Dynamic strategic corporate finance: A tug of war with financial frictions*

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Abstract

This paper develops and analyzes a benchmark theoretical setting that integrates the basic corporate finance insights about the impact of financial frictions on investment under perfect competition with the literature on dynamic industrial organization that implicitly assumes a perfect financial market. We highlight three key findings. First, we find that financial frictions generally lead firms to charge lower prices in equilibrium. Second, we find that financial frictions often slow down industry monopolization. In this sense, both of these findings show that financial frictions can act as a pro-competitive force. Finally, and perhaps most surprisingly, we find that even if financial frictions lead to a more concentrated industry in the long run, this often involves higher investment along the way, even though access to external funds is now costly.

Keywords: Price and Investment Competition, Financial Frictions

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1 Introduction

This paper develops and analyzes a model to understand how financial market imperfections impact dynamic strategic interactions between firms and industry dynamics. Our main goal is to construct a benchmark theoretical setting that integrates the basic corporate finance insights about the impact of financial frictions on investment (see Strebulaev & Whited (2011) for a survey) with the literature on dynamic industrial organization. This large and active literature (see Doraszelski & Pakes (2007) for an already outdated survey of the literature following Ericson & Pakes (1995)) implicitly assumes a perfect financial market in which firms can obtain financing for any NPV-positive investment project, an assumption that may not be appropriate for many real-world firms. At the same time, most of the vast literature on financial market imperfections abstracts from the strategic interactions that are at the forefront of dynamic industrial organization.

While still in the early, exploratory stage, our paper already shows that bridging these two literatures is a fruitful undertaking. It is widely believed that financial market imperfections exacerbate asymmetries between firms over time, in extreme cases allowing one firm to dominate the industry. This belief rests on the intuition that a firm that is ahead and already has a better product to offer costumers earns higher current profits and is thus better able to finance the investments required to further improve its product. Our preliminary results, however, show that this belief can be overturned in an environment that combines financial market imperfections with dynamic strategic interactions.

We highlight three key results. First, we find that financial frictions generally lead firms to charge lower prices in equilibrium. Second, we find that financial frictions often slow down industry monopolization. In this sense, both of these findings show that financial frictions can act as a pro-competitive force. Finally, and perhaps most surprisingly, we find that even if financial frictions lead to a more concentrated industry in the long run, this often involves higher investment along the way, even though access to external funds is now costly.

Related literature. Cooley & Quadrini (2001), Gomes (2001), Albuquerque & Hopenhayn (2004), and Clementi & Hopenhayn (2006) study the impact of financial frictions in the context of perfectly competitive industry models a la Hopenhayn (1992). In these economies, there is a continuum of firms and their policies have no impact on their rivals so there is no room for the strategic interactions among them. Hence they are not well suited to understand the impact of changing degrees of industry competition on firm level investment. All of them conclude that tightening financial frictions leads to lower investment.

The focus of our paper is shared by an older literature that studies the impact of debt and limited liability on product market competition in static models (Brander & Lewis 1986) or repeated games (Maksimovic 1988). Strategic interactions are similarly at the forefront of models of predatory pricing (Telser 1966, Bolton & Sharfstein 1990).

Absent financial frictions, our model is similar to Budd, Harris & Vickers (1993). Recent work by Liu, Mian & Sufi (2019) also borrow model elements from Budd et al. (1993) to investigate the impact of low interest rates on industry concentration and productivity growth.

Paper structure. Section 2 presents our model. Section 3 describes our main findings.

2 Model

We consider an infinite-horizon dynamic stochastic game between two forward-looking firms that discount future payoffs using the discount factor $\beta \in [0,1)$. Firms compete with one another through their pricing, p_1 and p_2, and investment decisions, x_1 and x_2. We assume that in every period firms make their decisions in two separate stages. They make pricing decisions first and these are then followed by their investment decisions.

The state space is summarized by a single state variable $\omega \in \Omega = \{-L, -L+1, \ldots, L\}$. We interpret ω as the state of competition: if $\omega > 0$, then firm 1 is the leader and firm 2 is the follower; if $\omega < 0$, then firm 1 is the follower and firm 2 is the leader; and if $\omega = 0$, then the two firms compete head-to-head. The size of the competitive advantage that the leader enjoys over the follower is $|\omega|$.[1]

2.1 Product Market Competition

We assume both firms sell a similar, but differentiated, good to consumers whose demand for good 1 is given by the expression

$$M \frac{1}{1 + \exp\left(\frac{-g(\omega)+\alpha(p_1-p_2)}{\nu}\right)}$$

where $M > 0$, $\alpha > 0$, $\nu > 0$ denote the overall market size, price sensitivity and the degree of horizontal product differentiation of the products. As $\nu \to 0$, goods become homogeneous, thus intensifying price competition between the two firms. [2]

The function $g(\omega)$ maps the state of competition into consumers' valuation and demand. While ω can reflect many potential sources of competitive advantages over competitors, such as branding or network effects, we refer to it here as simply the firm's overall productive capacity. Our assumption is that a firm's decisions can be completely summarized by its relative competitive advantage, ω.

We parameterize

$$g(\omega) = \begin{cases} \tau_0 \frac{\omega}{L} & \text{if} \quad \omega \geq 0, \\ \frac{\omega}{L} & \text{if} \quad \omega < 0, \end{cases} \tag{1}$$

where $\tau_0 \in [0,1]$ is a handicap parameter. Note that $g(-L) = -1$, $g(0) = 0$, and $g(L) = \tau_0$. A smaller value of τ_0 therefore imposes a disadvantage on the leader. At the extreme of $\tau_0 = 0$, $g(\omega) = 0$ for all $\omega \geq 0$ and the size of the competitive advantage that the leader enjoys over the follower is irrelevant for consumers' valuation and demand.

The aggregate demand for firm 1 and 2 is given by

$$M \frac{1}{1 + \exp\left(\frac{-g(\omega)+\alpha(p_1-p_2)}{\nu}\right)} + M \frac{1}{1 + \exp\left(\frac{-g(-\omega)-\alpha(p_1-p_2)}{\nu}\right)}$$

[1] Budd et al. (1993) study a similar model with a one-dimensional state space but assume that the state space is continuous but without financing frictions.

[2] The demand for good 2 is $M/1 + \exp\left((-g(-(\omega_1 - \omega_2)) - \alpha(p_1 - p_2))/\nu\right)$.

$$= M\frac{2+\exp\left(\frac{-g(\omega)+\alpha(p_1-p_2)}{\nu}\right)+\exp\left(\frac{-g(-\omega)-\alpha(p_1-p_2)}{\nu}\right)}{1+\exp\left(\frac{-g(\omega)+\alpha(p_1-p_2)}{\nu}\right)+\exp\left(\frac{-g(-\omega)-\alpha(p_1-p_2)}{\nu}\right)+\exp\left(\frac{-g(\omega)-g(-\omega)}{\nu}\right)}$$

and equals M if and only if $g(\omega) = -g(-\omega)$ or, equivalently, if and only if $\tau_0 = 1$. Aggregate demand falls short of M when $\tau_0 < 1$.

The per period profit of firm i is

$$\pi_i(\omega, p_1, p_2),$$

where $\pi_1(\omega, p_1, p_2)$ is increasing in ω, $\pi_2(\omega, p_1, p_2)$ is decreasing in ω, and p_1 and p_2 are the prices charged by the two firms. If we further impose symmetry, then the profit of firm 2 is related to the profit of firm 1 by

$$\pi_2(\omega, p_1, p_2) = \pi_1(-\omega, p_2, p_1).$$

Given the expression for demand, profits can be parameterized as

$$\pi_1(\omega, p_1, p_2) = M\frac{1}{1+\exp\left(\frac{-g(\omega)+\alpha(p_1-p_2)}{\nu}\right)}(p_1 - c), \quad (2)$$

$$\pi_2(\omega, p_1, p_2) = M\frac{1}{1+\exp\left(\frac{-g(-\omega)-\alpha(p_1-p_2)}{\nu}\right)}(p_2 - c), \quad (3)$$

where $c \geq 0$ is the marginal cost of production.

2.2 Static Nash equilibrium and normalizations

A static Nash equilibrium is a solution to the system of equations

$$\frac{\partial \pi_1(\omega, p_1, p_2)}{\partial p_1} = 0 \implies \exp\left(\frac{g(\omega) - \alpha(p_1 - p_2)}{\nu}\right) + 1 - (p_1 - c)\frac{\alpha}{\nu} = 0, \quad (4)$$

$$\frac{\partial \pi_2(\omega, p_1, p_2)}{\partial p_2} = 0 \implies \exp\left(\frac{g(-\omega) + \alpha(p_1 - p_2)}{\nu}\right) + 1 - (p_2 - c)\frac{\alpha}{\nu} = 0. \quad (5)$$

A number of normalizations enable us to reduce the parameter space:

Proposition 1 *Let $(p_1^\circ(\omega), p_2^\circ(\omega))$ be a static Nash equilibrium in state ω for market size $M = 1$, price sensitivity $\alpha = 1$, and marginal cost $c = 0$ and $\pi_i^\circ(\omega, p_1^\circ(\omega), p_2^\circ(\omega))$ the associated profit of firm i. Then $\left(p_1^N(\omega) = \frac{1}{\alpha}p_1^\circ(\omega) + c, p_2^N(\omega) = \frac{1}{\alpha}p_2^\circ(\omega) + c\right)$ is a static Nash equilibrium in state ω for market size $M > 0$, price sensitivity $\alpha > 0$, and marginal cost $c \geq 0$ and the associated profit of firm i is $\pi_i(\omega, p_1^N(\omega), p_2^N(\omega)) = \frac{M}{\alpha}\pi_i^\circ(\omega, p_1^\circ(\omega), p_2^\circ(\omega))$.*

Proof. Plug $\left(p_1^N(\omega), p_2^N(\omega)\right)$ into equations (4) and (5) that define a static Nash equilibrium in state ω and simplify. Then plug into equations (2) and (3).[3] ■

This proposition implies that we can normalize $\alpha = 1$ and $c = 0$ without loss of generality. In the remainder of the paper we thus restrict attention to the remaining demand parameters to explore are M, ν, and τ_0.

[3] As a corollary note that due to $g(0) = 0$, we have $p_i^\circ(0) = 2\nu$ and $\pi_i^\circ(0, p_1^\circ(0), p_2^\circ(0)) = \nu$ in a static Nash equilibrium in state $\omega = 0$.

2.3 Investment and Financing

By investing, a firm builds up its productive capacity and increases its relative advantage over its rival. As a result, investment choices will impact future demand.

Formally, we assume the law of motion for the state in this industry obeys

$$\omega' = \max\{-L, \min\{L, \omega + x_1 - x_2\}\}, \tag{6}$$

where $x_1 \in \{0,1\}$ and $x_2 \in \{0,1\}$ are the (binary) investment decisions of the two firms. Effectively, investment allows a firm to gain an advantage by moving the state to "its" side as in a tug of war.

Firm-level investment opportunities evolve stochastically over time. We assume that in each period a firm has the opportunity to expand its productive capacity by paying a one time cost $F_0 + \theta_i$, where θ_i is a privately observed random variable with probability density function $\psi(\cdot)$ and cumulative distribution function $\Psi(\cdot)$. Accordingly, θ_i is too high, the firm's current investment opportunities are poor and it will optimally choose not to invest. We further assume firm i observes θ_i only after its pricing decisions are made.

Crucially, in our model capital markets are not frictionless. If the firms current profits, π_i, are not sufficiently large, the firm must tap capital markets and incur additional funding costs. Gomes (2001) discusses several specific examples where access to external debt or equity issues give rise to a cost of external finance that is both increasing and weakly convex in the required amount of funding $F_0 - \pi_i$. Gomes, Yaron & Zhang (2006) provide more micro foundations for this cost function under very general cases of external funding. [4].

Thus, the overall cost of investing depends on the specific form of financing, with internal funds being cheaper than external ones. We parameterize their observable component as:

$$F(\pi_i) = F_0 + \zeta \max\{0, F_0 - \pi_i\}^{\kappa}, \tag{7}$$

where $\zeta \geq 0$ governs the severity of the financial frictions and $\kappa \in \mathbb{N}$ determines their smoothness. Note that financial frictions are absent if $\zeta = 0$. This implies that:

$$\begin{aligned} F'(\pi_i) &= -\kappa\zeta \max\{0, F_0 - \pi_i\}^{\kappa-1}, \\ F''(\pi_i) &= \kappa(\kappa-1)\zeta \max\{0, F_0 - \pi_i\}^{\kappa-2}. \end{aligned}$$

In what follows we set $\kappa = 3$ thus ensuring that $F(\pi_i)$ is twice continuously differentiable.

We further assume that the cost shock θ_i has mean zero and variance σ^2 and, for technical reasons, that $\psi'(\cdot)$ is continuous. In what follows we consider the case where we, $\theta_i \sim N(0, \sigma^2)$. Appendix A provides additional details and discusses the case where θ_i is governed by a $Beta(3,3)$ distribution.

Without loss of generality, we set $F_0 = 1$ in what follows. This corresponds to an appropriate choice of monetary units and leaves us with σ and ζ as the key cost parameters to explore.

[4]For a discussion of the broad application of these costs functions in corporate finance see Stein (2003).

2.4 Optimal investment strategies.

The dynamic stochastic game uses a two-stage setup: in each period, firms first decide on prices and then on investments. In what follows $V_i(\omega)$ is the beginning-of-period value function of firm i that is relevant when setting price and $U_i(\omega)$ the middle-of-period value function that is relevant when deciding on investment after prices have already been set.

Investment strategies. Overloading notation, we let $U_i(\omega, \theta_i)$ denote the value function of firm i in state ω *after* it has observed θ_i. Similarly, we let $x_i(\omega, \theta_i) \in \{0, 1\}$ denote the actual investment choice of firm i in state ω and use $x_i(\omega) \in [0, 1]$ to denote the probability that firm i invests before observing θ_i.

The Bellman equations of firms 1 and 2 are

$$\begin{aligned} U_1(\omega, \theta_1) = \max\Big\{ & - F(\pi_1) - \theta_1 + \beta \left[V_1(\omega^+)(1 - x_2(\omega)) + V_1(\omega) x_2(\omega)\right], \\ & \beta \left[V_1(\omega)(1 - x_2(\omega)) + V_1(\omega^-) x_2(\omega)\right] \Big\}, \end{aligned} \tag{8}$$

$$\begin{aligned} U_2(\omega, \theta_2) = \max\Big\{ & - F(\pi_2) - \theta_2 + \beta \left[V_2(\omega^-)(1 - x_1(\omega)) + V_2(\omega) x_1(\omega)\right], \\ & \beta \left[V_2(\omega)(1 - x_1(\omega)) + V_2(\omega^+) x_1(\omega)\right] \Big\}, \end{aligned} \tag{9}$$

where π_i is shorthand for $\pi_i(\omega, p_1(\omega), p_2(\omega))$ and we define $\omega^+ = \min\{L, \omega + 1\}$ and $\omega^- = \max\{-L, \omega - 1\}$.

The optimal investment strategy for firm 1, is characterized by a simple cutoff rule, where $x_1(\omega, \theta_1) = 1$ if and only if

$$\theta_1 + F(\pi_1) \leq \beta \left[\left(V_1(\omega^+) - V_1(\omega)\right)(1 - x_2(\omega)) + \left(V_1(\omega) - V_1(\omega^-)\right) x_2(\omega)\right].$$

Here the right side of the equation is effectively firm 1's marginal q as in Hayashi (1982). The implied investment probability is then

$$x_1(\omega) = \Psi\left(-F(\pi_1) + \beta \left[\left(V_1(\omega^+) - V_1(\omega)\right)(1 - x_2(\omega)) + \left(V_1(\omega) - V_1(\omega^-)\right) x_2(\omega)\right]\right), \tag{10}$$

The direct impact of an increase in financial costs is to raise the marginal cost of investment and reducing investment probabilities, $x_1(\omega)$. As Gomes (2001) points out, an increase in $F(\pi)$ indirectly impacts investment by also changing the value function $V_1(\omega)$. Importantly however, in this paper strategic considerations, create a novel transmission mechanism for the effect of financial frictions on firm level investment because we must now also account for their impact on firm's 2 optimal strategy, $x_2(\omega)$.

The investment decision for firm 2 can be defined analogously as

$$x_2(\omega) = \Psi\left(-F(\pi_2) + \beta \left[\left(V_2(\omega^+) - V_2(\omega)\right)(1 - x_1(\omega)) + \left(V_2(\omega) - V_2(\omega^-)\right) x_1(\omega)\right]\right), . \tag{11}$$

In general there may be multiple solutions to the system of equations (10) and (11).

Necessity and sufficiency. Substituting the optimal investment decision into the Bellman equation (8) of firm 1 and integrating both sides with respect to θ_1 yields

$$U_1(\omega) = -F(\pi_1)x_1(\omega) - x_1(\omega)E\left[\theta_1|\theta_1 \le \Psi^{-1}(x_1(\omega))\right]$$
$$+\beta\left[V_1(\omega^+)x_1(\omega)(1-x_2(\omega))\right.$$
$$\left.+V_1(\omega)(1-x_1(\omega)-x_2(\omega)+2x_1(\omega)x_2(\omega)) + V_1(\omega^-)(1-x_1(\omega))x_2(\omega)\right], \tag{12}$$

where

$$x_1(\omega)E\left[\theta_i|\theta_i \le \Psi^{-1}(x_1(\omega))\right] = \int_{-\infty}^{\Psi^{-1}(x_i(\omega))} \theta_1 d\Psi(\theta_1) \tag{13}$$

and $E\left[\theta_1|\theta_1 \le \Psi^{-1}(x_1(\omega))\right]$ is the expectation of the cost shock θ_1 conditional on firm 1 investing.

Using Leibniz's rule to differentiate $U_1(\omega)$ we can show that

$$\frac{\partial^2 U_1(\omega)}{\partial x_1^2} = -\frac{1}{\psi\left(\Psi^{-1}(x_1(\omega))\right)} < 0.$$

so that equation (10) is therefore necessary and sufficient for a maximum for firm 1. A similar result applies for firm 2's optimality condition.

Similarly, we can define firm 2's Bellman equation at the investment stage as

$$U_2(\omega) = -F(\pi_2)x_2(\omega) - x_2(\omega)E\left[\theta_2|\theta_2 \le \Psi^{-1}(x_2(\omega))\right]$$
$$+\beta\left[V_2(\omega^+)x_2(\omega)(1-x_1(\omega))\right.$$
$$\left.+V_2(\omega)(1-x_2(\omega)-x_1(\omega)+2x_2(\omega)x_1(\omega)) + V_2(\omega^-)(1-x_2(\omega))x_1(\omega)\right], \tag{14}$$

Strategic complementarities. Equation (10) defines the best response of firm 1 in state ω in the investment phase given continuation play. It implies that

$$\frac{\partial x_1(\omega)}{\partial x_2(\omega)} \propto \left[(V_1(\omega^+) - V_1(\omega)) - (V_1(\omega) - V_1(\omega^-))\right]$$

Hence, the best response of firm 1 to firm 2's investment decision is upward sloping, so that optimal investment are strategic complements if and only if the beginning-of-period value function of firm 1 is locally concave n state ω. Conversely, investment decisions are strategic substitutes if the value functions are locally convex.

Similarly, the best response of firm 2 defined in equation (11) is upward (downward) sloping so that its and its rival's investment are strategic complements (substitutes) if and only if the beginning-of-period value function of firm 2 is locally concave (convex) in state ω. In a symmetric equilibrium (formally defined below), this is equivalent to the beginning-of-period value function of firm 1 being locally concave (convex) in state $-\omega$.

Pricing strategies. We now turn towards the beginning of period when pricing strategies are determined At this stage, the Bellman equations of firms 1 and 2 are

$$V_1(\omega) = \max_{p_1} \pi_1(\omega, p_1, p_2(\omega)) + U_1(\omega), \tag{15}$$

$$V_2(\omega) = \max_{p_2} \pi_2(\omega, p_1(\omega), p_2) + U_2(\omega). \tag{16}$$

where $U_1(\omega)$ and $U_2(\omega)$ are defined in equations (12) and (14).

Recognizing the dependence of $x_1(\omega)$ and $x_2(\omega)$ on $\pi_1(\omega, p_1(\omega), p_2(\omega))$ and $\pi_2(\omega, p_1(\omega), p_2(\omega))$, the Nash equilibrium in prices $p_1(\omega)$ and $p_2(\omega)$ is the solution to the system of equations

$$\frac{\partial \pi_1(\omega, p_1(\omega), p_2(\omega))}{\partial p_1} \left(1 - F'\left(\pi_1(\omega, p_1(\omega), p_2(\omega))\right) x_1(\omega) + \frac{\partial U_1(\omega)}{\partial x_2} \frac{\partial x_2(\omega)}{\partial \pi_1}\right) + \frac{\partial \pi_2(\omega, p_1(\omega), p_2(\omega))}{\partial p_1} \frac{\partial U_1(\omega)}{\partial x_2} \frac{\partial x_2(\omega)}{\partial \pi_2} = 0, \tag{17}$$

$$\frac{\partial \pi_2(\omega, p_1(\omega), p_2(\omega))}{\partial p_2} \left(1 - F'\left(\pi_2(\omega, p_1(\omega), p_2(\omega))\right) x_2(\omega) + \frac{\partial U_2(\omega)}{\partial x_1} \frac{\partial x_1(\omega)}{\partial \pi_2}\right) + \frac{\partial \pi_1(\omega, p_1(\omega), p_2(\omega))}{\partial p_2} \frac{\partial U_2(\omega)}{\partial x_1} \frac{\partial x_1(\omega)}{\partial \pi_1} = 0. \tag{18}$$

where we have used the fact that $\frac{\partial U_i(\omega)}{\partial x_i} = 0$ from the envelope theorem.

There may be also multiple solutions to the system of equations (17) and (18). In contrast to equation (10), which is necessary and sufficient for a maximum for firm 1 in the investment phase, in the pricing phase equation (17) is necessary but in general not sufficient.[5]

2.5 Markov perfect equilibrium

Equilibrium We characterize the industry outcomes using the concept of Markov perfect equilibria. This is a solution to the system of equations consisting of the Bellman equations and optimality conditions in equations (10), (11), (12), (14), (15), (16), (17), and (18) for all $\omega \in \{-L, \ldots, L\}$. We focus on symmetric equilibria in which the value and policy functions of firm 2 are related to those of firm 1 by

$$\begin{aligned} V_2(\omega) &= V_1(-\omega), \quad & U_2(\omega) &= U_1(-\omega), \\ p_2(\omega) &= p_1(-\omega), \quad & x_2(\omega) &= x_1(-\omega). \end{aligned}$$

In a symmetric equilibrium, it suffices to compute the value and policy function for firm 1. We provide additional details on the resulting system of equations and its Jacobian in Appendix B. Because equations (17) and (18) are necessary but not sufficient, a solution is not necessarily an equilibrium. We thus check that for every computed solution there is no unilateral profitable deviation as detailed in Appendix C.

Equilibrium state dynamics. In equilibrium the law of motion for the state in equation (6) can be written as

$$\omega' = \begin{cases} \omega^- = \max\{-L, \omega - 1\} & \text{with prob.} \quad (1 - x_1(\omega))x_2(\omega), \\ \omega^+ = \min\{L, \omega + 1\} & \text{with prob.} \quad x_1(\omega)(1 - x_2(\omega)), \\ \omega & \text{with prob.} \quad 1 - x_1(\omega) - x_2(\omega) + 2x_1(\omega)x_2(\omega). \end{cases} \tag{19}$$

[5] Absent financial costs, $\pi_1(\omega, p_1, p_2)$ is strictly quasiconcave in p_1 (see Appendix B), ensuring sufficiency.

Equation (19) defines the $(2L+1) \times (2L+1)$ state-to-state transition probability matrix P of a Markov chain. We provide details on its limiting distribution μ^∞ in Appendix D.

Equation (19) implies

$$E[\omega'|\omega] = \begin{cases} -L + x_1(\omega)(1 - x_2(\omega)) & \text{if} \quad \omega = -L, \\ \omega + x_1(\omega) - x_2(\omega) & \text{if} \quad -L < \omega < L, \\ L - (1 - x_1(\omega))x_2(\omega) & \text{if} \quad \omega = L. \end{cases} \tag{20}$$

Hence, state is expected to increase at $\omega = -L$ and to decrease at $\omega = L$: the boundaries are repulsive in the terminology of Budd et al. (1993) although the degree of repulsion ($x_1(\omega)(1 - x_2(\omega)$ at $\omega = -L$ and $(1 - x_1(\omega))x_2(\omega)$ at $\omega = L$) is endogenous in our model. Away form the boundaries the dynamics of the state is linear on wether the difference in investment probabilities across firms. In particular ω is expected to increase over time whenever firm 1 is more likely to invest than firm 2.

3 Results

We are now ready to investigate the properties of our model. To do so we compute the solutions across several thousand parameterizations which are discussed in detail below. In most cases we compute multiple equilibria. For a few parameterizations no equilibria was found.

3.1 Parameterization.

Table 1 summarizes the range of values for the parameters of the model. As discussed above several of these can be normalized without loss of generality. The value of β has no real impact on our findings and is picked to imply a 5% real rate of return on investment, a plausible average between the historical average rates of return on equity and debt finance. The value of L is chosen to be 15, a value large enough so that it has no impact on our findings. The value of τ_0 is naturally bounded between 0 and 1.

The key parameters M, ν are not not naturally bounded from above. To rule out uninteresting parameterizations, we consider the static Nash equilibrium $(p_1^N(\omega), p_2^N(\omega))$ in state ω described in Section 2.2. From equation (7) we say that financial frictions matter if $\pi_1(\omega, p_1^N(\omega), p_2^N(\omega)) < F_0$; otherwise we say that financial frictions do not matter. We discard any parameterizations for which the financial frictions do not matter for any ω. This bounds the market size, M, and the degree of price competition, ν from above.

To construct a plausible lower bound for M we discard parameterizations for which the cost of external funds $\zeta \max\left\{0, F_0 - \pi_1(\omega, p_1^N(\omega), p_2^N(\omega))\right\}^\kappa$, evaluated at the static Nash equilibrium $(p_1^N(\omega), p_2^N(\omega))$ exceeds 90% of the expected investment outlay, F_0 for all ω. [6]

Finally, we bound σ from above by 0.5 to limit the probability that the actual project outlays $F_0 + \theta_i$ become negative.

[6] This is an extremely conservative value. The true "financing cost" would be the spread between the actual rate of borrowing (cost of using external funds) and the rate of time preference (the implicit rate of return on internal funds) which we set at 5% per year. A reasonably high empirical estimate of this spread ranges between 3% and 7% per year. Since total investment outlays also include real/depreciation costs, usually estimated at around 10% per year. A plausible ratio for the ratio of excess funding costs due to financial market imperfections and frictionless investment expenses is then $5/(5 + 10) = 1/3$.

	parameter	range	value/grid
state space:			
Maximum value of ω	L	$\mathbb{N}$	15
discounting:			
discount factor	β	$[0,1)$	0.95
product market competition:			
market size	M	$(0,\infty)$	10^{-2}, $10^{-1.9}$, $10^{-1.8}$, …, $10^{1.9}$, 10^{2}
price sensitivity	α	$(0,\infty)$	1 (normalization)
degree of horizontal product differentiation	ν	$(0,\infty)$	0.025, 0.075, 0.125,…, 1.925, 1.975
marginal cost	c	$[0,\infty)$	0 (normalization)
handicap	τ_0	$[0,1]$	0, 0.05, 0.1, …, 0.95, 1
investment:			
fixed cost	F_0	$(0,\infty)$	1 (normalization)
severity of financial frictions	ζ	$[0,\infty)$	1
smoothness of financial frictions	κ	$\mathbb{N}$	3
standard deviation of θ_i	σ	$(0,\infty)$	0.05, 0.1, 0.15, …, 0.45, 0.5

Table 1: Parameterization.

3.2 Baseline model without financial frictions

We have computed a total of 25,040 equilibria over a subset of 13,665 out of 13,794 parameterizations (see again Table 1). The parameterizations for which we have been unable to compute an equilibrium tend to involve high values of M, low values of ν, low values of τ_0, and/or low values of σ. The number of equilibria ranges between 0 and 3 for a given parameterization, with an average of 1.82 equilibria per parameterization. Multiple equilibria occur across the entire parameter space.

Absent financial frictions, $F(\pi_i) = F_0$ and $F'(\pi_i) = 0$ and $\frac{\partial x_1(\omega)}{\partial \pi_1} = \frac{\partial x_1(\omega)}{\partial \pi_2} = \frac{\partial x_2(\omega)}{\partial \pi_1} = \frac{\partial x_2(\omega)}{\partial \pi_2} = 0$ so that the optimal investment policies $x_1(\omega)$ and $x_2(\omega)$ do nmot depend on $\pi_1(\omega, p_1(\omega), p_2(\omega))$ and $\pi_2(\omega, p_1(\omega), p_2(\omega))$.

In this case the system of equations (17) and (18) that determines prices $p_1(\omega)$ and $p_2(\omega)$ reduces to the system of equations (4) and (5). Thus, $p_1(\omega) = p_1^N(\omega)$ and $p_2(\omega) = p_2^N(\omega)$, where $(p_1^N(\omega), p_2^N(\omega))$ is the static Nash equilibrium in state ω described in Section 2.2. Because the prices charged by the two firms do not impact the evolution of the industry, we view the associated profits $\left(\pi_1(\omega, p_1^N(\omega), p_2^N(\omega)), \pi_2(\omega, p_1^N(\omega), p_2^N(\omega))\right)$ as an exogenously given input into the Markov perfect equilibrium in what follows.

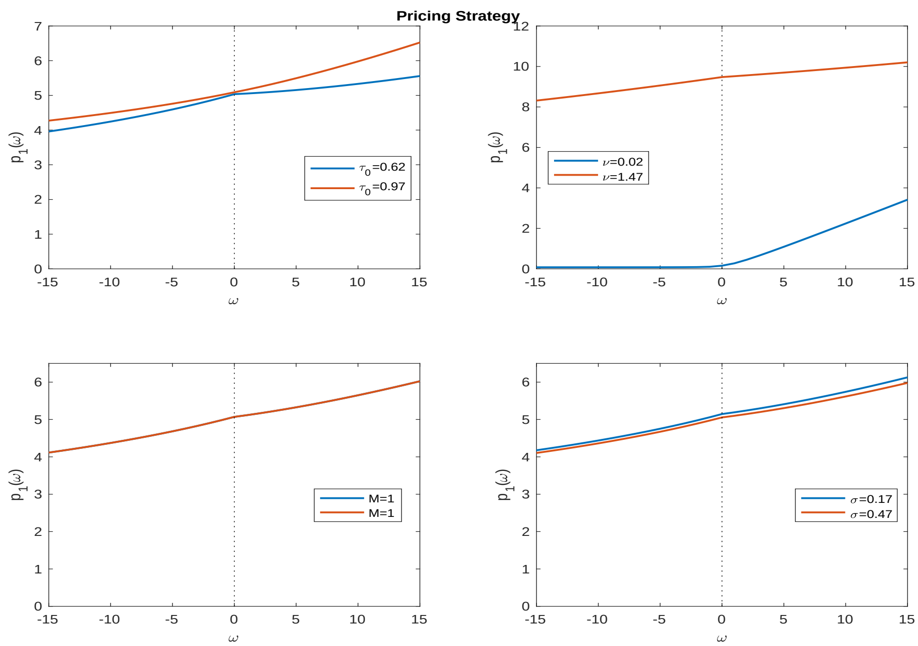
Pricing and investment strategies. Figure 1 shows the equilibrium pricing behavior of a firm across the state space for the baseline model without financial frictions. To do so, we average equilibrium policies across the various computed equilibria and parameterizations. Unsurprisingly, we find that leaders ($\omega > 0$) set higher prices than followers and prices are consistently increasing with the size of the lead.

The figure also illustrates how the pricing behavior changes for different parameter values.

The first panel shows that the equilibrium price function is "kinked" around $\omega = 0$ when τ_0 is low. Equation (1) implies that, in this case, the leader's advantage is significantly lessened and this reduction in demand translates in optimally setting lower prices.

The price policies are also quite sensitive to the degree of product differentiation ν. Prices are always lower when the firm's products are more homogenous (low ν). However, when differentiation is low, followers set very low prices while leaders tend to increase them quite rapidly as their advantage (ω) increases. The two bottom panels show that while prices are naturally higher when market size, M increases, they are largely insensitive to the value of σ which controls the volatility of investment costs but does not directly impact the demand function.

Figure 1: Pricing strategy - Baseline Model



Pricing strategy across alternative parameter values for the baseline model without financial frictions.

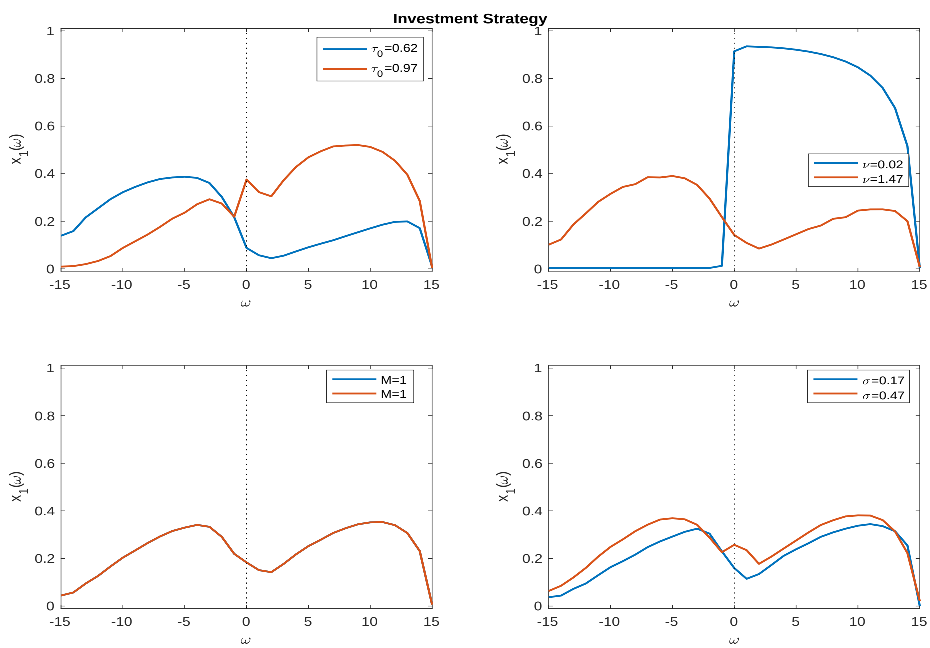
Figure 2 shows the equilibrium optimal investment behavior of a firm across the state space. As the figure shows the behavior of investment is much more complex. Generally leaders invest more than followers but only up to a point. In some cases followers essentially stop to invest altogether when their disadvantage becomes too large to overcome (very low values of ω). Leaders on the other hand often invest more to preserve their competitive advantage and their currently large profits. Importantly, investment generally spikes around

$\omega = 0$, suggesting the incentive to invest and gain an advantage increases significantly in a symmetric industry.

The first panel also shows that the equilibrium investment policy is much more sensitive to $\omega = 0$ when τ_0 is large. In this case leaders have a much stronger incentive to invest than followers. More investment by the leaders seemingly discourages followers who invest even less than when τ_0 is small.

Similarly, investment is very responsive to ω when ν is low and product substitution is very large. In this case leaders have a much greater incentive to preserve even a small competitive advantage and almost always invest with very high probability. This, in turn, seemingly discourages followers who almost never invest (recall that they are also charging extremely low prices and thus earning very low profits). By contrast, the two bottom panels show that the values of M and σ do not materially change our results. Investment naturally increases across the entire state space when M, and thus market size, is large. Investment is also generally larger when the volatility of investment costs increases except when ω is already very large.

Figure 2: Investment strategy - Baseline Model



Investment strategy across alternative parameter values for the baseline model without financial frictions.

Stationary Equilibria To examine the long run properties of our model we look at most likely long-run industry structure

$$\widehat{\omega}^{\infty} = \arg \max_{\omega \in \{0,\dots,L\}} \mu^{\infty}(\omega),$$

where again μ^{∞} is the limiting distribution of the Markov chain characterizing equilibrium state dynamics.

Joint payoff effect. Equation (20) implies that the dynamics of ω (away from the boundaries) is driven solely by the difference in investment probabilities

$$\begin{aligned} x_1(\omega) &= \Psi\left(-F_0 + A_1(1 - x_2(\omega)) + B_1 x_2(\omega)\right), \\ x_2(\omega) &= \Psi\left(-F_0 + A_2(1 - x_1(\omega)) + B_2 x_1(\omega)\right). \end{aligned}$$

where both $A_1 = \beta\left[V_1(\omega^+) - V_1(\omega)\right]$ and $B_1 = \beta\left[V_1(\omega) - V_1(\omega^-)\right]$ measure the slope of the beginning-of-period value function of firm 1 around state ω, while $A_2 = -\beta\left[V_2(\omega) - V_2(\omega^-)\right]$ and $B_2 = -\beta\left[V_2(\omega^+) - V_2(\omega)\right]$ capture *the negative of* the slope of the beginning-of-period value function of firm 2 around state ω.

Consequently, firm 1 tends to invest more than firm 2 in state ω if the slope of its value function around ω is larger than *the negative of* the slope of value function of firm 2 so that

$$\begin{aligned} A_1 = \beta\left[V_1(\omega^+) - V_1(\omega)\right] > B_2 = -\beta\left[V_2(\omega^+) - V_2(\omega)\right] &\iff V_1(\omega^+) + V_2(\omega^+) > V_1(\omega) + V_2(\omega), \\ B_1 = \beta\left[V_1(\omega) - V_1(\omega^-)\right] > A_2 = -\beta\left[V_2(\omega) - V_2(\omega^-)\right] &\iff V_1(\omega) + V_2(\omega) > V_1(\omega^-) + V_2(\omega^-). \end{aligned}$$

Similarly, the inequalities are reversed when firm 2 invests more than firm 1 at ω. It follows that the industry state tends to evolve into the direction where joint firm value $V_1(\omega) + V_2(\omega)$ is maximized.

Budd et al. (1993) study a similar model without financing frictions but with a continuous state space. They show that in equilibrium the industry tends to evolve in the direction where the joint payoff of the two firms is larger. As in our model this joint payoff combines both joint profits from product market competition and joint investment costs, so that there are two classes of effects. Budd et al. (1993) illuminate the various joint-cost effects through a combination of asymptotic expansions (around both myopic firms and infinitely noisy state-to-state transitions) and numerical analysis.[7]

To quantify the importance of this joint-payoff effect in our model we define the state where the joint payoff is largest, $\widehat{\omega}^V$, as[8]

$$\widehat{\omega}^{V} = \arg \max_{\omega \in \{0,\dots,L\}} V_1(\omega) + V_2(\omega)$$

and compare this with $\widehat{\omega}^{\infty}$.

[7] Also, in our discrete-time model, $x_1(\omega)$ in equation (10) depends on a weighted average of A_1 and B_1 where these weights, in turn, depend on $x_2(\omega)$. In the continuous-time model of Budd et al. (1993), the investment decision of firm 1 depends entirely on the slope of its value function.

[8] In a symmetric equilibrium, restricting attention to $\omega \in \{0, \dots, L\}$ instead of $\omega \in \{-L, \dots, L\}$ is without loss of generality.

		Joint-payoff effect			Joint-profit effect			Joint-profit net of joint-cost effect		
	$\widehat{\omega}^{\infty}$	$\widehat{\omega}^{V} < \widehat{\omega}^{\infty}$	$\widehat{\omega}^{V} = \widehat{\omega}^{\infty}$	$\widehat{\omega}^{V} > \widehat{\omega}^{\infty}$	$\widehat{\omega}^{\pi} < \widehat{\omega}^{\infty}$	$\widehat{\omega}^{\pi} = \widehat{\omega}^{\infty}$	$\widehat{\omega}^{\pi} > \widehat{\omega}^{\infty}$	$\widehat{\omega}^{\pi-F} < \widehat{\omega}^{\infty}$	$\widehat{\omega}^{\pi-F} = \widehat{\omega}^{\infty}$	$\widehat{\omega}^{\pi-F} > \widehat{\omega}^{\infty}$
0	5.12%		4.88%	0.24%		4.90%	0.22%		4.89%	0.23%
1	0.44%		0.43%	0.00%	0.43%		0.00%		0.43%	0.00%
2	0.06%		0.05%	0.01%	0.06%		0.00%		0.05%	0.01%
4	0.00%		0.00%		0.00%				0.00%	
15	94.38%	2.71%	91.67%		4.20%	90.18%		2.99%	91.39%	
	100.00%	2.71%	97.04%	0.25%	4.69%	95.08%	0.23%	2.99%	96.77%	0.24%

Table 2: Joint-payoff, joint-profit, and joint-profit net of joint-cost effects. Baseline model without financial frictions.

Table 2 summarizes the results. The first panel shows the distribution of $\widehat{\omega}^{\infty}$ over the 25,040 equilibria. 94.38% of equilibria lead to a maximally asymmetric industry structure in the long run for the parameterizations that we have so far explored in our numerical analysis. The second panel shows the incidence of equilibria with $\widehat{\omega}^{V} \lesseqqgtr \widehat{\omega}^{\infty}$. In 97.04% of equilibria, the most likely long-run industry structure maximizes the joint payoff so that this effect overwhelmingly determines the evolution of the industry.

Joint profit and cost effects. The value function is an equilibrium construct rather than a primitive. In contrast, absent financial frictions profits can be regarded as a primitive for the Markov perfect equilibrium. To further parse the joint-payoff effect, we define

$$\widehat{\omega}^{\pi} = \arg \max_{\omega \in \{0,\dots,L\}} \pi_1(\omega, p_1(\omega), p_2(\omega)) + \pi_2(\omega, p_1(\omega), p_2(\omega))$$

to be the state where the joint profit is largest. We compare $\widehat{\omega}^{\pi}$ to $\widehat{\omega}^{\infty}$ to quantify the joint-profit effect. Further, from equation (13), in expectation firm i's cost of investing in state ω conditional on it investing is

$$\begin{aligned} &E\left[F(\pi_i(\omega, p_1(\omega), p_2(\omega))) + \theta_i | \theta_i \leq \Psi^{-1}(x_i(\omega))\right] \\ &= F(\pi_i(\omega, p_1(\omega), p_2(\omega)))x_i(\omega) + \int_{-\infty}^{\Psi^{-1}(x_i(\omega))} \theta_i d\Psi(\theta_i). \end{aligned}$$

We define

$$\begin{aligned} \widehat{\omega}^{\pi-F} = \arg \max_{\omega \in \{0,\dots,L\}} &\pi_1(\omega, p_1(\omega), p_2(\omega)) - E\left[F(\pi_1(\omega, p_1(\omega), p_2(\omega))) + \theta_1 | \theta_1 \leq \Psi^{-1}(x_1(\omega))\right] \\ &+\pi_2(\omega, p_1(\omega), p_2(\omega)) - E\left[F(\pi_2(\omega, p_1(\omega), p_2(\omega))) + \theta_2 | \theta_2 \leq \Psi^{-1}(x_2(\omega))\right] \end{aligned}$$

to be the state where the joint profit net of the joint cost is largest. We compare $\widehat{\omega}^{\pi-F}$ to $\widehat{\omega}^{\infty}$ to quantify the joint-profit net of joint-cost effect.

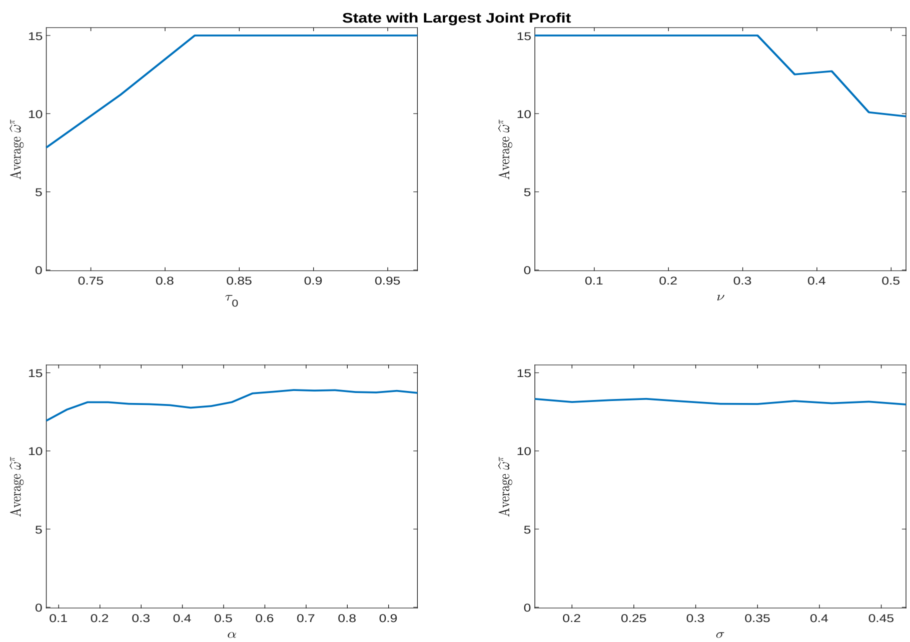
The last two panels in Table 2 summarize the results. The third panel shows that in 95.08% of equilibria, the most likely long-run industry structure also maximizes joint profits. Finally, the fourth panel shows that in 96.77% of equilibria, the most likely long-run industry structure maximizes joint profits net of joint investment expenses.

Figure 3 further illustrates the joint-profit effect across the various parameter combinations. As we can see, joint profits are generally maximized when the industry is more asymmetric. Only for low enough values of τ_0 or for high enough values of ν are joint profits higher in more symmetric industries. For example, for $\tau_0 = 0.72$ about half of the parameter combinations imply that the joint profit is maximized at $\omega = 0$ when the industry is perfectly symmetric. The same happens in approximately 2/3 of the parameter combinations when the degree of product differentiation is set at $\nu = 0.52$.

3.3 Model with costly external finance

We now turn to the full fledged model with costly external finance. We computed a total of 18,314 equilibria over a subset of 10,998 out of 13,794 parameterizations (see again Table 1). The parameterizations for which we have been unable to compute an equilibrium in

Figure 3: Maximal Joint Profit States



Average state for maximal joint profit across various parameter values for the baseline model.

models with financial frictions tend to involve high values of M, low values of ν, low values of τ_0, and/or low values of σ. Multiple equilibria occur across parameter space. The number of equilibria ranges between 0 and 5 for a given parameterization, with an average of 1.33 equilibria per parameterization.

To juxtapose the models with and without financial frictions and document how financial market imperfections impact the dynamic strategic interactions between firms and industry dynamics, we form all possible pairs of equilibria in the two models for each a given parameterization. This gives us 37,461 pairs of equilibria over a total of 10,907 parameterizations. To simplify the exposition, we will use the terminology equilibrium with (without) financial frictions as shorthand for the equilibrium of the corresponding models.

We say that financial frictions matter in an equilibrium if $\pi_1(\omega, p_1(\omega), p_2(\omega)) < F_0$ for some ω; otherwise we say that financial frictions do not matter in the equilibrium. We find that financial frictions matter in 16,718 out of 18,314 equilibria that we computed for the model with financial frictions. As a point of comparison, financial frictions matter (or rather would matter if they were present) in 23,387 out of computed 25,040 equilibria for the baseline model without financial frictions. Overall, we find that financial frictions matter in 33,791 out of 37,461 pairs of juxtaposed equilibria with and without financial frictions.

Pricing and investment strategies. Figure 4 compares the average equilibrium price and investment polices across the two versions of the model. These averages are computed across all equilibria and parameter values. As we can see, on average financial frictions lead to lower price and investment in equilibrium across the entire state space. A notable exception, however, occurs in state $\omega = 0$ where firms charge higher prices on average than in the baseline model without financial frictions. The pricing strategies are noticeably more "kinked" in the presence of financial frictions.

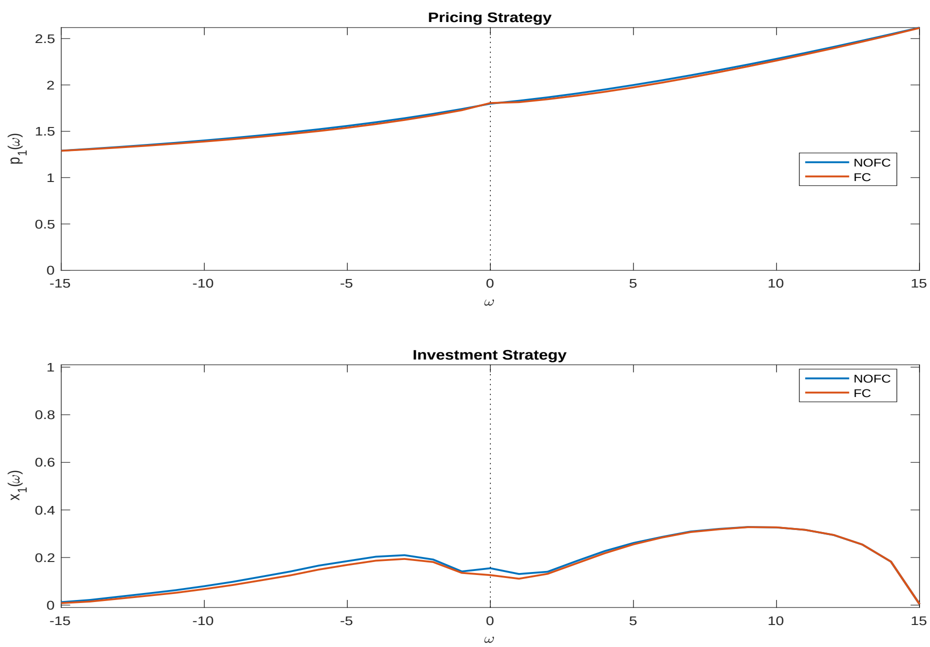
These patterns depend crucially on the values of the two key parameters, τ_0 and ν. As Figure 5 shows, when τ_0 is low, the equilibrium pricing strategies remain kinked but generally imply lower prices across the entire state space. Similarly prices are always lower in the equilibrium with financial frictions when the degree of product differentiation is small.

Long run industry structure. Table 3 provides initial evidence on how costly external finance impacts the long run structure of our industry. The first panel shows the distribution of $\widehat{\omega}^{\infty}$ over the 18,314 computed equilibria for the model with financial frictions. We see that 97.85% of equilibria lead to a maximally asymmetric industry structure in the long run.

The remaining panels investigate the role of maximal joint payoffs. The second panel shows that in 92.07% of equilibria, the most likely long-run industry structure maximizes the joint payoff. The third panel shows the incidence of equilibria with $\widehat{\omega}^{\pi} \lesseqgtr \widehat{\omega}^{\infty}$. We find that in 89.53% of equilibria, the most likely long-run industry structure also maximizes the joint profit. Finally, the fourth panel shows the incidence of equilibria with $\widehat{\omega}^{\pi-F} \lesseqgtr \widehat{\omega}^{\infty}$. In 91.68% of equilibria, the most likely long-run industry structure maximizes the joint profit net of the joint cost. Of course, in the model with financial frictions not even profits can be regarded as a primitive for the Markov perfect equilibrium.

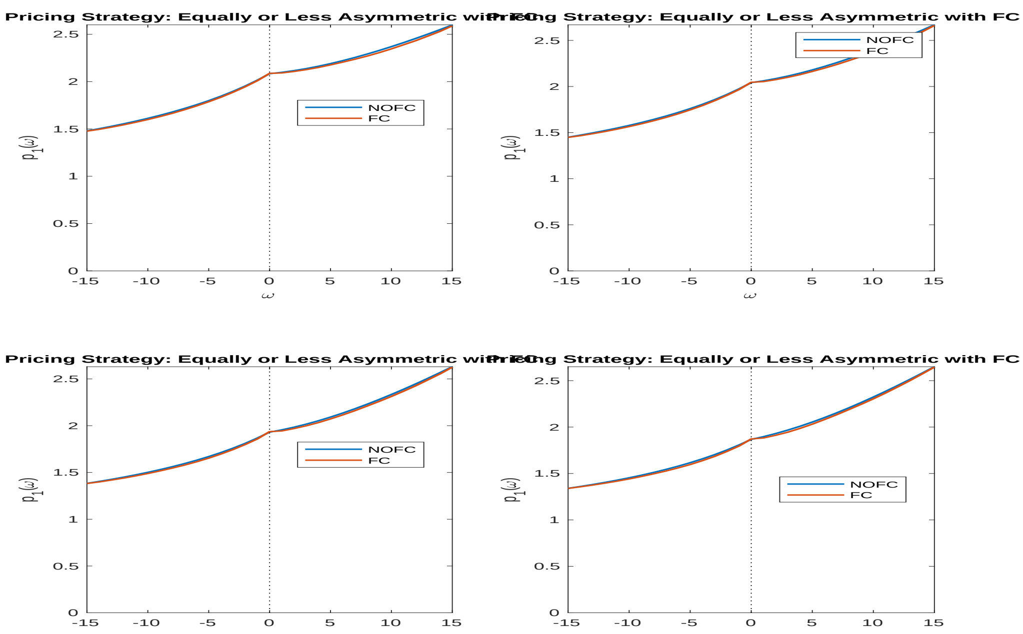
In comparison to the baseline model discussed above, financing frictions can disrupt industry evolution, preventing it, in some cases, from drifting towards the states that maximize

Figure 4: Pricing and investment strategies - with and without financial frictions



Average pricing and investment strategies across all parameter values for the baseline model and the model with costly external finance.

Figure 5: Pricing strategy - with and without financial frictions



Price strategy across different parameter values for the baseline model and the model with costly external finance.

joint payoffs. The joint-payoff effect however clearly remains a major force for the evolution of the industry.

Long run industry concentration. In 93.93% of pairs, the most likely long-run industry structure $\widehat{\omega}^{\infty}$ does not differ in the cases with and without financial frictions. In 4.91% of pairs, $\widehat{\omega}^{\infty}$ is smaller in the equilibrium without financial frictions than in the equilibrium with financial frictions so that financial frictions exacerbate long run asymmetries between firms. However in 1.27% of pairs, $\widehat{\omega}^{\infty}$ is larger in the equilibrium without financial frictions than in the equilibrium with financial frictions, meaning that in some cases financial frictions can actually increase long run competition.

Uli: do we have a table for this?

Figure 6 further illustrates how different parameter values generate differences in long run industry concentration levels. As before, we find that the two key parameters are the degree of product differentiation ν and the extent to which leaders are (dis)advantaged over followers. Only for low enough values of τ_0 or high enough values of ν do economies with financial frictions eventually lead to more concentrated industries. By contrast, when the degree of product differentiation is very low, the impact of financial frictions on industry concentration is virtually non-existent.

3.4 Financial frictions as a pro-competitive force

Speed of convergence. To better understand the role of financial frictions we next consider its impact on the speed of convergence to the limiting distribution μ^{∞}. To do this, we define the total variation distance between the transient distribution after T periods μ^T and the limiting distribution μ^{∞} as

$$\delta(\mu^T, \mu^{\infty}) = \frac{\sum_{\omega=-L}^{L} \left|\mu^T(\omega) - \mu^{\infty}(\omega)\right|}{2}.$$

A larger value of $\delta(\mu^T, \mu^{\infty})$ implies a slower speed of convergence to the long-run industry structure. To implement this distance in practice, we start from a perfectly symmetric industry at time 0 and set $T = 25$.

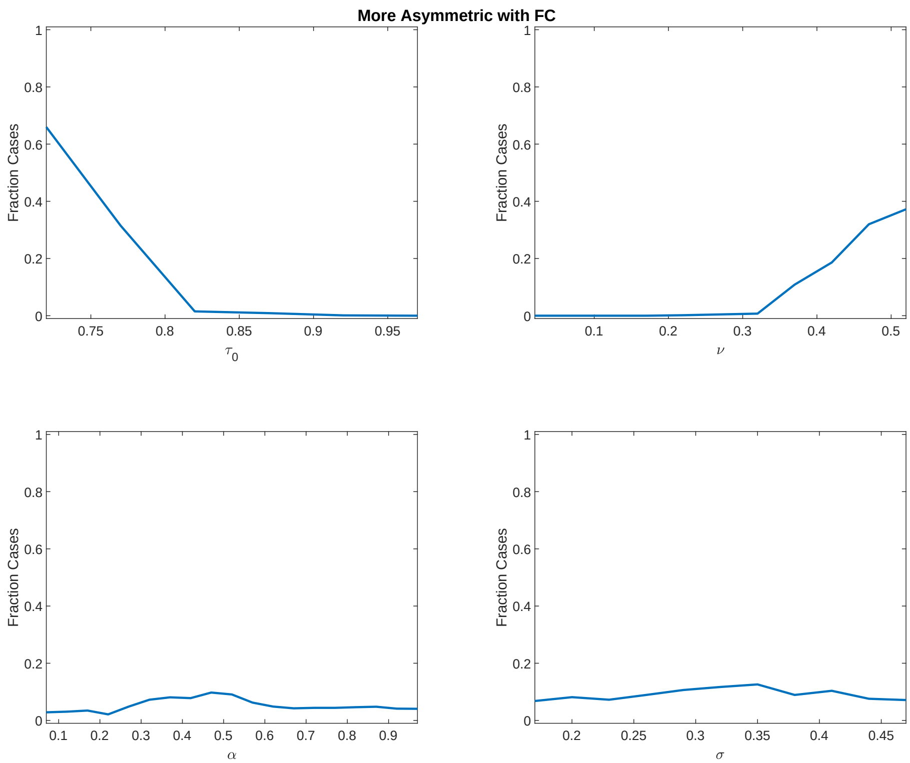
We find that the most likely industry structure $\widehat{\omega}^{25}$ is the same in the equilibria with and without financial frictions in 71.91% of pairs. In 10.21% of pairs, $\widehat{\omega}^{25}$ is smaller in the equilibrium without financial frictions than in the equilibrium with financial frictions. In 17.87% of pairs, $\widehat{\omega}^{25}$ is larger in the equilibrium without financial frictions than in the equilibrium with financial frictions. Comparing these percentages to the ones computed using the limiting distributions suggests that financial frictions can often act a pro-competitive force by slowing down the evolution of the industry, especially towards maximal asymmetry.

In 66.90% of pairs, $\delta(\mu^{25}, \mu^{\infty})$ is substantively the same (within 0.01) in the equilibria with and without financial frictions. However, in 20.83% of pairs, we find that $\delta(\mu^{25}, \mu^{\infty})$ is actually smaller (by at least 0.01) in the equilibrium without financial frictions than in the equilibrium with financial frictions, meaning that financial frictions significantly slow down the monopolization of the industry. In the remaining 12.27% of pairs however $\delta(\mu^{25}, \mu^{\infty})$ is larger (by at least 0.01) in the equilibrium without financial frictions than in the equilibrium with financial frictions.

		Joint-payoff effect			Joint-profit effect			Joint-profit net of joint-cost effect		
	$\widehat{\omega}^{\infty}$	$\widehat{\omega}^{V} < \widehat{\omega}^{\infty}$	$\widehat{\omega}^{V} = \widehat{\omega}^{\infty}$	$\widehat{\omega}^{V} > \widehat{\omega}^{\infty}$	$\widehat{\omega}^{\pi} < \widehat{\omega}^{\infty}$	$\widehat{\omega}^{\pi} = \widehat{\omega}^{\infty}$	$\widehat{\omega}^{\pi} > \widehat{\omega}^{\infty}$	$\widehat{\omega}^{\pi-F} < \widehat{\omega}^{\infty}$	$\widehat{\omega}^{\pi-F} = \widehat{\omega}^{\infty}$	$\widehat{\omega}^{\pi-F} > \widehat{\omega}^{\infty}$
0	1.92%		1.74%	0.19%		1.76%	0.16%		1.74%	0.18%
1	0.20%		0.20%	0.01%	0.20%				0.20%	0.01%
2	0.03%		0.02%		0.03%				0.02%	
4	0.01%		0.01%		0.01%				0.01%	
15	97.85%	7.74%	90.11%		10.07%	87.77%		8.13%	89.72%	
	100.00%	7.74%	92.07%	0.19%	10.30%	89.53%	0.16%	8.13%	91.68%	0.19%

Table 3: Joint-payoff, joint-profit, and joint-profit net of joint-cost effects. Model with financial frictions.

Figure 6: Determinants of Long-Run Industry Concentration



Fraction of cases where the level of industry concentration (long run modal state) is higher in the model with financial frictions across various parameter values.

Clearly, the increase the cost of investing induced by the presence of financial frictions (weakly) decrease the probability of investment and slows down the convergence to the long run equilibrium. Nevertheless, the fact that $\delta(\mu^{25}, \mu^{\infty})$ can be larger in the equilibrium without financial frictions than in the equilibrium with financial frictions suggests that more is going on. To sort this out, we conduct a counterfactual that hold the cost of investing fixed at its value in the equilibrium with financial frictions but forces firms to ignore any further dependence of $x_1(\omega)$ and $x_2(\omega)$ on $\pi_1(\omega, p_1(\omega), p_2(\omega))$ and $\pi_2(\omega, p_1(\omega), p_2(\omega))$ and thus to charge the same price as in a static Nash equilibrium.

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3.5 Financial frictions as a boost to investment

Differences in price policies lead to different long run outcomes. As Figure 7 shows, when price polices are significantly more kinked around $\omega = 0$ the two equilibria lead to substantively similar long run dynamics. Financial frictions lead to more concentrated industries only when the equilibrium price policies imply lower prices across the entire state space. In both cases prices drop the most when the competitive imbalance is relatively small.

Investment policies differ substantially more. Figure 8 shows that when financial frictions lead to lower equilibrium investment the long run outcome of the two economies either does not differ or, in very rare cases, implies a more concentrated industry in the case of no financial frictions.

By contrast, when the long run equilibrium implies a higher level of industry concentration in the presence of financial frictions, the associated equilibrium investment policies imply that firms invest more over at least a portion of the state space. In some cases, even though financial frictions matter in these economies, corporate investment can actually be higher in equilibrium. This is entirely for strategic reasons. Leaders in particular invest more to strengthen their position, lower followers' future profits and thus raise their investment costs.

The pertinent question is how relevant the states where firms invest more are in equilibrium. To answer this question, we compute the expected net present value of investment along the sequence of states induced by the state-to-state transition probability matrix P, starting from state 0 in period 0. From equation (13), in expectation firm i's investment in state ω is

$$\begin{aligned} I_i(\omega) &= E\left[F_0 + \theta_i | \theta_i \leq \Psi^{-1}(x_i(\omega))\right] \\ &= F_0 x_i(\omega) + \int_{-\infty}^{\Psi^{-1}(x_i(\omega))} \theta_i d\Psi(\theta_i). \end{aligned}$$

This is in monetary units and neglects the "hassle cost" $\zeta \max\{0, F_0 - \pi_i\}^{\kappa}$ from equation (7). We compute the expected net present value of the leader's investment $I_1(|\omega|)$, the follower's investment $I_1(-|\omega|)$, and the industry-wide investment. $I_1(\omega) + I_1(-\omega)$

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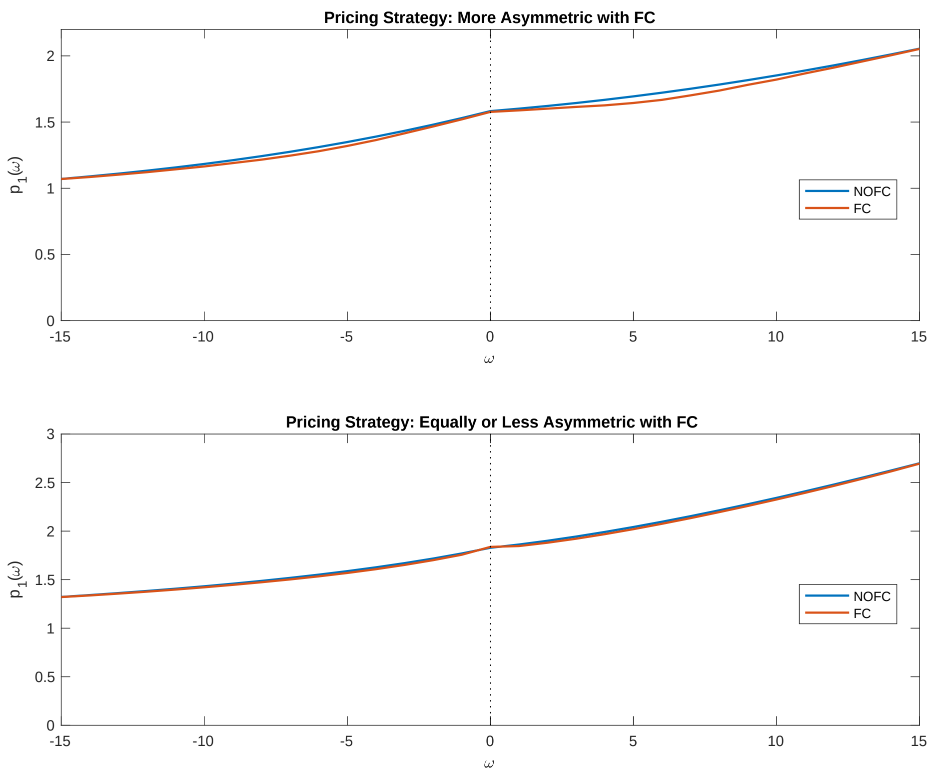
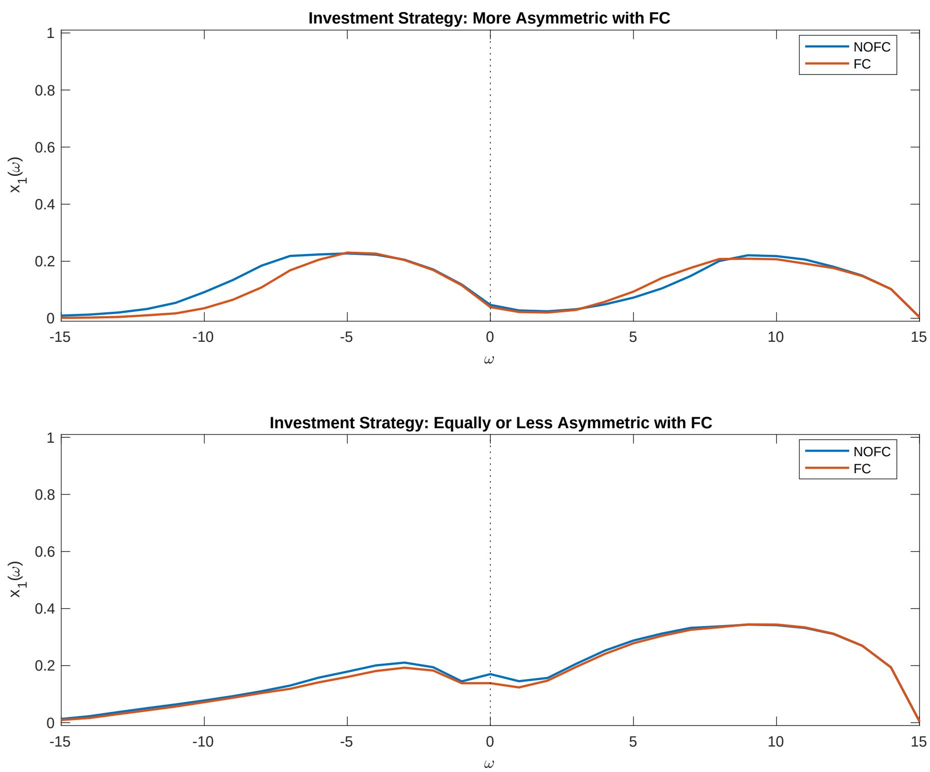
Figure 7: Long-Run Industry Concentration and Pricing Strategy



Pricing strategy across all parameter values for the baseline model and model with financial frictions.

Figure 8: Long-Run Industry Concentration and Investment Strategy



Average investment strategies for the baseline model and model with financial frictions.

4 Concluding remarks

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Appendix A Distributions

Appendix A.1 $N(0,\sigma^2)$ distribution

Let $\phi(z)$ and $\Phi(z)$ be the standard normal probability density and cumulative distribution functions. We have

$$\Psi(\theta_i) = \Phi\left(\frac{\theta_i}{\sigma}\right), \quad \psi(\theta_i) = \frac{1}{\sigma}\phi\left(\frac{\theta_i}{\sigma}\right), \quad \psi'(\theta_i) = -\frac{\theta_i}{\sigma^3}\phi\left(\frac{\theta_i}{\sigma}\right)$$

and

$$\Psi^{-1}(p) = \sigma\Phi^{-1}(p).$$

Finally, we have

$$\Upsilon(\bar{\theta}) = \int_{-\infty}^{\bar{\theta}} \theta_i d\Psi(\theta_i) = -\sigma\phi\left(\frac{\bar{\theta}}{\sigma}\right).$$

Appendix A.2 $Beta(3,3)$ distribution

Let $\phi(z)$ and $\Phi(z)$ be the probability density and cumulative distribution functions of a $Beta(3,3)$ distribution with support $[0,1]$. We have

$$\Phi(z) = \begin{cases} 0 & \text{if} \quad z < 0, \\ 30z^3\left(\frac{1}{5}z^2 - \frac{1}{2}z + \frac{1}{3}\right) & \text{if} \quad 0 \le z < 1, \\ 1 & \text{if} \quad z \ge 1, \end{cases}$$

$$\phi(z) = \begin{cases} 0 & \text{if} \quad z < 0, \\ 30z^2(1-z)^2 & \text{if} \quad 0 \le z < 1, \\ 0 & \text{if} \quad z \ge 1, \end{cases}$$

$$\phi'(z) = \begin{cases} 0 & \text{if} \quad z < 0, \\ 60(1-2z)z(1-z) & \text{if} \quad 0 \le z < 1, \\ 0 & \text{if} \quad z \ge 1. \end{cases}$$

Note that $\phi'(z)$ is continuous, including at $z=0$ and $z=1$.

Consider the linear transformation $\theta_i = \sigma\sqrt{7}(2Z-1)$. Then θ_i has support $\left[-\sigma\sqrt{7}, \sigma\sqrt{7}\right]$, mean zero, and variance σ^2. Moreover, we have

$$\Psi(\theta_i) = \Phi\left(\frac{\theta_i + \sigma\sqrt{7}}{2\sigma\sqrt{7}}\right), \quad \psi(\theta_i) = \frac{1}{2\sigma\sqrt{7}}\phi\left(\frac{\theta_i + \sigma\sqrt{7}}{2\sigma\sqrt{7}}\right), \quad \psi'(\theta_i) = \frac{1}{(2\sigma\sqrt{7})^2}\phi'\left(\frac{\theta_i + \sigma\sqrt{7}}{2\sigma\sqrt{7}}\right)$$

and

$$\Psi^{-1}(p) = \sigma\sqrt{7}\left(2\Phi^{-1}(p) - 1\right).$$

Finally, define

$$\Omega(\bar{z}) = \int_{-\infty}^{\bar{z}} z d\Phi(z) = \begin{cases} 0 & \text{if} \quad \bar{z} < 0, \\ \bar{z}^4 \left(5\bar{z}^2 - 12\bar{z} + \frac{15}{2}\right) & \text{if} \quad 0 \leq \bar{z} < 1, \\ \frac{1}{2} & \text{if} \quad \bar{z} \geq 1. \end{cases}$$

We have

$$\Upsilon(\bar{\theta}) = \int_{-\infty}^{\bar{\theta}} \theta_i d\Psi(\theta_i) = \int_{-\infty}^{\bar{\theta}} \theta_i \phi\left(\frac{\theta_i + \sigma\sqrt{7}}{2\sigma\sqrt{7}}\right) \frac{1}{2\sigma\sqrt{7}} d\theta_i = \int_{-\infty}^{\frac{\bar{\theta}+\sigma\sqrt{7}}{2\sigma\sqrt{7}}} \sigma\sqrt{7}(2z-1)\phi\left(z\right) dz$$

$$= \sigma\sqrt{7}\left(2\int_{-\infty}^{\frac{\bar{\theta}+\sigma\sqrt{7}}{2\sigma\sqrt{7}}} z\phi\left(z\right) dz - \int_{-\infty}^{\frac{\bar{\theta}+\sigma\sqrt{7}}{2\sigma\sqrt{7}}} \phi\left(z\right) dz\right) = \sigma\sqrt{7}\left(2\Omega\left(\frac{\bar{\theta}+\sigma\sqrt{7}}{2\sigma\sqrt{7}}\right) - \Phi\left(\frac{\bar{\theta}+\sigma\sqrt{7}}{2\sigma\sqrt{7}}\right)\right).$$

Appendix B System of equations and Jacobian

Appendix B.1 System of equations

A symmetric equilibrium is a solution to the system of equations

$$\mathcal{F}(\mathcal{X}) = 0,$$

where

$$\mathcal{X} = (V_1(-L), \ldots, V_1(L), U_1(-L), \ldots, U_1(L), p_1(-L), \ldots, p_1(L), x_1(-L), \ldots, x_1(L))$$

is a vector of $8L+4$ unknowns and $\mathcal{F}$ is defined by the following $8L+4$ equations:

$$-V_1(\omega) + \pi_1(\omega, p_1(\omega), p_2(\omega)) + U_1(\omega) = 0, \quad \omega \in \{-L, \ldots, L\}, \tag{21}$$

$$\begin{aligned} &-U_1(\omega) - F\left(\pi_1(\omega, p_1(\omega), p_2(\omega))\right) x_1(\omega) - \int_{-\infty}^{Z_1} \theta_1 d\Psi(\theta_1) \\ &+\beta\left[V_1(\omega^+)x_1(\omega)(1-x_2(\omega)) + V_1(\omega)(1-x_1(\omega)-x_2(\omega)+2x_1(\omega)x_2(\omega)) + V_1(\omega^-)(1-x_1(\omega))x_2(\omega)\right] \\ &= 0, \quad \omega \in \{-L, \ldots, L\}, \end{aligned} \tag{22}$$

$$\begin{aligned} &\frac{\partial \pi_1(\omega, p_1(\omega), p_2(\omega))}{\partial p_1}\left(1 - F'\left(\pi_1(\omega, p_1(\omega), p_2(\omega))\right) x_1(\omega) + \frac{\partial U_1(\omega)}{\partial x_2}\frac{\partial x_2(\omega)}{\partial \pi_1}\right) \\ &+\frac{\partial \pi_2(\omega, p_1(\omega), p_2(\omega))}{\partial p_1}\frac{\partial U_1(\omega)}{\partial x_2}\frac{\partial x_2(\omega)}{\partial \pi_2} = 0, \quad \omega \in \{-L, \ldots, L\}, \end{aligned} \tag{23}$$

$$-x_1(\omega) + \Psi(Z_1) = 0, \quad \omega \in \{-L, \ldots, L\}, \tag{24}$$

where,

$$\begin{aligned} \frac{\partial U_1(\omega)}{\partial x_2} &= -B_1 + (B_1 - A_1)x_1(\omega), \\ \frac{\partial x_2(\omega)}{\partial \pi_1} &= \frac{1}{Y}\left\{-\psi(Z_1)\psi(Z_2)F'(\pi_1)\left(B_2 - A_2\right)\right\}, \\ \frac{\partial x_2(\omega)}{\partial \pi_2} &= \frac{1}{Y}\left\{-\psi(Z_2)F'(\pi_2)\right\} \end{aligned}$$

and

$$
\begin{aligned}
A_1 &= \beta \left[V_1(\omega^+) - V_1(\omega) \right], \quad B_1 = \beta \left[V_1(\omega) - V_1(\omega^-) \right], \\
Z_1 &= -F\left(\pi_1(\omega, p_1(\omega), p_2(\omega))\right) + A_1 + (B_1 - A_1)x_2(\omega), \\
A_2 &= \beta \left[V_2(\omega^-) - V_2(\omega) \right], \quad B_2 = \beta \left[V_2(\omega) - V_2(\omega^+) \right], \\
Z_2 &= -F\left(\pi_2(\omega, p_1(\omega), p_2(\omega))\right) + A_2 + (B_2 - A_2)x_1(\omega), \\
Y &= 1 - \psi(Z_1)\psi(Z_2)\left(B_1 - A_1\right)\left(B_2 - A_2\right).
\end{aligned}
$$

Note that we have substituted equation (10) into equation (13) to obtain equation (22).[9] Throughout it is understood that we use the shorthands

$$
V_2(\omega) = V_1(-\omega), \quad U_2(\omega) = U_1(-\omega), \quad p_2(\omega) = p_1(-\omega), \quad x_2(\omega) = x_1(-\omega).
$$

Notation. To simplify the notation, without loss of generality we redefine the parameters α and $g(\omega)$ to be $\frac{\alpha}{\nu}$ and $\frac{g(\omega)}{\nu}$. This avoids having to carry along ν.

Transformations. To facilitate solving the system of equations, we multiply equation (23) by

$$
\left(1 + \exp(-g(\omega) + \alpha(p_1(\omega) - p_2(\omega)))\right)^2
$$

to avoid asymptotes.

Uli – Following the changes to the static Nash equilibrium, this should be updated to $\frac{(1+\exp(-g(\omega)+\alpha(p_1(}{\exp(-g(\omega)+\alpha(p_1(\omega}$

Appendix B.2 Jacobian

In what follows, we note only the non-zero derivatives. Note that for the time being we ignore any transformations as well as any adjustments arising from the symmetry restriction $V_1(0) = V_2(0)$, $U_1(0) = U_2(0)$, $p_1(0) = p_2(0)$, and $x_1(0) = x_2(0)$; we return to them below.

Preliminaries: Profit. The first and second derivatives of profit are

$$
\frac{\partial \pi_1(\omega, p_1, p_2)}{\partial p_1} = \frac{M\left(1 + (1 - (p_1 - c)\alpha)\exp(-g(\omega) + \alpha(p_1 - p_2))\right)}{\left(1 + \exp(-g(\omega) + \alpha(p_1 - p_2))\right)^2},
$$

$$
\frac{\partial \pi_1(\omega, p_1, p_2)}{\partial p_2} = \frac{M(p_1 - c)\alpha \exp(-g(\omega) + \alpha(p_1 - p_2))}{\left(1 + \exp(-g(\omega) + \alpha(p_1 - p_2))\right)^2},
$$

$$
\frac{\partial \pi_2(\omega, p_1, p_2)}{\partial p_1} = \frac{M(p_2 - c)\alpha \exp(-g(-\omega) - \alpha(p_1 - p_2))}{\left(1 + \exp(-g(-\omega) - \alpha(p_1 - p_2))\right)^2} = \frac{\partial \pi_1(-\omega, p_2, p_1)}{\partial p_2},
$$

$$
\frac{\partial \pi_2(\omega, p_1, p_2)}{\partial p_2} = \frac{M\left(1 + (1 - (p_2 - c)\alpha)\exp(-g(-\omega) - \alpha(p_1 - p_2))\right)}{\left(1 + \exp(-g(-\omega) - \alpha(p_1 - p_2))\right)^2} = \frac{\partial \pi_1(-\omega, p_2, p_1)}{\partial p_1},
$$

$$
\frac{\partial^2 \pi_1(\omega, p_1, p_2)}{\partial p_1^2}
$$

$$
= -\frac{M\alpha \exp(-g(\omega) + \alpha(p_1 - p_2))\left(2(1 + \exp(-g(\omega) + \alpha(p_1 - p_2))) + (p_1 - c)\alpha(1 - \exp(-g(\omega) + \alpha(p_1 - p_2)))\right)}{\left(1 + \exp(-g(\omega) + \alpha(p_1 - p_2))\right)^3},
$$

[9] If $\theta_i \sim N(0, \sigma^2)$, then this substitution avoids numerical issues that arise because $\Psi^{-1}(0) = -\infty$ and $\Psi^{-1}(1) = \infty$ and because $\Psi^{-1}(-\epsilon)$ and $\Psi^{-1}(1+\epsilon)$ are undefined for all $\epsilon > 0$.

$$\frac{\partial^2 \pi_1(\omega,p_1,p_2)}{\partial p_1 \partial p_2} = \frac{\partial^2 \pi_1(\omega,p_1,p_2)}{\partial p_2 \partial p_1}$$

$$= \frac{M\alpha \exp(-g(\omega)+\alpha(p_1-p_2))\left(1+\exp(-g(\omega)+\alpha(p_1-p_2)) + (p_1-c)\alpha\left(1-\exp(-g(\omega)+\alpha(p_1-p_2))\right)\right)}{\left(1+\exp(-g(\omega)+\alpha(p_1-p_2))\right)^3},$$

$$\frac{\partial^2 \pi_1(\omega,p_1,p_2)}{\partial p_2^2}$$

$$= -\frac{M\alpha^2 \exp(-g(\omega)+\alpha(p_1-p_2))(p_1-c)\left(1-\exp(-g(\omega)+\alpha(p_1-p_2))\right)}{\left(1+\exp(-g(\omega)+\alpha(p_1-p_2))\right)^3},$$

$$\frac{\partial^2 \pi_2(\omega,p_1,p_2)}{\partial p_1^2}$$

$$= -\frac{M\alpha^2 \exp(-g(-\omega)-\alpha(p_1-p_2))(p_2-c)\left(1-\exp(-g(-\omega)-\alpha(p_1-p_2))\right)}{\left(1+\exp(-g(-\omega)-\alpha(p_1-p_2))\right)^3}$$

$$= \frac{\partial^2 \pi_2(-\omega,p_2,p_1)}{\partial p_1^2},$$

$$\frac{\partial^2 \pi_2(\omega,p_1,p_2)}{\partial p_2 \partial p_1} = \frac{\partial^2 \pi_2(\omega,p_1,p_2)}{\partial p_1 \partial p_2}$$

$$= \frac{M\alpha \exp(-g(-\omega)-\alpha(p_1-p_2))\left(1+\exp(-g(-\omega)-\alpha(p_1-p_2)) + (p_2-c)\alpha\left(1-\exp(-g(-\omega)-\alpha(p_1-p_2)\right.\right.}{\left(1+\exp(-g(-\omega)-\alpha(p_1-p_2))\right)^3}$$

$$= \frac{\partial^2 \pi_1(-\omega,p_2,p_1)}{\partial p_1 \partial p_2} = \frac{\partial^2 \pi_1(-\omega,p_2,p_1)}{\partial p_2 \partial p_1},$$

$$\frac{\partial^2 \pi_2(\omega,p_1,p_2)}{\partial p_2^2}$$

$$= -\frac{M\alpha \exp(-g(-\omega)-\alpha(p_1-p_2))\left(2(1+\exp(-g(-\omega)-\alpha(p_1-p_2))) + (p_2-c)\alpha(1-\exp(-g(-\omega)-\alpha(p_1-p\right.}{\left(1+\exp(-g(-\omega)-\alpha(p_1-p_2))\right)^3}$$

$$= \frac{\partial^2 \pi_1(-\omega,p_2,p_1)}{\partial p_1^2}.$$

Note that $\frac{\partial \pi_i(\omega,p_1,p_2)}{\partial p_i} = 0$ implies

$$\frac{\partial^2 \pi_i(\omega,p_1,p_2)}{\partial p_i^2} = -\frac{M\alpha}{1+\exp(g(\omega)+\alpha(p_1-p_2))} < 0.$$

Hence, $\pi_i(\omega,p_1,p_2)$ is strictly quasiconcave in p_i.

Preliminaries: Comparative statics. Totally differentiating the system of equations (10) and (11) yields

$$-dx_1(\omega) + \psi(Z_1)\left(B_1 - A_1\right)dx_2 - \psi(Z_1)F'(\pi_1)d\pi_1 = 0,$$
$$-dx_2(\omega) + \psi(Z_2)\left(B_2 - A_2\right)dx_1 - \psi(Z_2)F'(\pi_2)d\pi_2 = 0,$$

where, again, π_i is shorthand for $\pi_i(\omega, p_1(\omega), p_2(\omega))$. Defining

$$Y = 1 - \psi(Z_1)\psi(Z_2)\left(B_1 - A_1\right)\left(B_2 - A_2\right),$$

the first-order comparative statics are

$$\frac{\partial x_1(\omega)}{\partial \pi_1} = \frac{1}{Y}\left\{-\psi(Z_1)F'(\pi_1)\right\}, \tag{25}$$

$$\frac{\partial x_1(\omega)}{\partial \pi_2} = \frac{1}{Y}\left\{-\psi(Z_1)\psi(Z_2)F'(\pi_2)\left(B_1 - A_1\right)\right\}, \tag{26}$$

$$\frac{\partial x_2(\omega)}{\partial \pi_1} = \frac{1}{Y}\left\{-\psi(Z_1)\psi(Z_2)F'(\pi_1)\left(B_2 - A_2\right)\right\}, \tag{27}$$

$$\frac{\partial x_2(\omega)}{\partial \pi_2} = \frac{1}{Y}\left\{-\psi(Z_2)F'(\pi_2)\right\}. \tag{28}$$

It is in general not possible to sign the first-order comparative statics in equations (25)–(28).

Next, to facilitate taking derivatives of the first-order comparative statics in equations (25)–(28), it is useful to note that

$$\frac{\partial Y}{\partial \pi_1} = \psi'(Z_1)\psi(Z_2)F'(\pi_1)(B_1 - A_1)(B_2 - A_2) = (1 - Y)F'(\pi_1)\frac{\psi'(Z_1)}{\psi(Z_1)},$$

$$\frac{\partial Y}{\partial \pi_2} = \psi(Z_1)\psi'(Z_2)F'(\pi_2)(B_1 - A_1)(B_2 - A_2) = (1 - Y)F'(\pi_2)\frac{\psi'(Z_2)}{\psi(Z_2)}.$$

Differentiating the first-order comparative statics in equations (25)–(28) yields

$$\frac{\partial^2 x_i(\omega)}{\partial \pi_i^2} = \frac{1}{Y}\left\{\psi'(Z_i)\left(F'(\pi_i)\right)^2 - \psi(Z_i)F''(\pi_i) - \frac{\partial x_i(\omega)}{\partial \pi_i}\frac{\partial Y}{\partial \pi_i}\right\},$$

$$\frac{\partial^2 x_i(\omega)}{\partial \pi_i \partial \pi_j} = \frac{1}{Y}\left\{-\frac{\partial x_i(\omega)}{\partial \pi_i}\frac{\partial Y}{\partial \pi_j}\right\}, \quad i \neq j,$$

$$\frac{\partial^2 x_i(\omega)}{\partial \pi_j \partial \pi_i} = \frac{1}{Y}\left\{\psi'(Z_i)\psi(Z_j)F'(\pi_i)F'(\pi_j)(B_i - A_i) - \frac{\partial x_i(\omega)}{\partial \pi_j}\frac{\partial Y}{\partial \pi_i}\right\}, \quad i \neq j,$$

$$\frac{\partial^2 x_i(\omega)}{\partial \pi_j^2} = \frac{1}{Y}\left\{\psi(Z_i)\psi'(Z_j)\left(F'(\pi_j)\right)^2(B_i - A_i) - \psi(Z_i)\psi(Z_j)F''(\pi_j)(B_i - A_i) - \frac{\partial x_i(\omega)}{\partial \pi_j}\frac{\partial Y}{\partial \pi_j}\right\}, \quad i \neq j.$$

Note that $\frac{\partial^2 x_i(\omega)}{\partial \pi_i \partial \pi_j} \neq \frac{\partial^2 x_i(\omega)}{\partial \pi_j \partial \pi_i}$.

It is also useful to note that

$$\frac{\partial Y}{\partial x_1} = -\psi(Z_1)\psi'(Z_2)(B_1 - A_1)(B_2 - A_2)^2 = (Y - 1)(B_2 - A_2)\frac{\psi'(Z_2)}{\psi(Z_2)},$$

$$\frac{\partial Y}{\partial x_2} = -\psi'(Z_1)\psi(Z_2)(B_1 - A_1)^2(B_2 - A_2) = (Y - 1)(B_1 - A_1)\frac{\psi'(Z_1)}{\psi(Z_1)}.$$

Differentiating the first-order comparative statics in equations (25)–(28) yields

$$\frac{\partial^2 x_i(\omega)}{\partial \pi_i \partial x_i} = \frac{1}{Y}\left\{-\frac{\partial x_i(\omega)}{\partial \pi_i}\frac{\partial Y}{\partial x_i}\right\},$$

$$\frac{\partial^2 x_i(\omega)}{\partial \pi_i \partial x_j} = \frac{1}{Y}\left\{-\psi'(Z_i)F'(\pi_i)(B_i - A_i) - \frac{\partial x_i(\omega)}{\partial \pi_i}\frac{\partial Y}{\partial x_j}\right\}, \quad i \neq j,$$

$$\frac{\partial^2 x_i(\omega)}{\partial \pi_j \partial x_i} = \frac{1}{Y}\left\{-\psi(Z_i)\psi'(Z_j)F'(\pi_j)(B_i - A_i)(B_j - A_j) - \frac{\partial x_i(\omega)}{\partial \pi_j}\frac{\partial Y}{\partial x_i}\right\}, \quad i \neq j,$$
$$\frac{\partial^2 x_i(\omega)}{\partial \pi_j \partial x_j} = \frac{1}{Y}\left\{-\psi'(Z_i)\psi(Z_j)F'(\pi_j)(B_i - A_i)^2 - \frac{\partial x_i(\omega)}{\partial \pi_j}\frac{\partial Y}{\partial x_j}\right\}, \quad i \neq j.$$

It is also useful to note that

$$\frac{\partial Y}{\partial A_1} = -\psi'(Z_1)\psi(Z_2)(B_1 - A_1)(B_2 - A_2)(1 - x_2(\omega)) + \psi(Z_1)\psi(Z_2)(B_2 - A_2),$$
$$\frac{\partial Y}{\partial B_1} = -\psi'(Z_1)\psi(Z_2)(B_1 - A_1)(B_2 - A_2)x_2(\omega) - \psi(Z_1)\psi(Z_2)(B_2 - A_2),$$
$$\frac{\partial Y}{\partial A_2} = -\psi(Z_1)\psi'(Z_2)(B_1 - A_1)(B_2 - A_2)(1 - x_1(\omega)) + \psi(Z_1)\psi(Z_2)(B_1 - A_1),$$
$$\frac{\partial Y}{\partial B_2} = -\psi(Z_1)\psi'(Z_2)(B_1 - A_1)(B_2 - A_2)x_1(\omega) - \psi(Z_1)\psi(Z_2)(B_1 - A_1)$$

or, more compactly,

$$\frac{\partial Y}{\partial A_i} = (Y-1)\left\{(1 - x_j(\omega))\frac{\psi'(Z_i)}{\psi(Z_i)} - \frac{1}{B_i - A_i}\right\}, \quad i \neq j,$$
$$\frac{\partial Y}{\partial B_i} = (Y-1)\left\{x_j(\omega)\frac{\psi'(Z_i)}{\psi(Z_i)} + \frac{1}{B_i - A_i}\right\}, \quad i \neq j.$$

Differentiating the first-order comparative statics in equations (25)–(28) yields

$$\frac{\partial^2 x_i(\omega)}{\partial \pi_i \partial A_i} = \frac{1}{Y}\left\{-\psi'(Z_i)F'(\pi_i)(1 - x_j(\omega)) - \frac{\partial x_i(\omega)}{\partial \pi_i}\frac{\partial Y}{\partial A_i}\right\}, \quad i \neq j,$$
$$\frac{\partial^2 x_i(\omega)}{\partial \pi_i \partial B_i} = \frac{1}{Y}\left\{-\psi'(Z_i)F'(\pi_i)x_j(\omega) - \frac{\partial x_i(\omega)}{\partial \pi_i}\frac{\partial Y}{\partial B_i}\right\}, \quad i \neq j,$$
$$\frac{\partial^2 x_i(\omega)}{\partial \pi_i \partial A_j} = \frac{1}{Y}\left\{-\frac{\partial x_i(\omega)}{\partial \pi_i}\frac{\partial Y}{\partial A_j}\right\}, \quad i \neq j,$$
$$\frac{\partial^2 x_i(\omega)}{\partial \pi_i \partial B_j}\frac{1}{Y}\left\{-\frac{\partial x_i(\omega)}{\partial \pi_i}\frac{\partial Y}{\partial B_j}\right\}, \quad i \neq j,$$
$$\frac{\partial^2 x_i(\omega)}{\partial \pi_j \partial A_i} = \frac{1}{Y}\left\{-\psi'(Z_i)\psi(Z_j)F'(\pi_j)(B_i - A_i)(1 - x_j(\omega)) + \psi(Z_i)\psi(Z_j)F'(\pi_j) - \frac{\partial x_i(\omega)}{\partial \pi_j}\frac{\partial Y}{\partial A_i}\right\}, \quad i \neq j,$$
$$\frac{\partial^2 x_i(\omega)}{\partial \pi_j \partial B_i} = \frac{1}{Y}\left\{-\psi'(Z_i)\psi(Z_j)F'(\pi_j)(B_i - A_i)x_j(\omega) - \psi(Z_i)\psi(Z_j)F'(\pi_j) - \frac{\partial x_i(\omega)}{\partial \pi_j}\frac{\partial Y}{\partial B_i}\right\}, \quad i \neq j,$$
$$\frac{\partial^2 x_i(\omega)}{\partial \pi_j \partial A_j} = \frac{1}{Y}\left\{-\psi(Z_i)\psi'(Z_j)F'(\pi_j)(B_i - A_i)(1 - x_i(\omega)) - \frac{\partial x_i(\omega)}{\partial \pi_j}\frac{\partial Y}{\partial A_j}\right\}, \quad i \neq j,$$
$$\frac{\partial^2 x_i(\omega)}{\partial \pi_j \partial B_j} = \frac{1}{Y}\left\{-\psi(Z_i)\psi'(Z_j)F'(\pi_j)(B_i - A_i)x_i(\omega) - \frac{\partial x_i(\omega)}{\partial \pi_j}\frac{\partial Y}{\partial B_j}\right\}, \quad i \neq j.$$

Jacobian: Beginning-of-period Bellman equation (21) with respect to $V_1(\omega)$. Differentiating equation (21) with respect to $V_1(\omega)$ yields

$$-1.$$

Jacobian: Beginning-of-period Bellman equation (21) with respect to $U_1(\omega)$. Differentiating equation (21) with respect to $U_1(\omega)$ yields

$$1.$$

Jacobian: Beginning-of-period Bellman equation (21) with respect to $p_1(\omega)$ and $p_2(\omega) = p_1(-\omega)$. Differentiating equation (21) with respect to $p_i(\omega)$ yields

$$\frac{\partial \pi_1(\omega, p_1(\omega), p_2(\omega))}{\partial p_i}, \quad i = 1, 2. \tag{29}$$

Jacobian: Middle-of-period Bellman equation (22) with respect to $V_1(\omega^-)$, $V_1(\omega)$, and $V_1(\omega^+)$. Recalling that $\omega^+ = \min\{L, \omega + 1\}$ and $\omega^- = \max\{-L, \omega - 1\}$ and that $V_1(\omega^-)$ enters B_1, $V_1(\omega)$ enters A_1 and B_1, and $V_1(\omega^+)$ enters A_1, differentiating equation (22) with respect to $V_1(\omega^-)$ yields

$$\begin{cases} 0 & \text{if} \quad \omega = -L, \\ -Z_1\psi(Z_1)\beta(-x_2(\omega)) + \beta(1 - x_1(\omega))x_2(\omega) & \text{if} \quad \omega > -L. \end{cases}$$

Differentiating equation (22) with respect to $V_1(\omega)$ yields

$$\begin{cases} -Z_1\psi(Z_1)\beta(-1 + x_2(\omega)) + \beta\left(1 - x_1(\omega) + x_1(\omega)x_2(\omega)\right) & \text{if} \quad \omega = -L, \\ -Z_1\psi(Z_1)\beta(-1 + 2x_2(\omega)) + \beta\left(1 - x_1(\omega) - x_2(\omega) + 2x_1(\omega)x_2(\omega)\right) & \text{if} \quad -L < \omega < L, \\ -Z_1\psi(Z_1)\beta x_2(\omega) + \beta\left(1 - x_2(\omega) + x_1(\omega)x_2(\omega)\right) & \text{if} \quad \omega = L. \end{cases}$$

Differentiating equation (22) with respect to $V_1(\omega^+)$ yields

$$\begin{cases} -Z_1\psi(Z_1)\beta(1 - x_2(\omega)) + \beta x_1(\omega)(1 - x_2(\omega)) & \text{if} \quad \omega < L, \\ 0 & \text{if} \quad \omega = L. \end{cases}$$

Jacobian: Middle-of-period Bellman equation (22) with respect to $U_1(\omega)$. Differentiating equation (22) with respect to $U_1(\omega)$ yields

$$-1.$$

Jacobian: Middle-of-period Bellman equation (22) with respect to $p_1(\omega)$ and $p_2(\omega) = p_1(-\omega)$. Differentiating equation (22) with respect to $p_i(\omega)$ yields

$$-F'(\pi_1(\omega, p_1(\omega), p_2(\omega)))\frac{\partial \pi_1(\omega, p_1(\omega), p_2(\omega))}{\partial p_i}\left(x_1(\omega) - Z_1\psi(Z_1)\right), \quad i = 1, 2. \tag{30}$$

Jacobian: Middle-of-period Bellman equation (22) with respect to $x_1(\omega)$ and $x_2(\omega) = x_1(-\omega)$. Differentiating equation (22) with respect to $x_1(\omega)$ yields

$$-F(\pi_1(\omega, p_1(\omega), p_2(\omega)) + A_1 + (B_1 - A_1)x_2(\omega). \tag{31}$$

Differentiating equation (22) with respect to $x_2(\omega)$ yields

$$-B_1 + (B_1 - A_1)\left(x_1(\omega) - Z_1\psi(Z_1)\right). \tag{32}$$

Jacobian: Optimal pricing equation (23) with respect to $V_1(\omega^-)$, $V_1(\omega)$, $V_1(\omega^+)$, $V_2(\omega^-) = V_1(-\omega^-)$, $V_2(\omega) = V_1(\omega^-)$, and $V_2(\omega^+) = V_1(-\omega^+)$. Recalling that $V_1(\omega^-)$ enters B_1, $V_1(\omega)$ enters A_1 and B_1, $V_1(\omega^+)$ enters A_1, $V_2(\omega^-)$ enters A_2, $V_2(\omega)$ enters A_2 and B_2, and $V_2(\omega^+)$ enters B_2, differentiating equation (23) with respect to $V_1(\omega^-)$ yields

$$\begin{aligned}&\frac{\partial \pi_1(\omega, p_1(\omega), p_2(\omega))}{\partial p_1}\left(\frac{\partial^2 U_1(\omega)}{\partial x_2 \partial V_1(\omega^-)}\frac{\partial x_2(\omega)}{\partial \pi_1} + \frac{\partial U_1(\omega)}{\partial x_2}\frac{\partial^2 x_2(\omega)}{\partial \pi_1 \partial V_1(\omega^-)}\right)\\ &+\frac{\partial \pi_2(\omega, p_1(\omega), p_2(\omega))}{\partial p_1}\left(\frac{\partial^2 U_1(\omega)}{\partial x_2 \partial V_1(\omega^-)}\frac{\partial x_2(\omega)}{\partial \pi_2} + \frac{\partial U_1(\omega)}{\partial x_2}\frac{\partial^2 x_2(\omega)}{\partial \pi_2 \partial V_1(\omega^-)}\right),\end{aligned}$$

where

$$\frac{\partial^2 U_1(\omega)}{\partial x_2 \partial V_1(\omega^-)} = \begin{cases} 0 & \text{if} \quad \omega = -L, \\ \beta\left(1 - x_1(\omega)\right) & \text{if} \quad \omega > -L, \end{cases}$$

$$\frac{\partial^2 x_2(\omega)}{\partial \pi_i \partial V_1(\omega^-)} = \begin{cases} 0 & \text{if} \quad \omega = -L, \\ -\beta\frac{\partial^2 x_2(\omega)}{\partial \pi_i \partial B_1} & \text{if} \quad \omega > -L, \end{cases} \qquad i = 1, 2.$$

Differentiating equation (23) with respect to $V_1(\omega)$ yields

$$\begin{aligned}&\frac{\partial \pi_1(\omega, p_1(\omega), p_2(\omega))}{\partial p_1}\left(\frac{\partial^2 U_1(\omega)}{\partial x_2 \partial V_1(\omega)}\frac{\partial x_2(\omega)}{\partial \pi_1} + \frac{\partial U_1(\omega)}{\partial x_2}\frac{\partial^2 x_2(\omega)}{\partial \pi_1 \partial V_1(\omega)}\right)\\ &+\frac{\partial \pi_2(\omega, p_1(\omega), p_2(\omega))}{\partial p_1}\left(\frac{\partial^2 U_1(\omega)}{\partial x_2 \partial V_1(\omega)}\frac{\partial x_2(\omega)}{\partial \pi_2} + \frac{\partial U_1(\omega)}{\partial x_2}\frac{\partial^2 x_2(\omega)}{\partial \pi_2 \partial V_1(\omega)}\right),\end{aligned}$$

where

$$\frac{\partial^2 U_1(\omega)}{\partial x_2 \partial V_1(\omega)} = \begin{cases} \beta x_1(\omega) & \text{if} \quad \omega = -L, \\ \beta\left(-1 + 2x_1(\omega)\right) & \text{if} \quad -L < \omega < L, \\ \beta\left(-1 + x_1(\omega)\right) & \text{if} \quad \omega = L, \end{cases}$$

$$\frac{\partial^2 x_2(\omega)}{\partial \pi_i \partial V_1(\omega)} = \begin{cases} -\beta\frac{\partial^2 x_2(\omega)}{\partial \pi_i \partial A_1} & \text{if} \quad \omega = -L, \\ \beta\left(-\frac{\partial^2 x_2(\omega)}{\partial \pi_i \partial A_1} + \frac{\partial^2 x_2(\omega)}{\partial \pi_i \partial B_1}\right) & \text{if} \quad -L < \omega < L, \\ \beta\frac{\partial^2 x_2(\omega)}{\partial \pi_i \partial B_1} & \text{if} \quad \omega = L, \end{cases} \qquad i = 1, 2.$$

Differentiating equation (23) with respect to $V_1(\omega^+)$ yields

$$\begin{aligned}&\frac{\partial \pi_1(\omega, p_1(\omega), p_2(\omega))}{\partial p_1}\left(\frac{\partial^2 U_1(\omega)}{\partial x_2 \partial V_1(\omega^+)}\frac{\partial x_2(\omega)}{\partial \pi_1} + \frac{\partial U_1(\omega)}{\partial x_2}\frac{\partial^2 x_2(\omega)}{\partial \pi_1 \partial V_1(\omega^+)}\right)\\ &+\frac{\partial \pi_2(\omega, p_1(\omega), p_2(\omega))}{\partial p_1}\left(\frac{\partial^2 U_1(\omega)}{\partial x_2 \partial V_1(\omega^+)}\frac{\partial x_2(\omega)}{\partial \pi_2} + \frac{\partial U_1(\omega)}{\partial x_2}\frac{\partial^2 x_2(\omega)}{\partial \pi_2 \partial V_1(\omega^+)}\right),\end{aligned}$$

where

$$\frac{\partial^2 U_1(\omega)}{\partial x_2 \partial V_1(\omega^+)} = \begin{cases} \beta\left(-x_1(\omega)\right) & \text{if} \quad \omega < L, \\ 0 & \text{if} \quad \omega = L, \end{cases}$$

$$\frac{\partial^2 x_2(\omega)}{\partial \pi_i \partial V_1(\omega^+)} = \begin{cases} \beta\frac{\partial^2 x_2(\omega)}{\partial \pi_i \partial A_1} & \text{if} \quad \omega < L, \\ 0 & \text{if} \quad \omega = L, \end{cases} \qquad i = 1, 2.$$

Differentiating equation (23) with respect to $V_2(\omega^-)$ yields

$$\frac{\partial \pi_1(\omega, p_1(\omega), p_2(\omega))}{\partial p_1} \frac{\partial U_1(\omega)}{\partial x_2} \frac{\partial^2 x_2(\omega)}{\partial \pi_1 \partial V_2(\omega^-)} + \frac{\partial \pi_2(\omega, p_1(\omega), p_2(\omega))}{\partial p_1} \frac{\partial U_1(\omega)}{\partial x_2} \frac{\partial^2 x_2(\omega)}{\partial \pi_2 \partial V_2(\omega^-)},$$

where

$$\frac{\partial^2 x_2(\omega)}{\partial \pi_i \partial V_2(\omega^-)} = \begin{cases} 0 & \text{if} \quad \omega = -L, \\ \beta \frac{\partial^2 x_2(\omega)}{\partial \pi_i \partial A_2} & \text{if} \quad \omega > -L, \end{cases} \qquad i = 1, 2.$$

Differentiating equation (23) with respect to $V_2(\omega)$ yields

$$\frac{\partial \pi_1(\omega, p_1(\omega), p_2(\omega))}{\partial p_1} \frac{\partial U_1(\omega)}{\partial x_2} \frac{\partial^2 x_2(\omega)}{\partial \pi_1 \partial V_2(\omega)} + \frac{\partial \pi_2(\omega, p_1(\omega), p_2(\omega))}{\partial p_1} \frac{\partial U_1(\omega)}{\partial x_2} \frac{\partial^2 x_2(\omega)}{\partial \pi_2 \partial V_2(\omega)},$$

where

$$\frac{\partial^2 x_2(\omega)}{\partial \pi_i \partial V_2(\omega)} = \begin{cases} -\beta \frac{\partial^2 x_2(\omega)}{\partial \pi_i \partial A_2} & \text{if} \quad \omega = -L, \\ \beta \left(-\frac{\partial^2 x_2(\omega)}{\partial \pi_i \partial A_2} + \frac{\partial^2 x_2(\omega)}{\partial \pi_i \partial B_2} \right) & \text{if} \quad -L < \omega < L, \\ \beta \frac{\partial^2 x_2(\omega)}{\partial \pi_i \partial B_2} & \text{if} \quad \omega = L, \end{cases} \qquad i = 1, 2.$$

Differentiating equation (23) with respect to $V_2(\omega^+)$ yields

$$\frac{\partial \pi_1(\omega, p_1(\omega), p_2(\omega))}{\partial p_1} \frac{\partial U_1(\omega)}{\partial x_2} \frac{\partial^2 x_2(\omega)}{\partial \pi_1 \partial V_2(\omega^+)} + \frac{\partial \pi_2(\omega, p_1(\omega), p_2(\omega))}{\partial p_1} \frac{\partial U_1(\omega)}{\partial x_2} \frac{\partial^2 x_2(\omega)}{\partial \pi_2 \partial V_2(\omega^+)},$$

where

$$\frac{\partial^2 x_2(\omega)}{\partial \pi_i \partial V_2(\omega^+)} = \begin{cases} -\beta \frac{\partial^2 x_2(\omega)}{\partial \pi_i \partial B_2} & \text{if} \quad \omega < L, \\ 0 & \text{if} \quad \omega = L, \end{cases} \qquad i = 1, 2.$$

Jacobian: Optimal pricing equation (23) with respect to $p_1(\omega)$ and $p_2(\omega) = p_1(-\omega)$. Recalling the dependence of the first-order comparative statics $\frac{\partial x_i(\omega)}{\partial \pi_j}$ in equations (25)–(28) on $p_1(\omega)$, $p_2(\omega)$, $x_1(\omega)$, and $x_2(\omega)$, differentiating equation (23) with respect to $p_1(\omega)$ yields

$$\begin{aligned} &\frac{\partial^2 \pi_1(\omega, p_1(\omega), p_2(\omega))}{\partial p_1^2} \left(1 - F'(\pi_1(\omega, p_1(\omega), p_2(\omega))) x_1(\omega) + \frac{\partial U_1(\omega)}{\partial x_2} \frac{\partial x_2(\omega)}{\partial \pi_1} \right) \\ &+ \frac{\partial \pi_1(\omega, p_1(\omega), p_2(\omega))}{\partial p_1} \left[-F''(\pi_1(\omega, p_1(\omega), p_2(\omega))) \frac{\partial \pi_1(\omega, p_1(\omega), p_2(\omega))}{\partial p_1} x_1(\omega) \right. \\ &\left. + \frac{\partial U_1(\omega)}{\partial x_2} \left(\frac{\partial^2 x_2(\omega)}{\partial \pi_1^2} \frac{\partial \pi_1(\omega, p_1(\omega), p_2(\omega))}{\partial p_1} + \frac{\partial^2 x_2(\omega)}{\partial \pi_1 \partial \pi_2} \frac{\partial \pi_2(\omega, p_1(\omega), p_2(\omega))}{\partial p_1} \right) \right] \\ &+ \frac{\partial^2 \pi_2(\omega, p_1(\omega), p_2(\omega))}{\partial p_1^2} \frac{\partial U_1(\omega)}{\partial x_2} \frac{\partial x_2(\omega)}{\partial \pi_2} \\ &+ \frac{\partial \pi_2(\omega, p_1(\omega), p_2(\omega))}{\partial p_1} \frac{\partial U_1(\omega)}{\partial x_2} \left(\frac{\partial^2 x_2(\omega)}{\partial \pi_2^2} \frac{\partial \pi_2(\omega, p_1(\omega), p_2(\omega))}{\partial p_1} + \frac{\partial^2 x_2(\omega)}{\partial \pi_2 \partial \pi_1} \frac{\partial \pi_1(\omega, p_1(\omega), p_2(\omega))}{\partial p_1} \right), \end{aligned} \tag{33}$$

where we are careful not to interchange $\frac{\partial^2 x_1(\omega)}{\partial\pi_1\partial\pi_2}$ with $\frac{\partial^2 x_1(\omega)}{\partial\pi_2\partial\pi_1}$ and $\frac{\partial^2 x_2(\omega)}{\partial\pi_1\partial\pi_2}$ with $\frac{\partial^2 x_2(\omega)}{\partial\pi_2\partial\pi_1}$. Differentiating equation (23) with respect to $p_2(\omega)$ yields

$$\begin{aligned}
&\frac{\partial^2\pi_1(\omega,p_1(\omega),p_2(\omega))}{\partial p_1\partial p_2}\left(1-F'(\pi_1(\omega,p_1(\omega),p_2(\omega)))x_1(\omega)+\frac{\partial U_1(\omega)}{\partial x_2}\frac{\partial x_2(\omega)}{\partial\pi_1}\right)\\
&+\frac{\partial\pi_1(\omega,p_1(\omega),p_2(\omega))}{\partial p_1}\left[-F''(\pi_1(\omega,p_1(\omega),p_2(\omega)))\frac{\partial\pi_1(\omega,p_1(\omega),p_2(\omega))}{\partial p_2}x_1(\omega)\right.\\
&\left.+\frac{\partial U_1(\omega)}{\partial x_2}\left(\frac{\partial^2 x_2(\omega)}{\partial\pi_1^2}\frac{\partial\pi_1(\omega,p_1(\omega),p_2(\omega))}{\partial p_2}+\frac{\partial^2 x_2(\omega)}{\partial\pi_1\partial\pi_2}\frac{\partial\pi_2(\omega,p_1(\omega),p_2(\omega))}{\partial p_2}\right)\right]\\
&\qquad+\frac{\partial^2\pi_2(\omega,p_1(\omega),p_2(\omega))}{\partial p_1\partial p_2}\frac{\partial U_1(\omega)}{\partial x_2}\frac{\partial x_2(\omega)}{\partial\pi_2}\\
&+\frac{\partial\pi_2(\omega,p_1(\omega),p_2(\omega))}{\partial p_1}\frac{\partial U_1(\omega)}{\partial x_2}\left(\frac{\partial^2 x_2(\omega)}{\partial\pi_2^2}\frac{\partial\pi_2(\omega,p_1(\omega),p_2(\omega))}{\partial p_2}+\frac{\partial^2 x_2(\omega)}{\partial\pi_2\partial\pi_1}\frac{\partial\pi_1(\omega,p_1(\omega),p_2(\omega))}{\partial p_2}\right).
\end{aligned}$$

Jacobian: Optimal pricing equation (23) with respect to $x_1(\omega)$ and $x_2(\omega)=x_1(-\omega)$. Recalling the dependence of the first-order comparative statics $\frac{\partial x_i(\omega)}{\partial\pi_j}$ in equations (25)–(28) on $p_1(\omega)$, $p_2(\omega)$, $x_1(\omega)$, and $x_2(\omega)$, differentiating equation (23) with respect to $x_1(\omega)$ yields

$$\begin{aligned}
&\frac{\partial\pi_1(\omega,p_1(\omega),p_2(\omega))}{\partial p_1}\left(-F'(\pi_1(\omega,p_1(\omega),p_2(\omega)))+(B_1-A_1)\frac{\partial x_2(\omega)}{\partial\pi_1}+\frac{\partial U_1(\omega)}{\partial x_2}\frac{\partial^2 x_2(\omega)}{\partial\pi_1\partial x_1}\right)\\
&\qquad+\frac{\partial\pi_2(\omega,p_1(\omega),p_2(\omega))}{\partial p_1}\left((B_1-A_1)\frac{\partial x_2(\omega)}{\partial\pi_2}+\frac{\partial U_1(\omega)}{\partial x_2}\frac{\partial^2 x_2(\omega)}{\partial\pi_2\partial x_1}\right).
\end{aligned}$$

Differentiating equation (23) with respect to $x_2(\omega)$ yields

$$\frac{\partial\pi_1(\omega,p_1(\omega),p_2(\omega))}{\partial p_1}\frac{\partial U_1(\omega)}{\partial x_2}\frac{\partial^2 x_2(\omega)}{\partial\pi_1\partial x_2}+\frac{\partial\pi_2(\omega,p_1(\omega),p_2(\omega))}{\partial p_1}\frac{\partial U_1(\omega)}{\partial x_2}\frac{\partial^2 x_2(\omega)}{\partial\pi_2\partial x_2}.$$

Jacobian: Optimal investment equation (24) with respect to $V_1(\omega^-)$, $V_1(\omega)$, and $V_1(\omega^+)$. Differentiating equation (24) with respect to $V_1(\omega^-)$ yields

$$\begin{cases} 0 & \text{if} \quad \omega=-L,\\ \psi(Z_1)\beta\left(-x_2(\omega)\right) & \text{if} \quad \omega>-L. \end{cases}$$

Differentiating equation (24) with respect to $V_1(\omega)$ yields

$$\begin{cases} \psi(Z_1)\beta\left(-1+x_2(\omega)\right) & \text{if} \quad \omega=-L,\\ \psi(Z_1)\beta\left(-1+2x_2(\omega)\right) & \text{if} \quad -L<\omega<L,\\ \psi(Z_1)\beta x_2(\omega) & \text{if} \quad \omega=L. \end{cases}$$

Differentiating equation (24) with respect to $V_1(\omega^+)$ yields

$$\begin{cases} \psi(Z_1)\beta\left(1-x_2(\omega)\right) & \text{if} \quad \omega<L,\\ 0 & \text{if} \quad \omega=L. \end{cases}$$

Jacobian: Optimal investment equation (24) with respect to $p_1(\omega)$ and $p_2(\omega) = p_1(-\omega)$. Differentiating equation (24) with respect to $p_i(\omega)$ yields

$$-\psi(Z_1)F'(\pi_1(\omega, p_1(\omega), p_2(\omega)))\frac{\partial \pi_1(\omega, p_1(\omega), p_2(\omega))}{\partial p_i}, \quad i = 1, 2. \tag{34}$$

Jacobian: Optimal investment equation (24) with respect to $x_1(\omega)$ and $x_2(\omega) = x_1(-\omega)$. Differentiating equation (24) with respect to $x_1(\omega)$ yields

$$-1. \tag{35}$$

Differentiating equation (24) with respect to $x_2(\omega)$ yields

$$\psi(Z_1)\left(B_1 - A_1\right). \tag{36}$$

Adjustments. Because $V_1(0) = V_2(0)$, $U_1(0) = U_2(0)$, $p_1(0) = p_2(0)$, and $x_1(0) = x_2(0)$, we add the derivatives pertaining to $p_1(0)$ and $p_2(0)$ as well as those pertaining to $x_1(0)$ and $x_2(0)$.

Appendix C Check

In the pricing phase, firm 1 anticipates that changing its price changes its investment as well as the investment of firm 2 in the investment phase. To check if there is a unilateral profitable deviation from a candidate solution for firm 1 in state ω, we proceed in two steps.

Appendix C.1 Local deviations

First, restricting attention to local deviations and not using the envelope theorem, we examine the derivative of the optimal pricing equation (**??**) with respect to $p_1(\omega)$. This derivative is

$$\begin{aligned}
&\frac{\partial^2 \pi_1(\omega, p_1(\omega), p_2(\omega))}{\partial p_1^2}\left(1 - F'(\pi_1(\omega, p_1(\omega), p_2(\omega)))x_1(\omega) + \frac{\partial U_1(\omega)}{\partial x_1}\frac{\partial x_1(\omega)}{\partial \pi_1} + \frac{\partial U_1(\omega)}{\partial x_2}\frac{\partial x_2(\omega)}{\partial \pi_1}\right) \\
&\quad + \frac{\partial \pi_1(\omega, p_1(\omega), p_2(\omega))}{\partial p_1}\left[-F''(\pi_1(\omega, p_1(\omega), p_2(\omega)))\frac{\partial \pi_1(\omega, p_1(\omega), p_2(\omega))}{\partial p_1}x_1(\omega)\right. \\
&\quad\quad - F'\left(\pi_1(\omega, p_1(\omega), p_2(\omega))\right)\frac{\partial \pi_1(\omega, p_1(\omega), p_2(\omega))}{\partial p_1}\frac{\partial x_1(\omega)}{\partial \pi_1} \\
&\quad + \frac{\partial U_1(\omega)}{\partial x_1}\left(\frac{\partial^2 x_1(\omega)}{\partial \pi_1^2}\frac{\partial \pi_1(\omega, p_1(\omega), p_2(\omega))}{\partial p_1} + \frac{\partial^2 x_1(\omega)}{\partial \pi_1 \partial \pi_2}\frac{\partial \pi_2(\omega, p_1(\omega), p_2(\omega))}{\partial p_1}\right) \\
&\quad \left. + \frac{\partial U_1(\omega)}{\partial x_2}\left(\frac{\partial^2 x_2(\omega)}{\partial \pi_1^2}\frac{\partial \pi_1(\omega, p_1(\omega), p_2(\omega))}{\partial p_1} + \frac{\partial^2 x_2(\omega)}{\partial \pi_1 \partial \pi_2}\frac{\partial \pi_2(\omega, p_1(\omega), p_2(\omega))}{\partial p_1}\right)\right] \\
&\quad\quad + \frac{\partial^2 \pi_2(\omega, p_1(\omega), p_2(\omega))}{\partial p_1^2}\left(\frac{\partial U_1(\omega)}{\partial x_1}\frac{\partial x_1(\omega)}{\partial \pi_2} + \frac{\partial U_1(\omega)}{\partial x_2}\frac{\partial x_2(\omega)}{\partial \pi_2}\right) \\
&\quad + \frac{\partial \pi_2(\omega, p_1(\omega), p_2(\omega))}{\partial p_1}\left[-F'\left(\pi_1(\omega, p_1(\omega), p_2(\omega))\right)\frac{\partial \pi_1(\omega, p_1(\omega), p_2(\omega))}{\partial p_1}\frac{\partial x_1(\omega)}{\partial \pi_2}\right.
\end{aligned}$$

$$
\frac{\partial U_1(\omega)}{\partial x_1}\left(\frac{\partial^2 x_1(\omega)}{\partial \pi_2^2}\frac{\partial \pi_2(\omega,p_1(\omega),p_2(\omega))}{\partial p_1}+\frac{\partial^2 x_1(\omega)}{\partial \pi_2\partial \pi_1}\frac{\partial \pi_1(\omega,p_1(\omega),p_2(\omega))}{\partial p_1}\right)
$$

$$
+\frac{\partial U_1(\omega)}{\partial x_2}\left(\frac{\partial^2 x_2(\omega)}{\partial \pi_2^2}\frac{\partial \pi_2(\omega,p_1(\omega),p_2(\omega))}{\partial p_1}+\frac{\partial^2 x_2(\omega)}{\partial \pi_2\partial \pi_1}\frac{\partial \pi_1(\omega,p_1(\omega),p_2(\omega))}{\partial p_1}\right)\Bigg], \tag{37}
$$

where the third and seventh line arise because $\frac{\partial U_1(\omega)}{\partial x_1}$ in equation (**??**) depends on $F\left(\pi_1(\omega, p_1(\omega), p_2(\omega))\right)$ and we neglect the dependence of $\frac{\partial U_i(\omega)}{\partial x_j}$ and $\frac{\partial x_i(\omega)}{\partial \pi_j}$ on $x_1(\omega)$ and $x_2(\omega)$. Firm 1 has a unilateral profitable local deviation in state ω if the derivative in equation (37) evaluated at the candidate solution is positive.

Imposing the envelope condition $\frac{\partial U_1(\omega)}{\partial x_1} = 0$, the derivative in equation (37) simplifies to

$$
\frac{\partial^2 \pi_1(\omega,p_1(\omega),p_2(\omega))}{\partial p_1^2}\left(1-F'(\pi_1(\omega,p_1(\omega),p_2(\omega)))x_1(\omega)+\frac{\partial U_1(\omega)}{\partial x_2}\frac{\partial x_2(\omega)}{\partial \pi_1}\right)
$$

$$
+\frac{\partial \pi_1(\omega,p_1(\omega),p_2(\omega))}{\partial p_1}\Bigg[-F''(\pi_1(\omega,p_1(\omega),p_2(\omega)))\frac{\partial \pi_1(\omega,p_1(\omega),p_2(\omega))}{\partial p_1}x_1(\omega)
$$

$$
+\frac{\partial U_1(\omega)}{\partial x_2}\left(\frac{\partial^2 x_2(\omega)}{\partial \pi_1^2}\frac{\partial \pi_1(\omega,p_1(\omega),p_2(\omega))}{\partial p_1}+\frac{\partial^2 x_2(\omega)}{\partial \pi_1\partial \pi_2}\frac{\partial \pi_2(\omega,p_1(\omega),p_2(\omega))}{\partial p_1}\right)\Bigg]
$$

$$
+\frac{\partial^2 \pi_2(\omega,p_1(\omega),p_2(\omega))}{\partial p_1^2}\frac{\partial U_1(\omega)}{\partial x_2}\frac{\partial x_2(\omega)}{\partial \pi_2}
$$

$$
+\frac{\partial \pi_2(\omega,p_1(\omega),p_2(\omega))}{\partial p_1}\frac{\partial U_1(\omega)}{\partial x_2}\left(\frac{\partial^2 x_2(\omega)}{\partial \pi_2^2}\frac{\partial \pi_2(\omega,p_1(\omega),p_2(\omega))}{\partial p_1}+\frac{\partial^2 x_2(\omega)}{\partial \pi_2\partial \pi_1}\frac{\partial \pi_1(\omega,p_1(\omega),p_2(\omega))}{\partial p_1}\right) \tag{38}
$$

$$
-F'\left(\pi_1(\omega,p_1(\omega),p_2(\omega))\right)\frac{\partial \pi_1(\omega,p_1(\omega),p_2(\omega))}{\partial p_1}\left[\frac{\partial \pi_1(\omega,p_1(\omega),p_2(\omega))}{\partial p_1}\frac{\partial x_1(\omega)}{\partial \pi_1}+\frac{\partial \pi_2(\omega,p_1(\omega),p_2(\omega))}{\partial p_1}\frac{\partial x_1(\omega)}{\partial \pi_2}\right] \tag{39}
$$

Interestingly enough, taking the derivative with respect to $p_1(\omega)$ and then using the envelope theorem yields a different expression than using the envelope theorem and then taking the derivative of the optimal pricing equation (17).

Appendix C.2 Global deviations

Second, turning to global deviations, we solve the saddle point problem

$$
\max_{p_1}\min_{x_1,x_2} \pi_1(\omega,p_1,p_2(\omega))-F(\pi_1(\omega,p_1,p_2(\omega)))x_1-\int_{-\infty}^{\Psi^{-1}(x_1)}\theta_1 d\Psi(\theta_1)
$$

$$
+\beta\left[V_1(\omega^+)x_1(1-x_2)+V_1(\omega)(1-x_1-x_2+2x_1x_2)+V_1(\omega^-)(1-x_1)x_2\right] \tag{40}
$$

subject to equations (10) and (11) with $x_1(\omega)$ and $x_2(\omega)$ replaced by x_1 and x_2. In the spirit of simple penal codes (Abreu 1988), we assume that after deviating in the pricing phase firm 1 faces the worst possible continuation in the investment phase. This allows the model to generate the widest set of possible equilibrium behaviors. Accounting for numerical error, we say that firm 1 has a unilateral profitable global deviation in state ω if the value of the saddle point problem in equation (40) is larger than the value of the objective function evaluated at the candidate solution and if p_1, x_1, and x_2 in the saddle point problem are sufficiently different from the candidate solution.

To solve the saddle point problem, we nest the inner minimization problem given p_1 into the outer maximization problem over p_1. Starting with the inner minimization problem given p_1, we substitute equation (11) into equation (10) and aim to obtain all solutions to the resulting univariate equation in x_1 by a combination of a grid search and a derivative-free bisection algorithm. We select the solution that is associated with the worst possible continuation for firm 1. Turning to the outer maximization problem over p_1, we use a derivative-free golden section search algorithm.

Appendix D Limiting distribution

Let P denote the $(2L+1)\times(2L+1)$ state-to-state transition probability matrix constructed in equation (19) with typical element $P_{\omega,\omega'}$. The assumption $\theta_i \sim N(0,\sigma^2)$ ensures $x_i(\omega) \in (0,1)$ and thus $P_{\omega,\omega-1} > 0$, $P_{\omega,\omega} < 1$, and $P_{\omega,\omega+1} > 0$. It follows that the entire state space is one closed communicating class. The $1 \times (2L+1)$ limiting distribution μ^∞ is a solution to the system of linear equations

$$\mu^\infty P = \mu^\infty \Longleftrightarrow \mu^\infty (P - I) = 0,$$

where I is the $(2L+1)\times(2L+1)$ identity matrix. Because the system of linear equations is homogenous, if μ^∞ is a solution, then so is $\alpha\mu^\infty$ for any $\alpha \in \mathbb{R}$. We are therefore free to fix the scale of μ^∞ (or normalize any solution after obtaining it).

We develop a recursive formula for computing μ^∞. To reduce the number of unknowns and equations, we exploit that P is symmetric in the sense that $P_{-\omega,-\omega'} = P_{\omega,\omega'}$ for all $\omega,\omega' \in \{0,1,\ldots,L\}$. We thus have

$$\begin{pmatrix} \mu_0^\infty & \mu_1^\infty & \ldots & \mu_\omega^\infty & \ldots & \mu_{L-1}^\infty & \mu_L^\infty \end{pmatrix}$$
$$\cdot \begin{pmatrix} P_{0,0}-1 & P_{0,1} & 0 & 0 & 0 & \cdots & 0 \\ 2P_{1,0} & P_{1,1}-1 & P_{1,2} & 0 & 0 & \cdots & 0 \\ 0 & P_{2,1} & P_{2,2}-1 & P_{2,3} & 0 & \cdots & 0 \\ 0 & \cdots & \ddots & \ddots & \ddots & \cdots & 0 \\ 0 & \cdots & P_{\omega,\omega-1} & P_{\omega,\omega}-1 & P_{\omega,\omega+1} & \cdots & 0 \\ 0 & \cdots & \ddots & \ddots & \ddots & \cdots & 0 \\ 0 & \cdots & 0 & P_{L-2,L-3} & P_{L-2,L-2}-1 & P_{L-2,L-1} & 0 \\ 0 & \cdots & 0 & 0 & P_{L-1,L-2} & P_{L-1,L-1}-1 & P_{L-1,L} \\ 0 & \cdots & 0 & 0 & 0 & P_{L,L-1} & P_{L,L}-1 \end{pmatrix} = 0,$$

where the multiplication of $P_{1,0}$ by 2 in the second row and first column is the necessary adjustment for the dropped equations. Using that each row of P sums to 1, this can be rewritten as

$$\begin{pmatrix} \mu_0^\infty & \mu_1^\infty & \ldots & \mu_\omega^\infty & \ldots & \mu_{L-1}^\infty & \mu_L^\infty \end{pmatrix}$$

$$
\cdot \begin{pmatrix}
-2P_{0,1} & P_{0,1} & 0 & 0 & 0 & \cdots & 0 \\
2P_{1,0} & -(P_{1,0}+P_{1,2}) & P_{1,2} & 0 & 0 & \cdots & 0 \\
0 & P_{2,1} & -(P_{2,1}+P_{2,3}) & P_{2,3} & 0 & \cdots & 0 \\
 & & \ddots & \ddots & \ddots & & \\
0 & \cdots & & & & \cdots & 0 \\
0 & \cdots & P_{\omega,\omega-1} & -(P_{\omega,\omega-1}+P_{\omega,\omega+1}) & P_{\omega,\omega+1} & \cdots & 0 \\
 & & \ddots & \ddots & \ddots & & \\
0 & \cdots & & & & \cdots & 0 \\
0 & \cdots & 0 & P_{L-2,L-3} & -(P_{L-2,L-3}+P_{L-2,L-1}) & P_{L-2,L-1} & 0 \\
0 & \cdots & 0 & 0 & P_{L-1,L-2} & -(P_{L-1,L-2}+P_{L-1,L}) & P_{L-1,L} \\
0 & \cdots & 0 & 0 & 0 & P_{L,L-1} & -P_{L,L-1}
\end{pmatrix} = 0,
$$

where the multiplication of $P_{0,1}$ by 2 in the first row and first column is the necessary adjustment for the dropped equations, or

$$
\begin{gathered}
-2P_{0,1}\mu_0^\infty + 2P_{1,0}\mu_1^\infty = 0, \\
P_{0,1}\mu_0^\infty - (P_{1,0}+P_{1,2})\mu_1^\infty + P_{2,1}\mu_2^\infty = 0, \\
P_{1,2}\mu_1^\infty - (P_{2,1}+P_{2,3})\mu_2^\infty + P_{3,2}\mu_3^\infty = 0, \\
\vdots \\
P_{\omega-1,\omega}\mu_{\omega-1}^\infty - (P_{\omega,\omega-1}+P_{\omega,\omega+1})\mu_\omega^\infty + P_{\omega+1,\omega}\mu_{\omega+1}^\infty = 0, \\
\vdots \\
P_{L-3,L-2}\mu_{L-3}^\infty - (P_{L-2,L-3}+P_{L-2,L-1})\mu_{L-2}^\infty + P_{L-1,L-2}\mu_{L-1}^\infty = 0, \\
P_{L-2,L-1}\mu_{L-2}^\infty - (P_{L-1,L-2}+P_{L-1,L})\mu_{L-1}^\infty + P_{L,L-1}\mu_L^\infty = 0, \\
P_{L-1,L}\mu_{L-1}^\infty - P_{L,L-1}\mu_L^\infty = 0.
\end{gathered}
$$

Fixing μ_0^∞, we obtain the recursion

$$\mu_1^\infty = \frac{P_{0,1}\mu_0^\infty}{P_{1,0}}$$

together with

$$\mu_{\omega+1}^\infty = \frac{(P_{\omega,\omega-1}+P_{\omega,\omega+1})\mu_\omega^\infty - P_{\omega-1,\omega}\mu_{\omega-1}^\infty}{P_{\omega+1,\omega}}, \quad \omega = 1, 2, \ldots, L-1.$$

Alternatively, fixing μ_L^∞, we obtain the recursion

$$\mu_{L-1}^\infty = \frac{P_{L,L-1}\mu_L^\infty}{P_{L-1,L}}$$

together with

$$\mu_{\omega-1}^\infty = \frac{(P_{\omega,\omega-1}+P_{\omega,\omega+1})\mu_\omega^\infty - P_{\omega+1,\omega}\mu_{\omega+1}^\infty}{P_{\omega-1,\omega}}, \quad \omega = L-1, L-2, \ldots, 1.$$

To prevent a catastrophic cancellation in the numerator, we execute the recursion using symbolic math with infinite precision arithmetic.

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