Duopoly with Individual Demand for Variety

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Abstract

A literature in microeconomics explores specifications, if any, under which discrete individual choice among varieties of a differentiated product generates linear aggregate demands. Discrete individual choice is an appealing specification for goods that make up a large fraction of individual expenditure. The case of "small ticket" items, for which individuals may consume several varieties, is also of interest. I derive aggregate demand when there is such individual demand for variety, and explore characteristics of duopoly equilibrium.

JEL codes: D4, L1

Keywords: product differentiation, demand for variety

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Contents

1 Introduction

The linear aggregate demand product differentiation specification is widely used in industrial economics. It is used by Sutton (1998 and elsewhere) in his pathbreaking analysis of market structure. It is increasingly used in the international trade literature. But it has lacked micro foundations in a specification of individual demand, of the kind that has been common since Hotelling (1929). Linear aggregate demand equations for differentiated products have either been posited *a priori* or derived from a quadratic aggregate welfare function. The quadratic aggregate welfare function is itself typically simply assumed.

Clear micro foundations have the merit of making explicit the assumptions about individual demand that underpin a given model of aggregate demand (Kirman, 1992). The conditions, if any, under which linear aggregate demands for differentiated varieties are implied by discrete models of individual choice — that is, under the assumption that any individual either buys one unit of one variety or does not buy any variety is a subject of active research.[1] Jaffe and Weyl (2012) show that discrete individual demand cannot imply linear aggregate demands if the distribution of reservation prices in the population is everywhere nonnegative. Bos and Vermeulen (2019) derive aggregate demands that are linear in a region of price space from discrete individual choice behavior if there are "captive" groups in the population — individuals who will not purchase some varieties at any set of prices.

The individual discrete choice specification is appealing for cases where aggregate demands are built up from individual demand for single varieties, with the choice differing from individual to individual. Products for which price is a large part of most individual budgets provide examples — the market for new cars is a case in point. For other markets, discrete choice may be inherent in the nature of the product — choice of mode of transportation, for example. But there are many products where there is demand for variety at the individual level — e.g., food products such as breakfast cereal or fresh fruit.

In this paper, rather than seeking micro foundations for an assumed linear aggregate demand function, I assume linear individual demand for two varieties and derive the implied aggregate demands. This makes it possible

[1] Hendel (1999) estimates a model of individual demand for multiple discrete for multiple varieties.

to analyze the nature of duopoly equilibrium when individual demands are price-elastic and when individuals have a taste for variety.

In Section 2 I briefly discuss existing work that is related to this paper. In Section 3, I begin with a review of the derivation from micro foundations of a linear aggregate demand function for the case of a homogeneous product. I show how results change if individual demand for the homogeneous good is elastic, and solve for perfect competition, monopoly, and Cournot duopoly outcomes. for that case. Section 4 derives aggregate demands when individual demands are linear in prices, finds best-response price functions and equilibrium prices, and explores other aspects of equilibrium. It emerges from this analysis that equilibrium prices with linear individual demands are lower than equilibrium prices in a model of aggregate linear demand, all else equal. Section 6 concludes. Details of proofs are in the appendices.

2 Related Work

The quadratic representative consumer, linear aggregate demand specification has a long history in economics. Edgeworth (1897, p. 26/1925, p. 122) uses a quadratic utility function and linear inverse demand equations in discussion of complementary goods. Bowley (1924, p. 56) uses a quadratic aggregate welfare function. He does not include among its arguments a composite good with constant marginal utility; the implied demand equations are not linear. Despite these distinguished antecedents, it is to Spence (1976) and Dixit (1979) that the modern economics literature owes the linear aggregate demand for differentiated goods specification and the observation that it can be derived from a quadratic representative consumer welfare function. One should also highlight Singh and Vives (1984).[2] Häckner (2000), who allows for many varieties and quality differences, provides what is probably the most complete generalization of Dixit (1979).[3]

Hotelling (1929) assumed inelastic unit demand without a reservation price, effectively discrete choice individual demand. A few papers in the spatial literature modify Hotelling's specification to allow for elastic demand.[4]

[2] In what follows, I refer to the linear aggregate demand product differentiation model as the S-D-V specification.

[3] For a review of the specification, see Vives (1999). Foster *et al.* (2008) use a quadratic aggregate welfare function that allows for a continuum of varieties.

[4] Hoover (1937), Smithies (1941), Anderson and Neven (1991), and Rath and Zhao

These papers address different research questions from the present work, from which they are distinguished as well by the role of transportation cost in Hotelling-like models.

3 Inelastic vs. Elastic Individual Demand: the Homogeneous Good Case

3.1 Inelastic Individual Demand

Linear aggregate demand for a homogeneous good can be derived from a discrete choice model of individual demand in a straightforward way, which we make explicit to allow comparison with results that follow.

There are two goods, good 1 and a composite all other goods, "money". The composite good is produced in a perfectly competitive market, with a constant returns to scale technology. There is constant marginal utility of money, and the units in which all other goods are measured are chosen so consumption of one unit of all other goods yields utility 1.[5]

Individuals have identical incomes y. Each individual buys either 0 or 1 units of good 1, with the choice of individual i denoted by

$$\beta_i = \begin{cases} 1 & \text{buys one unit of good 1} \\ 0 & \text{does not buy the good} \end{cases} . \tag{1}$$

Individual utility for the alternative decisions is

$$v_i\left(r,p\right) = \begin{cases} y-p+r & \beta_i = 1 \\ y & \beta_i = 0 \end{cases} . \tag{2}$$

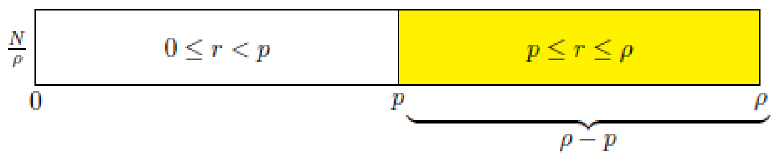
Utility-maximizing individuals with $r \geq p$ will each buy one unit of good 1. Those with $r < p$ will consumer only the composite good.

Now assume there are N potential buyers, with reservation prices r uniformly distributed on $0 \leq r \leq \rho$ (Figure 3.1). Demand for good 1 comes from those with reservation prices on the interval $p \leq r \leq \rho$, yielding linear demand

$$Q\left(p\right) = \frac{N}{\rho}\left(\rho - p\right) = N - \frac{N}{\rho}p \qquad p = \rho - \frac{\rho}{N}Q. \tag{3}$$

(2001).

[5] On the assumption of a constant marginal utility of money, see Marshall (1890), Hicks (1941), and Vives (1987).



3.2 Linear Individual Demand

Keeping other aspects of the previous specification unchanged, let the quantity demanded by an individual be linear in price,

$$q = \begin{cases} r - p & p \leq r \\ 0 & p > r \end{cases}. \tag{4}$$

Then demand and inverse demand equations are[6]

$$Q = \frac{N}{\rho} \int_{r=p}^{r=\rho} (r-p)\, dr = \frac{1}{2} \frac{N}{\rho} (\rho - p)^2 \tag{5}$$

and

$$p = \rho - \sqrt{2 \frac{\rho}{N} Q}, \tag{6}$$

respectively.

The corresponding marginal revenue equation is

$$MR = \rho - \frac{3}{2} \sqrt{2 \frac{\rho}{N} Q} \tag{7}$$

The demand and marginal revenue curves are shown in Figure 1.

Competitive equilibrium and monopoly equilibrium values of interest for the case of constant marginal cost c are shown in columns 2 and 4 of Table 1.

For Cournot duopoly, the objective function of (say) firm 1 is

$$\pi_1 = \left(\rho - c - \sqrt{2 \frac{\rho}{N} (q_1 + q_2)} \right) q_1. \tag{8}$$

The first-order condition to maximize π_1 can be written

$$9q_1^2 + 12 q_1 q_2 + 4 q_2^2 - Z^2 (q_1 + q_2) = 0, \tag{9}$$

[6] For details of derivations, see Appendix I.

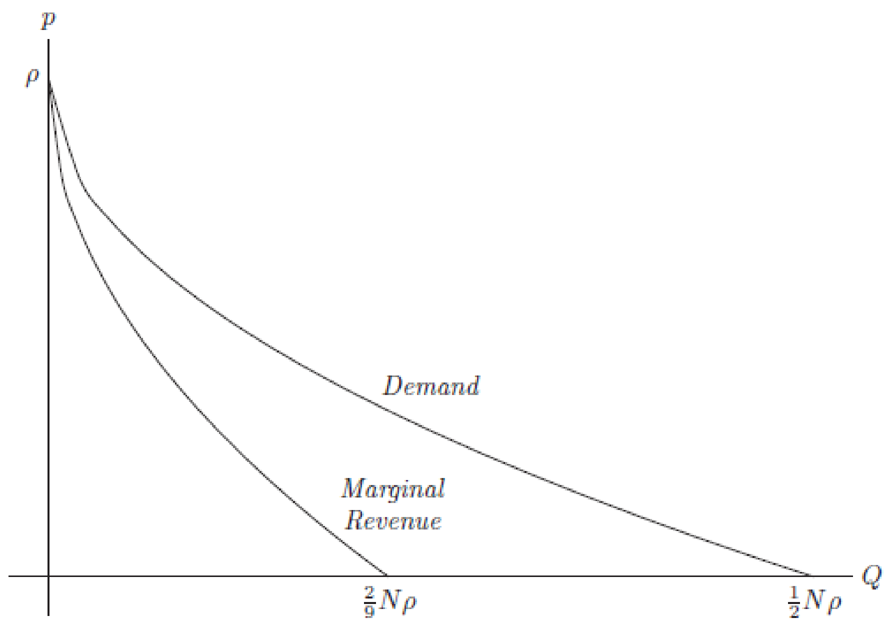


Figure 1: Market demand and marginal revenue curves, linear individual demand, homogeneous product.

	Competition	Duopoly	Monopoly
p	c	$c + \frac{1}{5}(\rho - c)$	$c + \frac{1}{3}(\rho - c)$
Q	$\frac{1}{2}\frac{N}{\rho}(\rho - c)^2$	$\frac{8}{25}\frac{N}{\rho}(\rho - c)^2$	$\frac{2}{9}\frac{N}{\rho}(\rho - c)^2$
π	0	$\frac{8}{125}\frac{N}{\rho}(\rho - c)^3$	$\frac{2}{27}\frac{N}{\rho}(\rho - c)^3$
CS	$\frac{1}{6}\frac{N}{\rho}(\rho - c)^3$	$\frac{32}{375}\frac{N}{\rho}(\rho - c)^3$	$\frac{4}{81}\frac{N}{\rho}(\rho - c)^3$

Table 1: Selected equilibrium values, competition, Cournot duopoly, monopoly: homogeneous product, linear individual demand, constant marginal cost c. Duopoly output and profit are totals, not per firm.

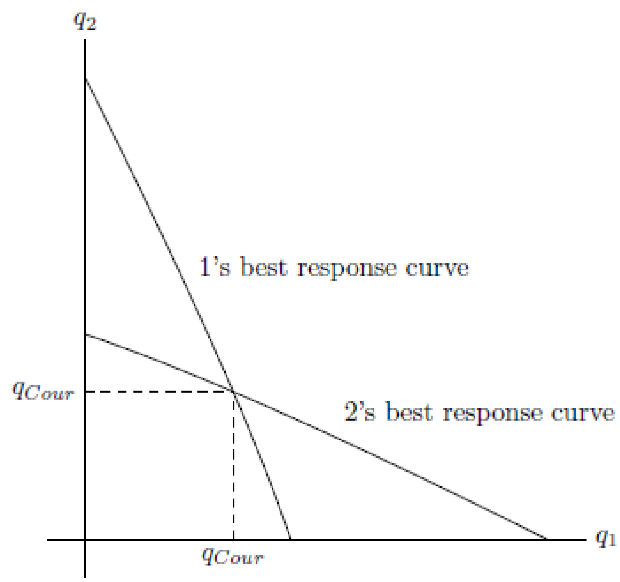


Figure 2: Cournot duopoly best-response curves, linear individual demand.

for

$$Z^2 = 2\frac{N}{\rho}(\rho - c)^2 . \tag{10}$$

(9) is the equation of firm 1's best-response function, written in implicit form. It is the equation of a rotated parabola (see Figure 2).

Equilibrium output per firm can be obtained by setting $q_1 = q_2$ in (9) and simplifying. This leads to the duopoly values shown in Table 1. The qualitative relationships between the equilibrium values given in Table 1 for the three market types is as expected.

4 Elastic Individual Demand: the Differentiated Goods Case

4.1 Individual demand

We now suppose that there are two varieties of a product, and that there is individual demand for variety: all individuals have preferences of the Spence-

Dixit-Vives form. An individual "of type r" has a utility function of the form

$$U = m + r\left(q_1 + q_2\right) - \frac{1}{2}\left(q_1^2 + 2\sigma q_1 q_2 + q_2^2\right). \tag{11}$$

We assume that r is uniformly distributed over the interval $[0, \rho]$, with density N/ρ. As before, ρ is the maximum reservation price.

The individual's constrained optimization problem is

$$\max_{q_1, q_2} U = m + r\left(q_1 + q_2\right) - \frac{1}{2}\left(q_1^2 + 2\sigma q_1 q_2 + q_2^2\right). \tag{12}$$

subject to the budget constraint

$$m + p_1 q_1 + p_2 q_2 \leq Y. \tag{13}$$

The Kuhn-Tucker conditions for this problem imply that if demand for both varieties is nonnegative, demand equations are

$$q_1\left(r\right) = \frac{1}{1+\sigma}\left(r - \frac{p_1 - \sigma p_2}{1-\sigma}\right) \tag{14}$$

and

$$q_2\left(r\right) = \frac{1}{1+\sigma}\left(r - \frac{p_2 - \sigma p_1}{1-\sigma}\right). \tag{15}$$

The expressions for individual demands $q_1\left(r\right)$ and $q_2\left(r\right)$ are familiar from S-D-V aggregate demand models, written in a way that is useful in what follows.

If the individual's demand for variety 2 is zero, demand for variety 1 is

$$q_1 = r - p_1 \tag{16}$$

for $p_1 \leq r$, and zero otherwise. If the individual's demand for variety 1 is zero, demand for variety 2 is

$$q_2 = r - p_2 \tag{17}$$

for $p_2 \leq r$, and zero otherwise.

Individual demand for variety 1 is the building block to derive aggregate demand. To characterize individual demand for variety 1, fix p_2 at an arbitrary value within the range of interest that emerges from the analysis, $0 \leq p_2 \leq (1-\sigma)\rho$.

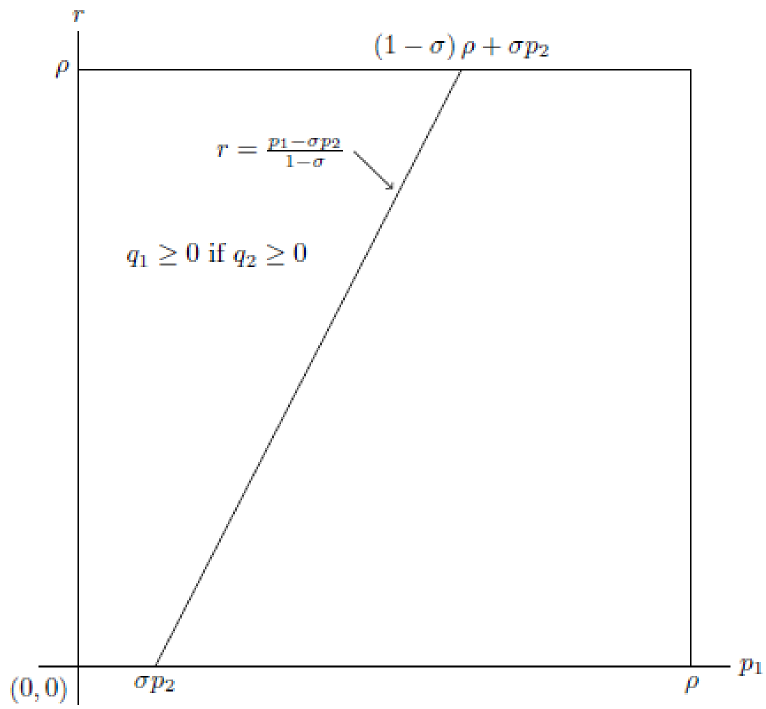


Figure 3: $q_1 \geq 0$ if $q_2 \geq 0$, (p_1, r)-space ($\rho = 8$, $\sigma = \frac{1}{2}$, $p_2 = 2$).

Figure 3 shows a square with sides of length ρ in (p_1, r)-space. On the horizontal axis, p_1 range from 0 to ρ. Higher prices would never to chosen in equilibrium. On the vertical axis, reservation prices also range from 0 to ρ.

The line in Figure 3 shows (p_1, r) combinations that make $q_1(r)$ from (14) just equal to 0 (for the given value of p_2). Above the line, $q_1(r) > 0$ if $q_2(r) > 0$, and below the line $q_1(r) = 0$,

In the same way, the line in Figure 4 shows (p_1, r) combinations that make $q_2(r)$ from (15) just equal to 0. Above the line, $q_2(r) > 0$ if $q_1(r) > 0$. Below the line, $q_2(r) = 0$.

Combining the two figures, as in Figure 5, divides the (p_1, r)-square into four regions, indicating the type of demand for an individual of type r for different values of p_1, holding p_2 fixed.

The configuration of regions shown in Figure 5 is valid provided point B is below point C on the vertical axis, and this defines the low-p_2 case, $0 \leq p_2 \leq (1 - \sigma)\rho$, that we now consider. The pattern of aggregate

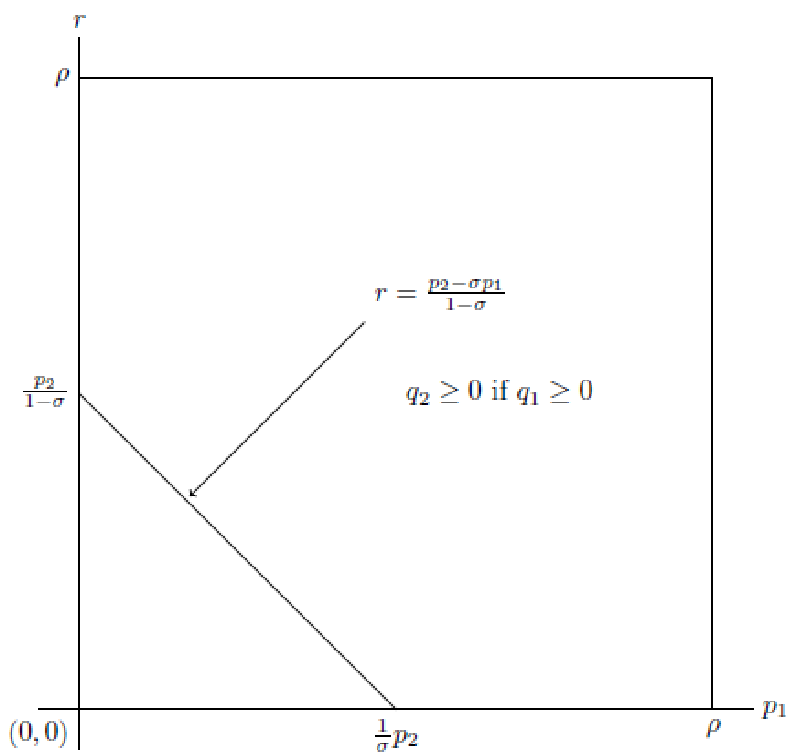


Figure 4: $q_2 \geq 0$ if $q_1 \geq 0$, (p_1, r)-space ($\rho = 8$, $\sigma = \frac{1}{2}$, $p_2 = 2$).

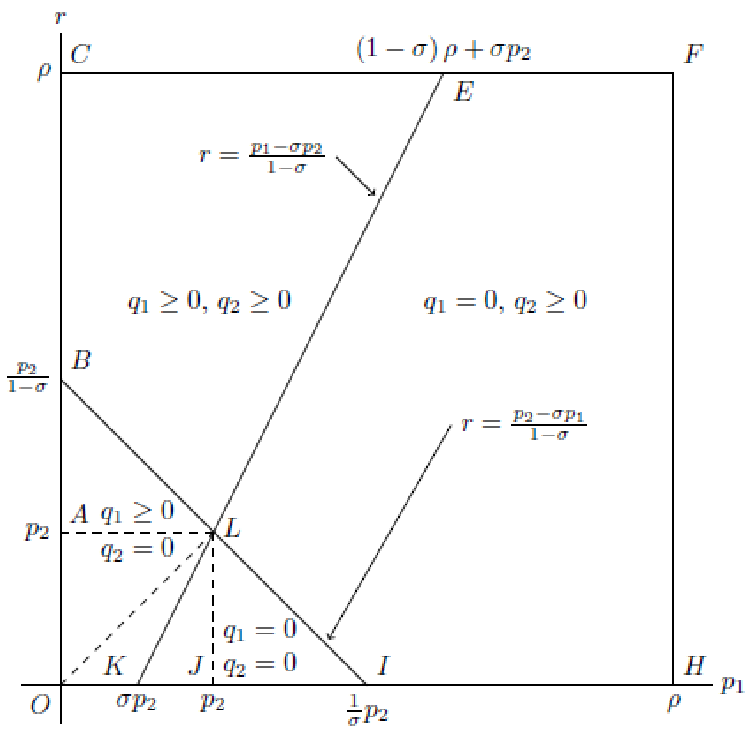


Figure 5: Nature of demand by region in (p_1, r)-space ($\rho = 8$, $\sigma = \frac{1}{2}$, $p_2 = 2$).

demand for higher p_2 differs from the expressions derived below, but such values do not occur in equilibrium.[7]

4.2 Aggregate demand for variety 1.

For $0 \leq p_1 \leq p_2$ (between the origin and point J on the horizontal axis), some demand for variety 1 is from those who buy only variety 1 and some demand for variety 1 is from those who buy both varieties. The lower bound of demand for the first group is the diagonal $r = p_1$, the upper bound for the second group is $r = \rho$. The boundary between the two grounds is the $q_2 = 0$ line. For $p_2 \leq (1-\sigma)\,\rho$, $0 \leq p_1 \leq p_2$, aggregate demand for variety 1 is

$$\frac{\rho}{N} Q_{1A} = \int_{p_1}^{\frac{p_2-\sigma p_1}{1-\sigma}} (r - p_1)\, dr + \frac{1}{1+\sigma} \int_{\frac{p_2-\sigma p_1}{1-\sigma}}^{\rho} \left(r - \frac{p_1 - \sigma p_2}{1-\sigma} \right) dr. \qquad (18)$$

[7] See footnote 8.

Region	Firm 1's best response price
D	$\frac{1}{4}(\rho + p_2)$
A	$\frac{1}{3\sigma^3}\left(-2(\alpha - \gamma p_2) + \sqrt{(4+3\sigma^3)\alpha^2 - 2(4\gamma - 3\sigma^4)\alpha p_2 + (4\gamma^2 - 3\beta\sigma^3)p_2^2}\right)$
B	$\frac{1}{3}[(1-\sigma)\rho + \sigma p_2]$

Table 2: Firm 1's best-response price, by region in price space..

For $p_2 \leq p_1 \leq (1-\sigma)\rho + \sigma p_2$ (between point J and the horizontal-axis coordinate of point E), demand for variety 1 is entirely from those who buy both varieties. The lower bound of the demand integral is the $q_1 = 0$ line. For $p_2 \leq (1-\sigma)\rho$, $p_2 \leq p_1 \leq (1-\sigma)\rho + \sigma p_2$, aggregate demand for variety 1 is

$$\frac{\rho}{N}Q_{1B} = \frac{1}{1+\sigma}\int_{\frac{p_1 - \sigma p_2}{1-\sigma}}^{\rho}\left(r - \frac{p_1 - \sigma p_2}{1-\sigma}\right)dr. \qquad (19)$$

For notational compactness, define the parameters

$$\alpha = (1-\sigma)\rho$$

$$\beta = 1 - \sigma^2 + \sigma^3$$

$$\gamma = 1 - \sigma + \sigma^3.$$

Then evaluating the integrals in (18) and (19) gives aggregate demand for variety 1 for low p_2:

$$2(1+\sigma)\frac{\rho}{N}Q_1 =$$

$$\begin{cases} \frac{\left(\alpha^2 - \sigma^3 p_1^2 - \beta p_2^2 + 2\gamma p_1 p_2 - 2\alpha p_1 + 2\sigma\alpha p_2\right)}{(1-\sigma)^2} & 0 \leq p_1 \leq p_2 \\ \left(\rho - \frac{p_1 - \sigma p_2}{1-\sigma}\right)^2 & p_2 \leq p_1 \leq (1-\sigma)\rho + \sigma p_2 \\ 0 & (1-\sigma)\rho + \sigma p_2 < p_1 \end{cases}. \qquad (20)$$

4.3 Best response prices

Assume that marginal cost is constant, and normalize its value to be zero. Assume also that there are no fixed costs.

The objective functions are then obtained multiplying the expressions for aggregate demand in (20) by price, and the first-order conditions to maximize profit give firm 1's best-response price. The best response equation is

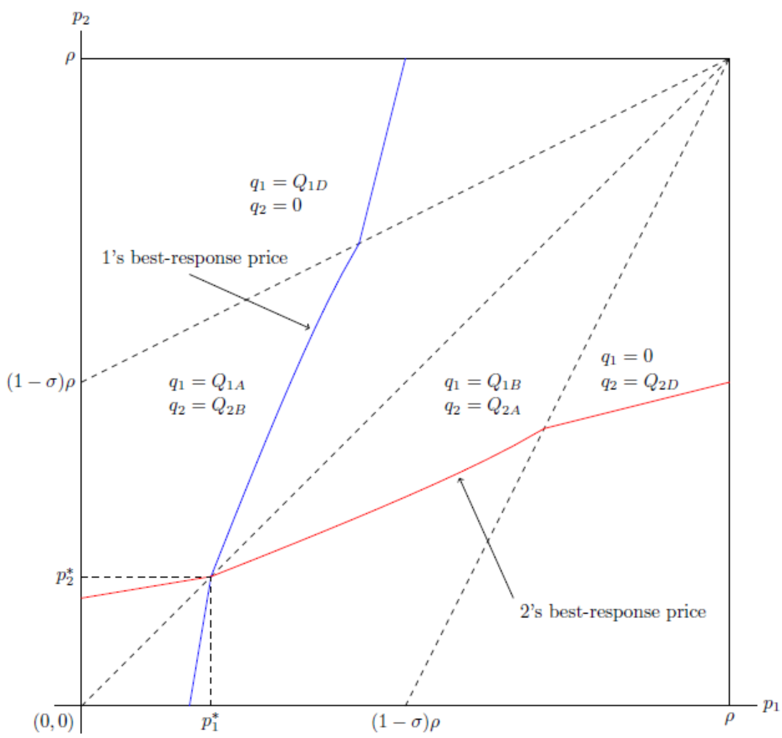


Figure 6: Price best-response functions.

given in Table 2.[8] Expressions for firm 2's best-response price have forms corresponding to those of firm 1. The price best-response curves are shown in Figure 6.

4.4 Equilibrium

Given the underlying symmetry of the model, we expect the best-response curves to intersect where prices are the same, as shown in Figure **??**. All sales of variety 2 in region A are to individuals who purchase both varieties. All sales of variety 1 in region B are to individuals who purchase both varieties. On the diagonal line in Figure **??**, the boundary between region A and region B, all sales of both varieties are to individuals who purchase both varieties.

[8]If $p_2 > (1-\sigma)\,\rho$, that is, if p_2 is outside the low-price range), firm 1's best response is to set the highest price it can without making demand for variety 2 positive. This price is given in row D of the table.

p	$\frac{1-\sigma}{3-\sigma}\rho$
Q	$\frac{2\rho N}{(1+\sigma)(3-\sigma)^2}$
π	$\frac{2(1-\sigma)\rho^2 N}{(1+\sigma)(3-\sigma)^3}$
CS	$\frac{8}{3}\frac{\rho^2 N}{(1+\sigma)(3-\sigma)^3}$

Table 3: Duopoly equilibrium prices, outputs and payoffs per firm, and consumers' surplus, linear individual demands.

That is, at identical prices, no one in the population buys one variety and not the other: an individual either buys both varieties or neither.

Duopoly equilibrium values for the individual S-D-V demands are shown in Table 3. Equilibrium comparative statics — for example, $\frac{\partial p}{\partial \sigma} < 0$ — are as expected.

In the aggregate S-D-V specification, duopoly equilibrium price is

$$p_{ag\ SDV} = \frac{1-\sigma}{2-\sigma}\rho > \frac{1-\sigma}{3-\sigma}\rho. \tag{21}$$

With elastic individual demand, equilibrium price is closer to marginal cost than the S-D-V representative consumer specification suggests.

5 Vertical as well as Horizontal Differentiation

Equilibrium in the horizontal product differentiation model developed in the previous section is symmetric. This aspect of equilibrium changed if there is vertical as well as horizontal product differentiation.

To see this, modify the individual utility function (11) by making the individual reservation price for variety 1 greater than the reservation price for variety 2. The utility function of an individual of type r becomes

$$U = m + \chi r q_1 + r q_2 - \frac{1}{2}\left(q_1^2 + 2\sigma q_1 q_2 + q_2^2\right), \tag{22}$$

where $\chi \geq 1$ parameterizes variety 1's quality relative to that of variety 2.

The individual's problem is

$$\max_{q_1,q_2} U = m + \chi r q_1 + r q_2 - \frac{1}{2}\left(q_1^2 + 2\sigma q_1 q_2 + q_2^2\right).$$

such that

$$m + p_1 q_1 + p_2 q_2 \leq y.$$

Proceeding as in the previous model, if demand for both varieties is positive, individual demands are

$$q_1 = \frac{\chi - \sigma}{1 - \sigma^2} \left(r - \frac{p_1 - \sigma p_2}{\chi - \sigma} \right) \tag{23}$$

$$q_2 = \frac{1 - \sigma \chi}{1 - \sigma^2} \left(r - \frac{p_2 - \sigma p_1}{1 - \sigma \chi} \right). \tag{24}$$

If demand for variety 2 is zero, demand for variety 1 is

$$q_1 = \chi r - p_1, \tag{25}$$

for $p_1 \leq \chi r$, and otherwise zero.

If demand for variety 1 is zero, demand for variety 2 is given by (17).

Demand regions in (p_1, r)-space for fixed p_2 are shown in Figure 7. The configuration of demand regions shown in the figure is valid for low $p_2 \leq (1 - \sigma\chi)\,\rho$, which defines the low-$p_2$ case in the presence of vertical product differentiation.

In preparation.

6 Conclusion

In this paper, I derive expressions for differentiated-good duopoly aggregate demands if there is linear individual demand for variety of the S-D-V form. In symmetric equilibrium, individuals either buy both varieties or neither. Equilibrium aggregate demand is a quadratic function of the difference between the maximum reservation price ρ and a weighted difference in prices, the weight being the product-differentiation parameter. Equilibrium duopoly prices with elastic individual demand are lower than equilibrium prices in the S-D-V aggregate demand model, keeping other aspects of the two specifications the same.

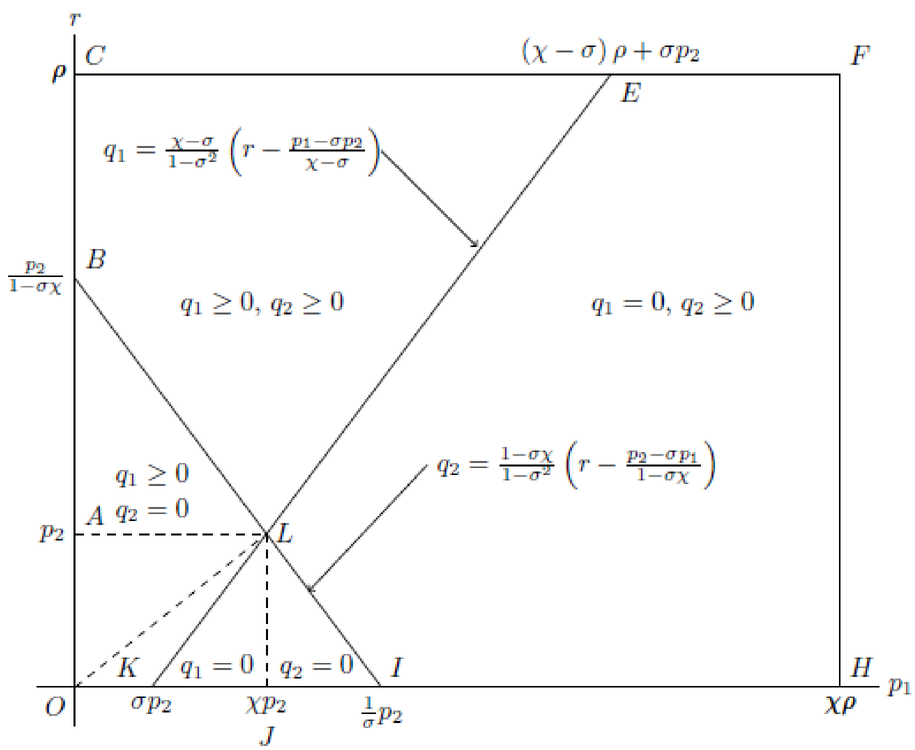


Figure 7: Nature of demand by region in (p_1, r)-space ($\rho = 8$, $\sigma = \frac{1}{2}$, $\chi = \frac{5}{4}$, $p_2 = 2$).

7 Appendix I

7.1 Competition

If there is constant marginal cost c, then from (5), long-run competitive equilibrium output is

$$Q = \frac{1}{2}\frac{N}{\rho}\left(\rho - c\right)^2. \tag{26}$$

At price p the consumer surplus of an individual of type r is

$$\frac{1}{2}\left(r - p\right)^2 = \frac{1}{2}q_r^2\left(p\right).$$

Consumers' surplus of all individuals who purchase at price p is then

$$\frac{N}{\rho}\int_{r=p}^{r=\rho}\left(\frac{1}{2}\right)\left(r - p\right)^2 dr = \frac{1}{6}\frac{N}{\rho}\left(\rho - p\right)^3, \tag{27}$$

and for competitive equilibrium price c, consumers' surplus is

$$CS_c = \frac{1}{6}\frac{N}{\rho}\left(\rho - c\right)^3. \tag{28}$$

7.2 Monopoly

The monopoly objective function is

$$\pi_m = \left(\rho - c - \sqrt{2\frac{\rho}{N}Q}\right)Q. \tag{29}$$

The first-order condition to maximize π_m is

$$\rho - c - \sqrt{2\frac{\rho}{N}Q} - \frac{1}{2}\sqrt{2\frac{\rho}{N}Q} = 0. \tag{30}$$

(30) implies that for output Q_m satisfying the first-order condition, monopoly profit is

$$\pi_m = \frac{1}{2}\sqrt{2\frac{\rho}{N}}Q_m^{3/2}. \tag{31}$$

From the first-order condition, monopoly output is

$$Q_m = \frac{2}{9}\frac{N}{\rho}(\rho - c)^2. \tag{32}$$

Then monopoly price and profit are

$$p_m = c + \frac{1}{3}(\rho - c) \tag{33}$$

and

$$\pi_m = \frac{2}{81}\frac{N}{\rho}(\rho - c)^3, \tag{34}$$

respectively.

Using (27) and (33), consumers' surplus under monopoly is

$$\frac{1}{6}\frac{N}{\rho}\left[\rho - c - \frac{1}{3}(\rho - c)\right]^3 = \frac{4}{81}\frac{N}{\rho}(\rho - c)^3. \tag{35}$$

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