabla^2 L(\theta, \sigma)$ *a.s.* $\forall \theta \in \Theta$. (iii) $\nabla^2 L(\theta, \sigma)$ is a nonrandom function that is uniformly continuous in $\theta \in \Theta$, and $|\nabla^2 L_M(\theta, \sigma) - \nabla^2 L_M(\theta', \sigma')| \leq |\nabla^2 L_M(\theta, \sigma) - \nabla^2 L_M(\theta^*, \sigma^*)| + |\nabla^2 L_M(\theta', \sigma') - \nabla^2 L_M(\theta^*, \sigma^*)| \leq 2\delta \leq \delta' d((\theta, \sigma), (\theta', \sigma'))$ for some small δ and δ', $\forall \theta', \theta \in \mathcal{N}$ a.s. since $d(\cdot) < \varepsilon$ and $L_M(\theta, \sigma)$ is continuous. Then, by Theorem 2 and Lemma 1 in Andrews (1992), the results follows.

$$
\begin{aligned}
\nabla_{\theta\theta'} L_M(\hat{\theta},\hat{\sigma}) &= -\Omega_{\theta\theta'} + O(M^{-1/2}) \\
\nabla_{\theta\sigma'} L_M(\hat{\theta},\hat{\sigma}) &= -\Omega_{\theta\sigma'} + O(M^{-1/2}) \\
\nabla_{\theta}\Psi(\hat{\theta},\hat{\sigma}) &= \Psi^*_{\theta} + O(M^{-1/2}) \\
\nabla_{\sigma}\Psi(\hat{\theta},\hat{\sigma}) &= \Psi^*_{\sigma} + O(M^{-1/2})
\end{aligned} \tag{22}
$$

We use the second step of NPL estimation $\hat{\sigma}^l = \Psi(\hat{\theta}^l, \hat{\sigma}^{l-1})$ and expand twice around $(\hat{\theta},\hat{\sigma})$.

$$
\begin{aligned}
\hat{\sigma}^l - \hat{\sigma} &= \nabla_{\theta}\Psi(\hat{\theta},\hat{\sigma})(\hat{\theta}^l - \hat{\theta}) + \nabla_{\sigma}\Psi(\hat{\theta},\hat{\sigma})(\hat{\sigma}^{l-1} - \hat{\sigma}) + O(||\hat{\sigma}^{l-1} - \hat{\sigma}||^2) \\
&= \Psi^*_{\theta}(\hat{\theta}^l - \hat{\theta}) + \Psi^*_{\sigma}(\hat{\sigma}^{l-1} - \hat{\sigma}) + O(||\hat{\sigma}^{l-1} - \hat{\sigma}||^2) + O(M^{-1/2}||\hat{\sigma}^{l-1} - \hat{\sigma}||).
\end{aligned} \tag{23}
$$

where the first equality follows from Step 1 and $\sup_{(\theta,\sigma)\in\mathcal{N}(\varepsilon_1)} \nabla^3\Psi(\theta,\sigma) < \infty$ in assumption (b) and the second equality comes from Equation (22).

We expand $\nabla_{\theta\theta'} L_M(\bar{\theta},\bar{\sigma})$ around $(\hat{\theta},\hat{\sigma})$. Since $||\bar{\theta} - \hat{\theta}|| \leq ||\hat{\theta}^l - \hat{\theta}||$, $||\bar{\sigma} - \hat{\sigma}|| \leq ||\hat{\sigma}^{l-1} - \hat{\sigma}||$ and $\hat{\theta}^{l-1} - \hat{\theta} = O(||\sigma^{l-1} - \hat{\sigma}||)$ from Step 1, we have $\nabla_{\theta\theta'} L_M(\bar{\theta},\bar{\sigma}) = \nabla_{\theta\theta'} L_M(\hat{\theta},\hat{\sigma}) + O(||\hat{\sigma}^{l-1} - \hat{\sigma}||)$ a.s. Using Equation (22) and repeating the similar process for $\nabla_{\theta\sigma'} L_M(\bar{\theta},\bar{\sigma})$,

$$
\begin{aligned}
\nabla_{\theta\theta'} L_M(\bar{\theta},\bar{\sigma}) &= -\Omega_{\theta\theta'} + O(M^{-1/2}) + O(||\hat{\sigma}^{l-1} - \hat{\sigma}||) \\
\nabla_{\theta\sigma'} L_M(\bar{\theta},\bar{\sigma}) &= -\Omega_{\theta\sigma'} + O(M^{-1/2}) + O(||\hat{\sigma}^{l-1} - \hat{\sigma}||)
\end{aligned}
$$

Then, applying Equation (21),

$$
\hat{\theta}^l - \hat{\theta} = -\Omega^{-1}_{\theta\theta'}\Omega_{\theta\sigma'}(\hat{\sigma}^{l-1} - \hat{\sigma}) + O(M^{-1/2}||\hat{\sigma}^{l-1} - \hat{\sigma}|| + ||\hat{\sigma}^{l-1} - \hat{\sigma}||^2).
$$

Substituting the equation above into Equation (23), we have the result.

Step 3. Now we need to show that $(\hat{\theta}^l, \hat{\sigma}^{l-1}) \in \mathcal{N}(\varepsilon_1)$. We first prove that $||\hat{\theta}^l - \hat{\theta}^0|| < \varepsilon/2$ and then, show that $||\hat{\sigma}^{l-1} - \hat{\sigma}|| < \varepsilon/2$ if we set $\mathcal{N}_1$ sufficiently small.

Let $\mathcal{N}_{\theta} \equiv \{\theta : ||\theta - \theta^*|| < \varepsilon_1/2\}$ and define $\Delta = L(\theta^*, \sigma^*) - \sup_{\theta\in\mathcal{N}^c_{\theta}\cap\Theta} L(\theta, \sigma^*) > 0$. The inequality follows from information inequality, compactness of $\mathcal{N}^c_{\theta} \cap \Theta$, and continuity of $L(\theta,\sigma)$. Then, if $\hat{\theta}^l \notin \mathcal{N}_{\theta}$, then $L(\theta^*,\sigma^*) - L(\hat{\theta}^l, \sigma^*) \geq \Delta$. We can also derive that

$$
\begin{aligned}
&L(\theta^*,\sigma^*) - L(\hat{\theta}^l,\sigma^*) \\
&\leq L_M(\theta^*,\sigma^*) - L_M(\hat{\theta}^l,\sigma^*) + 2\sup_{(\theta,\sigma)\in\Theta\times\Sigma}|L_M(\theta,\sigma) - L(\theta,\sigma)| \\
&\leq L_M(\theta^*,\hat{\sigma}^{l-1}) - L_M(\hat{\theta}^l,\hat{\sigma}^{l-1}) + 2\sup_{(\theta,\sigma)\in\Theta\times\Sigma}|L_M(\theta,\sigma) - L(\theta,\sigma)| + 2\sup_{\theta\in\Theta}|L(\theta,\sigma^*) - L(\theta,\hat{\sigma}^{l-1})| \\
&\leq 2\sup_{(\theta,\sigma)\in\Theta\times\Sigma}|L_M(\theta,\sigma) - L(\theta,\sigma)| + 2\sup_{\theta\in\Theta}|L(\theta,\sigma^*) - L(\theta,\hat{\sigma}^{l-1})|.
\end{aligned}
$$

The last equality follows from $\hat{\theta}^l = \text{argmax}_{\theta\in\Theta} L_M(\theta,\hat{\sigma}^{l-1})$. We know that $2\sup_{(\theta,\sigma)\in\Theta\times\Sigma}|L_M(\theta,\sigma) - L(\theta,\sigma)| = o(1)$ a.s. from Step 1 of Proof for consistency in Proposition 3. Since $L(\theta,\sigma)$ is continuous, there exists ε_Δ such that $2\sup_{\theta\in\Theta}|L(\theta,\sigma^*) - L(\theta,\hat{\sigma}^{l-1})| < \Delta/2$ if $||\sigma^* - \hat{\sigma}^{l-1}|| \leq \varepsilon_\Delta$. It follows that if $||\sigma^* - \hat{\sigma}^{l-1}|| \geq \varepsilon_\Delta$, $L(\theta^*,\sigma^*) - L(\hat{\theta}^l,\sigma^*) \leq \Delta$ which means $\hat{\theta}^l \in \mathcal{N}_\theta$.

Then, if we set $\mathcal{N}_1 = \{\sigma : ||\sigma - \sigma^*|| \leq \min\{\varepsilon_1/2, \varepsilon_\Delta\}\}$, we have $||\sigma - \sigma^*|| < \varepsilon/2$ which gives $(\hat{\theta}^l,\hat{\theta}^{l-1}) \in \mathcal{N}(\varepsilon_1)$ a.s. □

We now present a proof for Proposition 4. This is analogous to Proposition 2 in Kasahara and Shimotsu (2012).

Proof. Let $b > 0$ be a constant such that

$$\rho(M_{\Psi_\theta}\Psi_\sigma) + 2b < 1.$$

From Lemma 5.6.10 of Horn (1995), there is a matrix norm $||\cdot||_\alpha$ such that

$$||M_{\Psi_\theta}\Psi_\sigma||_\alpha \leq \rho(M_{\Psi_\theta}\Psi_\sigma) + b.$$

Define a vector norm $||\cdot||_\beta$ for $x \in \mathbb{R}^{NJK}$ as $||x||_\beta \equiv ||[x\ 0\cdots 0]||_\alpha$, then a direct calculation gives

$$||Ax||_\beta = ||A[x\ 0\cdots 0]||_\alpha \leq ||A||_\alpha ||x||_\beta$$

for any matrix A.
From the equivalence of vector norms in $\mathbb{R}^{NJK}$, we can restate Proposition 6 in terms of β: there exists $c > 0$ such that

$$\hat{\sigma}^l - \hat{\sigma} = M_{\Psi_\theta}\Psi_\sigma(\hat{\sigma}^{l-1} - \hat{\sigma}) + O(M^{-1/2}||\hat{\sigma}^{l-1} - \hat{\sigma}||_\beta + ||\hat{\sigma}^{l-1} - \hat{\sigma}||_\beta^2)$$

a.s. holds uniformly in $\hat{\sigma}^{l-1} \in \{\sigma : ||\sigma - \sigma^*||_\beta < c\}$.
We rewrite this statement further so that it is amenable to recursive substitution.

(i) $||M_{\Psi_\theta}\Psi_\sigma(\hat{\sigma}^{l-1}-\hat{\sigma})||_\beta \leq ||M_{\Psi_\theta}\Psi_\sigma||_\alpha ||\hat{\sigma}^{l-1}-\hat{\sigma}||_\beta \leq (\rho(M_{\Psi_\theta}\Psi_\sigma)+b)||\hat{\sigma}^{l-1}-\hat{\sigma}||_\beta$.
(ii) The remainder term can be written as $O(M^{-1/2}+||\hat{\sigma}^{l-1}-\hat{\sigma}||_\beta)||\hat{\sigma}^{l-1}-\hat{\sigma}||_\beta$. Then setting $c<b$ and using consistency of $\hat{\sigma}$, this term is smaller than $b||\hat{\sigma}^{l-1}-\hat{\sigma}||_\beta$ a.s.
(iii) Since $\hat{\sigma}$ is consistent, $\{\sigma : ||\hat{\sigma}-\hat{\sigma}||_\beta < c/2\} \subset \{\sigma : ||\sigma-\sigma^*||_\beta < c\}$ a.s.
From (i) to (iii),

$$||\hat{\sigma}^{l-1}-\hat{\sigma}||_\beta \leq (\rho(M_{\Psi_\theta}\Psi_\sigma)+2b)||\hat{\sigma}^{l-1}-\hat{\sigma}||_\beta$$

holds a.s. for all $\hat{\sigma}^{l-1} \in \{\sigma : ||\hat{\sigma}^{l-1}-\hat{\sigma}||_\beta < c/2\}$. Because each NPL updating of (θ,σ) uses the same pseudo-likelihood function, we may recursively substitute for the $\hat{\sigma}_j$'s.

$$\begin{aligned}
||\hat{\sigma}^{l-1}-\hat{\sigma}||_\beta &\leq (\rho(M_{\Psi_\theta}\Psi_\sigma)+2b)||\hat{\sigma}^{l-1}-\hat{\sigma}||_\beta \\
&\leq (\rho(M_{\Psi_\theta}\Psi_\sigma)+2b)^2||\hat{\sigma}^{l-2}-\hat{\sigma}||_\beta \\
&\vdots \\
&\leq (\rho(M_{\Psi_\theta}\Psi_\sigma)+2b)^l||\hat{\sigma}^{0}-\hat{\sigma}||_\beta
\end{aligned}$$

Then, $\lim_{l\to\infty}\hat{\sigma}^l = \hat{\sigma}$ a.s. if $||\hat{\sigma}^{l-1}-\hat{\sigma}||_\beta < c/2$. Applying the equivalence of vector norms in $\mathcal{R}^L$ to $||\hat{\sigma}^0-\hat{\sigma}||_\beta$ and $||\hat{\sigma}^0-\hat{\sigma}||$ and consistency of $\hat{\sigma}$, the result follows. □

B.2 Proof for Proposition 5

We introduce notations in ABBE (2016) and prove the zero Jacobian property in a single agent dynamic choice model in continuous time. We follow the proofs in Appendix of Aguirregabiria and Mira (2002).

For choice j in state k, let $v_{jk} = \psi_{jk} + V_{l(j,k)}$ denote an arbitrary choice-specific valuation and $v_k = (v_{0k}, \ldots, v_{J-1,k}) \in \mathbb{R}^J$ denote a J-vector of valuations in state k. Define $\tilde{v}_k$ as a $(J-1)$-vector of normalized valuations based on choice 0 on the $(J-1)$-dimensional space $\mathcal{V} = \{\tilde{v}_k \in \mathbb{R}^{J-1} : \tilde{v}_{0k} = 0\}$.

Now consider the mapping $\tilde{H}_k : \mathcal{V} \to \mathcal{P}_k$ where $\mathcal{P}_k \in \Delta^{J-1}$ is the space of all CCP vectors σ_k in state k. By Proposition 1 of Hotz-Miller (1993), $\tilde{H}_k$ is one-to-one with $\tilde{H}_k(\sigma_k) = \tilde{v}_k$ and invertible.

Lemma 1. *Let σ_k^0 denote an arbitrary vector of $J-1$ choice probabilities in state k.*

$$\frac{\partial}{\partial \sigma_k} \sum_j \sigma_{jk}^0 e_{jk}(\theta, \sigma_k^0) = -\tilde{H}_k^{-1}(\sigma_k^0) \tag{24}$$

where $e_{jk}(\theta,\sigma)$ is the expected value of ε_{jk} given that choice j is optimal at state k.

Proof. Let $W_{jk}(\tilde{v})$ represent the expectation e_{jk} as a function of $J-1$ normalized valuations. Since $\tilde{H}(\cdot)$ is invertible, we can write $e_{jk}(\sigma^0) = W(\tilde{H}_k^{-1}(\sigma^0))$.

$$\begin{aligned}\sum_{j=1}^{J}\sigma_j^0 e_{jk}(\theta,\sigma^0) &= \sum_{j=1}^{J}\sigma_{jk}^0 W_{jk}(\tilde{H}^{-1}(\sigma^0))\\ &= \sigma^0\cdot W(\tilde{H}^{-1}(\sigma^0)) \quad \text{in matrix form}\end{aligned}$$

Differentiating with respect to $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_J)$,

$$\frac{\partial}{\partial\sigma}\sigma^0\cdot W(\tilde{H}^{-1}(\sigma^0)) = \frac{\partial\sigma^0}{\partial\sigma}W(\tilde{v}) + \frac{\partial W(\tilde{v})}{\partial\tilde{H}^{-1}(\sigma)}\frac{\partial\tilde{H}^{-1}(\sigma)}{\partial\sigma}\frac{\partial\sigma}{\partial\sigma^0}\sigma^0$$

Each term becomes

1. $\dfrac{\partial\sigma^0}{\partial\sigma}W(\tilde{v}) = \dfrac{\partial(\sigma_1,\cdots,\sigma_{J-1})'}{\partial(\sigma_0,\cdots,\sigma_{J-1})'}W(\tilde{v}) = \begin{bmatrix}-1 & -1 & -1 & \cdots & -1\\ 1 & 0 & 0 & \cdots & 0\\ 0 & 1 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & 1\end{bmatrix}W(\tilde{v}) = [-i;I]W(\tilde{v})$
2. $\dfrac{\partial W(\tilde{v})}{\partial\tilde{H}^{-1}(\sigma)} = \dfrac{\partial W(\tilde{v})}{\partial\tilde{v}}$
3. $\dfrac{\partial\tilde{H}^{-1}(\sigma)}{\partial\sigma} = \dfrac{\partial\tilde{v}}{\partial\tilde{H}(\tilde{v})} = \left[\dfrac{\partial\tilde{H}(\tilde{v})}{\partial\tilde{v}}\right]^{-1}$
4. $\dfrac{\partial\sigma}{\partial\sigma^0}\sigma^0 = [-i;I]\sigma^0 = \begin{bmatrix}-\sigma_1-\cdots-\sigma_{J-1}\\ \sigma_1\\ \vdots\\ \sigma_{J-1}\end{bmatrix} = \begin{pmatrix}1-i'\sigma^0\\ \sigma^0\end{pmatrix}$

where i is a $1\times(J-1)$ vector of ones and I is $(J-1)\times(J-1)$ identity matrix. From 1∼4, we get an analogous equation to (Ap.1) in Aguirregabiria and Mira (2002).

$$\frac{\partial}{\partial\sigma}\sum_{j=1}^{J}\sigma_j^0 e_j = [-i;I]W(\tilde{v}) + \left[\frac{\partial\tilde{H}(\tilde{v})}{\partial\tilde{v}}\right]^{-1}\frac{\partial W(\tilde{v})'}{\partial\tilde{v}}\begin{pmatrix}1-i'\sigma^0\\ \sigma^0\end{pmatrix} \tag{25}$$

From the surplus function,

$$S(\tilde{v}) = [1-i'\tilde{H}(\tilde{v});\tilde{H}(\tilde{v})](v+W(\tilde{v})).$$

Differentiating with respect to $\tilde{v}$,

$$
\begin{aligned}
\sigma = \frac{\partial S(\tilde{v})}{\partial \tilde{v}} &= \begin{bmatrix} -i' \frac{\partial \tilde{H}(\tilde{v})}{\partial \tilde{v}} \\ \frac{\partial \tilde{H}(\tilde{v})}{\partial \tilde{v}} \end{bmatrix} [v + W(\tilde{v})] + \left[\frac{\partial v}{\partial \tilde{v}} + \frac{\partial W(\tilde{v})}{\partial \tilde{v}}\right] \begin{bmatrix} 1 - i'\tilde{H}(\tilde{v}) \\ \tilde{H}(\tilde{v}) \end{bmatrix} \\
&= \frac{\partial \tilde{H}(\tilde{v})}{\partial \tilde{v}} \begin{bmatrix} -i \\ I \end{bmatrix} [v + W(\tilde{v})] + \left[\frac{\partial v}{\partial \tilde{v}} + \frac{\partial W(\tilde{v})}{\partial \tilde{v}}\right] \begin{bmatrix} 1 - i'\sigma^0 \\ \sigma^0 \end{bmatrix} \\
\left[\frac{\partial \tilde{H}(\tilde{v})}{\partial \tilde{v}}\right]^{-1} \sigma &= \begin{bmatrix} -i \\ I \end{bmatrix} [v + W(\tilde{v})] + \left[\frac{\partial \tilde{H}(\tilde{v})}{\partial \tilde{v}}\right]^{-1} \left[\frac{\partial v}{\partial \tilde{v}} + \frac{\partial W(\tilde{v})}{\partial \tilde{v}}\right] \begin{bmatrix} 1 - i'\sigma^0 \\ \sigma^0 \end{bmatrix} \\
&= \begin{bmatrix} -i \\ I \end{bmatrix} v + \left[\frac{\partial \tilde{H}(\tilde{v})}{\partial \tilde{v}}\right]^{-1} \frac{\partial v}{\partial \tilde{v}} \begin{bmatrix} 1 - i'\sigma^0 \\ \sigma^0 \end{bmatrix} + \begin{bmatrix} -i \\ I \end{bmatrix} W(\tilde{v}) + \left[\frac{\partial \tilde{H}(\tilde{v})}{\partial \tilde{v}}\right]^{-1} \frac{\partial W(\tilde{v})'}{\partial \tilde{v}} \begin{bmatrix} 1 - i'\sigma^0 \\ \sigma^0 \end{bmatrix} \\
&= \begin{bmatrix} -i \\ I \end{bmatrix} v + \left[\frac{\partial \tilde{H}(\tilde{v})}{\partial \tilde{v}}\right]^{-1} \begin{bmatrix} -i \\ I \end{bmatrix} \begin{bmatrix} 1 - i'\sigma^0 \\ \sigma^0 \end{bmatrix} + \frac{\partial}{\partial \sigma} \sum_{j=1}^{J} \sigma_j^0 e_j \text{ from Equation (25)} \\
&= \begin{bmatrix} -i \\ I \end{bmatrix} v + \left[\frac{\partial \tilde{H}(\tilde{v})}{\partial \tilde{v}}\right]^{-1} \sigma + \frac{\partial}{\partial \sigma} \sum_{j=1}^{J} \sigma_j^0 e_j
\end{aligned}
$$

Rearranging, we have the stated result:

$$
\frac{\partial}{\partial \sigma} \sum_{j=1}^{J} \sigma_j^0 e_j(\theta, \sigma^0) = -\tilde{v} = -\tilde{H}^{-1}(\sigma^0).
$$

□

Lemma 2. *Let σ^0 be an arbitrary set of conditional choice probabilities, and define the mapping*

$$
G(\sigma^0, V) = [(\rho + \lambda)I - \lambda\Sigma(\sigma) - Q_0]^{-1}[\lambda\Sigma(\sigma)V + u + \lambda E(\theta, \sigma^0)]
$$

Let σ_k^0 be $(J-1)$ column vector of CCPs in σ^0 associated with state k. Then (i) $\partial G_l / \partial \sigma_k^0 = 0$ for $l \neq k$ and (ii) $\partial G_k / \partial \sigma_k^0 = [(\rho + \lambda)I - \lambda\Sigma(\sigma) - Q_0]^{-1}[\lambda[\tilde{v} - \tilde{H}_k^{-1}(\sigma^0)]]$

Proof. (i) Given that G_k does not depend on the probabilities for states different to k, (i) follows trivially.
(ii) Using the results from Lemma 1 and denoting $\tilde{v}_k$ as $(K-1)$ column vector of differ-

enced values corresponding to state k,

$$\begin{aligned}\frac{\partial G_k}{\partial \sigma_k^0} &= [(\rho+\lambda)I - \lambda\Sigma(\sigma) - Q_0]^{-1}\Big[\lambda \frac{\partial \Sigma_k(\theta,\sigma)}{\partial \sigma_k^0} V_k + \frac{\partial}{\partial \sigma}\sum_{j=1}^{J} \sigma_{jk}^0 e_{jk}\Big] \\ &= [(\rho+\lambda)I - \lambda\Sigma(\sigma) - Q_0]^{-1}\Big[\lambda[-i;I]V_{l(j,k)} - \lambda \tilde{H}_k^{-1}(\sigma^0)\Big] \\ &= [(\rho+\lambda)I - \lambda\Sigma(\sigma) - Q_0]^{-1}\big[\lambda[\tilde{v}_k - \tilde{H}_k^{-1}(\sigma^0)]\big]\end{aligned}$$

□

We present the proof for Proposition 5 below.

Proof. Rearranging Equation (2) and using the value function operator,

$$\Upsilon(\theta,\sigma) = [(\rho+\lambda)I - \lambda\Sigma(\sigma) - Q_0]^{-1}[\lambda\Sigma(\sigma)\Upsilon(\theta,\sigma) + u + \lambda E(\sigma)]$$

Therefore, $\Upsilon(\theta,\sigma) = G(\sigma,\Upsilon(\theta,\sigma))$. Differentiating on both sides with respect to σ,

$$\begin{aligned}\frac{\partial \Upsilon(\sigma)}{\partial \sigma} &= \frac{\partial G(\cdot)}{\partial \sigma} + \frac{\partial G(\cdot)}{\partial \Upsilon(\cdot)}\frac{\partial \Upsilon(\sigma)}{\partial \sigma} \\ \Big[I - \frac{\partial G(\cdot)}{\partial \Upsilon(\cdot)}\Big]\frac{\Upsilon(\sigma)}{\partial \sigma} &= \frac{\partial G(\cdot)}{\partial \sigma} \\ \frac{\partial \Upsilon(\sigma)}{\partial \sigma} &= \Big[I - \frac{\partial G(\cdot)}{\partial \Upsilon(\cdot)}\Big]^{-1}\frac{\partial G(\cdot)}{\partial \sigma}\end{aligned}$$

By Lemma 2, $\partial G_l/\partial\sigma_k = 0$ for $l \neq k$ and $\partial G_k/\partial\sigma_k = [(\rho+\lambda)I - \lambda\Sigma(\sigma) - Q_0]^{-1}\big[\lambda[\tilde{v}_k - \tilde{H}_k^{-1}(\sigma)]\big]$. In the latter case, let $\tilde{\sigma}$ be the fixed point of $\Upsilon(\theta,\sigma)$. Then $\tilde{v}_k = \tilde{H}^{-1}(\Upsilon(\theta,\tilde{\sigma})) = \tilde{H}^{-1}(\tilde{\sigma})$. So at the fixed point $\tilde{v}_k = \tilde{H}_k^{-1}(\sigma)$ which means $\partial G_k/\partial\sigma_k = 0$, and hence, $\partial G(\cdot)/\partial\sigma^0 = 0$. Then, at the fixed point σ^0, $\partial\Upsilon(\sigma^0)/\partial\sigma^0 = 0$.

Since $\Gamma(v)$ is continuous, and $\Psi(\theta,\sigma) = \Gamma(\Upsilon(\theta,\sigma))$, $\Psi_\sigma = 0$ at the fixed point σ. □

Since the Jacobian matrix Ψ_σ is zero at the fixed point, $\rho(M_\Psi\Psi_\sigma) = 0$. Then, by Proposition 4, the NPL estimator always converges to $(\hat{\theta},\hat{\sigma})$ in a single agent dynamic choice model in continuous time if the assumptions hold.

C Convergence condition in two player game example

In this section, we present a simple example of two player game to show the local convergence condition which is a continuous version of Example 2 in Kasahara and Shimotsu (2012). There exist two firms competing in a market with low demand (L) or high demand (H). They can choose to open or close a store ($j = 1$) or remain at the previous state ($j = 0$). Each state can be represented with a 1×3 vector $x_k = (x_{1k}, x_{2k}, x_{3k})$ where x_{1k} and x_{2k} are activity indicators for firm 1 and 2 and x_{3k} is market size which is exogenously determined. Then, we can write the state space as

$$\mathcal{X} = \{(0,0,L),(1,0,L),(0,1,L),(1,1,L),(0,0,H),(1,0,H),(0,1,H),(1,1,H)\}.$$

For exogenous state changes, let $q = (q_1, q_2)$ be a transition rate for nature where q_1 is the rate at which demand changes from L to H and q_2 is the rate at which demand decreases from H to L. When a move arrival occurs with parameter λ, an agent i chooses its action j which changes the state from k to $l(i, j, k)$.

$$x_{l(i,j,k)} = \begin{cases} (x_{-ik}, 1, x_{3k}) & \text{if } j = 1 \text{ and } x_{ik} = 0 \\ (x_{-ik}, 0, x_{3k}) & \text{if } j = 1 \text{ and } x_{ik} = 1 \\ x_k & \text{otherwise} \end{cases}$$

where x_{-ik} is the activity indicator for the other firm and x_{3k} is the market demand at state k.

We define the flow payoff u_{ik} as in Equation (1):

$$u_{ik} = \theta_{RS} \ln(x_{3k}) - \theta_{RN} \ln(1 + x_{-ik}) - \theta_{FC,i}.$$

$$\psi_{ijk} = \begin{cases} -\theta_{EC} & \text{if } j = 1 \text{ and } x_{ik} = 1 \\ 0 & \text{otherwise} \end{cases}$$

where θ_{EC} is the entry cost.

We now present two important equations for solving the equilibrium.

1. Bellman optimality

$$\Upsilon_i(\theta, \sigma) \equiv [(\rho + \lambda)I - \lambda\Sigma_1(\sigma_1) - \lambda\Sigma_2(\sigma_2) - Q_0]^{-1}[u + \lambda E(\sigma)]$$

2. Conditional choice probability

$$\Gamma_{ijk}(\theta,\sigma) \equiv \int \{\varepsilon_{ij'k} - \varepsilon_{ijk} \leq \psi_{ijk} - \psi_{ij'k} + V_{i,l(i,j,k)}(\theta,\sigma) - V_{i,l(i,j',k)}(\theta,\sigma)\ \forall j'\} f(\varepsilon_k) d\varepsilon_k$$

$$\Gamma_{i1k}(\theta,\sigma) \equiv \int \{\varepsilon_{i0k} - \varepsilon_{i1k} \leq \psi_{i1k} + V_{i,k'}(\sigma_i) - V_{i,k}(\theta,\sigma)\ \forall j'\} f(\varepsilon_k) d\varepsilon_k$$
$$= \frac{\exp(\Psi_{i1k} + V_{l(i,1,k)})}{1 + \exp(\Psi_{i1k} + V_{l(i,1,k)})}$$

where k' is $l(i,1,k)$ and assuming ε_{ijk} follows T1EV[6]. We can also write Γ_{i2k} similarly.

Then, substituting the value function operator into the best response function gives the mapping $\Psi(\theta,\sigma)$. The mapping Ψ and its Jacobian matrix evaluated at a fixed point (θ^0,σ^0) are given by

$$\Psi(\theta,\sigma) = \begin{pmatrix} \Psi_1(\theta,\sigma) \\ \Psi_2(\theta,\sigma) \end{pmatrix} = \begin{pmatrix} \Gamma_1(\theta, \Upsilon_1(\theta,\sigma_1),\sigma_2) \\ \Gamma_2(\theta, \Upsilon_2(\theta,\sigma_2),\sigma_1) \end{pmatrix}$$

and

$$\Psi_\sigma^0 = \begin{pmatrix} 0 & \nabla_{\sigma_2'} \Psi_1(\theta^0,\sigma^0) \\ \nabla_{\sigma_1'} \Psi_2(\theta^0,\sigma^0) & 0 \end{pmatrix}$$

where $\nabla_{\sigma_i'} \Psi_i(\theta^0,\sigma^0) = 0$ follows from Proposition 5. So, as suggested in Proposition 4 in Kasahara and Shimotsu (2012), when there are small dynamic interactions between agents, which implies that $\nabla_{\sigma_{-i}'} \Psi_i(\theta^0,\sigma^0)$ is small, it is more likely that $\rho(M_{\Psi_\theta} \Psi_\sigma^*) < 1$ is satisfied.

Since Ψ_σ^0 is a symmetric matrix, we focus on $\nabla_{\sigma_2'} \Psi_1(\theta^0,\sigma^0)$.

$$Q_1^L = \lambda \begin{bmatrix} -\sigma_{111} & \sigma_{111} & 0 & 0 \\ \sigma_{112} & -\sigma_{112} & 0 & 0 \\ 0 & 0 & -\sigma_{113} & \sigma_{113} \\ 0 & 0 & \sigma_{114} & -\sigma_{114} \end{bmatrix} \quad Q_1^H = \begin{bmatrix} -\sigma_{115} & \sigma_{115} & 0 & 0 \\ \sigma_{116} & -\sigma_{116} & 0 & 0 \\ 0 & 0 & -\sigma_{117} & \sigma_{117} \\ 0 & 0 & \sigma_{118} & -\sigma_{118} \end{bmatrix}$$

[6] If we assume ε_{i0} and ε_{i1} are normally distributed with zero means and define $\sigma^2 \equiv \text{var}(\varepsilon_{i0} - \varepsilon_{i1})$, the equation above becomes

$$\Gamma_{i1k}(\theta,\sigma) = \Phi\Big(\frac{1}{\sigma}\Big(\psi_{i1k} + V_{i,k'}(\sigma_i) - V_{ik}(\sigma_i)\Big)\Big)$$

$$Q_2^L = \lambda \begin{bmatrix} -\sigma_{211} & 0 & \sigma_{211} & 0 \\ 0 & -\sigma_{212} & 0 & \sigma_{212} \\ \sigma_{213} & 0 & -\sigma_{213} & 0 \\ 0 & \sigma_{214} & 0 & -\sigma_{214} \end{bmatrix} \quad Q_2^H = \begin{bmatrix} -\sigma_{215} & 0 & \sigma_{215} & 0 \\ 0 & -\sigma_{216} & 0 & \sigma_{216} \\ \sigma_{217} & 0 & -\sigma_{217} & 0 \\ 0 & \sigma_{218} & 0 & -\sigma_{218} \end{bmatrix}$$

$$Q_1 = \begin{bmatrix} Q_1^L & 0_4 \\ 0_4 & Q_1^H \end{bmatrix} \quad Q_2 = \begin{bmatrix} Q_2^L & 0_4 \\ 0_4 & Q_2^H \end{bmatrix}$$

$$Q_0 = \begin{bmatrix} -q_1 I_4 & q_1 I_4 \\ q_2 I_4 & -q_2 I_4 \end{bmatrix}$$

where I_4 is 4×4 identity matrix. Now we calculate the inverse part of Bellman operator which we denote by B.

$$B \equiv \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

where $B_{12} = q_1 I_4$, $B_{21} = q_2 I_4$,

$$B_{11} = \begin{bmatrix} \rho + 2\lambda + \lambda \sum_i \sigma_{i11} + q_1 & -\lambda\sigma_{111} & -\lambda\sigma_{211} & 0 \\ -\lambda\sigma_{112} & \rho + 2\lambda + \lambda \sum_i \sigma_{i12} + q_1 & 0 & -\lambda\sigma_{212} \\ -\lambda\sigma_{213} & 0 & \rho + 2\lambda + \lambda \sum_i {}_{i13} + q_1 & -\lambda\sigma_{113} \\ 0 & -\lambda\sigma_{214} & -\lambda\sigma_{114} & \rho + 2\lambda + \lambda \sum_i \sigma_{i14} + q_1 \end{bmatrix}$$

$$B_{22} = \begin{bmatrix} \rho + 2\lambda + \lambda \sum_\sigma {}_{i15} + q_2 & -\lambda\sigma_{115} + q_2 & -\lambda\sigma_{215} & 0 \\ -\lambda\sigma_{116} & \rho + 2\lambda + \lambda \sum_i \sigma_{i16} + q_2 & 0 & -\lambda\sigma_{216} \\ -\lambda\sigma_{217} & 0 & \rho + 2\lambda + \lambda \sum_i \sigma_{i17} + q_2 & -\lambda\sigma_{117} \\ 0 & -\lambda\sigma_{218} & -\lambda\sigma_{118} & \rho + 2\lambda + \lambda \sum_i \sigma_{i18} + q_2 \end{bmatrix}.$$

Substituting back to the Bellman equation,

$$\Upsilon_1(\theta, \sigma) = B^{-1}[u_1 + \lambda\sigma_1(\psi_1 + e_1(\theta, \sigma_1))]$$

where u_1, ψ_1, and $e_1(\sigma)$ are $K \times 1$ vectors. Differentiating with respect to σ_2 gives

$$\nabla_{\sigma_2'} \Upsilon_1(\theta, \sigma) = \frac{\partial B^{-1}}{\partial \sigma_2}[u_1 + \sum_j \sigma_{1j}(\psi_{1j} + e_{1j}(\theta, \sigma_1))]$$

Noting that $\frac{\partial B}{\partial \sigma_2} = \left(\frac{\partial B}{\partial \sigma_{211}} \quad \frac{\partial B}{\partial \sigma_{212}} \cdots \quad \frac{\partial B}{\partial \sigma_{21K}} \right)$ and $\frac{\partial B^{-1}}{\partial \sigma_2} = B^{-1} \frac{\partial B}{\partial \sigma_2} B^{-1\prime}$,

$$\frac{\partial B^{-1}}{\partial \sigma_2} = B^{-1} \left[\begin{pmatrix} \lambda & 0 & \lambda & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \lambda & 0 & -\lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \quad \cdots \quad \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \cdots & \lambda \end{pmatrix} \right] \begin{pmatrix} B^{-1} \\ B^{-1} \\ B^{-1} \\ B^{-1} \end{pmatrix}$$

If $\lambda = 0$ (or $\lambda_2 = 0$), $\nabla_{\sigma_2'} \Upsilon_1(\theta, \sigma)$ becomes a matrix of zeros which makes $\nabla_{\sigma_2'} \Psi_1(\theta, \sigma) = 0$. This means that receiving less chances to move can increase the possibility of convergence.

D Estimation details

In this section, we provide more details on estimation process. Our goal is to estimate $\theta = (\theta_{FC,1}, \ldots, \theta_{FC,5}, \theta_{RS}, \theta_{RN}, \theta_{EC})$ using the NPL algorithm. We first review two steps needed for estimating:

1. Given $\hat{\sigma}^{l-1}$, update $\hat{\theta}$ by

$$\hat{\theta}^l = \underset{\theta\in\Theta}{\text{argmax}}\, L_M(\theta, \hat{\sigma}^{l-1}) = \underset{\theta\in\Theta}{\text{argmax}}\, \frac{1}{M} \sum_{m=1}^{M} \sum_{n=1}^{T} \ln P_{k_{m,n-1},k_{mn}}(\Delta; \Psi(\theta, \hat{\sigma}^{l-1})). \tag{11}$$

2. Update $\hat{\sigma}$ using the equilibrium condition, i.e.

$$\hat{\sigma}^l = \Psi(\hat{\theta}^l, \hat{\sigma}^{l-1}) \tag{12}$$

   where

$$\hat{\sigma}_{jk} = \frac{\exp(\psi_{jk} + V_{l(j,k)})}{\sum_j' \exp(\psi_{j'k} + V_{l(j',k)})}.$$

We need to express the likelihood function in terms of θ to maximize it with regard to θ. First, we start from calculating the value function in Equation (4) using value function below:

$$V_i(\theta, \sigma) = \left[(\rho + N\lambda)I - \lambda \sum_{m=1}^{N} \Sigma_m(\sigma_m) - Q_0\right]^{-1} [u_i(\theta) + \lambda_i E_i(\theta, \sigma)] \tag{2}$$

where $\Sigma_m(\sigma_m)$ is the $K \times K$ state transition matrix induced by the actions of player m given the choice probabilities σ_m and where $E_i(\theta, \sigma)$ is a $K \times 1$ vector where each element k is the ex-ante expected value of the choice-specific payoff in state k, $\sum_j \sigma_{ijk}[\psi_{ijk} + e_{ijk}(\theta, \sigma)]$.

We can define the first parenthesis in Equation (2) as Ξ. We assume $\rho_i = 1$, $\lambda_{ik} = 1$ for all $i = 1, \ldots, N$ and $k = 1, \ldots, K$, and that Q_0 is known. We can also calculate $\Sigma_m(\sigma_m)$ using $\hat{\sigma}_m^{l-1}$. We start from true probabilities σ^* for 2S-True estimator and random draws for $\hat{\sigma}^0$ for NPL-random estimator.

Then, we rewrite the second parenthesis in Equation (2), in terms of θ. The flow payoff u_{ik} can be rewritten as:

$$\begin{aligned} u_{ik} &= \theta_{RS} \ln(s_k) - \theta_{RN} \ln\big(1 + \sum_{m \neq i} a_{mk}\big) - \theta_{FC,i} \\ &= z_{ik}^u \theta^u \end{aligned}$$

where $z^u_{ik} = [I_{ik} \ \ln(s_k) \ \ln\left(1 + \sum_{m \neq i} a_{mk}\right)]$ where I_{ik} is a 1×5 vector with 1 in i-th position and zero elsewhere, and $\theta^u = (\theta_{FC,1}, \ldots, \theta_{FC,5}, \theta_{RS}, \theta_{RN})'$.

Now we express $E_{ik}(\theta, \sigma)$, corresponding element of $E(\theta, \sigma)$ in term of θ. The determinant part of instantaneous payoff has the structure:

$$\psi_{ijk} = z^{\psi}_{ijk} \theta_{EC}$$

where

$$z^{\psi}_{ijk} = \begin{cases} -1 & \text{if } j = 1 \\ 0 & \text{otherwise} \end{cases}$$

Expected value of deterministic instantaneous payoff z^{ψ}_{ijk} differs by the choice, so we can write

$$\begin{aligned} Ez^{\psi}_{ik} &= \sum_{j \in \mathcal{A}} \sigma_{ijk} z^{\psi}_{ijk} \\ &= \lambda[\sigma_{0k} \times 0 + \sigma_{1k} \times (-1) + \sigma_{-1,k} \times 0] \end{aligned}$$

The stochastic part ε_{ijk} has the expected value e_{ijk}:

$$\begin{aligned} e_{ijk} &= E\varepsilon_{ijk} \\ &= \lambda[\sigma_{0k}(\gamma - \log(\sigma_{0k})) + \sigma_{1k}(\gamma - \log(\sigma_{1k})) + \sigma_{-1,k}(\gamma - \log(\sigma_{-1,k}))] \end{aligned}$$

using T1EV distribution of ε_{ijk}.

Collecting all terms for the value function, we can rewrite the value function as

$$\begin{aligned} V_{ij}(\theta, \sigma) &= \Xi^{-1}[u(\theta) + \lambda E(\theta, \sigma)] \\ &= \Xi^{-1}[z^u_{ij} \theta^u + \lambda E(z^{\psi}_{ij} \theta_{EC} + \varepsilon_{ij})] \\ &= \Xi^{-1} E[z_{ij} \theta + \varepsilon_{ij}] \\ &= W_{ij} \theta' + \tilde{e}_{ij}. \end{aligned}$$

where z_{ij} is a $K \times |\theta|$ vector with each kth element $(z^u_{ik}, z^{\psi}_{ijk})$, W_{ijk} is a $K \times 1$ vector which equals $\Xi^{-1} E z_{ij}$, and $\tilde{e}_{ij}$ is $\Xi^{-1} e_{ij}$. Now we can write conditional choice probabilities in terms of θ,

$$\hat{\sigma}_{ijk} = \frac{\exp(W_{ijk} \theta' + \tilde{e}_{ijk})}{\sum_{j'} \exp(W_{ij'k} \theta' + \tilde{e}_{ij'k})}$$

Since we can calculate W_{ijk} and $\tilde{e}_{ijk}$ from known parameters or conditional choice probabilities, the only unknown term in $\hat{\sigma}$, hence the likelihood function, is θ. Then, we can form Q in terms of θ using Q_0 and $\hat{\sigma}$, and maximize the likelihood with regard to θ. We use starting values of $\theta^0 = (1,1,1,1,1,1,1,1)$. We use the solver `fminunc` in MATLAB.

Once $\hat{\theta}^l$ is estimated, the conditional choice probability $\hat{\sigma}^l$ is updated. The entire process is repeated 20 times (L= 20). For Monte Carlo experiments, we generate 100 different data set using same parameters and estimate separately for each data set. Then, we report the mean of estimates as estimates for each estimator and the standard deviation from 100 estimates as standard error.

E Estimation results for discrete time model

Table 8 and 9 are the estimation results for a five-player discrete game example in Section 4 of Aguirregabiria and Mira (2007) (corresponding to Table IV and V).

Table 8: Monte Carlo experiment results (Discrete time data)

Exp.	Estimator	Parameters			
		$\theta_{FC,1}$	θ_{RS}	θ_{EC}	θ_{RN}
1	**True values**	**-1.9000**	**1.0000**	**1.0000**	**0.0000**
	2S-True	-1.9223 (0.2690)	1.0100 (0.1292)	1.0095 (0.1470)	0.0128 (0.3744)
	2S-Freq	-0.4418 (0.3175)	0.3537 (0.1483)	1.1478 (0.2036)	0.1580 (0.3900)
	2S-Logit	-1.9355 (0.2740)	1.0091 (0.1322)	1.0035 (0.1473)	0.0012 (0.3861)
	2S-Random	-2.2928 (0.3620)	1.1746 (0.0871)	1.0151 (0.1521)	-0.0492 (0.2920)
	NPL	-1.9138 (0.2668)	1.0186 (0.1353)	1.0061 (0.1472)	0.0379 (0.3842)
2	**True values**	**-1.9000**	**1.0000**	**1.0000**	**1.0000**
	2S-True	-1.9196 (0.1828)	1.0232 (0.1778)	0.9976 (0.1240)	1.0649 (0.5380)
	2S-Freq	-0.9187 (0.2238)	0.3392 (0.1172)	0.8985 (0.1371)	0.0768 (0.3296)
	2S-Logit	-1.9489 (0.1983)	0.9965 (0.1830)	0.9907 (0.1290)	0.9644 (0.5348)
	2S-Random	-2.5371 (0.3205)	0.8425 (0.0495)	1.0014 (0.1219)	-0.0077 (0.2620)
	NPL	-1.9161 (0.1991)	1.0473 (0.2130)	0.9875 (0.1272)	1.1381 (0.6340)
3	**True values**	**-1.9000**	**1.0000**	**1.0000**	**2.0000**
	2S-True	-1.9512 (0.1953)	0.9992 (0.2218)	1.0123 (0.1054)	1.9656 (0.8316)
	2S-Freq	-1.1521 (0.1842)	0.2822 (0.0961)	0.7940 (0.1057)	0.0071 (0.3249)
	2S-Logit	-1.9769 (0.2515)	1.0173 (0.3253)	0.9981 (0.1455)	2.0193 (1.1889)
	2S-Random	-2.5380 (0.3123)	0.5994 (0.0476)	1.0782 (0.1031)	0.0094 (0.2481)
	NPL	-1.9767 (0.2410)	0.9426 (0.2098)	1.0215 (0.1124)	1.7337 (0.7419)
4	**True values**	**-1.9000**	**1.0000**	**0.0000**	**1.0000**
	2S-True	-1.8646 (0.5459)	1.0447 (0.3312)	0.0135 (0.1147)	1.1795 (1.3573)
	2S-Freq	-0.9384 (0.2381)	0.3457 (0.1027)	0.2443 (0.1151)	0.1403 (0.3273)
	2S-Logit	-2.0710 (0.4666)	0.9030 (0.2665)	0.0121 (0.1125)	0.5917 (1.0819)
	2S-Random	-2.2555 (0.3179)	0.7475 (0.0508)	0.0266 (0.1099)	-0.0104 (0.2662)
	NPL	-1.8761 (0.5051)	1.0238 (0.2799)	0.0130 (0.1118)	1.1103 (1.1559)
5	**True values**	**-1.9000**	**1.0000**	**2.0000**	**1.0000**
	2S-True	-1.9289 (0.1737)	1.0190 (0.1361)	2.0251 (0.1326)	1.0361 (0.3341)
	2S-Freq	-0.8575 (0.2145)	0.3742 (0.1202)	1.5920 (0.1556)	0.1703 (0.2565)
	2S-Logit	-1.9414 (0.1988)	1.0078 (0.1546)	2.0193 (0.1382)	0.9935 (0.3619)
	2S-Random	-2.9207 (0.3876)	0.9711 (0.0674)	2.0053 (0.1294)	0.0112 (0.3531)
	NPL	-1.9447 (0.1963)	1.0271 (0.1637)	2.0173 (0.1364)	1.0462 (0.3923)
6	**True values**	**-1.9000**	**1.0000**	**4.0000**	**1.0000**
	2S-True	-1.9183 (0.2113)	0.9955 (0.1131)	4.0165 (0.1872)	0.9711 (0.2067)
	2S-Freq	-0.5668 (0.2412)	0.3366 (0.1375)	2.7380 (0.1972)	0.2183 (0.2455)
	2S-Logit	-1.9161 (0.2383)	0.9894 (0.1295)	4.0175 (0.1897)	0.9570 (0.2226)
	2S-Random	-3.9519 (0.6248)	1.2934 (0.1029)	3.8196 (0.1878)	0.0677 (0.5169)
	NPL	-1.9358 (0.2359)	1.0017 (0.1329)	4.0132 (0.1884)	0.9738 (0.2314)

Note: Values in parentheses are standard deviations.

Table 9: Square-root MSEs relative to the two-step PML with true CCPs (Discrete time data)

Exp.	Estimator	Parameters $\theta_{FC,1}$	θ_{RS}	θ_{EC}	θ_{RN}
1	2S-Freq	5.5287	5.1166	1.7075	1.1232
	2S-Logit	1.0234	1.0224	0.9999	1.0304
	2S-Random	1.9789	1.5055	1.0372	0.7903
	NPL	0.9897	1.0541	0.9997	1.0304
2	2S-Freq	5.4741	3.7424	1.3746	1.8089
	2S-Logit	1.1107	1.0210	1.0422	0.9892
	2S-Random	3.8787	0.9207	0.9823	1.9214
	NPL	1.0863	1.2165	1.0301	1.1974
3	2S-Freq	3.8145	3.2654	2.1813	2.4261
	2S-Logit	1.3026	1.4691	1.3708	1.4286
	2S-Random	3.5178	1.8189	1.2190	2.4102
	NPL	1.2522	0.9808	1.0782	0.9471
4	2S-Freq	1.8109	1.9818	2.3382	0.6719
	2S-Logit	0.9085	0.8486	0.9800	0.8447
	2S-Random	0.8718	0.7707	0.9791	0.7632
	NPL	0.9243	0.8406	0.9746	0.8481
5	2S-Freq	6.0445	4.6379	3.2359	2.5845
	2S-Logit	1.1534	1.1263	1.0340	1.0771
	2S-Random	6.2011	0.5340	0.9596	3.1247
	NPL	1.1431	1.2080	1.0187	1.1755
6	2S-Freq	6.3885	5.9859	6.7986	3.9264
	2S-Logit	1.1261	1.1482	1.0138	1.0863
	2S-Random	10.1139	2.7469	1.3860	5.1088
	NPL	1.1253	1.1742	1.0054	1.1162