Search for Bidders by a Deadline*

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Abstract

Empirical evidence shows that the dominant selling process in many markets, such as M&As, is not competitive. This paper provides a possible explanation for this puzzle, by modelling it as a seller's sequential search for bidders by a finite deadline. We show that the seller's optimal search outcomes can be implemented by a sequential search auction, which is characterized by declining reserve prices and increasing search intensities (sample sizes) over time. These monotonicity results are robust in both cases of long-lived and short-lived bidders, yet a seller with short-lived bidders sets lower reserve prices and searches more intensively. We further examine the efficient search auction, and show that it has both lower reserve prices and search intensities than the optimal search auction. Therefore, the inefficiency of an optimal search auction can stem from its inefficient search rule.

Keywords sequential search; search auction; deadline; search intensity; reserve prices

JEL classification: D44; D82; D83

1 Introduction

It is puzzling that many important selling processes in markets seem not competitive, where no obvious competition among buyers is observed. For instance, in mergers and acquisitions (M&As), it is well-documented that the dominant selling process is one-on-one negotiation.

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That is, when the board of directors decides to sell a firm, in most cases, they just contact one potential buyer. Betton et al. (2008) report that 95% of their sample deals in the US market, during the period from 1980 to 2005, are classified as non-competitive negotiation, and Andrade et al. (2001) also describe the prototypical M&As in the 1990s as friendly transactions, where normally there was just one bidder. These observations go against the conventional wisdom that competition among bidders not only helps to raise bid premiums, but also enhances allocative efficiency in markets.

There have been various empirical explanations for this puzzle. Boone and Mulherin (2007) present a new measure of competition in M&As, and show that many deals classified as negotiation are auction indeed. After reconstructing a new sample using their measure, however, they still have half of the sample deals classified as non-competitive negotiation. On the other hand, Aktas et al. (2010) argue that M&A involves a seller's sequential decisions, and a negotiation in an early stage is usually under the threat of following-up auctions. For example, if the seller fails to achieve a good deal in the negotiation stage, she may invite more bidders and run auctions among them in the following stages. Therefore, the one-on-one negotiation is not insulated from competition at all, and the pressure of following-up auctions can drive up bid premiums in the negotiation stage. This argument is supported by the empirical evidence that there is no significant difference in bid premiums across the two selling processes of negotiation and auction (Boone and Mulherin, 2007, 2008).

Depending on how many bidders to contact in the first stage, Boone and Mulherin (2009) classify the selling processes in M&As into three categories: *one-on-one negotiation*, where a seller contacts a single most likely buyer first; *private controlled sale* (auction), where a seller screens and first invites a small number of qualified bidders to an auction; and *public full-scale auction*, where a seller announces and runs a public simultaneous auction, and all interested bidders can submit bids. Besides the sequential nature, a typical M&A selling process also involves a finite transaction deadline, and a search cost for a seller to contact bidders. For instance, the search cost could be an information cost of the target firm (the seller), due to the loss of its proprietary information to potential acquirers (bidders) in the process of due diligence.

In fact, M&As can be thought of as an example of the following more general problem. A seller wants to allocate an indivisible product among a number of potential bidders; to contact a bidder, she needs to incur a search cost; and the seller has to complete the transaction by a finite deadline. There are many variants of this problem in the real world, such as matching in marriage markets where women are normally more sensitive to age deadlines, academic recruitment in the UK by the Research Excellence Framework deadline, sequential R&D investment over a bounded time horizon, and so on.

This paper proposes a unified framework for studying this problem. Specifically, we model it as a seller's sequential search for potential bidders by a finite deadline. Due to the presence of search costs, a simultaneous full-scale auction is usually not optimal. For instance, if the seller searches bidders sequentially and happens to get an ideal offer, she may not still have incentives to contact other bidders, as search is costly. Likewise, due to the presence of a finite deadline, one-by-one sequential search may not be optimal either, as too few bidders might be sampled. Therefore, facing a finite deadline, a seller may conduct compound search where she searches both sequentially and simultaneously, i.e., she may search multiple bidders simultaneously in one period. We show that the optimal search rule, in this case, is featured by declining reserve prices and increasing search intensities (sample sizes) over time. Our result of increasing search intensities may explain why, in many markets, the dominant selling process can be non-competitive negotiation.

In our model, a (female) seller wants to allocate an indivisible product among a set of potential (male) bidders. To contact a bidder, she needs to incur a search cost, and she has to complete the transaction within T periods. A search mechanism is composed of a sampling rule and a sequence of stage mechanisms. The sampling rule specifies a sequence of bidder samples that the seller will search in each period, and a stage mechanism defines the payment and allocation rules for the stage transaction in each period. If the product is not allocated in a period, the seller then continues searching another bidder sample in the next period, till the end of period T.

We assume the seller fully commits to the announced search mechanism. The standard results in mechanism design show that, when the mechanism is *incentive feasible*, a bidder's expected payment is equal to his virtual value (Myerson, 1981), and therefore, the seller's expected revenue from a bidder sample is equal to the highest virtual value of them. Under the commitment assumption, we can transform a compound search problem into a sequential one, by taking a sample of bidders as an aggregate bidder, who is characterized by their highest virtual value and their gross search cost.

We consider both cases of *long-lived* and *short-lived* bidders. A long-lived bidder, once invited, will stay in the transaction thereafter until the end of period T. With long-lived bidders, a seller can reclaim a previously declined bid at no extra search costs. The case of long-lived bidders is analogous to sequential search with *full* recall. In contrast, a short-lived bidder participates in the transaction only once, when invited, and then goes away. Therefore, a declined bid of a short-lived bidder can not be reclaimed in later periods. The case of short-lived bidders corresponds to sequential search with *no* recall.

We show that, in both cases, the outcomes of an optimal search mechanism can be implemented by a sequential search auction. To be specific, the auction rule is as follows: in period 1, the seller invites a sample M_1 of bidders to a second-price auction with reserve price r_1; if any bidder submits an effective bid, e.g., a bid higher than the reserve price, then the transaction ends, and the payment and allocation are implemented according to the auction rule; if no effective bid submitted, then the seller moves on to period 2 and invites a sample M_2 of new bidders, and runs an auction with a new reserve price r_2; the seller continues with this process until the end of period T.

Our model generates several interesting results. Firstly, we show that an optimal sequential search auction is characterized by decreasing reserve prices and increasing search intensities (sample sizes) over time, and the monotonicity results are robust in both cases of long-lived and short-lived bidders. For decreasing reserve prices, the intuition is that an optimal reserve price reflects the continuation value of following an optimal search procedure from that period on, which naturally gets smaller when the deadline approaches. As a result, the sequence of optimal reserve prices is decreasing over time.

The result of increasing search intensities is more interesting. When bidders are *ex-ante* homogeneous, the level of search intensity is simply measured by the number of bidders sampled (sample size) in each period. We show that, in an optimal search auction, a seller will search increasingly more bidders in each period when the deadline approaches. In other words, the seller will contact the fewest number of bidders in the first period. Our result of increasing search intensities may help explain why, in M&As and other related markets, the dominant selling process can be non-competitive negotiation.

Secondly, we examine the efficient sequential search auction, and show that it is also featured by decreasing reserve prices and increasing search intensities. Comparing with the optimal one, the efficient search auction has both lower reserve prices and search intensities *ceteris paribus*. The result then demonstrates a new source of the inefficiency in an optimal search auction, e.g., due to its inefficient search rule. For instance, an optimal search auction may exclude some bidders who are otherwise valuable for welfare improving; or, a profit-maximizing seller may have excessive incentives to invite more than socially efficient number of bidders in some stages of the transaction.

Thirdly, in the case of *long-lived* bidders, our formula for optimal reserve prices generalizes the well-known result of Weitzman (1979), who studies sequential search without a deadline. Our model differs from his in at least two aspects: first, the targets for search in our model are strategic bidders, while in his paper, they are non-strategic boxes; second, we consider compound search where a seller searches both sequentially and simultaneously, which takes his one-by-one sequential search as a special case. Therefore, our result on optimal reserve prices can be applied to a larger set of search problems. Moreover, we provide a simple formula of the maximum expected search profit, which also takes the standard result in stationary and infinite-horizon (SIH) search problems as a special case.

Finally, we compare the optimal sequential search auctions across the two cases of long-lived and short-lived bidders. Our comparative results show that, for a given sampling rule, the optimal reserve prices for short-lived bidders are lower than those for long-lived bidders.

Second, for a given sequence of reserve prices, a seller with short-lived bidders will search more intensively than one with long-lived bidders. The intuition is straightforward. With short-lived bidders, a seller can not reclaim a previously declined bid, and her fall-back revenue turns to 0 if she decides to search. In consequence, the seller is willing to accept lower reserve prices, and search bidders more intensively *ceteris paribus*.

The remainder of this paper is organized as follows. Section 2 provides a review of the related literature. Section 3 setups the basic model, where we define the search mechanism and characterize the optimal search rule. Section 4 proposes a sequential search auction, and shows that it implements the outcomes of the optimal search mechanism. Section 5 studies an efficient search auction. Section 6 further investigates the case of short-lived bidders. Section 7 is a short conclusion. All proofs appear in Appendix.

2 Related Literature

Our paper is related to the following strands of literature: (1) sequential search and sequential search mechanisms, (2) negotiation versus auction as a selling mechanism, (3) auctions with buy-price options, and (4) sequential auctions and revenue management.

First, for the literature on sequential search, Weitzman (1979) is a seminal paper that studies the so-called Pandora's problem of infinite sequential search with *full* recall. In the model, Pandora faces a number of closed boxes, each containing a random prize; she needs to incur a search cost to open a box, and opens just one box in a single period; Pandora's objective is to maximize the expected value of the prize discovered, net of the total search costs. Weitzman provides a nice solution to this problem, known as Pandora's Rule. First, we can derive a cutoff prize for each box, at which she is indifferent between keeping the cutoff prize and inspecting that box at a cost. Second, the selection rule specifies that if Pandora intends to open a box, it should be the box with the highest cutoff prize among all the remaining unopened boxes. Third, the stopping rule suggests that Pandora should stop searching whenever the highest prize discovered is greater than the highest cutoff prize of all the remaining unopened boxes. Pandora's rule then indicates that, under an optimal search rule, the cutoff prizes are necessarily declining over time.[1]

When there is a deadline, Pandora may intend to open multiple boxes in a single period, and it becomes a compound search problem. Gal et al. (1981) is an early paper that studies this kind of problems in labor markets, by introducing a finite deadline into the classic sequential search model of Lippman and McCall (1976). In their model, job offers are homogeneous in terms of search cost and wage distribution, and the number of offers sampled in a single period thus measures search intensity. They show that, when search is with no recall, a searcher's optimal search rule is featured by decreasing reservation wages and increasing search intensities (sample sizes) over time.[2]

Morgan (1983) extend the work of Gal et al. (1981) to the case of search with full recall, and show that the sequence of optimal search intensities is in general a stochastic process. This is because a searcher chooses search intensity *adaptively* in each period, depending on the realizations of the previous search outcomes which are *ex-ante* uncertain. Morgan and Manning (1985) further investigate the case where a searcher chooses not only the search intensity in each period but the number of periods she may engage in search, and present some results on the existence and properties of optimal search rules.

In the abovementioned literature, the targets for search are non-strategic, e.g., boxes. In contrast, the literature of search mechanism studies the optimal search for strategic agents, who behave rationally and strategically, such as buyers. Crémer et al. (2007) study a search mechanism where a seller of a indivisible product searches for potential bidders sequentially and runs an auction among the invited bidders in each period.[3] They adopt a mechanism design approach and assume the seller is fully committed to the announced mechanism, like in our paper. They show that the Pandora's rule of Weitzman (1979) is still valid in their

[1] Armstrong (2017) provides some additional results on Weitzman's model, and reviews its relevant applications and progress in the literature of consumer search.

[2] Benhabib and Bull (1983) also study search intensity in job markets, where job offers are homogeneous and the search is with no recall. They derive the similar monotonicity results on the optimal search rule as Gal et al. (1981), but they further consider the on-the-job search and provide some further results.

[3] McAfee and McMillan (1988) precede Crémer et al. (2007) in considering search mechanism. In their model of procurement, a buyer seeks to buy an indivisible product from one of a set of producers, who are *ex-ante* homogeneous. The buyer searches the producers sequentially, each at a constant cost. They show that the optimal search mechanism is a combination of *constant* reservation-price search and auction.

search auction. Yet they just consider long-lived bidders, and their sequential search problem is not bounded by a finite deadline.

Our paper contributes to this small literature of search mechanism. Compared with other papers studying sequential search auctions (Crémer et al., 2007; McAfee and McMillan, 1988), our paper differs in at least two aspects. First, we consider a sequential search auction with a finite deadline, where a seller will search both sequentially and simultaneously. It takes those sequential search auctions without a deadline as a special case. Second, we investigate both cases of long-lived and short-lived bidders in this paper, while those papers focus solely on long-lived bidders.

Second, our paper is related to the persistent debates on the choice of optimal selling mechanisms in markets, particularly those between negotiation and auction. Bulow and Klemperer (1996) show that an English auction with $n+1$ bidders yields strictly higher revenue than an optimal auction with n bidders. Therefore, the value of competition by inviting one more bidder strictly dominates the value of bargaining power. In a later paper, Bulow and Klemperer (2009) compare the two selling mechanisms of sequential negotiation and simultaneous auction. They show that a simultaneous auction yields higher expected revenue than a sequential negotiation, though the latter is more efficient as more bidder information is exploited. In their model, bidders need to pay positive entry costs to participate in the transaction, and a seller is unable to commit to a take-it-or-leave-it offer. Under sequential negotiation, an already entered bidder can make a jump-bid so as to deter further entry of outside bidders, which may harm the seller. As a result, a seller usually prefers simultaneous auction to sequential negotiation when selling a product.

But the empirical evidence does not support their results in general. For example, in M&As, the dominant selling process is one-on-one negotiation, not competitive auction. The empirical evidence also shows that there is no significant difference in bid premiums across the two selling processes of negotiation and auction in M&As (Boone and Mulherin, 2007, 2009). In this paper, we propose a new explanation for this puzzle, by modelling the selling process as a seller's sequential search for bidders by a finite deadline. Our result of increasing search intensities may explain why negotiation, not simultaneous auction, can be a dominant selling process in many important markets.

Third, our paper is related to the literature on buy-price auctions. Reynolds and Wooders (2009) study a static buy-price auction with risk-averse bidders, and derive the condition for bidders' cutoff strategies, like in our paper. Kirkegaard and Overgaard (2008) study a dynamic environment where bidders have multi-unit demands and two competing sellers sell products sequentially. They show that an early seller has an incentive to use a buy-price auction. Zhang (2017) studies the optimal sequence of posted-price and auction in a sequential mechanism, where a population of short-lived bidders enters the market periodically, and in each period, the seller chooses between a posted-price and an auction mechanism. He shows that, when there is a deadline and the auction cost is moderate, the optimal mechanism sequence takes the form of posted-prices then auctions.

Finally, our paper is also related to the growing literature on sequential auctions and revenue management. Skreta (2015) investigates optimal sequential auctions with limited commitment, where the same population of bidders participate in each round of the auction. She shows that, in the case of no commitment, the optimal mechanism is qualitatively similar to one under full commitment, e.g., with decreasing reserve prices over time. Said (2011) studies sequential auctions of multi-unit product with changing population, yet in a different environment to our model. Liu et al. (2019) study sequential auctions in the case of limited commitment. Other recent literature on revenue management includes Board and Skrzypacz (2016) with forward-looking buyers, in the case of full commitment, and Dilme and Li (2017), who study revenue management with the arrivals of strategic buyers in the case of no commitment. In this paper, we study a sequential search auction with changing population and under the assumption of full commitment.

3 The Model

A (female) seller wants to allocate an indivisible product among a set $N = \{1, 2, \cdots, n\}$ of potential (male) bidders within T periods. Bidder i's value of the product, denoted by V_i, is distributed according to distribution F on $[0,1]$, with strictly positive density $f > 0$ on $(0,1)$, and V_i's are independent across bidders. F is of increasing failure rate (IFR), and therefore the virtual value function

$$\psi(v) = v - \frac{1 - F(v)}{f(v)}$$

is strictly increasing in v. The distribution F is common knowledge, yet the realization of V_i is observed only by bidder i. The seller's value of the product is normalized to 0. We assume both the seller and the bidders are risk-neutral, and there is no time discounting.

A bidder can not submit a bid if not invited. To invite bidder i to participate in the transaction, the seller needs to incur a non-refundable search cost $c_i \geq 0$. We assume c_i is small enough, such that all the bidders are valuable for the seller, i.e., for all $i \in N$,

$$\int_{r^*}^{1} \psi(v) dF(v) > c_i, \tag{1}$$

where $\psi(r^*) = 0$. Note that the LHS of (1) is the maximum expected revenue the seller can obtain from a truthful bidder, and the RHS is the search cost. The gross search cost for inviting a sample $M \subseteq N$ of bidders is $c_M = \sum_{i \in M} c_i$.

We assume bidders are long-lived in this section. A long-lived bidder, once invited, will stay in the transaction till the end of period T. It also implies that the seller can reclaim a previously declined bid in later periods, without the need of paying extra search costs. With long-lived bidders, the seller's search problem is analogous to sequential search with *full* recall in the search literature. Later in Section 6, we will consider the other case of short-lived bidders, which corresponds to sequential search with *no* recall.

3.1 Compound Search Mechanism

Due to the presence of positive search costs, a simultaneous search, where a seller invites a number of bidders simultaneously to participate in a spot transaction, is usually not optimal. Similarly, due to the presence of a finite deadline, a one-by-one sequential search may not be optimal either. We here consider the more general procedure of compound search, where a seller searches both sequentially and simultaneously, i.e., she may invite multiple bidders simultaneously in a single period. It is clear that compound search takes simultaneous search and sequential search procedure as it special cases.

We define a compound search mechanism as a combination of a sampling rule and a sequence of stage mechanisms. The sampling rule specifies an ordered sequence of bidder samples that a seller will search in each period. To be specific, we define $\mathbf{M}^T = \left\{M^1, M^2, \cdots, M^T\right\}$ as a family of disjoint subsets of N, such that $M^j \cap M^{j'} = \emptyset$ for $j \neq j'$ and $\bigcup_{j=1}^{T} M^j \subseteq N$. A sampling rule is then a permutation of the set $\mathbf{M}^T$, denoted by $\mathbf{M} = (M_1, M_2, \cdots, M_T)$. For instance, if $M_t = M^j$, then the seller will search the bidder sample M^j in period t. We further denote $N_t = \bigcup_{\tau=0}^{t} M_\tau$ as the set of bidder samples that the seller has searched till the end of period t, with $N_0 \equiv \emptyset$, and $N_t^c = \mathbf{M}^T \backslash N_t$ denotes the set of bidder samples the seller has not searched till the end of period t.

The other component is a sequence of stage mechanisms, which specifies a pair of allocation rule $\mathbf{Q}_t$ and payment rule $\mathbf{P}_t$ for each period. Specifically, the seller offers a stage mechanism of $(\mathbf{Q}_t, \mathbf{P}_t)$ to the set N_t of bidders in period t.[4] The allocation rule $\mathbf{Q}_t$ and the payment rule $\mathbf{P}_t$ are respectively a mapping from the N_t bidders' reports to the allocation probabilities and the corresponding payments induced. The sequence of the stage mechanisms is denoted by $(\mathbf{Q}, \mathbf{P}) = \{(\mathbf{Q}_t, \mathbf{P}_t)\}_{1 \leq t \leq T}$.

The compound search mechanism, denoted by $(\mathbf{Q}, \mathbf{P}) \circ \mathbf{M}$, then specifies the following search rules for the seller. In period $t < T$, the seller invites the sample M_t of bidders, and the set $N_t = N_{t-1} \cup M_t$ of bidders participate in the stage mechanism $(\mathbf{Q}_t, \mathbf{P}_t)$; if the product is allocated, then the seller stops searching, and the allocation and the payment are implemented accordingly; if the product is not allocated, then the seller moves on to period $t + 1$, and continues the process till the end of period T.

We assume that the seller announces the search mechanism $(\mathbf{Q}, \mathbf{P}) \circ \mathbf{M}$ in period 0, and fully commits to it thereafter. Under this full commitment assumption, we can restrict our

[4]Remember, under our assumption of long-lived bidders, there will be a set $N_t = \bigcup_{\tau=0}^{t} M_\tau$ of bidders participating in the stage transaction in period t.

attention to a direct mechanism. A well-known result in mechanism design (Myerson, 1981) is that, when the search mechanism is *incentive feasible*, the expected product revenue a seller can obtain from a truthful bidder i is equal to his virtual value $\psi(v_i)$.

3.2 Optimal Search Mechanism

Under the full commitment assumption, we can reformulate the seller's compound search problem as Pandora's problem *a la* Weitzman (1979). Specifically, given a family $\mathbf{M}^T$ of bidder samples, we can think of the bidder sample M^j as an aggregate bidder j, who is characterized by the gross search cost c_{M^j}, and their highest virtual value

$$\psi(V_{M^j}^{(1)}) = \psi\left(\max_{i \in M^j} \{V_i\}\right), \tag{2}$$

where $V_{M^j}^{(k)}$ denotes the k-th highest order statistics of the M^j bidders' values, which follows the distribution $F_{M^j}^{(k)}(v)$. If we denote m^j as the number of bidders in M^j, it then follows that $F_{M^j}^{(1)}(v) = F^{m^j}(x)$. As mentioned above, when the search mechanism is incentive feasible, the maximum revenue a seller can obtain from the M^j bidders is just equal to the highest virtual value of them, which is $\psi(V_{M^j}^{(1)})$.

Therefore, when the search mechanism is incentive feasible, we may think of a bidder sample M^j as a single 'box' in Pandora's problem, except that the prize of the 'box' is now replaced by the highest virtual value $\psi(V_{M^j}^{(1)})$, and the inspection cost replaced by the gross search cost c_{M^j}. Analogous to Pandora's problem, the seller's search rule also involves both a selection rule, e.g., $\mathbf{M} = (M_t)_{1 \leq t \leq T}$, and a stopping rule, e.g., when to stop searching.

We can similarly formulate the seller's problem as a dynamic programming (DP) problem. At the end of period t, suppose the highest value of the N_t already sampled bidders is v. It follows that the fall-back revenue the seller can claim is $\psi(v)$, given that the mechanism is incentive feasible. Taking v as a state variable, the seller then faces a decision between the following two options. If she stops searching, she can claim the fall-back revenue $\psi(v)$ and the product is allocated. Otherwise, if she continues searching, she needs to decide which bidder sample to search in period $t+1$.

We denote $J_t(v)$ as the value of having a fall-back revenue $\psi(v)$ at the end of period t, which the seller can either accept or reject. It is clear that $J_{T+1}(v) = 0$ and $J_T(v) = \max\{\psi(v), 0\}$. For $t < T$, the Bellman equation for this DP problem is thus

$$J_t(v) = \max_{M^j \in N_t^c} \left\{ \psi(v), -c_{M^j} + \mathbb{E}J_{t+1}\left[\max\left\{v, X_{M^j}^{(1)}\right\}\right]\right\}, \tag{3}$$

where $\psi(v) \geq 0$ is the seller's fall-back revenue if she stops searching,[5] and the other term in the curly braces is the maximum expected payoff of continuing searching the bidder sample M^j in period $t+1$. Note that, for long-lived bidders, the seller can reclaim a previously declined revenue in later periods, and therefore, the new state variable at the end of period $t+1$ is the realized value of $\max\left\{v, X_{M^j}^{(1)}\right\}$, as shown in (3).

Analogous to the cutoff prize of a 'box' in Pandora's problem, we can define a unique cutoff value for a bidder sample $M \in \mathbf{M}^T$, denoted by $\xi^*(M)$, at which the seller is indifferent between the following two options: to stop searching and keep the fall-back revenue $\psi(\xi^*(M))$; and to continue searching the bidder sample M at the cost of c_M and then stop right away. The Bellman equation (3) then implies that the cutoff value $\xi^*(M)$ satisfies

$$\psi(\xi^*(M)) = \int_0^1 \max\{\psi(\xi^*(M)), \psi(x)\}\, dF_M^{(1)}(x) - c_M. \tag{4}$$

The LHS of (4) is the current fall-back revenue $\psi(\xi^*)$, and the RHS is the expected revenue net of search cost if the seller continues searching the bidder sample M and then stops. It is interesting to observe that the cutoff value $\xi^*(M)$ is solely determined by the characteristics of M, e.g., the gross search cost c_M and the distribution $F_M^{(1)}(x)$.

For a given family $\mathbf{M}^T$ of bidder samples, the optimal search rule is given as follows:

- Calculate the optimal cutoff value, $\xi^*(M^j)$, for each bidder sample $M^j \in \mathbf{M}^T$;

- Selection rule: At the end of period $t < T$, if a bidder sample is to be searched in period $t+1$, it must be the sample with the highest cutoff value in N_t^c. The optimal sampling rule, denoted by $\mathbf{M}^* = (M_t^*)_{1 \leq t \leq T}$, is such that

$$\xi^*(M_1^*) \geq \xi^*(M_2^*) \geq \cdots \geq \xi^*(M_T^*).$$

[5] The assumption of $\psi(v) \geq 0$ generally holds, as the seller can always stop searching and realize a zero revenue. In another word, without loss of generality, we can assume the fall-back value $v \geq r^*$.

- Stopping rule: At the end of period $t < T$, if the fall-back value $v \geq \xi^*(M^j)$ for any $M^j \in N_t^c$, then stop; otherwise, continue to search in period $t+1$.

The optimal search rule demonstrates the *one-step-ahead* property. That is, under an optimal sample rule $\mathbf{M}^*$, if $v = \xi^*(M_{t+1}^*)$, then the seller is indifferent between stopping now, and continuing to search the bidder sample M_{t+1}^* and then stopping *right away*. This is because, when the search is with *full* recall, searching for one more period will certainly increase the fall-back value. As a result, with a higher fall-back value, the seller will prefer stopping to continuing searching at the end of the next period.

4 Sequential Search Auction

In this section, we will show that the outcomes of the above optimal search mechanism are implementable through a sequence of second price auctions with properly set reserve prices. Specifically, the rule of this sequential search auction is as follows:

- At the beginning of period 1, the seller invites a sample $M_1 \in N_0^c$ of bidders to the auction at the search cost $c_{M_1} = \sum_{i \in M_1} c_i$, and runs a second-price auction among the set $N_1 = M_1$ of bidders, with a reserve price r_1;

- If an effective bid, e.g., a bid higher than the reserve price, is submitted by any bidder $i \in N_1$ in period 1, then the game ends, and the payment and allocation are implemented according to the auction rule. If no effective bid submitted, the seller then continues searching in period 2, by inviting a sample $M_2 \in N_1^c$ of bidders at the search cost c_{M_2}. She then runs an auction among the accumulated set $N_2 = N_1 \cup M_2$ of bidders, with a new reserve price r_2;

- The seller continues with this process, until the end of period T. For example, in any period $t < T$, if there is no effective bid submitted, then the seller continues searching in period $t+1$, by inviting a set $M_{t+1} \in N_t^c$ of bidders at the search cost $c_{M_{t+1}}$, and runs an auction among the accumulated set $N_{t+1} = N_t \cup M_{t+1}$ of bidders, with a updated reserve price r_{t+1}.

Here are some comments. First, the sequential search auction is defined by a sequence of reserve prices, $\mathbf{r} = (r_t)_{1\leq t\leq T}$, and a sequence of bidder samples, $\mathbf{M} = (M_t)_{1\leq t\leq T}$. Second, we assume the seller announces the sequential search auction $(\mathbf{r}, \mathbf{M})$ in period 0, and fully commits to it thereafter. Third, we focus on the sealed-bid second price auction, which is strategically equivalent to a standard ascending open-auction. Under the full commitment assumption, if the sequential auction is incentive compatible, then bidding true value is a weakly dominant strategy for a bidder, if he intends to bid at all.

4.1 Incentive Compatibility

We first investigate the incentive compatible conditions for bidders, and will study equilibria in the form of cutoff strategies. A cutoff strategy is characterized by a vector of cutoff values, $\boldsymbol{\xi} = (\xi_t)_{1\leq t\leq T}$, such that a bidder will bid his true value v in period t if $v \geq \xi_t$, and wait otherwise. Given that bidders are *ex-ante* homogeneous according to their value distributions, we will focus on symmetric equilibria. A symmetric cutoff equilibrium is an equilibrium where all the bidders adopt the same cutoff strategy.

Let $\bar{U}_t(v)$ be the maximum expected payoff of a bidder with value v at the beginning of period t, and it is clear that $\bar{U}_{T+1}(v) = 0$ as the game terminates at the end of period T. Let $U_t^b(v)$ be the expected payoff of the bidder by submitting an effective bid in the stage auction of period t. It then follows that, for $t = 1, 2, \cdots, T$,

$$U_t^b(v) = F^{(1)}_{N_t\backslash\{i\}}(\xi_t)(v - r_t) + \mathbb{I}_{\{v\geq\xi_t\}} \int_{\xi_t}^{v} (v - x)dF^{(1)}_{N_t\backslash\{i\}}(x), \tag{5}$$

where $\mathbb{I}$ is an indicator function, and $F^{(1)}_{N_t\backslash\{i\}}$ is the distribution of the highest value of the set $N_t\backslash\{i\}$ of bidders. On the RHS of (5), the first term is the expected payoff when no other bidders in N_t bid, and the second term is that when some other bidders in N_t submit bids in the stage auction. The envelope theorem gives

$$\bar{U}_t(v) = \max\{U_t^b(v), \bar{U}_{t+1}(v)\},$$

which is non-decreasing, convex, and right-hand differentiable for all $v \in (0, 1]$.[6] The properties of $\bar{U}_t(v)$ imply that a bidders' optimal strategy is necessarily in the form of cut-off strategies, and hence our assumption of cutoff strategies is without loss of generality. As an direct application of envelope theorem, Lemma 1 below shows that there exists a one-to-one mapping between the sequence of reserve prices, $\mathbf{r}$, and that of the cutoff values, $\boldsymbol{\xi}$.

Lemma 1. *Given a sequential search auction* $(\mathbf{r}, \mathbf{M})$*, in each period* $t \leq T$*, there exists a unique* ξ_t *such that each bidder* $i \in N_t$ *bids if and only if his value* $v \geq \xi_t$*. Furthermore, the maximum expected payoff of a bidder with value* v *at the beginning of period* t *is:*

$$\bar{U}_t(v) = \begin{cases} \bar{U}_{t+1}(v) & \text{if } v < \xi_t, \\ \bar{U}_{t+1}(\xi_t) + \int_{\xi_t}^{v} F^{(1)}_{N_t \setminus \{i\}}(x)dx & \text{if } v \geq \xi_t. \end{cases} \tag{6}$$

For example, in (6), if $v < \xi_t$, a bidder will not bid in period t and hence his maximum expected payoff remains the same till the beginning of the next period, that is, $\bar{U}_t(v) = \bar{U}_{t+1}(v)$. From Lemma 1, the sequential search auction $(\mathbf{r}, \mathbf{M})$ can be equivalently represented by $(\boldsymbol{\xi}, \mathbf{M})$, and hereinafter we use the latter notation instead.

We have known from the previous section that, in an optimal sequential search mechanism, the sequence of cutoff values are necessarily declining over time. Lemma 2 below characterizes a bidder's equilibrium cutoff strategy in this case.

Lemma 2 (Cutoff Condition)**.** *Given a sequential search auction* $(\boldsymbol{\xi}, \mathbf{M})$ *with decreasing cutoff values,* ξ_t *is uniquely determined by*

$$F^{(1)}_{N_t \setminus \{i\}}(\xi_t)(\xi_t - r_t) = F^{(1)}_{N_{t+1} \setminus \{i\}}(\xi_{t+1})(\xi_{t+1} - r_{t+1}) + \int_{\xi_{t+1}}^{\xi_t} F^{(1)}_{N_{t+1} \setminus \{i\}}(x)dx, \tag{7}$$

for $t < T$*, and* $\xi_T = r_T$*. Moreover, the reserve prices* $(r_t)_{1 \leq t \leq T}$ *is also decreasing in* t*.*

It is intuitive that $\xi_T = r_T$, as it is the last period to submit a bid, and a bidder will bid whenever his value is greater than the reserve price. For $t < T$, it is implied in (7) that $\xi_t \geq r_t$, and a bidder with value higher than the reserve price may wait, e.g., not bid. In

[6]The derivation of the cut-off strategy is standard. It also appears in the literature of buy-price auction (Chen et al., 2017; Reynolds and Wooders, 2009) and sequential auctions with information acquisition costs (Crémer et al., 2009). Here we apply the envelope theorem.

fact, with decreasing reserve prices, a bidder faces the basic trade-off between bidding now with a higher reserve price and less competition, or bidding later with a lower reserve price yet more competition when more bidders will enter the auction.

4.2 Optimal Cutoff Values

The expected auction profit for the seller is equal to the expected auction revenue minus the expected gross search costs. Given a sequential search auction $(\boldsymbol{\xi}, \mathbf{M})$ with declining cutoff values, the expected auction profit can be represented by

$$\pi(\boldsymbol{\xi}, \mathbf{M}) = \sum_{t=1}^{T} F_{N_{t-1}}^{(1)}(\xi_{t-1}) \left[R_t(N_t) - c_{M_t} \right], \tag{8}$$

where $F_{N_0}^{(1)}(\xi_0) \equiv 1$ and $R_t(N_t)$ is the expected revenue of the stage auction in period t, conditional on it happens. It is worthy of attention that, in period t, there are M_t strong bidders and N_{t-1} weak bidders in the auction. To be specific, the values of the M_t new bidders are independent draws from F on $[0, 1]$, while those of the N_{t-1} weak bidders are independent draws from the truncated distribution $F(v \,|\xi_{t-1}) \equiv \Pr(V \leq v \,|V \leq \xi_{t-1})$.

Substituting the bidders' equilibrium cutoff strategies of (7) into (8), we then get the following expression of the expected auction profit.

Lemma 3. *Given a sequential search auction* $(\boldsymbol{\xi}, \mathbf{M})$ *with declining cutoff values, the expected auction profit is*

$$\pi(\boldsymbol{\xi}, \mathbf{M}) = \sum_{t=1}^{T} \int_{\xi_t}^{\xi_{t-1}} \psi(x) dF_{N_t}^{(1)}(x) + \sum_{t=1}^{T} F_{N_{t-1}}^{(1)}(\xi_{t-1}) \left[\int_{\xi_{t-1}}^{1} \psi(x) dF_{M_t}^{(1)}(x) - c_{M_t} \right], \tag{9}$$

where $\xi_0 \equiv 1$ *and* $F_{N_0}^{(1)}(\cdot) \equiv 1$.

An equivalent yet more intuitive expression of (9) is that

$$\pi(\boldsymbol{\xi}, \mathbf{M}) = \sum_{t=1}^{T} F_{N_{t-1}}^{(1)}(\xi_{t-1}) \left[\int_{\xi_t}^{1} \psi(x) dG_{N_t}^{(1)}(x) - c_{M_t} \right], \tag{10}$$

where $G_{N_t}^{(1)}(x)$ is the distribution function of the highest value of the N_t bidders.[7] The expression of (9), as well as (10), demonstrates that the revenue equivalence theorem still

[7] Among the N_t bidders, N_{t-1} bidders' values are independent draws from the truncated distribution $F(v \,|\xi_{t-1})$, and M_t bidders' are from the distribution $F(v)$. We then have $G_{N_t}^{(1)}(x) = F_{N_{t-1}}^{(1)}(x \,|\xi_{t-1}) F_{M_t}^{(1)}(x)$.

holds here, that is, the expected revenue of a stage auction is equal to the highest virtual value of the N_t bidders (Myerson, 1981).

However, different from the standard results of static auctions with symmetric bidders, (9) provides a nice formula of the expected revenue of a sequential auction with asymmetric bidders. In our model, as bidders are *ex-ante* homogeneous, their virtual value functions are the same, e.g., $\psi(v)$. The stage auction in period t happens with probability $F^{(1)}_{N_{t-1}}(\xi_{t-1})$, and there are M_t strong bidders and N_{t-1} weak bidders whose values are truncated above from ξ_{t-1}. When the highest value of the N_t bidders is between ξ_t and ξ_{t-1}, all the bidders are competing for the product. This corresponds to the first term of $\int_{\xi_t}^{\xi_{t-1}} \psi(x) dF^{(1)}_{N_t}(x)$ in (9). However, when the highest value is greater than ξ_{t-1}, the N_{t-1} weak bidders are strictly dominated by the M_t strong bidders, and the effective competition is among the M_t strong bidders. This corresponds to the second term of $\int_{\xi_{t-1}}^{1} \psi(x) dF^{(1)}_{M_t}(x)$ in (9).

Proposition 1 (Optimal Cutoffs)**.** *Given a sequential search auction $(\boldsymbol{\xi}, \mathbf{M})$ with declining cutoff values, the expected auction profit, $\pi(\boldsymbol{\xi}, \mathbf{M})$, is quasi-concave in ξ_t. Moreover, the sequence of optimal cutoff values, $(\xi_t^*)_{1 \le t \le T}$, satisfies*

$$c_{M_{t+1}} = \int_{\xi_t^*}^{1} \left[\psi(x) - \psi(\xi_t^*)\right] dF^{(1)}_{M_{t+1}}(x), \text{ for } 1 \le t < T, \tag{11}$$

and $\psi(\xi_T^*) = 0$ for $t = T$.

The optimal cutoff value in (11) also specifies the condition for optimal stopping, as also shown in (4). For instance, given the current fall-back revenue $\psi(\xi_t^*)$, the RHS of (11) is the increment in expected auction revenue if the seller continues searching the M_{t+1} bidders in period $t+1$ and then stops, while the LHS is the gross search cost $c_{M_{t+1}}$ of doing that. If the RHS is smaller than $c_{M_{t+1}}$, then the seller will stop searching at the end of period t.

The solution of ξ_t^* is unique, as the RHS of (11) is strictly decreasing in ξ_t^*. Moreover, ξ_t^* reveals a one-step-ahead property, in the sense that it just depends on the bidder sample M_{t+1}, that is to be searched in the right next period $t+1$, not further. This property is implied by the fact that the optimal cutoff values are decreasing, while the fall-back values with long-lived bidders are necessarily increasing over time.

Our formula (11) generalizes the well-known result of Weitzman (1979). In Pandora's problem, Pandora faces a number of closed boxes; insides each box, there is a random prize distributed according to F_i on $[0, 1]$; to open a box i, she needs to incur a search cost c_i. Pandora inspects boxes one-by-one sequentially, and her objective is to maximize the expected prize discovered, net of the gross search costs. Weitzman shows that the optimal search rule involves a unique cutoff prize ξ_i^* to each box i, which is the unique solution to

$$c_i = \int_{\xi_i}^{1} (x - \xi_i) dF_i(x). \tag{12}$$

Supposing Pandora already has a fall-back prize ξ_i, then the RHS of (12) is the increment in expected utility if she inspects box i, and the LHS is the search cost. When her fall-back prize $v = \xi_i^*$, Pandora is indifferent between stopping and continuing inspecting box i.

Our formula (11) generalizes Weitzman's result of (12) in at least two aspects. First, in our model, the target for search are strategic bidders, rather than non-strategic boxes. The extension is not trivial, as in many real circumstances, the targets for search are strategic agents, such as bidders, job candidates, potential partners in marriage markets, and so on. Second, our formula (11) extends Weitzman's result of one-by-one sequential search to the more general case of compound search, where a seller may search multiple bidders simultaneously in a single period. As a result, our formula takes Weitzman's formula of (12) as a special case, and can be applied to a larger set of search problems.

For our following discussion, it is helpful to define $\xi^*(M)$ as the optimal cutoff value for searching a bidder sample M. To be specific, for any $M \subseteq N$ with $M \neq \emptyset$, applying condition (11), $\xi^*(M)$ is the implicit function defined by

$$c_M = \int_{\xi^*(M)}^{1} \left[1 - F_M^{(1)}(x)\right] d\psi(x). \tag{13}$$

From the definition, $\xi^*(M)$ measures a seller's cutoff value of searching the bidder sample M, for the purpose of profit maximization. For instance, if the seller's fall-back value v is greater than the cutoff $\xi^*(M)$, she will not have incentive to further search the bidder sample M. The following corollary provides some interesting properties of $\xi^*(M)$.

Corollary 1. *For any two bidder samples $M, M' \subseteq N$,*

1. *if the cardinality $|M| = |M'|$, then*

$$c_M < c_{M'} \Longrightarrow \xi^*(M) > \xi^*(M');$$

2. *if $c_M = c_{M'}$, then*

$$|M| < |M'| \Longrightarrow \xi^*(M) < \xi^*(M').$$

Corollary 1 shows the intuitive results that, for any two bidder samples, if they have the same sample size (e.g. the same number of bidders), then the sample with a smaller gross search cost is more valuable for the seller. Second, if their gross search costs are the same, then the sample with more bidders is more valuable for the seller.

We next provide a simple formula of the maximum expected auction profit by following the optimal search rule. We will show that this formula also takes the standard result of stationary and infinite horizon (SIH) search problems as a special case.

First, by comparing (11) with (13), it is clear that the optimal cutoff value of $\xi_t^* = \xi^*(M_{t+1})$ for $1 \leq t < T$. This result coincides with the optimal cutoff condition of (4). It then implies that the outcomes of an optimal search mechanism can be implemented by a sequential search auctions with the cutoff value ξ_t^* set properly. For instance, given a family $\mathbf{M}^T$ of bidder samples, the optimal sampling rule is a permutation of $\mathbf{M}^T$ such that $\xi^*(M_t) \geq \xi^*(M_{t+1})$ for $1 \leq t < T$. We denote this optimal sampling rule by $\mathbf{M}^* = (M_t^*)_{1 \leq t \leq T}$, such that $\xi^*(M_t^*)$ is decreasing in t. Then given a family $\mathbf{M}^T$ of bidder samples, the sequential search auction $(\boldsymbol{\xi}^*, \mathbf{M}^*)$ implements the outcomes of the optimal search mechanism, where $\boldsymbol{\xi}^* = (\xi^*(M_{t+1}^*))_{1 \leq t \leq T}$ and $\xi^*(M_{T+1}) \equiv r^*$.

Second, by substituting the optimal cutoff values of (11) into the expected auction profit of (9), we have the maximum expected auction profit as follows.

Lemma 4. *Given a family $\mathbf{M}^T$ of bidder samples, a seller's maximum expected auction profit by following an optimal search rule is*

$$\pi^*(\mathbf{M}^T) = \pi(\boldsymbol{\xi}^*, \mathbf{M}^*) = \sum_{t=1}^{T} \int_{\xi^*(M_{t+1}^*)}^{\xi^*(M_t^*)} \left[1 - F_{N_t}^{(1)}(x)\right] d\psi(x), \tag{14}$$

where $\xi^*(M_{T+1}) \equiv r^*$.

A well-known result in an SIH search problem is that, when following an optimal search rule, the maximum expected search profit is equal to the value of the optimal reservation value $\psi(\xi^*)$ that solves $c = \int_{\xi^*}^{1} [1 - F(x)]\, d\psi(x)$ (Lippman and McCall, 1976). If we denote the optimal search profit of an SIH search problem by $\pi^{SIH}(\xi^*) = \psi(\xi^*)$, our result of (14) then incorporates $\pi^{SIH}(\xi^*)$ as a special case.

To see this, let us consider the case of homogeneous bidders, e.g., $c_i = c$, and one-by-one sequential search, e.g., $|M_t| = 1$. In this case, when $T \to \infty$, our compound search problem converges to an SIH search problem *a la* Lippman and McCall (1976). Applying the formula of (14) and taking the limit $T \to \infty$, the maximum expected search profit is thus

$$\lim_{T\to\infty} \pi^*(\mathbf{M}^T) = \lim_{T\to\infty} \int_{r^*}^{\xi^*} \left[1 - F^T(x)\right] d\psi(x) = \psi(\xi^*) = \pi^{SIH}(\xi^*). \tag{15}$$

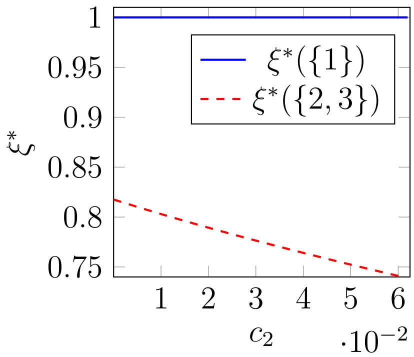
To end this section, we provide an example of 2-period search with 3 long-lived bidders. The example shows that, depending on the specific values of c_i's, a seller may invite either less or more bidders in the first period, under an optimal search rule. Therefore, all of the three selling processes in M&As, e.g., negotiation, control sale and full-scale auction, can be optimal under certain conditions. For example, when there is no cost of inviting bidders, i.e., $c_i = 0$ for all i, a simultaneous full-scale auction is obviously optimal.

Example 1 (2-period with 3 heterogenous bidders)**.** Consider an example where $F(x) = x$ and $c_1 = 0 \le c_2 \le c_3 = \frac{1}{16}$. By (13), the optimal cutoff $\xi^*(\{i\})$ for inviting a single bidder i is $\xi^*(\{i\}) = 1 - \sqrt{c_i}$, and the optimal cutoff $\xi^*(\{i,j\})$ for inviting two bidders $\{i,j\}$ is the solution to $c_i + c_j = \frac{2}{3}(1-\xi)^2(2+\xi)$. The cutoffs for various bidder samples are plotted in the upper panel of Figure 1, where c_2 changes from 0 to 1/16. As the cutoffs for inviting any 2-bidder samples are greater than $r^* = 1/2$, the candidates of an optimal sampling rule are the permutations of $\{\{1\},\{2,3\}\}$, $\{\{2\},\{1,3\}\}$, and $\{\{3\},\{1,2\}\}$.[8]

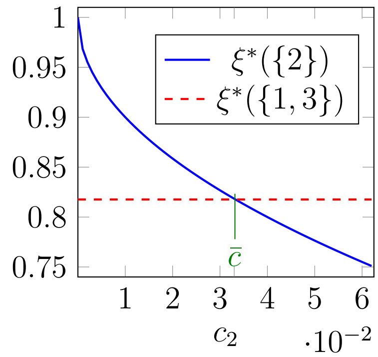
Remember that, in an optimal search rule, the cutoff of the bidder sample in the first period must be greater than that of the second period. First, observing that $\xi^*(\{1\}) >$

[8] We note that the sampling rule of $(\{1,2,3\},\emptyset)$, e.g. a full-scale auction, is never optimal as $c_3 > 0$.

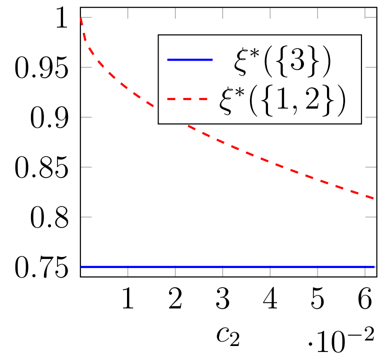
Figure 1: Three Partitions and Four Sampling Strategies over Two Periods



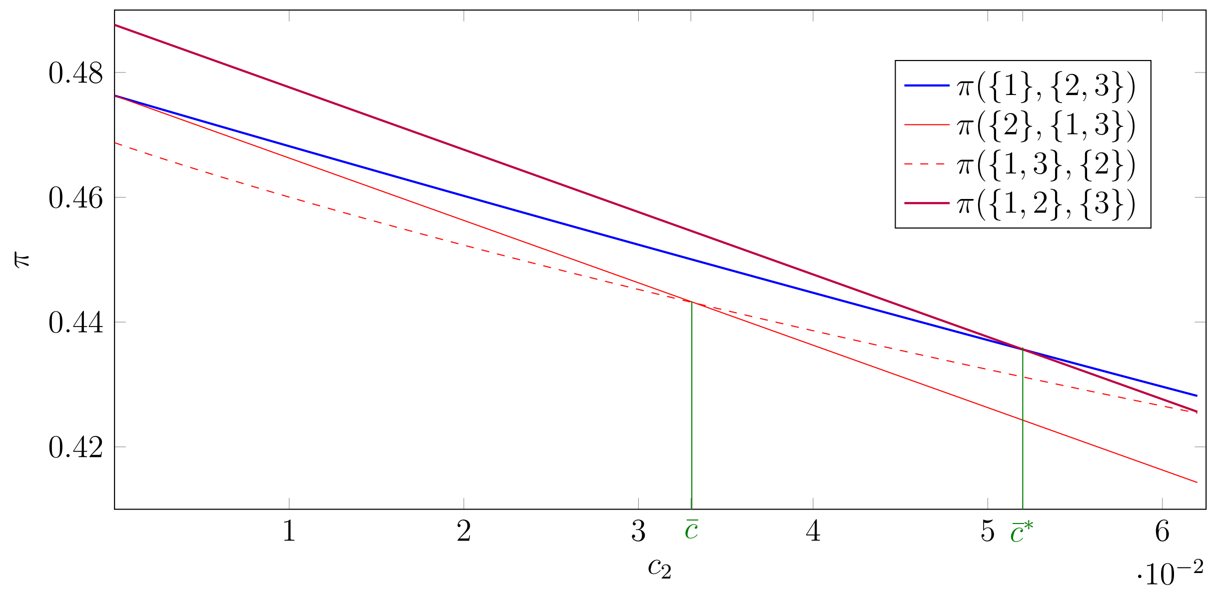
(a) Partition $\{\{1\},\{2,3\}\}$



(b) Partition $\{\{2\},\{1,3\}\}$



(c) Partition $\{\{3\},\{1,2\}\}$



(d) Four Sampling Strategies

See Example 1. Consider the heterogeneous costs of $c_1 = 0 \leq c_2 \leq c_3 = 1/16$. As $c_i \leq 1/16$, the cutoff value for any 2-bidder sample is greater than r^*, and therefore the three partitions of bidder samples are: $\{\{1\},\{2,3\}\}$, $\{\{2\},\{1,3\}\}$, and $\{\{3\},\{1,2\}\}$. The upper panels show the optimal cutoff values of the corresponding bidder samples of the three partitions, where c_2 changes from 0 to 1/16. Under an optimal search procedure, the optimal cutoff values are decreasing over time, and therefore, $(\{1\},\{2,3\})$ and $(\{1,2\},\{3\})$ are two candidates for optimal sampling rules, as shown in (a) and (c). For the other partition of $\{\{2\},\{1,3\}\}$, as shown in (b), the optimal sampling rule depends on the value of c_2. Specifically, when $c_2 \leq \bar{c} \approx 0.03327$, $(\{2\},\{1,3\})$ is the optimal sampling rule; otherwise, $(\{1,3\},\{2\})$ is. Subfigure (d) depicts the expected profits of the four possible sampling rules. For $c_2 < \bar{c}^* \approx 0.05208$, inviting $\{1,2\}$ in the first period and $\{3\}$ in the second period is optimal. On the other hand, if $c > \bar{c}^*$, inviting only $\{1\}$ in the first period and inviting $\{2,3\}$ in the second period is optimal.

$\xi^*(\{2,3\})$ and $\xi^*(\{3\}) < \xi^*(\{1,2\})$ for any c_2 as in Figure 1 (a) and (c), two candidates of the optimal sampling rule are $(\{1\},\{2,3\})$ and $(\{1,2\},\{3\})$. From Lemma 4, the expected profits of the two sampling rules can be computed as follows:

$$\pi(\{1\},\{2,3\}) = \int_{\xi^*(\{1\})}^{\xi^*(\{2,3\})} (1-F(x))d\psi(x) + \int_{r^*}^{\xi^*(\{2,3\})} (1-F^3(x))d\psi(x) \tag{16}$$

$$\pi(\{1,2\},\{3\}) = \int_{\xi^*(\{1,2\})}^{\xi^*(\{3\})} (1-F^2(x))d\psi(x) + \int_{r^*}^{\xi^*(\{3\})} (1-F^3(x))d\psi(x) \tag{17}$$

Here, to ease notation, $\pi(M) = \pi(\xi^*, M)$ in (14). Other candidates are permutations of the partition $\{\{2\},\{1,3\}\}$. As in Figure 1 (b), if $c_2 < \bar{c} \approx 0.03327$, then $\xi^*(\{2\}) > \xi^*(\{1,3\})$ and $\pi(\{2\},\{1,3\}) > \pi(\{1,3\},\{2\})$; otherwise $\pi(\{2\},\{1,3\}) \leq \pi(\{1,3\},\{2\})$. As Figure 1 (d) illustrates, however, both $(\{2\},\{1,3\})$ and $(\{1,3\},\{2\})$ are dominated by either $(\{1\},\{2,3\})$ or $(\{1,2\},\{3\})$ Comparing (16) and (17), the optimal sampling rule $\mathbf{M}^*$ is $(\{1,2\},\{3\})$ for $c_2 < \bar{c}^*$ and $(\{1\},\{2,3\})$ otherwise, where $\bar{c}^* \approx 0.05208$. Therefore, depending on the value of c_2, the seller may invite either more or less bidders in the first period, and either control sale or negotiation can be optimal under certain conditions.

4.3 Optimal Sampling Rule

In the previous section, we characterize the optimal cutoff values. Now we turn to the other important question – the optimal sampling rule. We have known that, given a family $\mathbf{M}^T$ of bidder samples, the optimal sampling rule $\mathbf{M}^*$ is such that $\xi^*(M_t^*)$ is decreasing over time. In this section, we will provide more characterizations of the optimal sampling rule, and our focus is on the optimal search intensity in each period.

To fix the idea of search intensity, we consider the case of *ex-ante* homogeneous bidders, where all bidders have not only the same value distribution F but the same unit search cost, i.e., $c_i = c$ for all $i \in N$. For ease of notation, we denote $m_t = |M_t|$ and $n_t = |N_t|$ as the cardinality of M_t and N_t respectively, and $\xi^*(m)$ then as the optimal cutoff value for searching the bidder sample M with a sample size m.

In the case of homogeneous bidders, the sampling rule is simply characterized by a sequence of sample sizes, denoted by $\mathbf{m} = (m_1, m_2, \cdots, m_T)$. Intuitively, the level of search intensity in period t is measured by the sample size m_t. We say a seller searches more intensively in period t' than in period t, if and only if $m_{t'} \geq m_t$. The following result shows that the optimal cutoff value $\xi^*(m)$ is decreasing in m.

Lemma 5. *Suppose $c_i = c$ for all $i \in N$. The optimal cutoff value of $\xi^*(m)$ for inviting a sample M of bidders is strictly decreasing in its sample size of m. That is, for any two sets of bidders, $M, M' \subseteq N$, if $m < m'$, then*

$$\xi^*(m) > \xi^*(m').$$

The result is more striking than it first looks. We already know from Corollary 1 that, when the sample size $|M|$ is given, the cutoff value $\xi^*(M)$ is decreasing in the gross search cost c_M. On the other hand, when the gross search cost is given, $\xi^*(M)$ is increasing in the sample size $|M|$. Lemma 5 then shows that, when bidders are homogeneous, the benefit of increasing competition by inviting one more bidder is strictly dominated by the search cost of doing that. In other words, removing one bidder from the sample will strictly increase the cutoff value of the bidder sample. Therefore, $\xi(m)$ achieves its maximum when the seller just samples one bidder. It also implies that, when a seller is not constrained by a finite deadline, like in the SIH problems, it is optimal for her to search the bidders one-by-one sequentially, as it generates the highest expected profit of $\psi(\xi^*(1))$, as shown in (15).

We next show that the optimal search intensity is increasing over time. This result is a direct implication of Lemma 5 and the fact that, under an optimal search rule, the optimal cutoff value ξ_t^* is necessarily decreasing in t. If we denote $\mathbf{m}^* = (m_1^*, m_2^*, \cdots, m_T^*)$ as the sequence of optimal sample sizes, it then follows that:

Proposition 2 (Optimal Sampling)**.** *Suppose $c_i = c$ for all $i \in N$. The optimal search intensity (sample size) is increasing in t, that is, for $1 \leq t < T$,*

$$m_t^* \leq m_{t+1}^*.$$

Proposition 2 is a second main result of this paper, which states that, in the presence of positive search costs and a finite deadline, a seller will search increasingly more intensively when the deadline approaches. Therefore, in the first period of the transaction, a seller will search the smallest number of potential buyers.

Proposition 2 may help explain why, in many important markets, such as M&As, the dominant selling processes can be non-competitive negotiation. For instance, if we interpret M&As as a finite sequential search process for a seller, then it is optimal for her to contact relatively fewer bidders and set higher reserve prices in the early stages of the transaction. However, if she fails to secure a good offer, it is then better for the seller to lower the reserve prices and invite more bidders in the next period when the deadline gets closer.

Our model may also explain why, in M&A markets, the difference in bid premiums across the two selling processes of negotiation and auction is generally not significant. In our model, though a bidder expects that the reserve prices in the following period will decrease, he would still be willing to accept a higher reserve price in an earlier period, so as to avoid the increased competition in the following-up stage auctions, as more bidders will enter. Therefore, the negotiation is not insulated from competition at all, and the pressure of following-up competition will drive up the bid premiums in the early (negotiation) periods.

At the end of this section, we provide an example of 2-period problem with 3 homogeneous long-lived bidders. The results show that, under an optimal search rule, the optimal sample size is necessarily increasing over time, that confirms the result of Proposition 2.

Example 2 (2-period with 3 homogeneous bidders)**.** Let $F(x) = x$ and $c_i = c \in (0, 1/4)$ for all i. The virtual value function is $\psi(x) = 2x - 1$. We denote the sampling rule by $\mathbf{m} = (m_1, m_2)$, where m_t is the bidder sample size in period t. We know that, under an optimal search procedure, it must be true that $m_1 \leq m_2$, and the candidates for optimal sampling rule is thus either $\mathbf{m} = (1, 1)$ or $(1, 2)$, depending on the value of c. From formula (13), the optimal cutoff values for different sample size are given by:

- If $m = 1$, then $\xi^*(1)$ is the solution to $c = (1 - \xi)^2$, e.g., $\xi^*(1) = 1 - \sqrt{c}$;
- If $m = 2$, then $\xi^*(2)$ is the solution to $2c = \frac{2}{3}(1 - \xi)^2(2 + \xi)$.

Case 1: If $c = \frac{1}{16}$, then $\xi^*(1) = \frac{3}{4}$ and $\xi^*(2) \approx 0.738$. For a sampling rule of $\mathbf{m} = (1, 1)$, the expected profit, from (14), is

$$\pi(1,1) = \int_{\xi^*(1)}^{\xi^*(1)} [1 - F(x)]\, d\psi(x) + \int_{r^*}^{\xi^*(1)} \left[1 - F^2(x)\right] d\psi(x) = \frac{29}{96} \approx 0.302.$$

The expected profit for a sampling rule of $\mathbf{m} = (1, 2)$ is

$$\pi(1,2) = \int_{\xi^*(2)}^{\xi^*(1)} [1 - F(x)]\, d\psi(x) + \int_{r^*}^{\xi^*(2)} \left[1 - F^3(x)\right] d\psi(x) \approx 0.365.$$

Therefore, the sampling rule of $(1, 2)$ is optimal.

Case 2: If $c = \frac{5}{24}$, then $\xi^*(1) \approx 0.544$ and $\xi^*(2) = 0.5$. For a sampling rule of $(1, 1)$, the expected profit, again from (14), is

$$\pi(1,1) = \int_{\xi^*(1)}^{\xi^*(1)} [1 - F(x)]\, d\psi(x) + \int_{r^*}^{\xi^*(1)} \left[1 - F^2(x)\right] d\psi(x) \approx 0.0634.$$

The expected profit for a sampling rule of $(1, 2)$ is

$$\pi(1,2) = \int_{\xi^*(2)}^{\xi^*(1)} [1 - F(x)]\, d\psi(x) + \int_{r^*}^{\xi^*(2)} \left[1 - F^3(x)\right] d\psi(x) \approx 0.0417.$$

Therefore, the sampling rule of $(1, 1)$ is now optimal.

Figure 2 in Section 5 illustrates how the optimal sampling rule changes with c. When $c \le c^* \approx 0.164$, the optimal sampling rule is $(1, 2)$, e.g., the dashed blue curve, and when $c \ge c^*$, the optimal sampling rule is $(1, 1)$, e.g., the solid blue curve.

5 Efficient Search Auction

It is well-known that an optimal static auction may lead to inefficient outcomes, either due to the possibility of no trade in some states, or due to biased allocations with heterogeneous bidders, where a bidder with the highest value may not win the auction. In our model of sequential search auctions with homogeneous bidders, there is a new source of inefficiency, that is, due to the inefficiency of the optimal sampling rule. First, an optimal search auction may exclude some bidders who would otherwise be valuable in improving social welfare.

Second, in an optimal search auction, a profit-maximizing seller may have excessive incentives to invite bidders. Third, the optimal sequence of bidder samples may be different from that of an efficient search auction.

5.1 Efficient Cutoffs

An efficient search mechanism is a mechanism that maximizes the expected social welfare. The value of social welfare is defined as the value of the winning bidder net of the gross search costs. Replacing the virtual value $\psi(v)$ by the true value v in (3), we can similarly set up the DP problem for welfare maximization. Following the same analysis as in Section 3.2, we can also show that an efficient search mechanism has declining cutoff values, and its outcomes are implementable by a sequential search auction.

To be specific, given a sequential search auction $(\boldsymbol{\xi}, \mathbf{M})$ with declining cutoff values, similar to (8), the *ex-ante* expected social welfare is

$$W(\boldsymbol{\xi}, \mathbf{M}) = \sum_{t=1}^{T} F_{N_{t-1}}^{(1)}(\xi_{t-1}) \left[W_t(N_t) - c_{M_t}\right], \tag{18}$$

where $W(\boldsymbol{\xi}, \mathbf{M})$ is the expected social welfare of the sequential auction, and by an abuse of notation, $W_t(N_t)$ is the expected value of the winning bidder in the stage auction of period t, conditional on it happens. As the bidder with the highest value wins in a second price auction, $W_t(N_t)$ is then equal to the expected highest value of the set N_t of bidders.

In addition, although the objective changes from profit to welfare maximization, the incentive problem for the bidders remains the same as in Section 4.1. Therefore, we have the same equilibrium cutoff strategies (7) for bidders in equilibrium. Furthermore, as in Section 5, by substituting the cutoff condition (7) into the expected social welfare function of (18), we have the following expression of the expected social welfare.

Lemma 6. *Given a sequential search auction $(\boldsymbol{\xi}, \mathbf{M})$ with declining cutoff values, the expected social welfare is*

$$W(\boldsymbol{\xi}, \mathbf{M}) = \sum_{t=1}^{T} \int_{\xi_t}^{\xi_{t-1}} x dF_{N_t}^{(1)}(x) + \sum_{t=1}^{T} F_{N_{t-1}}^{(1)}(\xi_{t-1}) \left[\int_{\xi_{t-1}}^{1} x dF_{M_t}^{(1)}(x) - c_{M_t}\right], \tag{19}$$

where $\xi_0 \equiv 1$ *and* $F_{N_0}^{(1)}(\xi_0) \equiv 1$.

Similar to Proposition 1, the following result characterizes the cutoff values in an efficient search mechanism, which also demonstrates the one-step-ahead property.

Proposition 3 (Efficient Cutoffs). *Given a sequential search auction* $(\boldsymbol{\xi}, \mathbf{M})$ *with declining cutoff values, the expected social welfare,* $W(\boldsymbol{\xi}, \mathbf{M})$*, is quasi-concave in* ξ_t*. The sequence of efficient cutoff values,* $(\xi_t^{**})_{1 \le t \le T}$*, is the unique solution to*

$$c_{M_{t+1}} = \int_{\xi_t^{**}}^{1} (x - \xi_t^{**}) dF_{M_{t+1}}^{(1)}(x), \text{ for } 1 \le t < T, \tag{20}$$

and $\xi_T^{**} = 0$ *for* $t = T$.

If we replace the true value v by its virtual value of $\psi(v)$, the formula for efficient cutoff value, (20), is identical to that for optimal cutoff value, (11). The connection between them is clear: the virtual value $\psi(v)$ is the maximum revenue a seller can obtain from a truthful bidder, while the value v measures the social welfare if that bidder wins the product.

Similarly, we can define a function of the efficient cutoff for a bidder sample M, denoted by $\xi^{**}(M)$. Specifically, for any $M \subseteq N$ with $M \neq \emptyset$, following the condition of (20), $\xi^{**}(M)$ is the implicit function defined by

$$c_M = \int_{\xi^{**}(M)}^{1} \left[1 - F_M^{(1)}(x)\right] dx. \tag{21}$$

$\xi^{**}(M)$ measures the value of searching the sample M of bidders, for the purpose of welfare maximization. Similar to the case of optimal cutoff values, the following properties of $\xi^{**}(M)$ are straightforward: for two samples of bidders, $M, M' \subseteq N$, 1) if the cardinality $|M| = |M'|$ and $c_M < c_{M'}$, then $\xi^{**}(M) > \xi^{**}(M')$; 2) if $c_M = c_{M'}$ and $|M| < |M'|$, then $\xi^{**}(M) < \xi^{**}(M')$. The proof is similar to that of Corollary 1, and therefore is omitted here.

Comparing (20) and (21), it follows that the efficient cutoff value of $\xi_t^{**} = \xi^{**}(M_{t+1})$ for $1 \le t < T$. We similarly denote $\mathbf{M}^{**} = (M_t^{**})_{1 \le t \le T}$ as the efficient sampling rule, such that $\xi^{**}(M_t^{**})$ is decreasing in t. Substituting (20) into (19), we then have the following formula for calculating the maximum expected social welfare.

Lemma 7. *Given a family* $\mathbf{M}^T$ *of bidder samples, the maximum expected social welfare is*

$$W^{**}(\mathbf{M}^T) = W(\boldsymbol{\xi}^{**}, \mathbf{M}^{**}) = \sum_{t=1}^{T} \int_{\xi^{**}(M_{t+1}^{**})}^{\xi^{**}(M_t^{**})} \left[1 - F_{N_t}^{(1)}(x)\right] dx \tag{22}$$

5.2 Efficient Sampling Rule

For efficient sampling rule, we again consider the case of *ex-ante* homogeneous bidders, by assuming that $c_i = c$ for all $i \in N$. In this homogeneous case, we denote $\xi^{**}(m)$ as the efficient cutoff value for searching a sample of m bidders, as given in (21). The sample size of m_t then measures the seller's search intensity in period t. The following result shows that the efficient cutoff value, $\xi^{**}(m)$, is also decreasing in m.

Lemma 8. *Suppose* $c_i = c$ *for all* $i \in N$*. The efficient cutoff value of* $\xi^{**}(m)$ *for inviting a sample* M *of bidders is strictly decreasing in its sample size of* m*. That is, for any two bidder samples,* $M, M' \subseteq N$*, if* $m < m'$*, then*

$$\xi^{**}(m) > \xi^{**}(m').$$

The following result is a direct implication of Lemma 8 and the fact that, in an efficient search mechanism, the efficient cutoffs are necessarily declining over time. Proposition 4 below shows the same result that, in an efficient search mechanism, it is optimal for the seller to search increasingly more intensively when the deadline approaches. Specifically, denoting $\mathbf{m}^{**} = (m_1^{**}, \cdots, m_T^{**})$ as the sequence of efficient sample sizes, we have

Proposition 4 (Efficient Sampling)**.** *Suppose* $c_i = c$ *for all* $i \in N$*. The efficient search intensity (sampling size) is increasing in* t*, that is, for* $t = 1, \cdots, T-1$*,*

$$m_t^{**} \leq m_{t+1}^{**}.$$

It is helpful to compare the cutoff values and the sample sizes between the optimal and the efficient search auction. The first result shows that, when the sequence of bidder samples is given, the optimal cutoff value is greater than the efficient one in each period.

Corollary 2. *Suppose $c_i = c$ for all $i \in N$. For a given sampling rule of* $\mathbf{M}$*, the optimal cutoff value is higher than the efficient one in each period, that is, for* $1 \leq t \leq T$*,*

$$\xi^*(m_t) > \xi^{**}(m_t).$$

This result is reminiscent of the related results in static auctions. In a symmetric static auction, the optimal reserve price r^* is such that the virtual value $\psi(r^*) = 0$, and the efficient reserve price is simply 0. It is clear that $r^* > 0$. Corollary 2 shows that the same comparative result holds in sequential search auctions. That is, for a given sequence of bidder samples, the optimal cutoff value is strictly greater than the efficient one in each period.

Second, as the optimal and efficient cutoff functions of $\xi^*(m)$ and $\xi^{**}(m)$ are both strictly decreasing in m, we can define their inverse functions, denoted by $m^*(\xi_t)$ and $m^{**}(\xi_t)$ respectively, which roughly measure the optimal and efficient search intensity, when the cutoff value ξ_t is fixed.

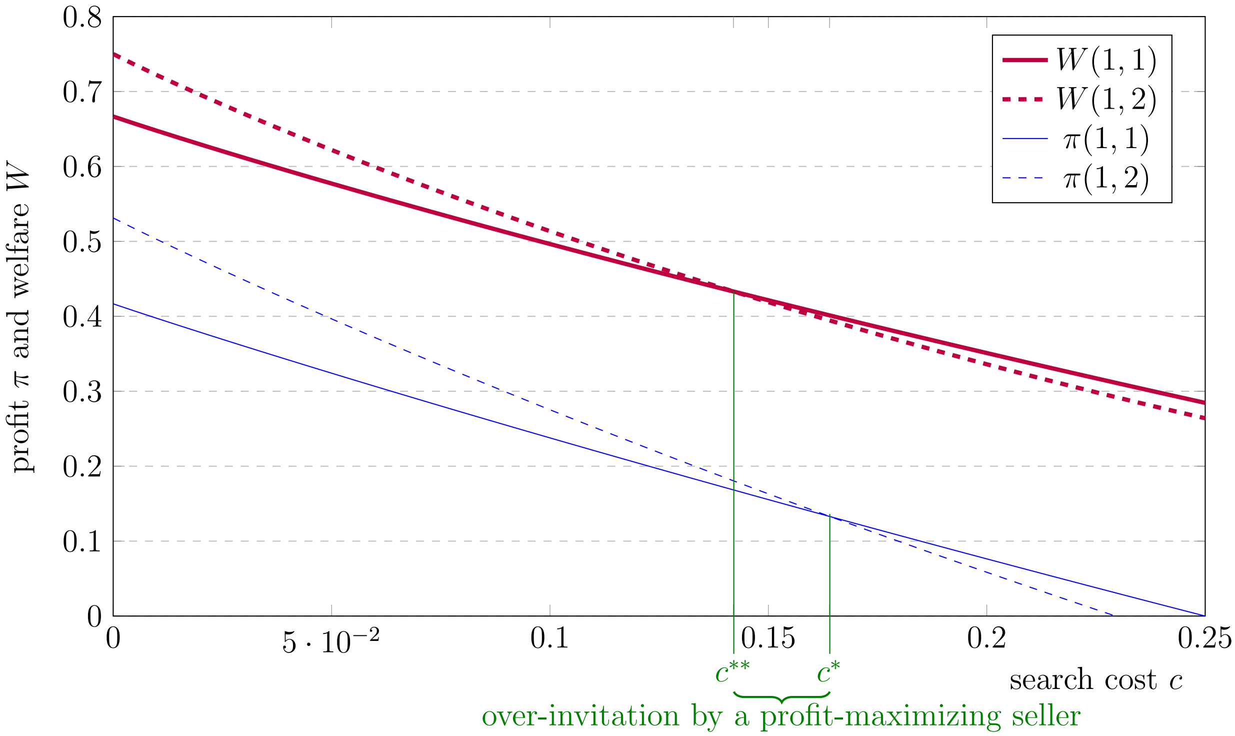
Proposition 5 (Efficient Sampling vs. Optimal Sampling)**.** *Suppose $c_i = c$ for all $i \in N$. For a given sequence of declining cutoff values* $\boldsymbol{\xi}$*, the optimal sample size is greater than the efficient one in each period, that is,*

$$m^*(\xi_t) > m^{**}(\xi_t), \text{ for } t = 1, \cdots, T.$$

Proposition 5 shows that, given a declining sequence of cutoff values, a profit-maximizing seller will invite more bidders in each period than a welfare-maximizing seller. As a result, the expected total number of participating bidders is also larger than that in an optimal search auction. This over-invitation result is also reported in models of static search auctions, where a profit-maximizing seller will invite more than the socially efficient number of bidders to the auction (Li, 2017; Szech, 2011; Xu and Li, 2019).

To end this section, we consider an example of efficient search auction, using the previous example of 2-period with 3 homogeneous bidders. Figure 2 illustrates how the efficient and the optimal sampling rules change with the unit search cost of c. It also reveals the sources

Figure 2: Optimal vs. Efficient Auction



See Example 2 for expected profit and Example 3 for expected welfare. For **optimal sampling rules**: when the unit search cost $c \leq c^* \approx 0.164$, the optimal sampling rule is $(1, 2)$, as shown by the dashed blue curve; when $c^* < c \leq 1/4$, it is $(1, 1)$, as shown by the solid blue curve; furthermore, when $c_2 > 1/4$, the net value of inviting a bidder is negative to a profit-maxmizing seller, and therefore she will not search any bidder. For **efficient sampling rules**: when the unit search cost $c \leq c^{**} \approx 0.142$, the efficient sampling rule is $(1, 2)$, as shown by the dashed red curve; when $c^{**} < c \leq 1/2$, it is $(1, 1)$, as shown by the solid red curve; moreover, when $c_2 > 1/2$, a welfare-maximizing seller will not conduct any search at all. It is also worth noting that the threshold value of $c^{**} \approx 0.142$ is smaller than that of $c^* \approx 0.164$. Therefore, when $c^{**} < c \leq c^*$, a welfare-maximizing seller invites just 1 bidder in the second period, while a profit-maximizing seller invites 2 instead. This example confirms the tendency of over-invitation by profit-maximizing sellers, which causes inefficiency.

of inefficiency in an optimal search auction. For example, when $1/4 < c \leq 1/2$, a profit-maximizing seller will not invite any bidder, yet it is still desirable for a welfare-maximizing seller to invite bidders. It indicates that socially valuable bidders may not be invited in an optimal search auction. Second, when $c^{**} < c \leq c^*$, a profit-maximizing seller will choose a sampling rule of $\mathbf{m}^* = (1, 2)$, while the efficient sampling rule is $\mathbf{m}^{**} = (1, 1)$. This shows that over-invitation can happen in an optimal search auction.

Example 3 (2-period with 3 homogeneous bidders)**.** Let $F(x) = x$ and $c_i = c \in (0, 1/2]$ for all i. We denote the sampling rule by $\mathbf{m} = (m_1, m_2)$, where m_t is the bidder sample size in period t. We know that, under an efficient search procedure, it must be true that $m_1 \leq m_2$, and the candidates for efficient sampling rule is thus $\mathbf{m} = (1,1)$ or $(1,2)$, depending on the value of c. From formula (21), the efficient cutoff value for different sample size is given by:

- If $m = 1$, then $\xi^{**}(1)$ is the solution to $c = \frac{1}{2}(1-\xi)^2$, e.g., $\xi^{**}(1) = 1 - \sqrt{2c}$;
- If $m = 2$, then $\xi^{**}(2)$ is the solution to $2c = \frac{1}{3}(1-\xi)^2(2+\xi)$.

Case 1: If $c = \frac{1}{16}$, then $\xi^{**}(1) \approx 0.646$ and $\xi^{**}(2) \approx 0.622$. For a sampling rule of $(1,1)$, the expected social welfare, from (22), is

$$W(1,1) = \int_{\xi^{**}(1)}^{\xi^{**}(1)} [1 - F(x)]\, dx + \int_0^{\xi^{**}(1)} \left[1 - F^2(x)\right] dx \approx 0.556.$$

The expected welfare for a sampling rule of $(1,2)$ is

$$W(1,2) = \int_{\xi^{**}(2)}^{\xi^{**}(1)} [1 - F(x)]\, dx + \int_0^{\xi^{**}(2)} \left[1 - F^3(x)\right] dx \approx 0.593.$$

Therefore, the sampling rule of $(1,2)$ is efficient.

Case 2: If $c = \frac{5}{24}$, then $\xi^{**}(1) \approx 0.355$ and $\xi^{**}(2) = 0.256$. For a sampling rule of $(1,1)$, the expected social welfare, from (22), is

$$W(1,1) = \int_{\xi^{**}(1)}^{\xi^{**}(1)} [1 - F(x)]\, dx + \int_0^{\xi^{**}(1)} \left[1 - F^2(x)\right] dx \approx 0.340.$$

The expected profit for a sampling rule of $(1,2)$ is

$$W(1,2) = \int_{\xi^{**}(2)}^{\xi^{**}(1)} [1 - F(x)]\, dx + \int_0^{\xi^{**}(2)} \left[1 - F^3(x)\right] dx \approx 0.323.$$

Therefore, the sampling rule of $(1,1)$ now is efficient.

Figure 2 plots how the efficient and optimal sampling rule change with the unit search cost c. For example, when $c \leq c^{**} \approx 0.142$, the efficient sampling rule is $(1,2)$; when $c^{**} < c \leq 1/2$, the efficient sampling rule is $(1,1)$; and when $c > 1/2$, athe welfare-maximizing seller will not conduct search at all. It is interesting to compare the two threshold values of $c^{**} \approx 0.142$

for the efficient sampling rule, and of $c^* \approx 0.164$ in Example 2 for the optimal sampling rule. Therefore, when $c^{**} < c \leq c^*$, a welfare-maximizing seller invites just 1 bidder in the second period, while a profit-maximizing seller invites 2 instead. This example confirms the tendency of over-invitation by profit-maximizing sellers, which causes inefficiency.

6 Optimal Search with Short-Lived Bidders

In this section, we turn to the other important case of *shorted-lived* bidders. A short-lived bidder, when invited, will just participate in the stage transaction of that period, and then goes away. The case of short-lived bidders is analogous to sequential search with *no* recall.

With short-lived bidders, we first prove that the optimal search mechanism is also featured by declining cutoff values and increasing search intensities (sample sizes). Therefore, these monotonicity results are robust across both cases of long-lived and short-lived bidders. Second, we show that the outcomes of an optimal search mechanism are also implementable by a sequence of second price auctions, yet with both lower reserve prices and greater search intensities if compared with those of long-lived bidders.

Like in Section 3, we can formulate a seller's sequential search problem with short-lived bidders as a DP problem. With the same notations, $\mathbf{M}^T$ is a family of bidder samples, and $N_t^c = \mathbf{M}^T \backslash N_t$ is the set of bidder samples that the seller has yet sampled till the end of period t. The key difference is that, in period t, only the sample M_t of bidders participate in the stage auction, as the previously invited bidders already go away.

Suppose at the end of period t the seller's fall-back revenue is $\psi(v)$. The seller then faces a decision between the following two options: she can stop searching and keep the revenue $\psi(v)$, or she can decline it and continue searching in the next period. However, when bidders are short-lived, if the seller declines the fall-back revenue $\psi(v)$, she can never reclaim it in the future. Therefore, with short-lived bidders, the seller's fall-back revenue will turn to $0 = \psi(r^*)$, if she continues to search in the next period.

In the case of short-lived bidders, we denote $\hat{J}_t(v)$ as the value of having an fall-back revenue $\psi(v)$ at the end of period t, which the seller can either accept or reject. It is obvious that $\hat{J}_{T+1}(v) = 0$ and $\hat{J}_T(v) = \max\{\psi(v), 0\}$. For $t < T$, the Bellman equation for this sequential search problem with short-lived bidders is thus

$$\hat{J}_t(v) = \max_{M_{t+1} \in N_t^c} \left\{ \psi(v), -c_{M_{t+1}} + \mathbb{E}\hat{J}_{t+1}\left[\max\left\{ r^*, X_{M_{t+1}}^{(1)} \right\} \right] \right\}. \tag{23}$$

Remember, when the seller declines the current offer of $\psi(v)$ and continues to search, her fall-back revenue turns to $\psi(r^*) = 0$, e.g., the virtual value of a bidder with value r^*.

The Bellman equation of (23) yields the following recurrence relation between the optimal cutoff values of $\hat{\xi}_t^*$ and $\hat{\xi}_{t+1}^*$,

$$\psi(\hat{\xi}_t^*) = \max_{M_{t+1} \in N_t^c} \left\{ \int_0^1 \max\left\{ \psi(\hat{\xi}_{t+1}^*), \psi(x) \right\} dF_{M_{t+1}}^{(1)}(x) - c_{M_{t+1}} \right\}, \tag{24}$$

where the LHS is the seller's payoff by stopping and keeping the current revenue $\psi(\hat{\xi}_t^*)$, and the RHS is her continuation payoff by following an optimal search procedure from period $t+1$ on. It is important to note that the current cutoff value $\hat{\xi}_t^*$ depends not only on M_{t+1}, the bidder sample of the next period, but also the next period cutoff value $\hat{\xi}_{t+1}^*$. In other words, the *one-step-ahead* property no longer holds here when bidders are short-lived, as the cutoff value $\hat{\xi}_t^*$ is recursively determined from the last period T.

Rearranging the terms of (24), it then follows that

$$\psi(\hat{\xi}_t^*) \;=\; \max_{M_{t+1} \in N_t^c} \left\{ \int_{\hat{\xi}_{t+1}^*}^1 \left[\psi(x) - \psi(\hat{\xi}_{t+1}^*) \right] dF_{M_{t+1}}^{(1)}(x) - c_{M_{t+1}} \right\} + \psi(\hat{\xi}_{t+1}^*). \tag{25}$$

By comparison with (11), it is clear that, for a given bidder sample M_{t+1}, the cutoff value of $\hat{\xi}_t^*$ with short-lived bidders is smaller than that with long-lived bidders. The following proposition characterizes the optimal cutoffs with short-lived bidders.

Proposition 6 (Optimal Cutoffs with Short-Lived Bidders)**.** *Given a sample rule of* $\mathbf{M}$*, the sequence of optimal cutoffs* $\hat{\xi}$ *is recursively determined by*

$$\psi(\hat{\xi}_t^*) - \psi(\hat{\xi}_{t+1}^*) = \max_{M_{t+1} \in N_t^c} \left\{ \int_{\hat{\xi}_{t+1}^*}^1 \left[1 - F_{M_{t+1}}^{(1)}(x) \right] d\psi(x) - c_{M_{t+1}} \right\}.$$

for any $t < T$ *and* $\psi(\hat{\xi}_T^*) = 0$.

We next prove that the optimal search mechanism with short-lived bidders is still featured by decreasing cutoff values and increasing search intensities. As before, to fix the idea of search intensity, we consider the case of homogeneous bidders, where $c_i = c$ for all $i \in N$. In the proof, we apply the Principle of Optimality, and derive the optimal search rule through backward induction. Specifically, at the end of period t, if a seller decides to search, she needs to select the optimal bidder sample in the next period, given that an optimal search rule will be followed in the following periods of $t+1, \cdots, T$.

Proposition 7 (Optimal Cutoffs and Sampling with Short-Lived Bidders)**.** *Suppose $c_i = c$ for all $i \in N$. In an optimal search mechanism with short-lived bidders, the optimal cutoff value $\hat{\xi}_t^*$ is decreasing, and the optimal sample size $\hat{m}_t^*$ is increasing over time. That is, for all $t = 0, 1, \cdots, T-1$, we have*

$$\hat{\xi}_t^* > \hat{\xi}_{t+1}^*, \qquad \hat{m}_t^* \leq \hat{m}_{t+1}^*.$$

The optimal cutoff values are decreasing over time. The intuition is that, if an offer is good enough to be acceptable in period t, it should also be acceptable in period $t+1$ when there is one less chance for improvement. Alternatively, we can interpret the revenue $\psi(\hat{\xi}_t^*)$ as the outside option for the seller, which is determined by the continuation value of following an optimal search rule. For any given sampling rule, the continuation value is decreasing over time, as there is fewer trial opportunities for the seller to improve her payoff.

Not surprisingly, the optimal search outcomes with short-lived bidders can also be implemented by a sequential search auction. In our proof of Proposition 7, we solve for the optimal search mechanism by backward induction. For example, in the last period of T, it is clear that the optimal reserve price $\hat{\xi}_T^* = r^*$, and there exists an optimal sample size of $\hat{m}_T^*$ that maximizes the expected auction profit.[9] With the optimal solution of $(\hat{\xi}_T^*, \hat{m}_T^*)$ in the last period T, we can calculate the continuation value, denoted by B_{T-1}, which is also the seller's reservation revenue at the end of period $T-1$. The optimal cutoff value in period

[9]This optimization problem is a well-defined convex problem (Li, 2017; Szech, 2011).

$T-1$ is such that $\psi(\hat{\xi}^*_{T-1}) = B_{T-1}$, with which we can derive the optimal sample size $\hat{m}^*_{T-1}$. Continuing this process, we then derive the optimal search mechanism of $(\hat{\boldsymbol{\xi}}^*, \hat{\mathbf{M}}^*)$.

In the sequential search auction, the incentive problems for short-lived bidders become much simpler, as the cutoff value for biding is always equal to the reserve price, e.g., $\xi_t = r_t$ for $t = 1, 2, \cdots, T$. This is because an invited short-lived bidder has but one chance to submit a bid, and therefore, he will bid if and only if his value is greater than the reserve price of the stage auction. Therefore, by setting a sequence of reserve prices $\hat{\mathbf{r}}^* = \hat{\xi}^*$, the sequential search auction of $(\hat{\mathbf{r}}^*, \hat{\mathbf{M}}^*)$ then implements the outcomes of an optimal sequential search mechanism with short-lived bidders.

When bidders are short-lived, we do not have an analytical solution of the optimal cutoff value $\hat{\xi}^*_t$, and the one-step-ahead property no longer holds. However, by comparing the two Bellman equations of (3) and (23), for *long-lived* and *short-lived* bidders respectively, we can show that, for a given sampling rule $\mathbf{M}$, the optimal cutoff value for short-lived bidders is always smaller than that for long-lived bidders in each period. The intuition is that, when bidders are short-lived, a seller can not reclaim any offer declined in previous periods, and as a result, she is willing to accept a lower reserve prize.

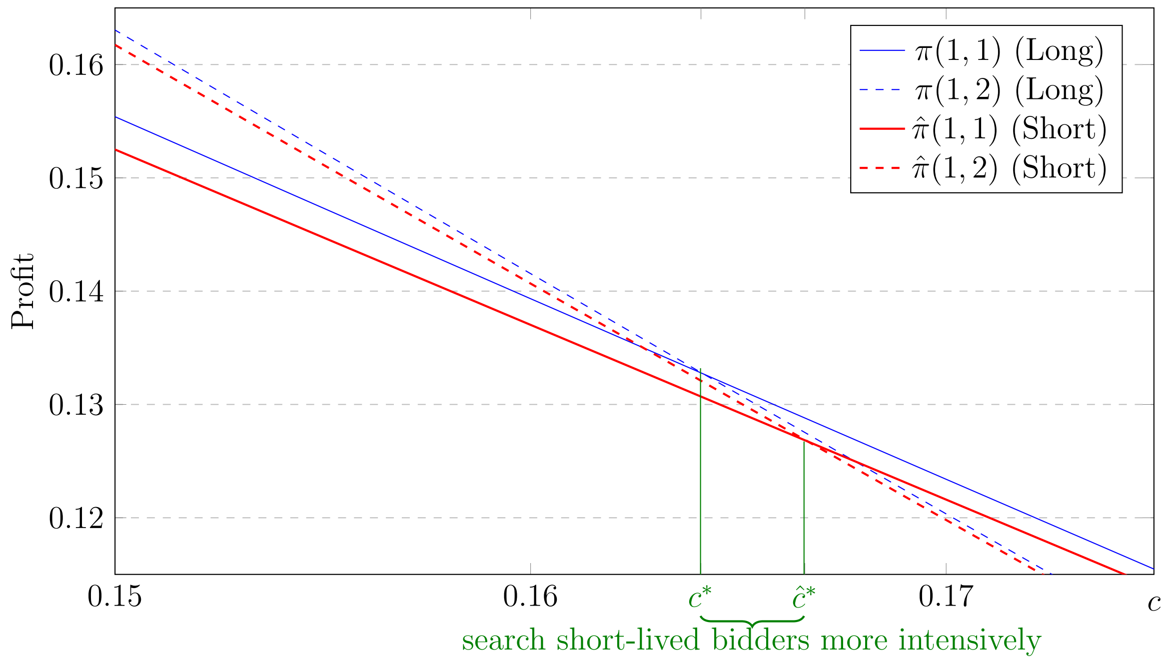
Proposition 8 (Long-Lived Bidders vs. Short-Lived Bidders: Optimal Cutoffs)**.** *For a given sampling rule* $\mathbf{M}$*, the optimal cutoff value for short-lived bidders is smaller than that for long-lived bidders. That is, for* $0 \leq t < T-1$*,*

$$\hat{\xi}^*_t < \xi^*_t,$$

and $\hat{\xi}^*_T = \xi^*_T = r^*$ *for* $t = T$*.*

Another intuitive result is that, for given cutoff values, the optimal search intensity (sample size) for short-lived bidders is greater than that for long-lived bidders. This is because a higher fall-back revenue will decrease the marginal value of search, and therefore dampen the seller's incentive to search bidders. When bidders are short-lived, a seller's fall-back revenue is always 0, which is smaller than that of a seller who searches long-lived

Figure 3: Long-Lived vs. Short-Lived Bidders



See Example 4. When bidders are short-lived and the unit search cost $c \leq \hat{c}^* \approx 0.167$, it is optimal to invite one bidder in the first period and two the remaining bidders in the second period, as shown by the dashed red curve. If $c > \hat{c}^*$, however, it is optimal to invite just one bidder sequentially in each period, as shown by the solid red curve. Compared to Example 2 where bidders are long-lived (as shown by the two blue curves), if c is in between $c^* \approx 0.164$ and $\hat{c}^* \approx 0.167$, the seller with short-lived bidders will invite more bidders in the second period than one with long-lived bidders.

bidders. Therefore, a seller with short-lived bidders will search bidders more intensively. The formal result is as follows.

Proposition 9 (Long-Lived Bidders vs. Short-Lived Bidders: Optimal Sampling)**.** *Given a sequence of cutoff values* $\boldsymbol{\xi}$ *that is declining in* t*, the optimal search intensity (sample size) for short-lived bidders is greater than that for long-lived bidders in each period* $t = 1, \cdots, T$.

Finally, let us consider an example of 2-period with 3 homogeneous short-lived bidders. The example first shows show how to use the recursive method to derive the optimal search auction of $(\hat{\xi}^*, \hat{\mathbf{M}}^*)$. Second, the numerical results in Figure 3 also illustrate the comparative results of Proposition 8 and 9 that a seller with short-lived bidders will set lower reserve prices and search more intensively than a seller with long-lived bidders.

Example 4 (2-period with 3 homogenous short-lived bidders)**.** Bidders are ex-ante homogeneous, with uniform value distribution on $[0,1]$ and unit search cost of $c > \underline{c} \approx 0.047$.[10] We denote the sampling rule by $\hat{\mathbf{m}} = (\hat{m}_1, \hat{m}_2)$, where $\hat{m}_t$ is the bidder sample size in period t. We consider two candidates for optimal sampling rule, $\hat{\mathbf{m}} = (1,1)$ and $(1,2)$.

We define B_t, $t = 1, 2$, as the continuation value of following an optimal search procedure after the end of period t, and the optimal cutoff value $\hat{\xi}_t^*$ satisfies $\psi(\hat{\xi}_t^*) = B_t$, as implied by the Bellman equation of (23). It is clear that $B_2 = 0$ and $\hat{\xi}_2^* = r^*$; and

$$B_1 = \max_{\hat{m}_2} \left\{ \int_{r^*}^1 \psi(x) dF^{\hat{m}_2}(x) - \hat{m}_2 c \right\}, \tag{26}$$

where the virtual value function $\psi(x) = 2x - 1$. Following an optimal search procedure of $\left\{ (\hat{\xi}_1^*, r^*), (\hat{m}_1^*, \hat{m}_2^*) \right\}$, the expected profit is

$$\hat{\pi}^* = \left[\int_{\hat{\xi}_1^*}^1 \psi(x) dF^{\hat{m}_1^*}(x) - \hat{m}_1^* c \right] + F^{\hat{m}_1^*}(\hat{\xi}_1^*) \left[\int_{r^*}^1 \psi(x) dF^{\hat{m}_2^*}(x) - \hat{m}_2^* c \right].$$

Case 1: $\underline{c} < c \leq \hat{c}^* = 1/6$

$\hat{m}_2^* = 2$ maximizes (26) with $B_1 = \frac{5}{12} - 2c$ and $\hat{\xi}_1^* = \frac{17}{24} - c$. The expected profit is $\hat{\pi}^* = c^2 - \frac{29c}{12} + \frac{289}{576}$.

Case 2: $\hat{c}^* < c < 1/4$

$\hat{m}_2^* = 1$ maximizes (26) with $B_1 = \frac{1}{4} - c$ and $\hat{\xi}_1^* = \frac{5}{8} - \frac{1}{2}c$. The expected profit is $\hat{\pi}^* = \frac{1}{64}(16c^2 - 104c + 25)$.

This example also verifies the result of Proposition 7, that is, for given sampling rule, the optimal cutoff value with short-lived bidders is smaller that with long-lived bidders, in each period. For instance, for $c = \frac{1}{16}$, it follows $\hat{m}_2^* = 2$, $B_1 = \frac{7}{24}$ and

$$\hat{\xi}_1^* = \frac{31}{48} \approx 0.646 < \xi_1^* \approx 0.738,$$

and for $c = \frac{5}{24}$, it follows $\hat{m}_2^* = 1$, $B_1 = \frac{1}{24}$, and

$$\hat{\xi}_1^* = \frac{25}{48} \approx 0.521 < \xi_1^* \approx 0.544,$$

where ξ_1^*'s are derived in Example 2.

[10] For a very low $c < \underline{c}$, it is optimal to invite all the three bidders in the first period, as $\hat{m}_2^* = 3$ maximizes (26). Note that inviting all bidders in one period is never optimal with long-lived bidders.

7 Conclusion

This paper proposes a framework that may explain why, in many important markets, the predominant selling process is non-competitive. Specifically, we model this kind of problems as a seller's sequential search for potential bidders by a finite deadline. We show that the optimal search outcomes can be implemented by a sequential search auction, which is characterized by declining reserve prices and increasing search intensities (sample sizes) over time. The monotonicity results are robust in both cases of long-lived and short-lived bidders, and across optimal and efficient search auctions. The result of increasing search intensities provides a possible explanation for the pervasiveness of non-competitive selling processes in many important markets, such as M&As.

We further show that an efficient search auction is featured by both lower reserve prices and greater search intensities than an optimal search auction, *ceteris paribus*. The result indicates a new source of inefficiency of the optimal search auction, e.g., due to the inefficient search rule. For example, an optimal search auction may exclude some socially valuable bidders, or a profit-maximizing seller may have excessive incentives to invite bidders in certain stages of the transaction.

Finally, we compare the two cases of long-lived and short-lived bidders, and show that, with short-lived bidders, a profit-maximizing seller will set lower reserve prices and search more intensively than in the case of long-lived bidders. The intuition is that, with short-lived bidders, a seller can not reclaim a previously declined offer, and therefore, she is willing to accept a lower reserve price and will search bidders more intensively.

This paper contributes to the small literature of search mechanism, and there are several key features of our model. First, we study a seller's sequential search for strategic bidders, who behave both rationally and strategically. Second, we study a sequential search problem that is bounded by a finite deadline, for which there many real world examples. Third, we provide complete characterizations of the optimal and efficient search auctions, as well as the two cases of long-lived and short-lived bidders. We believe our framework can be conveniently applied to the study of a large set of search problems, such as sequential matching in marriage markets, job recruitment by a deadline, or R&D tournament within a finite time horizon. These extensions and explorations may be left for future research.

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Appendix

PROOF OF LEMMA 1: It is obvious that $\bar{U}_t(0) = \bar{U}_{t+1}(0) = 0$. Since the product must be sold at a positive probability, there exists v° such that $\bar{U}_t(v^\circ) > \bar{U}_{t+1}(v^\circ)$. First, we show $\bar{U}_t(v) > \bar{U}_{t+1}(v)$ for any $v \geq v^\circ$. Suppose, for a contradiction, that there exists $\bar{U}_t(v) = \bar{U}_{t+1}(v)$ for some $v \geq v^\circ$. Let $\bar{v} = \min\{v \geq v^\circ \mid \bar{U}_t(v) = \bar{U}_{t+1}(v)\}$, which is well-defined as $\bar{U}_t$ and $\bar{U}_{t+1}$ are continuous. Then for any $\tilde{v} \in [v^\circ, \bar{v})$, it must be $\bar{U}_t(\tilde{v}) > \bar{U}_{t+1}(\tilde{v})$ and hence $\bar{U}'_t(\tilde{v}) = F^{(1)}_{N_t \setminus \{i\}}(\tilde{v})$, which is in turn strictly greater than $F^{(1)}_{N_{t+1} \setminus \{i\}}(\tilde{v}) \geq \bar{U}'_{t+1}(\tilde{v})$. It contradicts to the continuity of $\bar{U}_t$ and $\bar{U}_{t+1}$ and hence $\bar{U}_t(v) > \bar{U}_{t+1}(v)$ for any $v \geq v^\circ$. Then, $\xi_t = \max\{v \mid \bar{U}_t(v) = \bar{U}_{t+1}(v)\}$ is uniquely defined and the standard payoff equivalence argument yields the bidder's payoff function as (6). □

PROOF OF LEMMA 2: For $t < T$, a bidder with the cutoff value ξ_t is indifferent between bidding and waiting, and therefore $\bar{U}_t(\xi_t) = U^b_t(\xi_t) = \bar{U}_{t+1}(\xi_t)$. As $\xi_t \geq \xi_{t+1}$, he then prefers bidding to waiting in period $t+1$, and hence $\bar{U}_{t+1}(\xi_t) = U^b_{t+1}(\xi_t)$. It then follows that, for $t < T$,

$$\begin{aligned} U^b_{t+1}(\xi_t) &= F^{(1)}_{N_{t+1} \setminus \{i\}}(\xi_{t+1})(\xi_t - r_{t+1}) + \mathbb{I}_{\{\xi_t \geq \xi_{t+1}\}} \int_{\xi_{t+1}}^{\xi_t} (\xi_t - x) dF^{(1)}_{N_{t+1} \setminus \{i\}}(x) \\ &= F^{(1)}_{N_{t+1} \setminus \{i\}}(\xi_{t+1})(\xi_{t+1} - r_{t+1}) + \int_{\xi_{t+1}}^{\xi_t} F^{(1)}_{N_{t+1} \setminus \{i\}}(x) dx, \end{aligned}$$

and $U^b_t(\xi_t) = F^{(1)}_{N_t \setminus \{i\}}(\xi_t)(\xi_t - r_t)$. We then prove the result of (7), and $\xi_T = r_T$ obviously as $\bar{U}_{T+1}(v) = 0$. Next we show the reserve prices $\{r_t\}_{1 \leq t \leq T}$ are also decreasing in t. From (7),

$$\begin{aligned} F^{(1)}_{N_t \setminus \{i\}}(\xi_t)(\xi_t - r_t) &= F^{(1)}_{N_{t+1} \setminus \{i\}}(\xi_{t+1})(\xi_{t+1} - r_{t+1}) + \int_{\xi_{t+1}}^{\xi_t} F^{(1)}_{N_{t+1} \setminus \{i\}}(x) dx \\ &\leq F^{(1)}_{N_{t+1} \setminus \{i\}}(\xi_t)(\xi_{t+1} - r_{t+1}) + F^{(1)}_{N_{t+1} \setminus \{i\}}(\xi_t)(\xi_t - \xi_{t+1}) \\ &= F^{(1)}_{N_{t+1} \setminus \{i\}}(\xi_t)(\xi_t - r_{t+1}). \end{aligned}$$

The result then implies $r_t \geq r_{t+1}$, as desired. □

PROOF OF LEMMA 3:

In the stage auction of period t, there are total $N_t = N_{t-1} \cup M_t$ bidders. Among them, M_t bidders are strong whose values are independent draws from F on $[0,1]$, and the other N_{t-1} bidders are weak, whose values are independent draws from the truncated distribution of $F(v\,|\xi_{t-1}) = \Pr(V \leq v\,|V \leq \xi_{t-1})$. The reserve price is r_t, and bidders' cutoff value for submitting bid is ξ_t, with $\xi_t > r_t$ for $t < T$, as shown in Proposition 2.

We need to introduce some new notations here. We denote the truncated distribution by $F(x\,|\xi_{t-1}) = F(x)/F(\xi_{t-1})$. Moreover, let $G^{(k)}_{N_t}$, $k = 1, 2$, denote the distribution function of the k-th highest order statistics of the values of the N_t bidders'. Based on the properties of order statistics, we have the following expressions of $G^{(k)}_{N_t}$.

$$\begin{aligned} G^{(1)}_{N_t}(x) &= F^{(1)}_{N_{t-1}}(v\,|\xi_{t-1})F^{(1)}_{M_t}(v), \\ G^{(2)}_{N_t}(x) &= F^{(1)}_{N_{t-1}}(v\,|\xi_{t-1})F^{(2)}_{M_t}(v) + n_{t-1}\bar{F}(x\,|\xi_{t-1})F^{(1)}_{N_{t-1}-1}(v\,|\xi_{t-1})F^{(1)}_{M_t}(v), \end{aligned} \tag{27}$$

where $\bar{F}(x\,|\xi_{t-1}) = 1 - F(x\,|\xi_{t-1})$ is the survival function. The expected revenue of the stage auction in period t is thus

$$R_t(N_t) = r_t\left[G^{(2)}_{N_t}(\xi_t) - G^{(1)}_{N_t}(\xi_t)\right] + \int_{\xi_t}^1 x dG^{(2)}_{N_t}(x)$$

It is helpful to do the following transformation,

$$R_t(N_t) = \left\{\xi_t\left[G^{(2)}_{N_t}(\xi_t) - G^{(1)}_{N_t}(\xi_t)\right] + \int_{\xi_t}^1 x dG^{(2)}_{N_t}(x)\right\} - (\xi_t - r_t)\left[G^{(2)}_{N_t}(\xi_t) - G^{(1)}_{N_t}(\xi_t)\right], \tag{28}$$

where the part in the curly braces is the expected revenue of a one-shot auction with a reserve price of ξ_t. From Myerson (1981) and Kirkegaard (2012), it is equal to

$$\begin{aligned} &F^{(1)}_{N_{t-1}}(\xi_t\,|\xi_{t-1})\int_{\xi_t}^1 \psi(x) dF^{(1)}_{M_t}(x) \\ &\qquad + \int_{\xi_t}^{\xi_{t-1}}\left[\psi(v\,|\xi_{t-1})F^{(1)}_{M_t}(v) + \int_v^1 \psi(x) dF^{(1)}_{M_t}(x)\right] dF^{(1)}_{N_{t-1}}(v\,|\xi_{t-1}) \end{aligned} \tag{29}$$

where $\psi(v\,|\xi_{t-1})$ is the virtual value function of the N_{t-1} weak bidders. Substituting $\psi(v\,|\xi_{t-1}) = \psi(v) + \frac{\bar{F}(\xi_{t-1})}{f(v)}$ into (29) and integrating by parts, we then have the expected revenue of (29) is equal to

$$\int_{\xi_t}^{\xi_{t-1}} \psi(x) d\frac{F_{N_t}^{(1)}(x)}{F_{N_{t-1}}^{(1)}(\xi_{t-1})} + \int_{\xi_{t-1}}^{1} \psi(x) dF_{M_t}^{(1)}(x) + \frac{n_{t-1}\bar{F}(\xi_{t-1})}{F_{N_{t-1}}^{(1)}(\xi_{t-1})} \int_{\xi_t}^{\xi_{t-1}} F_{N_t-1}^{(1)}(x)dx \tag{30}$$

Second, from the cutoff condition of (7) for bidders' equilibrium strategies, we have

$$\int_{\xi_t}^{\xi_{t-1}} F_{N_t-1}^{(1)}(x)dx = F_{N_{t-1}-1}^{(1)}(\xi_{t-1})(\xi_{t-1} - r_{t-1}) - F_{N_t-1}^{(1)}(\xi_t)(\xi_t - r_t). \tag{31}$$

Moreover, the the property of order statistics implies that

$$G_{N_t}^{(2)}(\xi_t) - G_{N_t}^{(1)}(\xi_t) = m_t\bar{F}(\xi_t)\frac{F_{N_t-1}^{(1)}(\xi_t)}{F_{N_{t-1}}^{(1)}(\xi_{t-1})} + n_{t-1}\bar{F}(x\,|\xi_{t-1})\frac{F_{N_t-1}^{(1)}(\xi_t)}{F_{N_{t-1}-1}^{(1)}(\xi_{t-1})}. \tag{32}$$

Substituting the results of (29)-(32) into (28), we then have the *ex-ante* expected stage revenue in period t:

$$\begin{aligned}
F_{N_{t-1}}^{(1)}(\xi_{t-1})R_t(N_t) &= \int_{\xi_t}^{\xi_{t-1}} \psi(x) dF_{N_t}^{(1)}(x) + F_{N_{t-1}}^{(1)}(\xi_{t-1}) \int_{\xi_{t-1}}^{1} \psi(x) \mathrm{d}F_{M_t}^{(1)}(x) \\
&\quad + n_{t-1}\bar{F}(\xi_{t-1}) \left[F_{N_{t-1}-1}^{(1)}(\xi_{t-1})(\xi_{t-1} - r_{t-1}) - F_{N_t-1}^{(1)}(\xi_t)(\xi_t - r_t)\right] \\
&\quad - (\xi_t - r_t)\left[m_t\bar{F}(\xi_t)F_{N_t-1}^{(1)}(\xi_t) + n_{t-1}(F(\xi_{t-1}) - F(\xi_t))F_{N_t-1}^{(1)}(\xi_t)\right] \\
&= \int_{\xi_t}^{\xi_{t-1}} \psi(x) dF_{N_t}^{(1)}(x) + F_{N_{t-1}}^{(1)}(\xi_{t-1}) \int_{\xi_{t-1}}^{1} \psi(x) \mathrm{d}F_{M_t}^{(1)}(x) \\
&\quad + n_{t-1}\bar{F}(\xi_{t-1})F_{N_{t-1}-1}^{(1)}(\xi_{t-1})(\xi_{t-1} - r_{t-1}) - n_t\bar{F}(\xi_t)F_{N_t-1}^{(1)}(\xi_t)(\xi_t - r_t).
\end{aligned}$$

Summing all of them together, we then get the result of (9). □

PROOF OF PROPOSITION 1:

For $t < T$, due to Lemma 3, the partial derivative of $\pi(\xi, \mathbf{M})$ with respect to ξ_t is

$$\begin{aligned}
\frac{\partial \pi}{\partial \xi_t} &= \psi(\xi_t)\left[f_{N_{t+1}}^{(1)}(\xi_t) - f_{N_t}^{(1)}(\xi_t)\right] \\
&\quad + f_{N_t}^{(1)}(\xi_t)\left[\int_{\xi_t}^{1} \psi(x) dF_{M_{t+1}}^{(1)}(x) - c_{M_{t+1}}\right] - \psi(\xi_t)F_{N_t}^{(1)}(\xi_t)f_{M_{t+1}}^{(1)}(\xi_t) \\
&= f_{N_t}^{(1)}(\xi_t)\left[\int_{\xi_t}^{1} (\psi(x) - \psi(\xi_t)) dF_{M_{t+1}}^{(1)}(x) - c_{M_{t+1}}\right] = f_{N_t}^{(1)}(\xi_t) \cdot \eta(\xi_t) = 0.
\end{aligned}$$

Note that

$$\frac{\partial \eta}{\partial \xi_t} = -\psi'(\xi_t) \int_{\xi_t}^{1} dF_{M_{t+1}}^{(1)}(x) < 0,$$

and then $\frac{\partial \pi}{\partial \xi_t}$ changes its sign from positive to negative at most once. $\pi(\boldsymbol{\xi}, \mathbf{M})$ is then quasi-concave in ξ_t, and the first order condition is also sufficient. When $t = T$, $\frac{\partial \pi}{\partial \xi_T} = -\psi(\xi_T) f_{N_T}^{(1)}(\xi_T)$. It is obvious that π is concave in ξ_T given the IFR assumption, and thus $\psi(\xi_T^*) = 0$. □

PROOF OF COROLLARY 1:

Result 1) is straightforward from (13), as $F_M^{(1)}(x) = F_{M'}^{(1)}(x) = F^{|M|}(x)$. For result 2), as $c_M = c_{M'}$, then from (13),

$$\int_{\xi^*(M)}^1 \left[1 - F_M^{(1)}(x)\right] d\psi(x) = \int_{\xi^*(M')}^1 \left[1 - F_{M'}^{(1)}(x)\right] d\psi(x) > \int_{\xi^*(M')}^1 \left[1 - F_M^{(1)}(x)\right] d\psi(x)$$

where the inequality is due to $F_M^{(1)}(x) > F_{M'}^{(1)}(x)$. It then follows that $\xi^*(M') > \xi^*(M)$ as $\int_{\xi^*(M)}^{\xi^*(M')} \left[1 - F_M^{(1)}(x)\right] d\psi(x) > 0$. □

PROOF OF LEMMA 4:

From (13) and (11), we have $\xi^*(M_{t+1}) = \xi_t^*$ for $1 \le t < T$, and define $\xi^*(M_{T+1}) := \xi_T^*$ Substituting (11) into (9), we then get

$$\begin{aligned}
\pi^*(\mathbf{M}) &= \left[\int_{\xi_1^*}^{\xi_0} \psi(x) dF_{N_1}^{(1)}(x) - c_{M_1}\right] \\
&\quad + \sum_{t=2}^{T} \left[\int_{\xi_t^*}^{\xi_{t-1}^*} \psi(x) dF_{N_t}^{(1)}(x) + F_{N_{t-1}}^{(1)}(\xi_{t-1}^*) \int_{\xi_{t-1}^*}^{1} \psi(\xi_{t-1}^*) dF_{M_t}^{(1)}(x)\right] \\
&= \left[\int_{\xi_1^*}^{1} \psi(x) dF_{N_1}^{(1)}(x) - c_{M_1}\right] \\
&\quad + \sum_{t=2}^{T} \left[\psi(\xi_{t-1}^*) F_{N_{t-1}}^{(1)}(\xi_{t-1}^*) - \psi(\xi_t^*) F_{N_t}^{(1)}(\xi_t^*) - \int_{\xi_t^*}^{\xi_{t-1}^*} F_{N_t}^{(1)}(x) d\psi(x)\right] \\
&= \left[\int_{\xi_1^*}^{1} \psi(x) dF_{N_1}^{(1)}(x) - c_{M_1}\right] \\
&\quad + \psi(\xi_1^*) F_{N_1}^{(1)}(\xi_1^*) - \psi(\xi_1^*) + \sum_{t=2}^{T} \int_{\xi_t^*}^{\xi_{t-1}^*} \left[1 - F_{N_t}^{(1)}(x)\right] d\psi(x) \\
&= \int_{\xi^*(M_2)}^{\xi^*(M_1)} \left[1 - F_{N_1}^{(1)}(x)\right] d\psi(x) + \sum_{t=2}^{T} \int_{\xi^*(M_{t+1})}^{\xi^*(M_t)} \left[1 - F_{N_t}^{(1)}(x)\right] d\psi(x).
\end{aligned}$$

For the last equality, we apply the definition that $c_{M_1} = \int_{\xi^*(M_1)}^1 \left[1 - F_{M_1}^{(1)}(x)\right] d\psi(x)$. $\square$

PROOF OF LEMMA 5:

For any $M \subseteq N$, due to (13), we have

$$c = \int_{\xi^*(m)}^1 \frac{1}{m} \left(1 - F^m(x)\right) d\psi(x).$$

As $F(x) < 1$ for $x \in [0, 1)$, $(1 - F^m(x))/m$ is strictly decreasing in m. Therefore, when m increases, $\xi^*(m)$ must decrease so as to keep the above equation to hold. $\square$

PROOF OF PROPOSITION 2:

It is straightforward from Lemma 5. $\square$

PROOF OF LEMMA 6:

The conditional expected social welfare in period t is

$$W_t(N_t) = \int_{\xi_t}^{\xi_{t-1}} x dG_{N_t}^{(1)}(x) + \int_{\xi_{t-1}}^1 x dF_{M_t}^{(1)}(x).$$

where $G_{N_t}^{(1)}(x) = F_{N_{t-1}}^{(1)}(x \,|\xi_{t-1}) F_{M_t}^{(1)}(x)$. Summing up all the terms of $F_{N_{t-1}}^{(1)}(\xi_{t-1}) W_t(N_t)$, we then get the result of (19). $\square$

PROOF OF PROPOSITION 3:

For $1 \leq t < T$, from (19), the derivative of $W(\boldsymbol{\xi}, \mathbf{M})$ with respect to ξ_t is

$$\frac{\partial W}{\partial \xi_t} = f_{N_t}^{(1)}(\xi_t) \left[\int_{\xi_t}^1 (x - \xi_t) dF_{M_{t+1}}^{(1)}(x) - c_{M_{t+1}}\right] = f_{N_t}^{(1)}(\xi_t) \tilde{\eta}(\xi_t) = 0.$$

Note that $\tilde{\eta}(\xi_t)$ is decreasing in ξ_t, then W is quasi-concave in ξ_t, and the first order condition is also sufficient. Second, when $t = T$, the partial derivative of W with respect to ξ_t is $\frac{\partial W}{\partial \xi_T} = -\xi_T f_{N_t}^{(1)}(\xi_T) \leq 0$, and therefore $\xi_T^{**} = 0$. $\square$

PROOF OF LEMMA 7:

From (21) and (20), we have $\xi^{**}(M_{t+1}) = \xi_t^{**}$ for $1 \leq t < T$. For $t = T$, we define

$\xi^{**}(M_{T+1}) = \xi^{**}_T = 0$. We then have

$$\begin{aligned}
W^{**} &= \left[\int_{\xi_1^{**}}^{\xi_0} x dF_{N_1}^{(1)}(x) - c_{M_1}\right] + \sum_{t=2}^{T}\left[\int_{\xi_t^{**}}^{\xi_{t-1}^{**}} x dF_{N_t}^{(1)}(x) + F_{N_{t-1}}^{(1)}(\xi_{t-1}^{**})\int_{\xi_{t-1}^{**}}^{1} \xi_{t-1}^{**} dF_{M_t}^{(1)}(x)\right] \\
&= \left[\int_{\xi_1^{**}}^{1} x dF_{N_1}^{(1)}(x) - c_{M_1}\right] + \sum_{t=2}^{T}\left[\xi_{t-1}^{**} F_{N_{t-1}}^{(1)}(\xi_{t-1}^{**}) - \xi_t^{**} F_{N_t}^{(1)}(\xi_t^{**}) - \int_{\xi_t^{**}}^{\xi_{t-1}^{**}} F_{N_t}^{(1)}(x)dx\right] \\
&= \left[\xi_1^{**} - \xi_1^{**} F_{N_1}^{(1)}(\xi_1^{**}) + \int_{\xi_1^{**}}^{\xi^{**}(M_1)} \left[1 - F_{N_1}^{(1)}(x)\right] dx\right] \\
&\quad + \sum_{t=2}^{T}\left[\xi_{t-1}^{**} F_{N_{t-1}}^{(1)}(\xi_{t-1}^{**}) - \xi_t^{**} F_{N_t}^{(1)}(\xi_t^{**}) + \int_{\xi_t^{**}}^{\xi_{t-1}^{**}} \left[1 - F_{N_t}^{(1)}(x)\right] dx - (\xi_{t-1}^{**} - \xi_t^{**})\right] \\
&= \sum_{t=1}^{T} \int_{\xi^{**}(M_{t+1})}^{\xi^{**}(M_t)} \left[1 - F_{N_t}^{(1)}(x)\right] dx,
\end{aligned}$$

where in the third equality, we substitute $c_{M_1} = \int_{\xi^{**}(M_1)}^{1} \left[1 - F_{M_1}^{(1)}(x)\right] dx$. □

PROOF OF LEMMA 8:

From (21), the proof is the same as that of Lemma 5. □

PROOF OF PROPOSITION 4:

It is straightforward from Lemma 8. □

PROOF OF COROLLARY 2:

For $1 \le t < T$, $\boldsymbol{\xi}^*$ and $\boldsymbol{\xi}^{**}$ are given by (11) and (20) respectively. If we define

$$\tilde{\eta}(v) = \int_v^1 (x - v) dF_{M_t}^{(1)}(x) - c_{M_t} \text{ and } \eta(v) = \int_v^1 \left[\psi(x) - \psi(v)\right] dF_{M_t}^{(1)}(x) - c_{M_t},$$

then both $\tilde{\eta}(v)$ and $\eta(v)$ are decreasing in v. Note that

$$\eta(v) - \tilde{\eta}(v) = \int_v^1 \left[\frac{1 - F(v)}{f(v)} - \frac{1 - F(x)}{f(x)}\right] dF_{M_t}^{(1)}(x) > 0,$$

due to the IFR assumption. Finally, for $t = T$, we already know $r^* = \xi_T^* > \xi_T^{**} = 0$. □

PROOF OF PROPOSITION 5:

From (13) and (21), it follows that, for given ξ,

$$c = \int_{\xi}^{1} \frac{1 - F^{m^*}(x)}{m^*} \cdot \psi'(x) dx = \int_{\xi}^{1} \frac{1 - F^{m^{**}}(x)}{m^{**}} dx.$$

As $\psi'(x) > 1$ from the IFR assumption and $[1 - F^m(x)]/m$ is decreasing in m, we then get the result. □

PROOF OF PROPOSITION 6:

It is straightforward from the equation (25). □

PROOF OF PROPOSITION 7:

Denote $Z^{(1)}_{m_{t+1}} = \max\left\{r^*, X^{(1)}_{m_{t+1}}\right\}$, and define

$$B_t = \max_{m_{t+1}} \left\{ \mathrm{E}\hat{J}_{t+1}\left[Z^{(1)}_{m_{t+1}}\right] - m_{t+1}c \right\},$$

which is the continuation value of following an optimal search strategy after the end of period t. Temporarily, we define

$$\hat{J}_t(v) = \max_{m_{t+1}} \left\{ \psi(v), \mathrm{E}\hat{J}_{t+1}\left[Z^{(1)}_{m_{t+1}}\right] - m_{t+1}c \right\} = \max\left\{\psi(v), B_t\right\}, \tag{33}$$

which will later be shown is equivalent to (23). It is clear that $B_T = 0$, and for $t < T$,

$$B_t = \max_{m_{t+1}} \left\{ \mathrm{E}\hat{J}_{t+1}\left[Z^{(1)}_{m_{t+1}}\right] - m_{t+1}c \right\} = \max_{m_{t+1}} \left\{ \mathrm{E}\max\left\{\psi(Z^{(1)}_{m_{t+1}}), B_{t+1}\right\} - m_{t+1}c \right\}.$$

When $t = T - 1$,

$$B_{T-1} = \max_{m_T} \left\{ \mathrm{E}\hat{J}_T\left[Z^{(1)}_{m_T}\right] - m_T c \right\} = \max_{m_T} \left\{ \mathrm{E}\max\left\{\psi(Z^{(1)}_{m_T}), B_T\right\} - m_T c \right\} > 0 = B_T.$$

When $t = T - 2$, similarly,

$$\begin{aligned} B_{T-2} &= \max_{m_{T-1}} \left\{ \mathrm{E}\max\left\{\psi(Z^{(1)}_{m_{T-1}}), B_{T-1}\right\} - m_{T-1}c \right\} \\ &> \max_{m_{T-1}} \left\{ \mathrm{E}\max\left\{\psi(Z^{(1)}_{m_{T}-1}), B_{T}\right\} - m_{T-1}c \right\} = B_{T-1}. \end{aligned}$$

Continuing in this manner, we see that

$$B_t > B_{t+1} \text{ for } t = 0, 1, \cdots, T-1.$$

The result is then implied by the fact that the optimal cutoff satisfies $\psi(\hat{\xi}_t^*) = B_t$. Second, note that the optimal sample size m_t is the maximizer of

$$\varsigma(m_t, \hat{\xi}_t^*) = \mathrm{E}\max\left\{\psi(Z_{m_t}^{(1)}), B_t\right\} - m_t c.$$

Simple transformation gives

$$\varsigma(m_t, \hat{\xi}_t^*) = \mathrm{E}\max\left\{\psi(X_{m_t}^{(1)}), \psi(\hat{\xi}_t^*)\right\} - m_t c = \psi(\hat{\xi}_t^*) + \int_{\hat{\xi}_t^*}^1 \left[\psi(x) - \psi(\hat{\xi}_t^*)\right] dF_{m_t}^{(1)}(x) - m_t c$$

which is concave in m_t given the fact that the virtual value function, $\psi(x)$, is increasing and $\hat{\xi}_t^*$ is independent of m_t (Szech, 2011, Lemma 1). Furthermore,

$$\varsigma(m_t + 1, \hat{\xi}_t^*) - \varsigma(m_t, \hat{\xi}_t^*) = \int_{\hat{\xi}_t^*}^1 F_{m_t}^{(1)}(x)\left[1 - F(x)\right] d\psi(x) - c$$

is decreasing in $\hat{\xi}_t^*$. The optimization condition for m_t is

$$\varsigma(m_t - 1, \hat{\xi}_t^*) - \varsigma(m_t, \hat{\xi}_t^*) \geq 0 > \varsigma(m_t + 1, \hat{\xi}_t^*) - \varsigma(m_t, \hat{\xi}_t^*).$$

Given that $\hat{\xi}_t^* > \hat{\xi}_{t+1}^*$, then the concavity of $\varsigma(m_t, \hat{\xi}_t^*)$ in m_t then implies $\hat{m}_t^* \leq \hat{m}_{t+1}^*$. □

PROOF OF PROPOSITION 8:

For $t = T$, $\hat{J}_T(v) = \max\{\psi(v), 0\} = J_T(v)$, and therefore, $\hat{\xi}_T^* = \xi_T^* = r^*$. For $t = T-1$, for long-lived bidders, from (3),

$$J_{T-1}(v) = \max\left\{\psi(v), -c_{M_T} + \mathrm{E}J_T\left[\max\left\{v, X_{M_T}^{(1)}\right\}\right]\right\},$$

and $v \geq r^*$ as $\psi(v) \geq 0$. For short-lived bidders, from (23),

$$\hat{J}_{T-1}(v) = \max\left\{\psi(v), -c_{M_T} + \mathrm{E}\hat{J}_T\left[\max\left\{r^*, X_{M_T}^{(1)}\right\}\right]\right\}.$$

As $\hat{J}_T(v) = J_T(v)$ and both are increasing function, it is then clear that $J_{T-1}(v) \geq \hat{J}_{T-1}(v)$ with equality only when $v = 0$. Repeating this process, we then reaches the conclusion that

$J_t(v) \geq \hat{J}_t(v)$, for $0 \leq t < T-1$. The indifference condition for cutoff value then implies $\hat{\xi}_t^* < \xi_t^*$, for $0 \leq t < T-1$. □

PROOF OF PROPOSITION 9:

Recall the condition (4) for long-lived bidders and the recurrence equations (24) for short-lived bidders. Given a sequence of cutoff values $\boldsymbol{\xi}$ such that $\xi_t > \xi_{t+1}$, the above equations define the inverse real-value functions of $m_{t+1}^*(\xi_t)$ for long-lived bidders and $\hat{m}_{t+1}^*(\xi_t, \xi_{t+1})$ for short-lived bidders. That is, for given $\boldsymbol{\xi}$, $m^*(\xi_t)$ and $\hat{m}^*(\xi_t, \xi_{t+1})$ are respectively the optimal sample sizes for long-lived and short-lived bidders in period $t+1$. Our objective is to show $m_{t+1}^*(\xi_t) < \hat{m}_{t+1}^*(\xi_t, \xi_{t+1})$. We can define a new function

$$\eta(m, \xi) = \int_0^1 \max\{\psi(\xi), \psi(x)\} \, \mathrm{d}F^m(x) - mc$$

which is strictly concave in m (Szech, 2011), and obeys single-crossing difference in (m, ξ) given that, for $m' > m$, $\eta(m'', \xi) - \eta(m', \xi)$ is decreasing in ξ. The well-known result of Milgrom and Shannon (1994)(Theorem 4) gives that

$$\tilde{m}(\xi) := \arg\max_{\xi} \eta(m, \xi)$$

is strictly decreasing in ξ, and hence $\tilde{m}(\xi_t) < \tilde{m}(\xi_{t+1})$. In addition, from (4), it follows

$$\begin{aligned} \psi(\xi_t) &= \int_0^1 \max\{\psi(\xi_t), \psi(x)\} \, \mathrm{d}F^{m^*(\xi_t)}(x) - m^*(\xi_t)c \\ &\leq \int_0^1 \max\{\psi(\xi_t), \psi(x)\} \, \mathrm{d}F^{\tilde{m}(\xi_t)}(x) - \tilde{m}(\xi_t)c \end{aligned}$$

Therefore,

$$m^*(\xi_t) \leq \tilde{m}(\xi_t) < \tilde{m}(\xi_{t+1}) = \hat{m}^*(\xi_t, \xi_{t+1}),$$

where the last equality is implied by (24). □