IDENTIFICATION OF INTERDEPENDENT VALUES IN SEQUENTIAL FIRST-PRICE AUCTIONS
[PRELIMINARY AND INCOMPLETE, PLEASE DO NOT CIRCULATE]

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ABSTRACT. We revisit the (non-)identification of affiliated interdependent-value auctions from the perspective of sequential auctions introduced by Milgrom and Weber (2000). In contrast to static auctions, prices in early rounds affect bidding in later rounds in sequential auctions, generating enough variation for testing interdependent against private values and model identification based on the idea of the "loser's curse," without the usual assumption of exogenous variation in the auction data.

1. INTRODUCTION

This paper revisits the classic question of testing for interdependent values using auction data. Specifically, we examine the sequential auction framework introduced by Milgrom and Weber (2000), in which unit-demand bidders compete for objects that are sold sequentially, each via a sealed-bid first-price auction, and bidders observe the previous winning prices. This framework encompasses both the private and interdependent value paradigms with affiliated signals. We show that dynamic bidding in sequential auctions enables testing and identification of interdependent values, without exogenous variation in the auction data.

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Our analysis is motivated by two aspects of the existing auction literature. On the one hand, following the seminal contributions by Wilson (1977) and Milgrom and Weber (1982), the theory of interdependent-value static auctions has advanced on multiple fronts, ranging from abstract mechanism design to structural estimations using auction data. And it is well recognized that value interdependence plays a fundamental role in auctions and many other markets. However, in the latter literature, Laffont and Vuong (1996) point out a fundamental difficulty of nonparametrically identifying value interdependence in static sealed-bid auctions even when all the bids are observable.[1] Roughly speaking, statistical correlation of private signals can always substitute for the interdependence of values to account for the correlation in observed bids. In response to this negative result, subsequent research seeks tests of interdependence based on exogenous variation in the data, such as number of bidders (Athey and Haile (2002), Haile et al. (2004)), ex post information (Hendricks et al. (2003)), or additional exogenous information (Hortaçsu and Kastl (2012), Somaini (2019)), all in the static auction framework.[2]

On the other hand, in another seminal but less cited paper, Milgrom and Weber (2000) extend the static auction framework to dynamic settings with multiple identical objects and symmetric bidders.[3] While they argue that "rarely does one actually observe sales involving only one object" and many such sales are in sequential-formats, sequential auctions have received much less attention than their static counterparts in the literature. Within the empirical analysis of multi-object auctions,[4] few papers investigate sequential sales.[5] In particular, no work has examined identification and tests of interdependent values in sequential auctions to the best of our knowledge.

[1] Athey and Haile (2002) show that identification is even more challenging in open format auctions.

[2] See Athey and Haile (2007) and Hendricks and Porter (2007) for comprehensive surveys.

[3] Milgrom and Weber (2000), which was written in 1982 and later published in the book edited by Paul Klemperer, consider both simultaneous and sequential sales of the multiple objects when bidders have unit-demand.

[4] see Hortaçsu and McAdams (2018) for a recent survey.

[5] Exceptions are Jofre-Bonet and Pesendorfer (2003) (procurement auctions) who examine the dynamic linkages via capacity constraints in repeated first-price auctions with i.i.d. private values, Lamy (2010) (tobacco auctions) who studies two-round sequential English auctions with independent private values, and Kong (2017) (oil-and-gas lease auctions) who considers sequential auctions with synergies and value-affiliation across auctions under the private value paradigm.

In light of the negative result by Laffont and Vuong (1996), it is not obvious that a positive answer for testing interdependence exists under sequential auctions, since even though bidders bid multiple times in successive rounds, the bids correlation, both within and across rounds, may still be attributed to either interdependent values or affiliated private signals. However, the main insight of this paper is that winning prices in early rounds of the sequential auction affect bidding behavior in later rounds when bidders' signals are affiliated; moreover, their impacts differ significantly under private and interdependent value models.

Specifically, when the previous round's price is higher, the previous winner has a more favorable signal, which implies that a bidder will bid more aggressively in the current round, for two reasons. First, a more favorable signal indicates a stronger competition among the remaining bidders, as a result of signal affiliation. Second, a bidder's expected value of the object increases with a more favorable signal, due to value interdependence. While the first "competition effect" is present in both private and interdependent value settings, the second "value effect" exists only with interdependent values, which is the basis of the tests for interdependence.

Nonetheless, since both effects move in the same direction as the previous price varies, a reduced form test of the (positive) correlation between previous prices and current bids is not valid. Instead, to isolate the "value effect," we adapt the nonparametric method for static sealed-bid auctions introduced in the seminal paper by Guerre et al. (2000) to sequential auctions, with which we derive the "conditional pseudo values" associated with the bids in the final round conditioning on the previous winning prices. In the interdependent value model, the conditional pseudo values are increasing functions of the previous prices, whereas they are independent of the previous prices in the private value model. Based on this distinction, we provide a hypothesis testing of interdependent values (Proposition 3.1) with respect to the distributions of the "conditional pseudo values."

One advantage of the above test is that it only uses the winning prices in the penultimate round and bids in the final round instead of all the bids in sequential auctions, without exogenous variation of the data. Adapting a result from Athey and Haile (2002), we further show (Corollary 3.2) that the interdependent value model is testable whenever the winning prices in the last two rounds and the second-highest bid in the last round are observable. In addition, we establish (Propositions 3.3 and 3.4) that Corollary 3.2 essentially identifies the minimum data required to test interdependence. Finally, in Section 4, we illustrate the test with a parametric example.

It is instructive to compare our tests with the existing tests in static auctions using exogenous variations. The first such test is introduced by Paarsch (1992) under certain parametric assumptions of the distribution of bidders' values. Tests devised in later studies are most nonparametric. Hendricks et al. (2003) show that in first-price auctions with reserve prices, it is possible to distinguish the private and interdependent value models based on bids near the reserve price, although in practice such bids are rare to implement their test.[6] The test in Proposition 3.3 is similar in spirit to that in Hendricks et al. (2003) as it is based on the bids at (or near) the lower bound, which is not practical to execute. Hendricks et al. (2003) also show that a pure common value model (an extreme case of the interdependent value models) can be identified when ex post values are observable in addition to the bids. Athey and Haile (2002) and Haile et al. (2004) assume exogenous variation in the number of bidders in the data and develop tests based on the winner's curse. Intuitively, the winner's curse is more severe with more participants, which implies that bidders further shade their bids in equilibrium. In contrast, our tests of the "value effect" discussed above use the opposite of the winner's curse: losing an early round in a sequential auction yields a favorable signal to remaining bidders, which is in spirit of the "loser's curse," a property first discovered by Pesendorfer and Swinkels (1997) in the context of large uniform-price static auctions. More recent tests by Hortaçsu and Kastl (2012) and Somaini (2019) exploit different sources of exogenous variation in the data. In the context of Canadian Treasury auctions, Hortaçsu and Kastl (2012) test for value interdependence between dealers and customers by checking whether a dealer's marginal value remain constant after privately observing customer information.[7] Related to their idea, our test examines whether pseudo values are affected by the previous prices. An important distinction is that since past prices are observed by all remaining bidders, we need to deal with both the competition and value effects discussed above, whereas the

[6]Also see Hill and Shneyerov (2013) for a formal test based on the tail index of the bid distribution.
[7]Additional difficulties arise in Hortaçsu and Kastl (2012) due to multi-unit demand.

customer information is assumed to be exclusive for a given dealer in Hortaçsu and Kastl (2012) and hence there is no competition effect.

Besides testing private versus interdependent value models, we extend the procedure in Li et al. (2002) to sequential auctions with ex ante symmetric bidders to study nonparametric identification in the framework of Milgrom and Weber (2000) in Section 5. One interesting issue in this unit-demand framework is that winners of earlier rounds do not bid in later rounds. Therefore, in order to recover the joint distribution of all bidders' signals, we need to use the first-order conditions that link private signals with observed bids in all rounds. Specifically, we first consider the last round of the auction, which is essentially a static sealed-bid first-price auction among a subgroup of all bidders. The corresponding first-order condition translates the bids into pseudo values for these bidders, which generates the conditional distributions of the vector of pseudo values given any realized prices in previous rounds. Then we use these conditional distributions to proceed backward to obtain the pseudo values of all the previous winners. We find that the affiliated private model (Proposition 5.1) can be nonparametrically identified and imposes testable restrictions on the dynamic bidding data. Furthermore, although the general affiliated interdependent value model cannot be fully identified, we show that the joint distribution of signals and certain conditional expectations of the true values can be recovered uniquely from all bidding data.

Our work contributes to the literature on structural analysis of auctions. We extend the identification methodology pioneered by Guerre et al. (2000) to sequential auctions with symmetric and unit-demand bidders. While sequential auctions are challenging to analyze in general, we made some preliminary progress in the framework of Milgrom and Weber (2000), complementing and expanding the vibrant literature on structural estimations of static auctions. We show that variation of the prices in early rounds together with the nonparametric approach provides an affirmative answer to the classical question of testing interdependence, without exogenous data variation. In Section 6, we conclude with several remarks regarding the applicability and limitations of our testing and identification results.

2. The Setting

Consider the sequential auction framework in Milgrom and Weber (2000): $K(\geq 2)$ units of an identical objects are auctioned to $n(> K)$ bidders, each of whom demands for one unit. Bidder $i \in \{1, \ldots, n\}$ receives a payoff $U_i - p$ from winning an object at a price p and zero otherwise, where U_i is bidder i's value of the object. Denote $\mathbf{U} = (U_1, \ldots, U_n)$ a vector of values for all bidders. Let $F_{\mathbf{U}}(\cdot)$ be the joint distribution of $\mathbf{U}$ with a strictly positive density function $f_{\mathbf{U}}(\cdot)$. Each bidder i's private information is a real-valued signal $X_i \in [\underline{x}, \bar{x}]$ that is strictly affiliated with U_i. Denote $\mathbf{X} = (X_1, \ldots, X_n)$ a vector of private signals. Let $F_{\mathbf{X}}(\cdot)$ be the joint distribution of $\mathbf{X}$ with a strictly positive density function $f_{\mathbf{X}}(\cdot)$.[8]

We assume that bidders are *symmetric*, that is, the joint distribution of $(\mathbf{U}, \mathbf{X})$, denoted by $F_{\mathbf{U},\mathbf{X}}(\cdot)$, is exchangeable with respect to bidder indices. Denote $F_U(\cdot)$ and $F_X(\cdot)$ the marginal distributions of U_i and X_i, respectively. When $U_i = X_i$ for all i, bidders have private values; if, in addition, $\mathbf{X}$ is affiliated, then the model is said to be an *affiliated private values* (APV) model. On the other hand, when U_i and X_j are strictly affiliated for all i and j conditional on any subset of $\{X_k\}_{k \neq j}$, but are not perfectly correlated, then the model is said to be an *affiliated interdependent values* (AV) model.

In the baseline model we focus on sequential first-price auctions with price announcements, where the objects are sold one in each round via a sealed-bid first-price auction and the auctioneer reveals previous winning bids before bidding in the next round. Finally, we assume that the number of bidders is common knowledge and there is no reserve price.

2.1. Strategy and Equilibrium.

We consider weak perfect Bayesian equilibrium in symmetric monotone strategies. Throughout the paper, for a generic random vector $\mathbf{S} = (S_1, \ldots, S_n)$ drawn from a distribution $F_{\mathbf{S}}(\cdot)$, denote $S^{(j)}$ the j-th order statistic, for any $j = 1, \ldots, n$; in particular, denote $F_{\mathbf{S}}^{(1:j)}(\cdot)$ the joint distribution of the first j-highest order statistics $(S^{(1)}, \ldots, S^{(j)})$.

In each round $k = 1, 2, \ldots, K$, let p_k be the winning price in round k. The price history at the beginning of round k is a sequence of previous winning prices $\tilde{p}_{k-1} =$

[8]We follow Athey and Haile (2002) for the description of bidders' values and signals. See Wilson (1977) and Milgrom and Weber (1982) for other ways of defining values and signals.

$(p_1, \ldots, p_{k-1})$. A (symmetric) strategy, $\beta = \{\beta_1, \ldots, \beta_K\}$, is a sequence of bid functions, where for each $k = 1, \ldots, K$, $\beta_k(x_i, \tilde{p}_{k-1})$ is bidder i's bid in round k given her realized signal $X_i = x_i$ and the history of winning prices $\tilde{p}_{k-1}$. A strategy β is *monotone* if for each $k = 1, \ldots, K$, β_k is increasing in x_i for all $\tilde{p}_{k-1}$. A monotone strategy β is a *symmetric weak perfect Bayesian equilibrium* if for each i, x_i, $k \leq K-1$, and $\tilde{p}_{k-1}$, the equilibrium bid $\beta_k(x_i, \tilde{p}_{k-1})$ solves

$$\begin{aligned}\Pi_k(x_i, \tilde{p}_{k-1}) = \max_{b_k} \Bigg(& \mathbb{E}\left[U_i \middle| X_i = x_i, \max_{j \in \mathcal{N}_k, j \neq i} B_{j,k} \leq b_k, \tilde{p}_{k-1}\right] - b_k \Bigg) \\ & \cdot \Pr\left(\max_{j \in \mathcal{N}_k, j \neq i} B_{j,k} \leq b_k \middle| X_i = x_i, \tilde{p}_{k-1} \right) \\ & + \mathbb{E}\left[\Pi_{k+1}\left(x_i, \tilde{p}_{k-1}, \max_{j \in \mathcal{N}_k, j \neq i} B_{j,k}\right) \middle| X_i = x_i, \max_{j \in \mathcal{N}_k, j \neq i} B_{j,k} > b_k, \tilde{p}_{k-1} \right] \\ & \cdot \Pr\left(\max_{j \in \mathcal{N}_k, j \neq i} B_{j,k} > b_k \middle| X_i = x_i, \tilde{p}_{k-1} \right),\end{aligned}$$

where $\mathcal{N}_k$ is the set of active bidders in round k, $B_{j,k}$ is bidder j's equilibrium bid in round k, and $\Pi_K(x_i, \tilde{p}_{K-1})$ is given by

$$\begin{aligned}& \Pi_K(x_i, \tilde{p}_{K-1}) \\ = & \max_{b_K} \left(\mathbb{E}\left[U_i \middle| X_i = x_i, \max_{j \in \mathcal{N}_K, j \neq i} B_{j,K} \leq b_K, \tilde{p}_{K-1}\right] - b_K \right) \\ & \cdot \Pr\left(\max_{j \in \mathcal{N}_K, j \neq i} B_{j,K} \leq b_K \middle| X_i = x_i, \tilde{p}_{K-1} \right).\end{aligned}$$

To derive the symmetric equilibrium $\beta = (\beta_k)_{k=1}^K$, without loss of generality consider bidder 1 and let $Y_1 > \cdots > Y_{n-1}$ denote the order statistics of $X_2, \ldots, X_n$.[9] For each $m = 1, \ldots, n-1$, Denote $F_{Y_m}(\cdot)$ the distribution of Y_m and $f_{Y_m}(\cdot)$ the corresponding density function. Since each β_k is increasing in x_1, in round k bidder 1 can infer the realized values of $Y_1 = y_1, \ldots, Y_{k-1} = y_{k-1}$ from the prices in previous rounds $\tilde{p}_{k-1}$. With a slight abuse of notation, here we write β_k as a function of x_1 and $(y_1, \ldots, y_{k-1})$. It follows from Milgrom and Weber (2000) that β is the unique

[9] Throughout the paper, especially for tests and identification, we shall consider bidder 1's problem.

solution to the following system of differential equations:

$$\frac{\partial \beta_k(x_1,y_1,\dots,y_{k-1})}{\partial x_1} = [\beta_{k+1}(x_1,y_1,\dots,y_{k-1},x_1) - \beta_k(x_1,y_1,\dots,y_{k-1})]\,\frac{f_{Y_k}(x_1|x_1,\dots,y_1,\dots,y_{k-1})}{F_{Y_k}(x_1|x_1,\dots,y_1,\dots,y_{k-1})}, \tag{1}$$

for all $k=1,\dots,K$, where

$$\beta_{K+1}(x_1,y_1,\dots,y_K) = \mathbb{E}[U_1|X_1=x_1,Y_1=y_1,\dots,Y_K=y_K],$$

and $F_{Y_k}(\cdot|x_1,\dots,y_1,\dots,y_{k-1})$ is the conditional distribution of Y_k given $X_1=x_1$, $Y_1=y_1,\dots,Y_{k-1}=y_{k-1}$ (f_{Y_k} is the corresponding density), with boundary conditions

$$\beta_k(\underline{x},y_1,\dots,y_{k-1}) = \mathbb{E}[U_1|X_1=\underline{x},Y_1=y_1,\dots,Y_{k-1}=y_{k-1},Y_k=\underline{x}],$$

for all $k=1,\dots,K$.

2.2. **Tests and Identification.** Following the structural approach (e.g. Laffont and Vuong (1996) and Athey and Haile (2002)), we examine circumstances in which a sample of observations are generated from the equilibrium of the sequential auctions described in the previous section, where the joint distribution of $(\mathbf{U},\mathbf{X})$ is fixed but unknown to the researcher. For each k and $i\in\mathcal{N}_k$, let $B_{i,k}$ denote an active bidder i's bid in round k with a distribution $G_{B_{i,k}}(\cdot)$. Let $\mathbf{B}_k$ denote the collection of bids in round k and $\mathbf{B}$ the collection of bids in all rounds. Denote $G_{\mathbf{B}_k}(\cdot)$ the joint distribution of bids in round k and $G_{\mathbf{B}}(\cdot)$ the joint distribution of bids in all rounds.

We assume that the researcher observes a subset $\mathbf{H}$ of the set of all bids $\mathbf{B}$. In particular, the researcher knows the joint distribution of $\mathbf{H}$, denoted by $G_{\mathbf{H}}(\cdot)$, from independent observations of the same sequential auction. For instance, if the researcher has access to all the bidding data, then $\mathbf{H}=\mathbf{B}$; if the researcher only observes the winning prices in all rounds, then $\mathbf{H}=(B_k^{(1)})_{k=1}^K$, where $B_k^{(1)} = \max_{i\in\mathcal{N}_k} B_{i,k}$ is the highest bid in round k.

The researcher's problem is to make inferences about the joint distribution of values and signals of bidders from the observables. Formally, following Athey and Haile (2002), a *model* is a tuple $(\mathbb{F},\gamma)$, where $\mathbb{F}$ is a collection of joint distributions over bidders' values and signals and $\gamma:\mathbb{F}\to\mathbb{G}$ is a mapping from the set of latent random variables to the set of observable random variables. That is, $\mathbb{G}$ is the collection of all joint distributions over the observables $\mathbf{H}$. We say that a model $(\mathbb{F}, \gamma)$ is *falsifiable* if $\bigcup_{F \in \mathbb{F}} \gamma(F) \neq \mathbb{G}$, that is, one can refute the model based on the equilibrium distribution of the observables. A model $(\mathbb{F}, \gamma)$ is *testable* against another model $(\hat{\mathbb{F}}, \hat{\gamma})$ if there exists $\hat{F} \in \hat{\mathbb{F}}$ such that $\hat{\gamma}(\hat{F}) \notin \bigcup_{F \in \mathbb{F}} \gamma(F)$, that is, it is possible to distinguish $(\mathbb{F}, \gamma)$ from $(\hat{\mathbb{F}}, \hat{\gamma})$ based on observables. A model $(\mathbb{F}, \gamma)$ is *identifiable* if for each pair $(F, \tilde{F}) \in \mathbb{F}^2$, $\gamma(F) = \gamma(\tilde{F})$ implies $F = \hat{F}$, that is, one can uniquely determine the joint distribution of bidders' values and signals from the observables. Finally, a data distribution $G \in \mathbb{G}$ is *rationalized* by a model $(\mathbb{F}, \gamma)$ if there exists a $F \in \mathbb{F}$ such that $\gamma(F) = G$.

3. Tests of Interdependent versus Private Values

In this section, we show that sequential auction data enable testing between APV and AV models, without exogenous variation in the number of bidders. In particular, we will examine the minimal data requirement for the existence of such tests. For notational simplicity, here we assume that there are two ($K = 2$) units to be auctioned to n bidders. One can directly adapt the results below to the general case with arbitrarily many units by focusing on the last two rounds.

First suppose that all the bids are observed, i.e., $\mathbf{H} = \mathbf{B}$. Denote P_1 the winning price in the first round, which is a random variable given by $P_1 = \beta_1(X^{(1)})$. Define the "value function" of bidder 1 conditional on (losing the first round and) winning the second round as:

$$v(x_1, x_1, p_1) = \mathbb{E}\left[U_1 | X_1 = x_1, \max_{j \in \mathcal{N}_2} X_j = x_1, P_1 = p_1\right]. \tag{2}$$

A key observation is that $v(x, x, p_1)$ is independent of p_1 for any x in the APV model, but it is strictly increasing in p_1 in the AV model. Intuitively, since the bidding function β_1 is monotone, p_1 reveals the winner's signal $X^{(1)}$ in the first round, which conveys to the remaining bidders the favorable information about their expected values of the object, a form of the *loser's curse* first identified in Pesendorfer and Swinkels (1997) in the context of information aggregation in large common value auctions.

Recall that the bidding function $\beta_2(x_1, y_1)$ in the second round is given by

$$\frac{\beta_2(x_1, y_1)}{\partial x_1} = [v(x_1, x_1, \beta_1(y_1)) - \beta_2(x_1, y_1)] \frac{f_{Y_2}(x_1 | x_1, y_1)}{F_{Y_2}(x_1 | x_1, y_1)}. \tag{3}$$

Using a procedure similar to Guerre et al. (2000) and Li et al. (2002), we first apply a change of variables based on $b_{1,2} = \beta_2(x_1, y_1)$ and $p_1 = \beta_1(y_1)$ to obtain

$$G_{B^{(1)}_{-1,2}}(b_{1,2}|b_{1,2}, p_1) = F_{Y_2}(x_1|x_1, y_1), \tag{4}$$

and

$$g_{B^{(1)}_{-1,2}}(b_{1,2}|b_{1,2}, p_1) = f_{Y_2}(x_1|x_1, y_1) \cdot \frac{1}{\frac{\beta_2(x_1,y_1)}{\partial x_1}}, \tag{5}$$

where $B^{(1)}_{-1,2} = \max_{j \in \mathcal{N}_2, j \neq 1} B_{j,2}$ is the highest bid in the second round among all remaining bidders but bidder 1, $b_{1,2}$ is bidder 1's realized bid in the second round, and $G_{B^{(1)}_{-1,2}}(\cdot|b_{1,2}, p_1)$ is the conditional distribution of $B^{(1)}_{-1,2}$ given $b_{1,2}$ and p_1 ($g_{B^{(1)}_{-1,2}}$ is the corresponding density).

Substituting (4) and (5) into (3) yields

$$v(x_1, x_1, p_1) = b_{1,2} + \frac{G_{B^{(1)}_{-1,2}}(b_{1,2}|b_{1,2}, p_1)}{g_{B^{(1)}_{-1,2}}(b_{1,2}|b_{1,2}, p_1)}. \tag{6}$$

Since the right-hand side of (6) can be directly derived from bid data and the left-hand side is constant in the APV model and is strictly increasing in p_1 for all x_1 in the AV model, this equation gives a test of the two models, as long as there is variation of the first round price in the data. The result is summarized in the following proposition.

Proposition 3.1. *In the sequential first-price auction with price announcement, if all the bids are observable, that is,* $\mathbf{H} = \mathbf{B}$*, then the symmetric affiliated private values model is testable against the symmetric affiliated interdependent values model.*

Proof. For any observed price in the first round $P_1 = p_1$, we can construct the distribution $G_{B^{(1)}_{-1,2}}(\cdot|b_{1,2}, p_1)$ and its density $g_{B^{(1)}_{-1,2}}(\cdot|b_{1,2}, p_1)$ from the corresponding bid data from the second round. This gives

$$\xi(b_{1,2}, p_1) \triangleq b_{1,2} + \frac{G_{B^{(1)}_{-1,2}}(b_{1,2}|b_{1,2}, p_1)}{g_{B^{(1)}_{-1,2}}(b_{1,2}|b_{1,2}, p_1)}.$$

Since P_1 is a strictly increasing function of the highest order statistic of all bidders' signals, there is variation of the realized P_1 in the data $\mathbf{H}$. Therefore, we can test interdependent values based on whether$\xi(b_{1,2}, p_1)$ is increasing in p_1.

Let $F_{v,p_1}(\cdot)$ be the distribution of the random variable $v(X_1, X_1, p_1)$ for any given p_1. Then, $F_{v,p_1}(\cdot)$ is increasing in p_1 in the sense of first-order stochastic dominance in the AV model. However, since the equilibria are monotone in both AV and APV models, a larger p_1 implies that the upper bound of the remaining bidders' signals, which equals the signal of the winner in the first round, is also higher; thus, $F_{v,p_1}(\cdot)$ is also increasing in p_1 in the APV model. To deal with this issue, we instead consider the *conditional* distribution of $F_{v,p_1}(\cdot)$ with a fixed support as the price p_1 varies.

Formally, for any realized prices p_1' and $p_1'' > p_1'$, denote $\tilde{V}$ the common support of the distributions $F_{v,p_1'}(\cdot)$ and $F_{v,p_1''}(\cdot)$. Let $\tilde{F}_{v,p_1'}(z)$ and $\tilde{F}_{v,p_1''}(z)$ be the conditional distributions of $F_{v,p_1'}(z)$ and $F_{v,p_1''}(z)$ given that $z \in \tilde{V}$, respectively. In the APV model, $F_{v,p_1'}(z)$ and $F_{v,p_1''}(z)$ are identical, whereas $\tilde{F}_{v,p_1''}(\cdot)$ first-order stochastically dominates $\tilde{F}_{v,p_1'}(\cdot)$ in the AV model due to affiliation. Then we can test the follow hypothesis

$$H_0(\text{APV}) : \tilde{F}_{v,p_1'}(\cdot) = \tilde{F}_{v,p_1''}(\cdot)$$

against the alternative

$$H_1(\text{AV}) : \tilde{F}_{v,p_1'}(\cdot) > \tilde{F}_{v,p_1''}(\cdot).$$

□

Proposition 3.1 implies that the variation in the first round price, which is a unique feature in sequential auctions, disentangles the correlation of the second round bids coming from either statistical correlation (affiliation) or a common value component. In fact, the following corollary shows that as long as the winning prices in both rounds plus the second highest bid in the second round are observable, such a test can be constructed.

Corollary 3.2. *If the observables* $\mathbf{H}$ *contain the winning prices in both rounds and the second highest bid in the second round, that is,* $\{P_1, P_2, \max_{j \in \mathcal{N}_2}\{B_{j,2} : B_{j,2} < P_2\}\} \in \mathbf{H}$*, then the symmetric APV model is testable against the symmetric AV model.*

Proof. The proof extends the argument of Lemma 1 in Athey and Haile (2002). Without loss of generality, suppose that bidder 1 is the winner of the second auction with a bid $b_{1,2}$ given the price of the first round is p_1. Since $\mathbf{H} = \{P_1, B_{1,2}, B^{(1)}_{-1,2}\}$, we have

$$\begin{aligned}
\frac{G_{B^{(1)}_{-1,2}}(b_{1,2}|b_{1,2},p_1)}{g_{B^{(1)}_{-1,2}}(b_{1,2}|b_{1,2},p_1)} &= \frac{\Pr\left(B^{(1)}_{-1,2} \leq b_{1,2}|B_{1,2} = b_{1,2}, P_1 = p_1\right)}{\frac{\partial}{\partial z}\Pr\left(B^{(1)}_{-1,2} \leq z|B_{1,2} = b_{1,2}, P_1 = p_1\right)\Big|_{z=b_{1,2}}} \\
&= \frac{\frac{\partial}{\partial y}\Pr\left(B^{(1)}_{-1,2} \leq b_{1,2}, B_{1,2} \leq y|P_1 = p_1\right)\Big|_{y=b_{1,2}}}{\frac{\partial^2}{\partial z \partial y}\Pr\left(B^{(1)}_{-1,2} \leq z, B_{1,2} \leq y|P_1 = p_1\right)\Big|_{z=y=b_{1,2}}} \\
&= \frac{\frac{\partial}{\partial y}G_{\mathbf{H}}(y, b_{1,2}|p_1)\Big|_{y=b_{1,2}}}{\frac{\partial^2}{\partial z \partial y}G_{\mathbf{H}}(y, z|p_1)\Big|_{z=y=b_{1,2}}} \\
&= \frac{\frac{\partial^2}{\partial y \partial w}G_{\mathbf{H}}(w, y, b_{1,2})\Big|_{y=b_{1,2}, w=p_1}}{\frac{\partial^3}{\partial z \partial y \partial w}G_{\mathbf{H}}(w, y, z)\Big|_{z=y=b_{1,2}, w=p_1}},
\end{aligned}$$

where $G_{\mathbf{H}}(\cdot,\cdot|p_1)$ is the conditional distribution of the two highest bids in the second round given the price in the first round p_1. That is, the right-hand side of (6) is identified from the observables $\mathbf{H}$. The result then follows from the proof of Proposition 3.1. □

The above results exploit the joint information between the two highest bids in the second round and the first-round price. Next, motivated by the fact that bidding data may be limited in certain auctions to the researcher, we examine tests of APV against AV models when only winning bids (prices) are observable, i.e., $\mathbf{H} = \{P_1, P_2\}$. We first show (in Proposition 3.3) that, at least in theory, one can still test the model based on the impact of P_1 on P_2, without even referring to the first-order conditions. However, since the test we will construct exploits the lower bound of the conditional distribution of P_2 given P_1, it is rather "knife-edge" and impractical.

Proposition 3.3. *If $\mathbf{H} = \{P_1, P_2\}$, then the symmetric APV model is testable against the symmetric AV model.*

Proof. For any realized $P_1 = p_1$, let $\underline{p}_2(p_1)$ denote the lower bound of the support of the conditional distribution $G(P_2|p_1)$. Note that $\underline{p}_2(p_1)$ corresponds to the winning price of the bidder with the lowest possible signal $\underline{x}$. Furthermore, by a Bertrand-competition argument, in either APV or AV models the bidder with signal $\underline{x}$ in the second round will bid her conditionally expected value given price p_1 and the event that she will win the second round. In the APV model, since $\underline{x}$ is the private value of this bidder, $\underline{p}_2(p_1)$ is independent of p_1. In contrast, in the AV model, the bidder's conditional expectation of her value, $\mathbb{E}[U_1|X_1 = \underline{x}, \max_{j\in\mathcal{N}_2} X_j = \underline{x}, P_1 = p_1]$, is strictly increasing in p_1 by the affiliation assumption. Therefore, the APV model is testable against the AV model when winning prices are observable. □

Since in practice the joint distribution of prices $G_{\mathbf{H}}$ used for tests is first estimated from realized prices, the researcher can at most approximate the lower bound of the second round prices from an actual data set. In addition, since in the APV model the winning price in the second round is increasing in p_1 for all but the lowest possible value, the test in Proposition 3.3 can be sensitive to the estimation of $G_{\mathbf{H}}$. To rule out such tests, we modify the testability of a model as follows.

We say that a model $(\mathbb{F}, \gamma)$ is *robustly testable* against another model $(\hat{\mathbb{F}}, \hat{\gamma})$ if there exist $\varepsilon > 0$ and $\hat{F} \in \hat{\mathbb{F}}$ such that $\{G \in \mathbb{G} : \|G - \hat{\gamma}(\hat{F})\| \leq \varepsilon\} \cap (\bigcup_{F\in\mathbb{F}} \gamma(F)) = \emptyset$, where $\|\cdot\|$ is the sup-norm. That is, one model is robustly testable against another if it is possible to distinguish them even with small observation errors. Note that for the collection of conditional distributions of P_2 given any p_1 in the AV model, we can slightly perturb all the distributions such that the support of all the perturbed distributions have the same lower bound, which implies that the test in Proposition 3.3 is not robust.[10] To the contrary, since the test in Proposition 3.1 is based on first-order stochastic dominance relation of the conditional distributions, it is robust to perturbations of the distributions of the observables.

Next we show (Proposition 3.4) that observing only the winning prices in sequential auctions does not enable robust testings between APV and AV models.

[10]Another way to see this is to consider the limiting AV case in which a bidder's conditionally expected value is constant whenever she receives the lowest signal. Now the test based on the lower bound of the support is invalid, as in both AV and APV models, the lower bounds are constant.

The intuition, related to Laffont and Vuong (1996), is that the statistical correlation between the winning prices in different rounds could be due to either signal affiliation or value interdependence.

Proposition 3.4. *If* $\mathbf{H} = \{P_1, P_2\}$*, then the symmetric APV model is not robustly testable against the symmetric AV model.*

Proof. Note that in the AV model given any equilibrium price $p_1 = \beta_1(y_1)$ in the first round, the bid of bidder 1 with signal $x_1 (\leq y_1)$ in the second round is the following conditional expectation

$$\beta_2(x_1, y_1) = \int_{\underline{x}}^{x_1} v(z, z, \beta_1(y_1)) dL_2(z|x_1, y_1),$$

where

$$L_2(z|x_1, y_1) = \exp\left(-\int_z^{x_1} \frac{f_{Y_2}(t|t, y_1)}{F_{Y_2}(t|t, y_1)} dt\right)$$

is a distribution function with a support $[\underline{x}, x_1]$. Let Z_{y_1} be a random variable with a distribution $L_2(\cdot|x_1, y_1)$. Let $\tilde{Z}_{y_1} = v(Z_{y_1}, Z_{y_1}, \beta_1(y_1))$ and $Z_{y_1} = \phi(\tilde{Z}_{y_1}, \beta_1(y_1))$, where ϕ is the inverse of $v(\cdot, \cdot, \beta_1(y_1))$. Then distribution of $\tilde{Z}_{y_1}$ is given by

$$\begin{aligned}
\Pr(\tilde{Z}_{y_1} \leq \tilde{z}) &= \Pr(v(Z_{y_1}, Z_{y_1}, \beta_1(y_1)) \leq \tilde{z}) \\
&= \Pr(Z_{y_1} \leq \phi(\tilde{z}, \beta_1(y_1))) \\
&= L_2(\phi(\tilde{z}, \beta_1(y_1))|x_1, y_1) \\
&\triangleq \tilde{L}_2(\tilde{z}|x_1, y_1).
\end{aligned}$$

Therefore, $\beta_2(x_1, y_1)$ can be rewritten as

$$\beta_2(x_1, y_1) = \int_{v(\underline{x}, \underline{x}, y_1)}^{v(x_1, x_1, \beta_1(y_1))} w d\tilde{L}_2(w|x_1, y_1),$$

which, after a perturbation of the lower bound of the support to some fixed number $\tilde{x}_l$, is indistinguishable from the second round's bidding function in another APV model in which

$$\beta_2^{APV}(\tilde{x}_1, \tilde{y}_1) = \int_{\tilde{x}_l}^{\tilde{x}_1} w d\tilde{L}_2(w|\tilde{x}_1, \tilde{y}_1).$$

Next consider the first round's bidding function:

$$\beta_1(x_1) = \int_{\underline{x}}^{x_1} \beta_2(z, z) dL_1(z|x_1),$$

where

$$L_1(z|x_1) = \exp\left(-\int_z^{x_1} \frac{f_{Y_1}(t|t)}{F_{Y_1}(t|t)}dt\right)$$

is a distribution function with a support $[\underline{x}, x_1]$. Note that β_1 has the same functional form in both the AV and the APV models. Therefore, there exists a distribution function $\tilde{L}_1(\cdot|\tilde{x}_1)$ such that

$$\beta_1^{APV}(\tilde{x}_1) = \int_{\tilde{x}_l}^{\tilde{x}_1} \beta_2^{APV}(w,w)d\tilde{L}_1(w|\tilde{x}_1)$$

is indistinguishable from β_1. □

Note that the negative result in Proposition 3.4 extends to the general case with more than two units. Thus, it gives a precise characterization of the minimal data requirement for discriminating between the two models and highlights the importance of the first-order structural approach underlying the test procedures. Although we state the result for the case with two units, it extends straightforwardly to the general case with finitely many units. Thus, it shows that the researcher cannot use a reduced-form test based solely on realized price sequences to differentiate between APV and AV models in sequential first-price auctions.

4. A Numerical Example

This section presents an example adapted from Chapter 6 in Krishna (2009) to illustrate the tests discussed previously. Suppose $n = 3$ and $K = 2$. Bidder i gets a private signal $X_i = S_i + T \in [0,2]$, where both S_i and T have a support $[0,1]$. We assume S_1, S_2, S_3 and T are independently distributed and bidder i only observe X_i. Due to the common component T, the signals $\{X_i\}$ are affiliated. Given this signal structure, we can consider both APV and AV models. In particular, we assume

$$V_i(X_i, X_{-i}) = \begin{cases} X_i, & \text{in an APV model,} \\ \dfrac{\sum_{j=1}^{3} X_j}{3}, & \text{in an AV model.} \end{cases}$$

For the tests, we will focus on equilibrium bidding in the second round. Note that in the APV model, the equilibrium is

$$\beta_2^{APV}(v,y_1) = \int_0^v z dL_2(z|x,y_1),$$

and in the AV model, the equilibrium is

$$\beta_2^{AV}(v,y_1) = \int_0^v V(z,z,y_1) dL_2(z|x,y_1) = \frac{2}{3}\int_0^v z dL_2(z|x,y_1) + \frac{y_1}{3},$$

where in both cases bidders in the second round can back out the first winner's value y_1 from the winning price p_1 in the first round.

We assume both S_i and T are uniformly distributed over $[0,1]$. In this parametric example the above bid functions have closed forms.[11]. Given the closed-form equilibria, we can calculate the bids for any signal $x_1 \in [y_1 - 1, y_1]$ when $y_1 \geq 1$, where the support of the signals of the remaining bidders is due to the common component. In figures 1 and 2 we illustrate the bid functions in the APV and AV models for two price levels $p_1 = \beta_1(1.3)$ and $\hat{p}_1 = \beta_1(1.8)$. Clearly, higher prices lead to higher bids in both models. Thus, as we discussed before, the positive correlation between the first round prices and the bids in the second round does not allow the researcher to ascertain the correct model.

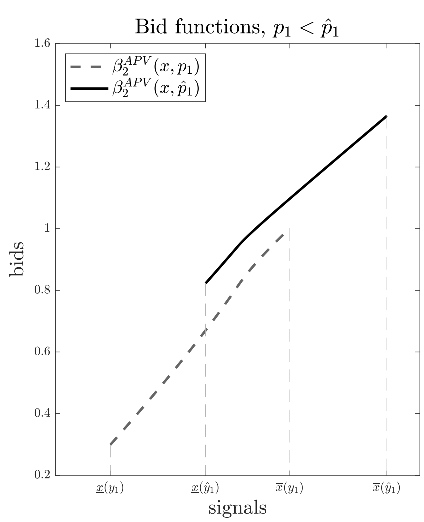

FIGURE 1. Bids in APV

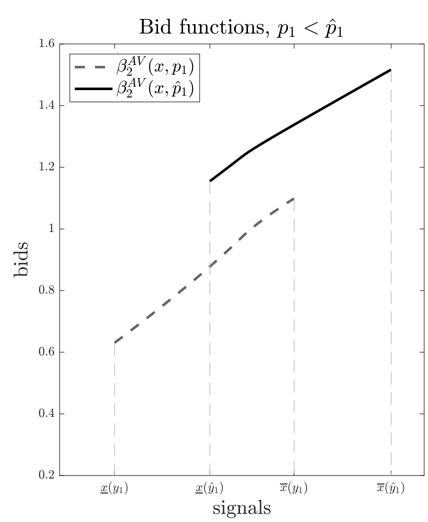

FIGURE 2. Bids in AV

[11]The derivation can be found in the supplemental appendix

From the bids, we can then calculate the pseudo values and the distribution of pseudo values for a given y_1 in both APV and AV models as described in equation (6). The crucial aspect of our identification is that the pseudo values are invariant to first round prices in the APV but not the AV model. We illustrate this in the figure 3, where we calculate the pseudo values as in equation (6) for three bidder signals in the second round of the auction.[12]

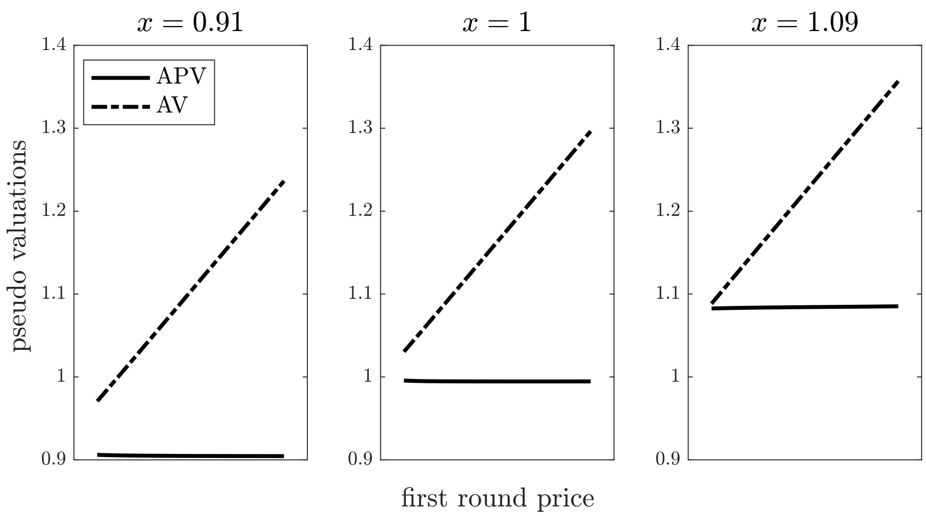

FIGURE 3. Pseudo values of a bidder with a fixed signal

Finally, to illustrate our identification, in figures 4 and 5 we draw the distribution of pseudo values and the conditional distributions (with the same support) for the two models. As before, we fix the first round prices to be $p_1 = \beta_1(1.3)$ and $\hat{p}_1 = \beta_1(1.8)$. The conditional distributions of pseudo values in the second round are indistinguishable from each other for different first round prices for APV. However in the case of AV there is a clear stochastic dominance relationship between the two distributions as shown by Proposition 3.1.

5. NONPARAMETRIC IDENTIFICATION

In this section, we examine nonparametric identification of the general AV model when all the bids are observable, i.e., $\mathbf{H} = \mathbf{B}$. We also provide testable restrictions of the APV and AV models, respectively. Again for notational simplicity, we consider the two-unit ($K = 2$) case with n bidders.

[12] We pick these signals such that $x \in [y_1 - 1, y_1]$ for both $y_1 = 1.3, 1.8$.

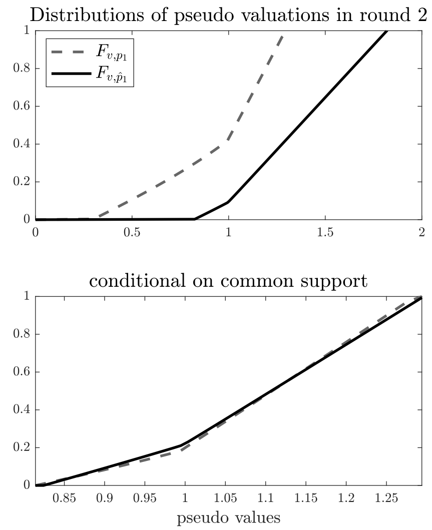

FIGURE 4. APV

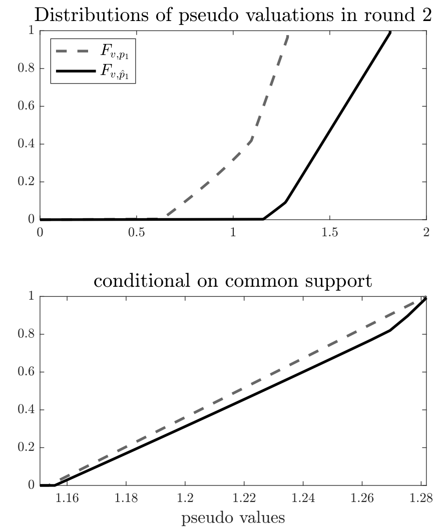

FIGURE 5. AV

First consider the symmetric APV model. Since the second round is essentially a static first-price auction, for any given winning price in the first round, we can apply the identification procedure in Li et al. (2002) to find the bidding function conditional on the price. Therefore, we can identify the equilibrium bidding function in the second round, from which we then identify the winner's value in the first round using the first-order condition. One additional subtlety here is that one cannot directly construct a symmetric and affiliated joint distribution of all the signals from the distribution of the first winner's signals and the conditional distributions of all other bidders' signals. We overcome this problem by investigating bids in the first round; in particular, we construct a joint distribution of signals from the first-order condition in the first round, which links bids in both rounds. Hence, the APV model is identified. The result is summarized in the following proposition.

Proposition 5.1. *A symmetric APV model is identified. Furthermore, a data distribution $G_{\mathbf{B}}$ can be rationalized by a symmetric APV model if and only if*

(i) *$G_{\mathbf{B}}$ is symmetric, both $G_{\mathbf{B}_1}$ and $G_{\mathbf{B}_2}$ are affiliated,* *(ii) for any winning price in the first round $p_1 \in [\underline{p}_1, \bar{p}_1]$, the function*

$$\xi_2(b_{1,2}, p_1) = b_{1,2} + \frac{G_{B^{(1)}_{-1,2}}(b_{1,2}|b_{1,2}, p_1)}{g_{B^{(1)}_{-1,2}}(b_{1,2}|b_{1,2}, p_1)}$$

is strictly increasing in $b_{1,2}$, where the support of $b_{1,2}$ is $[\underline{b}_2(p_1), \bar{b}_2(p_1)]$,

(iii) for any $b_{1,2}$, the function $\xi_2(b_{1,2}, p_1)$ is constant in p_1,

(iv) bidder 1's bid in the first round $b_{1,1} \in [\underline{b}_1, \bar{b}_1]$ satisfies

$$\bar{b}_2(b_{1,1}) = b_{1,1} + \frac{G_{B^{(1)}_{-1,1}}(b_{1,1}|b_{1,1})}{g_{B^{(1)}_{-1,1}}(b_{1,1}|b_{1,1})}.$$

Proof. The identifiability of the APV model follows from the same argument in Proposition 1 of Li et al. (2002) (page 188).

The necessity of conditions (i)-(iv): condition (i) follows from the assumption that $F_{\mathbf{X}}$ is symmetric and affiliated; condition (ii) follows from the fact the equilibrium is monotone and satisfies the first order condition in the second round; condition (iii) follows from the argument in Proposition 3.1; condition (iv) follows from the fact that the equilibrium satisfies the first order condition in the first round.

To prove sufficiency, let $G_{\mathbf{B}}$ be a data distribution satisfying conditions (i)-(iv). We also rewrite the function ξ_2 in condition (ii) as $\xi_2(b_{1,2}, p_1, G_{\mathbf{B}})$ to highlight that it is derived from the data $G_{\mathbf{B}}$. Let $\boldsymbol{\xi}_2 = (\xi_2, \ldots, \xi_2)$ denote an $(n-1)$-dimensional vector consisting of identical copies of ξ_2. We first construct, from the joint bid distribution $G_{\mathbf{B}_2}(\cdot|p_1)$ for any given $P_1 = p_1$ with support $[\underline{b}_2(p_1), \bar{b}_2(p_1)]^{n-1}$, an $(n-1)$-dimensional conditional joint distribution $\bar{F}(\cdot|p_1)$ as

$$\bar{F}(\mathbf{x}|p_1) = G_{\mathbf{B}_2}\left(\boldsymbol{\xi}_2^{-1}(\mathbf{x}, p_1, G_{\mathbf{B}})|p_1\right) \quad \text{for all } \mathbf{x} \in [\underline{x}(p_1), \bar{x}(p_1)]^{n-1},$$

where $\underline{x}(p_1) = \xi_2(\underline{b}_2(p_1), p_1, G_{\mathbf{B}})$ and $\bar{x}(p_1) = \xi_2(\bar{b}_2(p_1), p_1, G_{\mathbf{B}})$. Conditions (ii) and (iii) guarantee that $\bar{F}(\cdot|p_1)$ is indeed a distribution function. Condition (i) implies that $\bar{F}(\mathbf{x}|p_1)$ is symmetric.

Note that by construction the support of $\bar{F}(\cdot|p_1)$ is $[\underline{x}(p_1), \bar{x}(p_1)]^{n-1}$. Furthermore, the upper bound $\bar{x}(p_1)$ corresponds to the signal (or equivalently, private value) of the first round's winner with a bid p_1. That is, we have obtained a mapping from the first round's price p_1 to a signal $\bar{x}(p_1)$. Condition (iv) then implies that

$$\xi_2^{-1}(\bar{x}(b_{1,1}), b_{1,1}, G_{\mathbf{B}}) = b_{1,1} + \frac{G_{B_{-1,1}^{(1)}}(b_{1,1}|b_{1,1})}{g_{B_{-1,1}^{(1)}}(b_{1,1}|b_{1,1})},$$

which in turn leads to

$$\bar{x}(b_{1,1}) = \xi_2\left(b_{1,1} + \frac{G_{B_{-1,1}^{(1)}}(b_{1,1}|b_{1,1})}{g_{B_{-1,1}^{(1)}}(b_{1,1}|b_{1,1})}, b_{1,1}, G_{\mathbf{B}}\right) \triangleq \xi_1(b_{1,1}, G_{\mathbf{B}}). \tag{7}$$

It follows from conditions (ii) and (iii) that ξ_1 is strictly increasing in $b_{1,1}$. Let $\underline{x} = \xi_1(\underline{b}_1, G_{\mathbf{B}})$ and $\bar{x} = \xi_1(\bar{b}_1, G_{\mathbf{B}})$. Let $\boldsymbol{\xi}_1 = (\xi_1, \ldots, \xi_1)$ be an n-dimensional vector consisting of identical copies of ξ_1. Then, using (7) and bids in the first round, we can construct the following n-dimensional joint distribution $F(\cdot)$ as

$$F(\mathbf{x}) = G_{\mathbf{B}_1}\left(\boldsymbol{\xi}_1^{-1}(\mathbf{x}, G_{\mathbf{B}})\right) \quad \text{for all } \mathbf{x} \in [\underline{x}, \bar{x}]^n.$$

Since ξ_1 is strictly increasing in $b_{1,1}$, $F(\cdot)$ is strictly increasing on $[\underline{x}, \bar{x}]^n$. Since $G_{\mathbf{B}_1}$ is symmetric, $F(\cdot)$ is also symmetric. Since ξ_1^{-1} is strictly increasing, it follows from Theorem 3 of Milgrom and Weber (1982) that F is affiliated. Furthermore, let $F(\cdot|Y_1 = y_1)$ be the $(n-1)$-dimensional conditional distribution given the highest order statistic of $F(\cdot)$ is $Y_1 = y_1$. Then it follows from the construction of F that

$$F(\mathbf{x}|y_1) = \bar{F}(\mathbf{x}|\xi_1^{-1}(y_1, G_{\mathbf{B}})).$$

Now it remains to show that $F(\cdot)$ rationalizes the data $G_{\mathbf{B}}$ in the APV model, i.e.,

(a) $G_{\mathbf{B}_1}(\cdot) = F(\boldsymbol{\beta}_1^{-1}(\cdot))$, and
(b) $G_{\mathbf{B}_2}(\cdot|p_1) = F(\cdot|Y_1 = \beta_1^{-1}(p_1))$,

where $\boldsymbol{\beta}_1 = (\beta_1, \ldots, \beta_1)$ is an n-dimensional vector consisting of identical copies of β_1. We first prove (b). For any p_1, by the construction of F, we have $G_{\mathbf{B}_2}(\cdot|p_1) = \bar{F}(\boldsymbol{\xi}_2(\cdot, p_1, G_{\mathbf{B}})|p_1)$. Therefore, it suffices to show that $\xi_2^{-1}(\cdot, p_1, G_{\mathbf{B}})$ satisfies the first-order condition in the second round (equation (3)) with the boundary condition $\xi_2^{-1}(\underline{x}(p_1), p_1, G_{\mathbf{B}}) = \underline{x}(p_1)$. The boundary condition follows from the definition of ξ_2. For the first-order condition, note that by construction we have

$$F_{Y_2}(\cdot|x_1, \beta_1^{-1}(p_1)) = G_{B_{-1,2}^{(1)}}(\xi_2^{-1}(\cdot, p_1, G_{\mathbf{B}})|\xi_2^{-1}(x_1, p_1, G_{\mathbf{B}}), p_1)$$

and

$$\frac{f_{Y_2}(\cdot|x_1, \beta_1^{-1}(p_1))}{F_{Y_2}(\cdot|x_1, \beta_1^{-1}(p_1))} = \frac{g_{B_{-1,2}^{(1)}}(\xi_2^{-1}(\cdot, p_1, G_{\mathbf{B}})|\xi_2^{-1}(x_1, p_1, G_{\mathbf{B}}), p_1)}{G_{B_{-1,2}^{(1)}}(\xi_2^{-1}(\cdot, p_1, G_{\mathbf{B}})|\xi_2^{-1}(x_1, p_1, G_{\mathbf{B}}), p_1)} \cdot \xi_2^{-1\prime}(x_1, p_1, G_{\mathbf{B}}).$$

By the definition of ξ_2, we have

$$(x_1 - \xi_2^{-1}(x_1, p_1, G_{\mathbf{B}})) \frac{g_{B^{(1)}_{-1,2}}(\xi_2^{-1}(x_1, p_1, G_{\mathbf{B}}) | \xi_2^{-1}(x_1, p_1, G_{\mathbf{B}}), p_1)}{G_{B^{(1)}_{-1,2}}(\xi_2^{-1}(x_1, p_1, G_{\mathbf{B}}) | \xi_2^{-1}(x_1, p_1, G_{\mathbf{B}}), p_1)} = 1,$$

for all x_1. Therefore, ξ_2^{-1} satisfies the first-order condition.

Next we prove (a). By the construction of F, we have $G_{\mathbf{B}_1}(\cdot) = F(\boldsymbol{\xi}_1(\cdot, G_{\mathbf{B}}))$. Thus, it suffices to show that $\xi_1^{-1}(\cdot, G_{\mathbf{B}})$ solves the first-order condition in the first round (equation (1) for $k = 1$) with the boundary condition $\xi_1^{-1}(\underline{x}, G_{\mathbf{B}}) = \underline{x}$. The boundary condition follows from (b) and the fact that $\underline{x} = \underline{x}(\underline{p}_1)$. For the first-order condition, again by construction we have

$$F_{Y_1}(\cdot | x_1) = G_{B^{(1)}_{-1,1}}(\xi_1^{-1}(\cdot, G_{\mathbf{B}}) | \xi_1^{-1}(x_1, G_{\mathbf{B}}))$$

and

$$\frac{f_{Y_1}(\cdot | x_1)}{F_{Y_1}(\cdot | x_1)} = \frac{g_{B^{(1)}_{-1,1}}(\xi_1^{-1}(\cdot, G_{\mathbf{B}}) | \xi_1^{-1}(x_1, G_{\mathbf{B}}))}{G_{B^{(1)}_{-1,1}}(\xi_1^{-1}(\cdot, G_{\mathbf{B}}) | \xi_1^{-1}(x_1, G_{\mathbf{B}}))} \cdot \xi_1^{-1\prime}(\cdot, G_{\mathbf{B}}).$$

Furthermore, by the definition of ξ_1, we have

$$(\xi_2^{-1}(x_1, \xi_1^{-1}(x_1, G_{\mathbf{B}}), G_{\mathbf{B}}) - \xi_1^{-1}(x_1, G_{\mathbf{B}})) \cdot \frac{g_{B^{(1)}_{-1,1}}(\xi_1^{-1}(x_1, G_{\mathbf{B}}) | \xi_1^{-1}(x_1, G_{\mathbf{B}}))}{G_{B^{(1)}_{-1,1}}(\xi_1^{-1}(x_1, G_{\mathbf{B}}) | \xi_1^{-1}(x_1, G_{\mathbf{B}}))} = 1,$$

for all x_1, which proves that ξ_1^{-1} satisfies the first-order condition.

□

Now consider the symmetric AV model. Athey and Haile (2002) point out that a normalization of the signals is needed, since the information content of signals about values is invariant under monotone transformations. Here we first normalize the signals as follows:

$$\mathbb{E}[U_1 | X_1 = Y_1 = Y_2 = x_1] = x_1.$$

Unlike the APV special case, the first-order condition in the second round only yields the "pseudo" conditional expectation of a bidder's value. Nevertheless, as in the tests between AV and APV models, the variation in P_1 provides further information about the joint distribution of signals and values, $F_{\mathbf{X},\mathbf{U}}(\cdot)$. In particular, from the bid distribution we can construct a distribution of the right-hand side of equation (6) for any fixed p_1. Similar to the APV case, the upper bound of the support of this distribution, together with the signal normalization, will give us a conditional distribution of bidders' signals in the second round for any winning price p_1. Then we can follow the same argument in the proof of Proposition 5.1 to obtain the following result (Proposition 5.2).

Proposition 5.2. *In a symmetric AV model, the joint distribution of signals $F_{\mathbf{X}}(\cdot)$ and the distribution of the conditional expectation $\mathbb{E}[U_1|X_1, Y_2, Y_1, X_1 \leq Y_1]$ are identified. Furthermore, a data distribution $G_{\mathbf{B}}$ can be rationalized by a symmetric AV model if and only if*

(i) $G_{\mathbf{B}}$ is symmetric, both $G_{\mathbf{B}_1}$ and $G_{\mathbf{B}_2}$ are affiliated,

(ii) for any winning price in the first round p_1, the function

$$\xi_2(b_{1,2}, p_1) = b_{1,2} + \frac{G_{B^{(1)}_{-1,2}}(b_{1,2}|b_{1,2}, p_1)}{g_{B^{(1)}_{-1,2}}(b_{1,2}|b_{1,2}, p_1)}$$

is strictly increasing in $b_{1,2}$, where the support of $b_{1,2}$ is $[\underline{b}_2(p_1), \bar{b}_2(p_1)]$,

(iii) for any $b_{1,2}$, the function $\xi_2(b_{1,2}, p_1)$ is strictly increasing in p_1,

(iv) bidder 1's bid in the first round $b_{1,1}$ satisfies

$$\bar{b}_2(b_{1,1}) = b_{1,1} + \frac{G_{B^{(1)}_{-1,1}}(b_{1,1}|b_{1,1})}{g_{B^{(1)}_{-1,1}}(b_{1,1}|b_{1,1})}.$$

6. Concluding Remarks

TBA

- extensions
  - second-price auctions
  - sequential auctions of ranked objects
  - ipv is overidentified in sequential auctions
- limitations
  - data requirement
  - information available to bidders
  - asymmetric bidders

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