Unobserved Heterogeneity in Auctions under Restricted Stochastic Dominance

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September 30, 2018

Abstract

We study the identification of first-price auctions with nonseparable unobserved heterogeneity. In particular, we extend Hu, McAdams, and Shum (2013) by relaxing the first-order stochastic dominance condition. Instead, we assume restricted stochastic dominance relations among value quantile functions and show that the same relations pass to bid quantile functions. An ordered tree summarizes these relations and provides a total ordering. Relying on the proposed restricted stochastic dominance ordering, we extend a list of identification results in the empirical auction literature.

Keywords: Misclassification, Finite Mixture, Eigenvalue Decomposition, Multidimensional Unobserved Heterogeneity, Risk Aversion, Two-Sided Power Distribution

JEL: C14, D44

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I would like to thank the editor and anonymous referees for comments that have greatly improved this article. I am indebted to Isabelle Perrigne and Quang Vuong for their advice and encouragement. I also thank Victor Aguirregabiria, Yonghong An, Gaurab Aryal, Xiaohong Chen, Serafin Grundl, Jiaying Gu, Emmanuel Guerre, Marc Henry, Nail Kashaev, Joris Pinkse, Matthew Shum, Yuanyuan Wan, Ruli Xiao, Yu Zhu, and conference participants at the 2016 Shanghai Econometrics Workshop and the 2016 CESG for helpful comments. Jan Victor Dee, Cheok in Fok and Jiaqi Zou provided excellent research assistance. I thank SSHRC for financial support. All errors are my own.

1 Introduction

In this paper, we study the identification of first-price auctions with auction-specific unobserved heterogeneity,[1] which is prominent in many applications. There are two related methods for dealing with unobserved heterogeneity in auctions. The first is the deconvolution approach. See Li and Vuong (1998), Li, Perrigne, and Vuong (2000) and Krasnokutskaya (2011), among others. This method requires two bids in each auction and assumes that unobserved heterogeneity has a separable effect on bidders' values. The second is the misclassification approach of Hu, McAdams, and Shum (2013). They consider discrete one-dimensional unobserved heterogeneity and achieve identification with at least three bidders per auction relying on the results of Hu (2008).

We focus on the misclassification approach and consider finite unobserved heterogeneity. In particular, we generalize Hu, McAdams, and Shum (2013) by relaxing their first-order stochastic dominance (hereafter FOSD) ordering assumption. Since unobserved heterogeneity in auctions carries economic meaning, such as unknown competition, pinning down the ordering is crucial for identification and estimation.[2] While the FOSD ordering arises naturally (e.g., Aradillas-López, Gandhi, and Quint (2013)) and has been verified in some applications (e.g., An, Hu, and Shum (2010)), relaxing this assumption would further expand the breadth of applicability of their misclassification approach. Specifically, we show that the misclassification approach applies as long as a regular full-rank condition and a restricted stochastic dominance (hereafter RSD) ordering on the value distributions are satisfied.

One way to conceptualizing RSD ordering is through constructing an ordered tree for a set of distinct functions. This ordered tree simply describes how each function distinguishes itself from the rest. It contains points where two or more functions split and the ranking of the functions in the neighborhood of these splitting points. This concept allows a novel

[1] We use unobserved heterogeneity and state interchangeably.

[2] In contrast, the latent variable in finite mixture models carries no economic meaning. Identification up to a permutation is sufficient. Nevertheless, imposing an ordering is necessary for bootstrapping standard errors. See Hu and Xiao (2018).

way of ordering functions. First, it is well-defined for any set of distinct functions under the RSD condition. An ordered tree describes an ordering of the original functions, even in the absence of stochastic dominance of any order. In other words, our ordering of functions relies on a known functional that yields the ordering when applied to the set of functions. On the other hand, Hu, McAdams, and Shum (2013) rely on a monotonicity condition to order the value distributions conditional on the state. This requires a known functional that is applied to each function such that the resulting values are increasing in the state. Depending on the kind of unobserved heterogeneity, the exact functional would need to be determined on a case-by-case basis.

The advantage of RSD ordering is threefold. First, it is easy to interpret. Similar to FOSD ordering, RSD ordering is equivalent to expected utility ordering (in a limited range) with an increasing utility function. Second, it allows unambiguous ranking of distribution functions in a wider range of models. FOSD ordering focuses on one-dimensional unobserved heterogeneity. However, many commonly used distributions, such as the beta distribution, the two-sided power distribution, and the normal distribution, have two or more parameters. In empirical applications, there are potentially multiple sources of unobserved heterogeneity, such as unknown competition and unknown bidder types. Third, it leads to more convenient comparative statics in auctions. The misclassification approach identifies auctions with unobserved heterogeneity in two steps: first, it identifies the component bid distributions, and then it identifies the model primitives. Comparative statics are essential for linking the component bid distributions to the model primitives. Since RSD ordering only requires comparing functions in a limited range, comparative statics are normally clearer under much weaker conditions. On the other hand, FOSD ordering either excludes the model or requires more assumptions on the primitives.

Relying on RSD ordering, we extend a list of identification results in the empirical auction literature. First, we prove identification of first-price auction models with unobserved heterogeneity when bidder values are *i.i.d.* draws from Beta distributions.[3] Note that this family of distributions cannot be ranked by FOSD relations. However, we find that our RSD ordering is equivalent to a lexicographical ordering of the Beta distribution's two shape parameters. This leads to full identification of models with Beta-distributed values, rather than identification up to a permutation. Moreover, we obtain a new sufficient condition for nonparametric identification of the same model with covariates.

Second, we extend the identification results in Guerre, Perrigne, and Vuong (2009) by introducing nonseparable unobserved heterogeneity as in Hu, McAdams, and Shum (2013). We show novel comparative statics that the same relation of RSD passes from value quantile functions to bid quantile functions in first-price auctions. Therefore, the RSD ordering and the ordered tree are both preserved by the mapping from value quantile function to bid quantile function. Moreover, this property is invariant to the number of bidders as well as their utility function specification. We rely on this result to match the two bid distributions resulting from the same value distribution but different numbers of bidders. This allows us to identify the model primitives relying on Guerre, Perrigne, and Vuong (2009).

Third, we show that the identification results of Gentry and Li (2014) are extendable to unobserved heterogeneity under a smaller set of assumptions than called for in the paper. They study identification of the Affiliated-Signal model under risk neutrality. Point identification relies on a continuous cost shifter, which reduces to a problem of finite mixture with covariates and suffers from a "label switching" problem. Gentry and Li (2014) make two monotonicity assumptions on the value distributions and entry thresholds to ensure FOSD ordering of the component bid distributions. Relying on RSD ordering, we show that their identification results hold without the assumption on the ordering of value distributions.

Fourth, we consider an example of multidimensional unobserved heterogeneity.[4] Unobserved heterogeneity may arise for a wide variety of reasons. Examples include an unknown

[3]See, e.g., Jofre-Bonet and Pesendorfer (2003) and Bajari and Hortacsu (2005) for applications of Beta distributions in auctions.

[4]Krasnokutskaya (2012) studies such a problem using the deconvolution approach.

number of bidders (An, Hu, and Shum (2010)), implicit reserve prices or bidding costs (Hu, McAdams, and Shum (2013)), unknown bidder types or bounded rationality (An (2017)), and multiple equilibria (Xiao (2018)). While we can relabel multidimensional states into a one-dimensional one, there is no natural ordering of states, let alone arguing or proving that some feature of the distribution of bids is higher in higher states. Therefore, our RSD ordering is particularly relevant when unobserved heterogeneity is multidimensional. In particular, we extend the identification results of An (2017) by allowing for unobserved asymmetry in both bidder preferences and private values.

Related Literature

For dealing with unobserved heterogeneity, there are other important methods besides the deconvolution method and the misclassification approach. See, e.g., Haile, Hong, and Shum (2003) and Guerre, Perrigne, and Vuong (2009) for using the number of bidders; Roberts (2013) for using observed reserve price; and Armstrong (2013) for a partial identification approach. In English auctions where only winning bids are observed, Hernández, Quint, and Turansick (2018) exploit exogenous participation and additive separability of unobserved heterogeneity to achieve identification.

Since Samuelson, economists have been studying comparative statics predictions (Athey (2002)). For instance, the theoretical auction literature has been interested in comparative statics in the stochastic dominance sense. See, e.g., Krishna (2009). In this paper, we introduce the concept of RSD into auctions and study its properties. To the best of our knowledge, this seems to be new to the literature.

To learn about bidders' risk aversion, we consider the Guerre, Perrigne, and Vuong (2009) approach. For estimation methods based on this approach, see Zincenko (2018) and Kim (2015). For other approaches, see, e.g., Lu and Perrigne (2008) for exploiting an exogenous change in auction format; Campo, Guerre, Perrigne, and Vuong (2011) for a semiparametric method; and Fang and Tang (2014) for using entry behaviors. Considering risk aversion and unobserved heterogeneity simultaneously is nontrivial. On the one hand, applying the misclassification approach requires a condition to order the resulting elements after eigenvalue decomposition. The usual approach is to find a monotonicity condition with respect to the state. Previous applications often had to make extra assumptions to ensure monotonicity. See, e.g., Gentry and Li (2014). Under risk aversion, it becomes more involved to find a feature of bid distribution that is monotone in the state. On the other hand, applying the results in Guerre, Perrigne, and Vuong (2009) requires finding two bid distributions that are derived from the same value distribution but under different competition levels. See Grundl and Zhu (2018) for some results on this. They rely on this monotonicity condition to match the two bid distributions resulting from the same value distribution but different numbers of bidders.

The rest of the paper is organized as follows. Section 2 describes the first-price auction model with risk aversion. Section 3 introduces the concept of RSD and studies its properties in auctions. Moreover, we propose an ordering for a set of functions by defining its ordered tree. Section 4 applies these intermediate results in the identification of the first-price auction model with discrete unobserved heterogeneity and risk neutrality. Section 5 applies the RSD ordering to identification of auction models with Beta-distributed values, models with risk aversion, entry models, and models with asymmetric bidders. Section 6 contains an empirical application of our method to data from U.S. Forest Service timber auctions. Section 7 concludes. Proofs omitted from the main text are collected in Appendices.

2 First-Price Auction Model

We first introduce the first-price auction model in which bidders are risk averse. Suppose $I \geq 2$ symmetric bidders participate in a first-price auction with reserve price of zero. They are potentially risk averse. Let $U(\cdot)$ be their utility function with $U(0) = 0, U'(\cdot) > 0$ and $U''(\cdot) < 0$. Conditioning on an auction-specific state $k \in \mathcal{K} = \{1, \ldots, K\}$, their values are *i.i.d.* draws from the same distribution $F_k(\cdot)$ with support $[\underline{v}, \overline{v}_k]$. Denote $v_k(\cdot) = F_k^{-1}(\cdot)$ as the corresponding quantile function. For exposition purposes, we treat the graph of $v_k(\cdot)$ as directioned while letting $(0, v_k(0))$ and $(1, v_k(1))$ be our starting point and ending point, respectively. For convenience, we focus on functions with the same starting point $v_k(0) = \underline{v}$. Suppose $v_k(\cdot)$ is continuously differentiable and its first-order derivatives are bounded away from both zero and infinity.

In a state-k auction, a bidder with value v solves the following problem

$$\max_{b} \quad F_k\big(s_k^{-1}(b)\big)^{I-1} \cdot U(v - b),$$

where $s_k^{-1}(\cdot)$ is the inverse of his/her optimal bidding strategy in state-k auctions, and $F(s_k^{-1}(b))^{I-1}$ is the likelihood of winning, i.e., the probability of his/her bid being the highest. Guerre, Perrigne, and Vuong (2009) study the identification of $F_k(\cdot)$ and $U(\cdot)$ when k, I, and bid distribution $G_k(\cdot)$ are observed. To do so, they rewrite the equilibrium FOC in terms of the observed bid quantile function[5]

$$v_k(\alpha) = b_k(\alpha) + \lambda^{-1}\Big(\frac{1}{I-1}\alpha b_k'(\alpha)\Big), \tag{1}$$

where $\alpha \in [0,1], \lambda(\cdot) = U(\cdot)/U'(\cdot)$, and $v_k(\cdot)$ and $b_k(\cdot)$ are the quantile functions of values and bids, respectively. Moreover, the boundary condition is $b_k(0) = v_k(0) = \underline{v}$.

Consider the case where state k is known to the econometrician. If the bidders are risk neutral (i.e., $U(x) = x$), Equation (1) shows the identification of $v_k(\cdot)$ with knowledge of $b_k(\cdot)$ because $\lambda^{-1}(x) = x$. See Guerre, Perrigne, and Vuong (2000). On the other hand, if bidders are potentially risk averse, Guerre, Perrigne, and Vuong (2009) show that the model can always be rationalized by a first-price auction model where bidders have a CRRA or

[5]While $s(\cdot)$ defines the mapping from values to bids, $b(\cdot)$ represents the bidding strategy as a function of the quantile of the value. The latter deals with different supports for comparative statics analysis. In some cases, it allows for ordering the bidding behaviors of different bidders or in different auctions under weaker conditions.

CARA utility function. In view of this, they propose to exploit an exogenous participation condition. Consider two sets of auctions, which are homogeneous (i.e., they have the same value distribution $v_k(\cdot)$) except with different numbers of bidders. Without loss of generality, assume that $2 \leq I_1 < I_2$. They exploit exogenous variation in the number of bidders I and the compatibility condition:

$$b_{k,I_1}(\alpha) + \lambda^{-1}\Big(\frac{1}{I_1 - 1}\alpha b'_{k,I_1}(\alpha)\Big) = v_k(\alpha) = b_{k,I_2}(\alpha) + \lambda^{-1}\Big(\frac{1}{I_2 - 1}\alpha b'_{k,I_2}(\alpha)\Big),$$

where $b_{k,I}$ is the bid quantile function in state-k auctions when the number of bidders is I. This condition implies the identification of $\lambda^{-1}(\cdot)$, which leads to the identification of $v_k(\cdot)$. Therefore, the model is identified with two sets of auctions under different competition levels.

If state k is unknown to the econometrician, we call it unobserved heterogeneity. It represents auction-specific characteristics observed by the bidders but unobserved by the econometrician. Hu, McAdams, and Shum (2013) consider identification of the component value distributions and the distribution of unobserved heterogeneity when bidders are risk neutral. Suppose we observe the bids from a set of homogeneous auctions where bidders are risk neutral. There are at least three bidders in each auction, say $\imath = i, j, \ell$.

They first discretize the bids by defining a partition of $\mathbb{R}_+$ into K intervals, say $\mathcal{D}$: $\mathbb{R}_+ \to \{1, \ldots, K\}$. Denote $D_\imath = \mathcal{D}(B_\imath)$ to be the indicator function for the interval in which bidder $\imath$'s bid $B_\imath$ belongs, where $\imath = i, j, \ell$. To identify the auction model with unobserved heterogeneity, they assume a monotonicity condition and a full-rank condition.

Assumption 1 (UH monotonicity condition)**.** *There exists a known functional $M(\cdot)$ such that $M(v_k(\cdot))$ is strictly increasing in k.*

The existence and exact form of $M(\cdot)$ has to be determined on a case-by-case basis. They provide a list of examples where the maximum bid is increasing in the state. Other examples include the minimum bid and the mean equilibrium bid.

Assumption 2 (UH full-rank condition)**.** *There exists a discretization of bids such that the* $K \times K$ *matrix* $L_{D_i,D_j} = [\Pr\{D_i = i', D_j = j'\}]_{i',j' \in \{1,2,\dots,K\}}$ *has rank* K*.*

Hu, McAdams, and Shum (2013) provide a sufficient condition for both the UH monotonicity condition and the UH full-rank condition – the FOSD condition. For comparison, we rewrite it in terms of quantile functions.

Definition 1 (First-Order Stochastic Dominance)**.** $v_{k'}(\cdot)$ *first-order stochastically dominates* $v_k(\cdot)$ *if: (a)* $v_k(\alpha) \leq v_{k'}(\alpha), \forall \alpha \in [0,1]$*, and (b)* $\exists \alpha_* \in (0,1)$ *such that* $v_k(\alpha_*) < v_{k'}(\alpha_*)$*.*

Assumption 3 (Totality)**.** *For any two different states* $k, k' \in \mathcal{K}$*, either* $v_{k'}(\cdot)$ *first-order stochastically dominates* $v_k(\cdot)$ *or* $v_k(\cdot)$ *first-order stochastically dominates* $v_{k'}(\cdot)$*.*

Their method has two steps: first, identify the distribution of unobserved heterogeneity and the component bid distributions; second, apply Guerre, Perrigne, and Vuong (2000) to identify the underlying value distributions. While the second step is standard, they focus on the first one. Since bids are *i.i.d.* conditioning on the unobserved heterogeneity, the problem in the first step constitutes a finite mixture model with repeated measurements. Applying results from the measurement error literature (see Hu (2008)), they identify the component bid distributions up to a permutation. Assumption 3 ensures a unique mapping from the anonymous distributions to the elements of $\mathcal{K}$.

3 Restricted Stochastic Dominance Ordering

This section presents several important intermediate results. First, we introduce the concept of RSD in auctions, which generalizes the FOSD condition. Second, in first-price auctions, we show that the same relation of RSD passes from value quantile functions to bid quantile functions. Relying on these results, we propose an ordering for a set of value quantile functions and show that the corresponding set of bid quantile functions have the same ordering.

3.1 Definitions

In the auction literature, there are numerous results on FOSD passing from value distribution functions to bid distribution functions. See, e.g., Krishna (2009). However, in many cases, the FOSD condition fails to rank two distributions under consideration because it is not a total ordering.

To relax this condition, we introduce the concept of restricted stochastic dominance, which is particularly useful for situations where interests are focused on what happens below/above a certain threshold. In first-price auctions, the bidder cares about his/her value and bid only when he/she wins. Therefore, the bidder shall only consider the distribution below his/her value.

Definition 2 (Restricted Stochastic Dominance). *$v_{k'}(\cdot)$ dominates $v_k(\cdot)$ in the restricted sense if there exists an $x \in (0, 1]$ such that: (a) $v_k(\alpha) \leq v_{k'}(\alpha), \forall \alpha \in [0, x]$, and (b) $\exists \alpha_* \in (0, x]$ such that $v_k(\alpha_*) < v_{k'}(\alpha_*)$.*

The concept of RSD is introduced by Atkinson (1987) in the context of poverty measurement. See also Davidson and Duclos (2000) and Davidson and Duclos (2013). If one distribution FOSDs another, then the RSD condition is satisfied by letting $x = 1$. Therefore, Assumption 4 is weaker than the FOSD condition. While FOSD ordering is equivalent to expected utility ordering with an increasing utility function, RSD ordering has a similar interpretation applying well-known results (e.g., Bawa (1975)). In particular, if $v_2(\cdot)$ dominates $v_1(\cdot)$ up to x, we can find a z such that $F_2(\cdot)$ dominates $F_1(\cdot)$ up to z. Denote $\Pi_k(v) = \int_{\underline{v}}^{v} U(v - y)dF_k(y)$. $F_2(\cdot)$ dominates $F_1(\cdot)$ up to z if and only if $\Pi_1(v) \leq \Pi_2(v)$ for all $v \in [0, z]$ and any differentiable and increasing function $U(\cdot)$ satisfying $U(0) = 0$. To gain some intuition for this statement, consider ascending auctions where every bidder bids his/her private value. Conditioning on the bidder's private value v, his/her expected utility is higher in state-2 ascending auctions than state-1 ones for all $v \in [0, z]$.

Hereafter, we make a totality assumption.

Assumption 4 (Totality)**.** *For any two different states $k, k' \in \mathcal{K}$, either $v_{k'}(\cdot)$ dominates $v_k(\cdot)$ in the restricted sense or $v_k(\cdot)$ dominates $v_{k'}(\cdot)$ in the restricted sense.*

We can define a strict binary relationship between two functions at the infimum of the points below which there exist RSD relationships.

Definition 3 (Splitting Point)**.** *We call $\alpha_\dagger$ the splitting point of $v_k(\cdot)$ and $v_{k'}(\cdot)$ if $\alpha_\dagger = \inf\{x \in [0,1] | \forall \alpha \in [0,x], v_k(\alpha) \leq v_{k'}(\alpha),$ and $\exists \alpha_* \in (0,x], v_k(\alpha_*) < v_{k'}(\alpha_*)\}$.*

We denote this relationship as $v_k(\cdot) \underset{\alpha_\dagger}{\prec} v_{k'}(\cdot)$. By definition, for any $\epsilon > 0$, there exists $x \in [\alpha_\dagger, \alpha_\dagger + \epsilon)$ such that $\forall \alpha \in [0,x], v_k(\alpha) \leq v_{k'}(\alpha)$, and $\exists \alpha_* \in (0,x], v_k(\alpha_*) < v_{k'}(\alpha_*)$. In fact, for any $\alpha \in [0,x]$, $v_k(\alpha) < v_{k'}(\alpha)$ implies $\alpha > \alpha_\dagger$. Otherwise, if $\alpha \leq \alpha_\dagger$ and $v_k(\alpha) < v_{k'}(\alpha)$, by continuity of $v_k(\cdot)$ and $v_{k'}(\cdot)$, we can find an $x' < \alpha$ in the neighborhood of α such that $\forall \alpha \in [0,x'], v_1(\alpha) \leq v_2(\alpha)$, and that there exists $\alpha'_* \in (0,x'], v_k(\alpha'_*) < v_{k'}(\alpha'_*)$, a contradiction to the definition of $\alpha_\dagger$. Therefore, $v_k(\cdot)$ and $v_{k'}(\cdot)$ coincide below $\alpha_\dagger$ and split apart right after $\alpha_\dagger$.

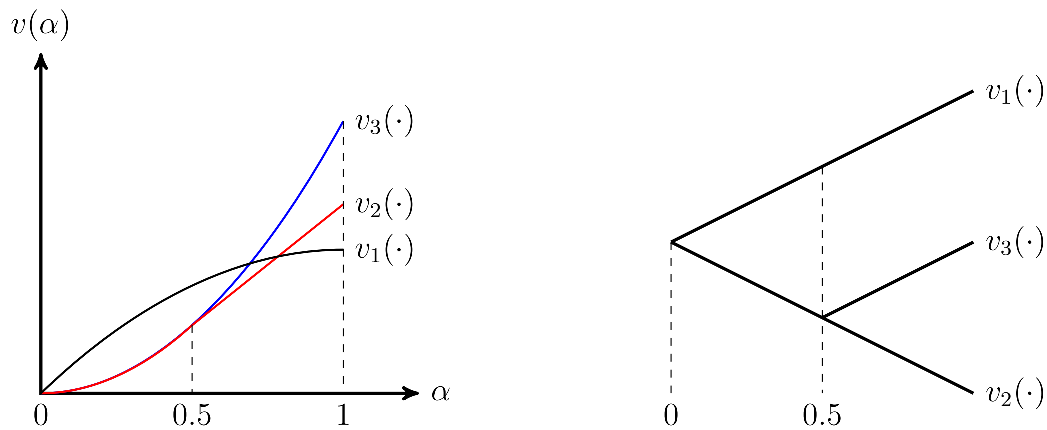
We remark that once two functions split at $\alpha_\dagger$, we allow them to cross or touch again above $\alpha_\dagger$. Consider the example shown in Figure 1a: $v_1(\cdot)$ and $v_2(\cdot)$ split apart at $\alpha = 0$, while $v_2(\cdot)$ and $v_3(\cdot)$ split apart at $\alpha = 0.5$. The graph of $v_1(\cdot)$ crosses the graphs of $v_2(\cdot)$ and $v_3(\cdot)$ on $(0.5, 1)$. Another example is $v_k(\alpha) = (2\alpha)^{\gamma_k}/2$, where $\gamma_k \in \{1/2, 1, 2\}$. The three functions split at 0 and cross again at 1/2.

Next, we prove that this binary relationship is transitive. This result will be useful when we define the ordering of a set of value/bid quantile functions.

Lemma 1 (Transitivity)**.** *If $v_1(\cdot) \underset{\alpha_\dagger}{\prec} v_2(\cdot)$ and $v_2(\cdot) \underset{\alpha_\dagger}{\prec} v_3(\cdot)$, then $v_1(\cdot) \underset{\alpha_\dagger}{\prec} v_3(\cdot)$.*

Since we have already assumed totality under Assumption 4, transitivity provides a total ordering of the group of functions that split at the same point. Note that the set of splitting points is totally ordered by the usual less than or greater than relations. Therefore, there exist total orderings on the set $\{v_k(\cdot)\}_{k=1,\ldots,K}$, one of which we define in Section 3.3.

Figure 1: Constructing An Ordered Tree from A Set of Distinct Functions



(a) Original Functions

(b) Ordered Tree

In fact, Lemma 1 can be strengthened in the following way so that the RSD relationships directly imply a total ordering, which we call the RSD ordering.[6]

Lemma 2 (Transitivity)**.** *If* $v_1(\cdot) \underset{\alpha_\dagger}{\prec} v_2(\cdot)$ *and* $v_2(\cdot) \underset{\alpha_{\dagger\dagger}}{\prec} v_3(\cdot)$*, then* $v_1(\cdot) \underset{\min\{\alpha_\dagger, \alpha_{\dagger\dagger}\}}{\prec} v_3(\cdot)$*.*

Intuitively, the bottom of the value distributions determine which dominates the others.

3.2 Restricted Stochastic Dominance in Auctions

To study RSD in the model described in Section 2, we propose two alternative representations of the equilibrium FOC. Rearranging terms in Equation (1) gives

$$b_k'(\alpha) = \frac{I-1}{\alpha}\lambda(v_k(\alpha) - b_k(\alpha)). \tag{2}$$

Therefore, $b_k(\cdot)$ satisfies the initial value problem if and only if it satisfies the integral problem

$$b_k(\alpha) = \underline{v} + (I-1)\int_0^\alpha \frac{\lambda(v_k(x) - b_k(x))}{x}dx.$$

Now we show that the same relation of RSD passes from value quantile functions to bid

[6]We thank a referee for this point.

quantile functions. Consider an arbitrary $x \in (0,1]$.

Lemma 3. *If $v_1(\alpha) \leq v_2(\alpha), \forall \alpha \in [0,x]$, then $b_1(\alpha) \leq b_2(\alpha), \forall \alpha \in [0,x]$.*

This result is most obvious for the case of risk neutrality because we have an explicit mapping from the value quantile function to the bid quantile function:

$$b_k(\alpha) = (I-1)\alpha^{1-I} \int_0^\alpha v_k(x) x^{I-2} dx.$$

The general case of risk aversion is less obvious due to a lack of explicit mapping. We include its proof in the Appendix. Similarly, we can show that $b_1(\alpha) \geq b_2(\alpha), \forall \alpha \in [0,x]$ if $v_1(\alpha) \geq v_2(\alpha), \forall \alpha \in [0,x]$. Therefore, if $v_1(\alpha) = v_2(\alpha), \forall \alpha \in [0,x]$, then $b_1(\alpha) = b_2(\alpha), \forall \alpha \in [0,x]$.

Lemma 4. *If $v_1(\alpha) \leq v_2(\alpha), \forall \alpha \in [0,x]$ and $v_1(\alpha_*) < v_2(\alpha_*)$, where $\alpha_* \in (0,x]$, then $b_1(\alpha) < b_2(\alpha), \forall \alpha \in [\alpha_*, x]$.*

Lemma 4 states a striking feature. Specifically, if $v_1(\alpha) \leq v_2(\alpha)$ on $[0,x]$, not only does the strict ordering pass from value quantile functions to bid quantile functions at the point of difference, it is maintained among the bid quantile functions until at least the cutoff point of the weak dominance relation. An immediate implication is that maximum bid $b_k(1)$ is increasing with respect to state k when Assumption 3 is true. Moreover, if $b_2(\cdot)$ dominates $b_1(\cdot)$ up to x, we can find a z such that $G_2(\cdot) = b_2^{-1}(\cdot)$ dominates $G_1(\cdot) = b_1^{-1}(\cdot)$ up to z. Denote $\Pi_k(v) = \int_{\underline{v}}^v U(v-b) dG_k(b)$. We have $\Pi_1(v) \leq \Pi_2(v)$ for all $v \in [0,z]$. Intuitively, conditioning on the bidder's private value v, his/her expected utility is higher in state-2 first-price auctions than state-1 ones.

Together, Lemmas 3 and 4 imply that if $\{v_k(\cdot)\}_{k=1,\ldots,K}$ satisfies Assumption 4, the induced $\{b_k(\cdot)\}_{k=1,\ldots,K}$ does as well. Therefore, there exist total orderings on the set $\{b_k(\cdot)\}_{k=1,\ldots,K}$, one of which is RSD ordering.

Theorem 1. *Under Assumption 4, the resulting RSD orderings are the same for $\{v_k(\cdot)\}_{k=1,2,\ldots,K}$ and $\{b_k(\cdot)\}_{k=1,2,\ldots,K}$ in independent private value first-price auctions with risk averse bidders. Moreover, this property is invariant to the number of bidders I or the utility function $U(\cdot)$.*

Discussion

Since the lower bound of the value distribution $v_k(0)$ is also totally ordered by the usual less than or greater than relations, we can extend our discussion to allow $v_k(0)$ to differ across states. Moreover, the lower bound of the bid distribution equals the lower bound of the value distribution in symmetric first-price auctions. Therefore, the ordering of the lower bound also passes from value quantile functions to bid quantile functions.

While we use probability $\alpha = 0$ as our starting/reference point, it can be useful to use other starting/reference points in some applications. For instance, in asymmetric auctions, bidders take value draws from potentially different distributions. Using probability $\alpha = 1$ as our starting/reference point, we arrive at similar results on the ordering of the set of value distributions and the set of bid distributions when bidders are risk neutral. In this RSD ordering, the top of the value distributions determine which dominates the others.[7]

Moreover, simply for the purpose of ordering distributions, we can define an RSD relation when the relevant inequality holds over some other restricted range of probability α rather than for an interval starting at 0, say $[0, x]$. RSD ordering can also be defined after some normalizations of the distributions, which may include but are not limited to the usual "demean" exercise. An example is the normal distribution $\mathcal{N}(\mu, \sigma^2)$, where μ is the mean/median and σ is the standard deviation. We can order a group of normal distributions first by mean/median using the usual less than or greater than relations and second by RSD relations on the right-hand side of the mean/median, i.e., $\alpha \in [0.5, 1]$, which is equivalent to a lexicographical ordering of the pairs (μ_k, σ_k). In this RSD ordering, the part of the value distributions above and closest to the mean/median determines which distribution dominates the others. Another example is the Gumbel distribution $\mathcal{G}(\mu, \beta)$, with mode μ and variance $\pi^2\beta^2/6$. We can order a group of Gumbel distributions first by mode and second by RSD relations on the right-hand side of the mode. In this RSD ordering, the part of the value distributions above and closest to the mode determines which distribution dominates

[7]For completeness, we include these results in Appendix C.

the others.

3.3 An Intuitive Representation of RSD ordering

In this section, we propose an intuitive representation of the proposed RSD ordering. In particular, we propose a simple way to summarize how each function in a set of functions becomes distinguished from the rest. Relying on Lemmas 3 and 4, we know that this ordering is preserved by the mapping from value quantile function to bid quantile function.

Consider $v_k(\cdot)$, where $k \in \mathcal{K} = \{1, \ldots, K\}$. We construct "splitting path" $\mathcal{P}_k$ as follows:

(1) We find the smallest splitting point α_1 where $v_k(\cdot)$ splits from some $v_{k'}(\cdot)$ in group $\mathcal{K}$. That is,

$$\alpha_1 = \min_{k' \neq k, k' \in \mathcal{K}} \inf\{x \in [0,1] | \forall \alpha \in [0,x], v_k(\alpha) \leq v_{k'}(\alpha), \& \exists \alpha_* \in [0,x], v_k(\alpha_*) < v_{k'}(\alpha_*)\}.$$

By Lemma 1, we can rank all quantile functions based on their rankings in a neighborhood of α_1, say $(\alpha_1, \alpha_1 + \Delta_1]$. Let the smallest ranking be 1. There is a group of functions that equal $v_k(\cdot)$ in $[0, \alpha_1 + \Delta_1]$. Let this group be $\mathcal{K}_1$. All the functions in this group have the same ranking in $(\alpha_1, \alpha_1 + \Delta_1]$, which we denote as R_1.

As shown in Figure 1a, if we consider $v_2(\cdot)$, we have $\alpha_1 = 0, R_1 = 1, \mathcal{K}_1 = \{2, 3\}$.

(2) We find the smallest splitting point α_2 where $v_k(\cdot)$ splits from some $v_{k'}$ in group $\mathcal{K}_1$. That is,

$$\alpha_2 = \min_{k' \neq k, k' \in \mathcal{K}_1} \inf\{x \in [0,1] | \forall \alpha \in [0,x], v_k(\alpha) \leq v_{k'}(\alpha), \& \exists \alpha_* \in [0,x], v_k(\alpha_*) < v_{k'}(\alpha_*)\}.$$

By Lemma 1, we can rank all quantile functions based on their rankings in a neighborhood of α_2, say $(\alpha_2, \alpha_2 + \Delta_2]$. Let the smallest ranking be 1. There is a group of functions that equal $v_k(\cdot)$ in $[0, \alpha_2 + \Delta_2]$. Let this group be $\mathcal{K}_2$. Denote the ranking of this group as R_2.

As shown in Figure 1a, if we consider $v_2(\cdot)$, we have $\alpha_2 = 0.5, R_2 = 1, \mathcal{K}_2 = \{2\}$.

(3) We continue this process until $v_k(\cdot)$ is the only function in the group.

We denote the list of pairs as $\mathcal{P}_k = \{(\alpha_1, R_1), \ldots, (\alpha_{T_k}, R_{T_k})\}$. We call $\mathcal{P}_k$ the splitting path of $v_k(\cdot)$. Note that $\mathcal{K}_{T_k} = \{k\}$. As shown Figure 1a, we have $\mathcal{P}_1 = \{(0, 2)\}$, $\mathcal{P}_2 = \{(0, 1), (0.5, 1)\}$, and $\mathcal{P}_3 = \{(0, 1), (0.5, 2)\}$.

Definition 4 (Ordered Tree)**.** *An ordered tree is defined by the set of splitting paths* $\mathcal{P} = (\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_K)$.

This ordered tree can be represented as in Figure 1b. The junctions are located at the list of splitting points. The branches at each junction represent the rankings of functions with the bottommost ranked as the first. A splitting path is thus a path through the tree from the root to a leaf. It describes how a function becomes distinguished from the rest. Since we specify an ordering for the branches at each junction, the leaves (i.e., $v_k(\cdot)$s) can be ordered from bottom to top.

Without loss of generality, we again denote the ordering in the ordered tree as $k = 1, 2, \ldots, K$. Theorem 1 implies the following result.

Corollary 1. *The resulting ordered trees and orderings are the same for* $\{v_k(\cdot)\}_{k=1,2,\ldots,K}$ *and* $\{b_k(\cdot)\}_{k=1,2,\ldots,K}$ *in first-price auctions.*

The ordered tree summarizes the pairwise relations of RSD on a set of functions. Theorem 1 says that it provides a total ordering on a set of value distributions that is passed to the bid distributions and stands unaffected by the utility function and the number of bidders.

4 Identification of Component Bid Distributions

Without loss of generality, let $k = 1, 2, \ldots, K$ represent the RSD ordering defined by the set of primitive value quantile functions. Hereafter we assume that K is known. In this section, we apply eigenvalue decomposition as in Hu, McAdams, and Shum (2013) to obtain the distribution of unobserved heterogeneity p_k and component bid distributions $G_k(\cdot)$.

Theorem 2. *Under Assumptions 2 and 4, the distribution of unobserved heterogeneity p_k and component bid distributions $G_k(\cdot)$ are identified if $I \geq 3$.*

Proof. We follow closely the identification procedure in Hu, McAdams, and Shum (2013). Consider three bidders ℓ, i, and j. We define a discretization of bids as a monotone onto mapping $\mathcal{D} : \mathbb{R}_+ \to \{1, \ldots, K\}$, which partitions the bid-space into K intervals. Denote $D_\imath = \mathcal{D}(B_\imath)$ to be the interval in which bidder i's bid belongs, where $\imath = i, j$. Fix the value of bidder ℓ's bid, say $\check{b}_\ell$. Define the following matrices:

$$
\begin{aligned}
L_{D_i|K} &= [g_{D_i|K}(i'|k)]_{i',k=1,2,\ldots,K}, \\
L_{D_j,D_i} &= [g_{D_j,D_i}(j',i')]_{j',i'=1,2,\ldots,K}, \\
L_{D_i,\check{b}_\ell,D_j} &= [g_{D_i,B_\ell,D_j}(i',\check{b}_\ell,j')]_{i',j'=1,2,\ldots,K}, \\
D_{\check{b}_\ell|K} &= \mathrm{diag}\{[g_{B_\ell|K}(\check{b}_\ell|k)]_{k=1,2,\ldots,K}\},
\end{aligned}
$$

where K represents the random variable of unobserved heterogeneity with a slight abuse of notation, $g_{D_i|K}(i'|k)$ is the conditional probability of observing $D_i = i'$ in state k, $g_{D_j,D_i}(j',i')$ is the probability of observing $(D_j, D_i) = (j', i')$, $g_{D_i,B_\ell,D_j}(i',\check{b}_\ell,j')$ is the joint density of (D_i, B_ℓ, D_j) at $(i', \check{b}_\ell, j')$, and $g_{B_\ell|K}(\check{b}_\ell|k)$ is the conditional bid density in state k.

Applying the results of Hu (2008), Hu, McAdams, and Shum (2013) obtain their key identification equation:

$$
L_{D_i,\check{b}_\ell,D_j}(L^T_{D_j,D_i})^{-1} = L_{D_i|K} D_{\check{b}_\ell|K} L^{-1}_{D_i|K},
$$

of which the left-hand side is identified from data and the terms on the right-hand side can be obtained through eigenvalue decomposition. Along the same lines, the K eigenvectors are uniquely determined, though the ordering is still arbitrary. That is, $L_{D_i|K}$ is identified up to a permutation of its columns.

We now depart from Hu, McAdams, and Shum (2013) by swapping their last two steps.

In particular, we fix an eigenvector matrix $\overline{L_{D_i|K}} = L_{D_i|K}Q$, which is a matrix generated by interchanging columns of the true eigenvector matrix $L_{D_i|K}$. Q is an unknown elementary matrix generated by interchanging columns of the identity matrix.

We now identify

$$\overline{D_{\check{b}_\ell|K}} = \overline{L_{D_i|K}}^{-1}\left[L_{D_i,\check{b}_\ell,D_j}(L^T_{D_j,D_i})^{-1}\right]\overline{L_{D_i|K}},$$

which is is a diagonal matrix. Therefore, by varying the value of $\check{b}_\ell$, we identify the component bid density functions (and quantile functions) up to a permutation because

$$\overline{D_{\check{b}_\ell|K}} = (L_{D_i|K}Q)^{-1}\left[L_{D_i,\check{b}_\ell,D_j}(L^T_{D_j,D_i})^{-1}\right]L_{D_i|K}Q = Q^{-1}D_{\check{b}_\ell|K}Q.$$

Note that the set of these “anonymous” component bid quantile functions is precisely the same as the set of primitive bid quantile functions. Thus, we can construct the same ordered tree using the former and the latter. Following Corollary 1, the kth function in the set of primitive value quantile functions is ranked the kth in the set of identified “anonymous” functions according to the RSD ordering. Therefore, it pins down $D_{\check{b}_\ell|K}$ and hence determines the matrix Q. Moreover, $L_{D_i|K} = \overline{L_{D_i|K}}Q^{-1}$ is identified. The remainder of the proof is identical to that of Hu, McAdams, and Shum (2013). □

A sufficient condition for Assumption 2 is that the component bid distribution functions are linearly independent. This condition is imposed on endogenous objects. Assumption 4 is satisfied if the value quantile functions are analytic (see Appendix B). Theorem 2 provides component bid distributions $G_k(\cdot)$ and the distribution of unobserved heterogeneity p_k. If bidders are risk neutral, component value distributions $F_k(\cdot)$ are identified according to Guerre, Perrigne, and Vuong (2000). In particular, we have the following FOC in quantile terms:

$$v_k(\alpha) = b_k(\alpha) + \frac{1}{I-1}\alpha b_k'(\alpha).$$

Therefore, the symmetric first-price auction model with discrete unobserved heterogeneity and risk neutrality is fully identified.

Numerical Example

We now use a numerical example to demonstrate Theorem 2. Consider unobserved heterogeneity $k = 1, 2, 3$, $p_k = 1/3$, and bid quantile functions $b_k(\alpha) = \alpha/(k+1)$, where $\alpha \in [0, 1]$.[8] The corresponding distribution functions are $G_k(b) = (k+1)b \cdot 1(b \in [0, 1/(k+1)]) + 1(b > 1/(k+1))$, and bid density functions are $g_k(b) = (k+1) \cdot 1(b \in [0, 1/(k+1)])$. Therefore, the resulting RSD rankings are $3, 2, 1$ for $k = 1, 2, 3$, respectively.

First, we partition the bid-space into three intervals: $[0, 1/4], (1/4, 1/3], (1/3, 1/2]$. The matrix for the conditional probabilities of observing D_i (D_j) in different states is

$$L_{D_i|K} = L_{D_j|K} = \begin{bmatrix} 1/2 & 3/4 & 1 \\ 1/6 & 1/4 & 0 \\ 1/3 & 0 & 0 \end{bmatrix}.$$

Moreover, $L^T_{D_j,D_i} = L_{D_j|K} \cdot \mathrm{diag}([p_1, p_2, p_3]) \cdot L^T_{D_i|K}$.

Second, take a point $\check{b}_\ell \in [0, 1/3]$, where conditional densities $g_{B_\ell|K}(\check{b}_\ell|k)$ are different, say $\check{b}_\ell = 7/24$.[9] Note that $L_{D_i,\check{b}_\ell,D_j}$ and L_{D_j,D_i} are observable. Eigenvalue decomposition of $L_{D_i,\check{b}_\ell,D_j}(L^T_{D_j,D_i})^{-1}$ using Matlab gives the eigenvector matrix

$$A = \begin{bmatrix} 1.0000 & -0.8018 & -0.9487 \\ 0.0000 & -0.2673 & -0.3162 \\ 0.0000 & -0.5345 & 0.0000 \end{bmatrix},$$

[8] The corresponding value quantile functions can be constructed using Guerre, Perrigne, and Vuong (2000).

[9] These conditional densities are the diagonal elements in the eigenvalue matrix $L_{\check{b}_\ell|K}$. In the case of duplicate eigenvalues, any linear combination of eigenvectors sharing an eigenvalue are themselves eigenvectors, causing indeterminacy (other than permutation and normalization) of $L_{D_i|K}$. See Hu (2008) for details.

and the diagonal eigenvalue matrix

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix},$$

respectively. The diagonal elements of D are conditional densities $g_{B_\ell|K}(\check{b}_\ell|k)$ but in a different order. Replacing all columns of A by their absolute values and normalizing column sum to 1 recovers

$$\overline{L_{D_i|K}} = \begin{bmatrix} 1 & 1/2 & 3/4 \\ 0 & 1/6 & 1/4 \\ 0 & 1/3 & 0 \end{bmatrix},$$

which is generated by interchanging columns of the true eigenvector matrix $L_{D_i|K}$.

Third, for any $b_\ell \in [0, 1/2]$, we calculate the component bid density functions up to a permutation

$$\overline{D_{b_\ell|K}} = \overline{L_{D_i|K}}^{-1} \cdot [L_{D_i,b_\ell,D_j}(L^T_{D_j,D_i})^{-1}] \cdot \overline{L_{D_i|K}},$$

where L_{D_i,b_ℓ,D_j} and $L^T_{D_jD_i}$ are observable. The diagonal elements of $\overline{D_{b_\ell|K}}$ for all $b_\ell \in [0, 1/2]$ constitute three "anonymous" density functions $g_{k_1}(b) = 4 \cdot 1(b \in [0, 1/4])$, $g_{k_2}(b) = 2 \cdot 1(b \in [0, 1/2]$, and $g_{k_3}(b) = 3 \cdot 1(b \in [0, 1/3]$. The corresponding bid quantile functions are $b_{k_1}(\alpha) = \alpha/4, b_{k_2}(\alpha) = \alpha/2$, and $b_{k_3}(\alpha) = \alpha/3$, respectively. Therefore, the RSD rankings of k_1, k_2, k_3 are $1, 3, 2$, respectively. In other words, the identified ranking is the same as the original ranking of primitives.

5 Applications

In this section, we demonstrate several applications that show how our relaxation of the FOSD condition further expands the breadth of applicability of the Hu, McAdams, and Shum (2013) misclassification approach. In particular, we apply RSD ordering to the identification of several auction models with unobserved heterogeneity: (a) values are *i.i.d.* draws from Beta distributions, (b) nonparametric utility function, (c) affiliated-signal entry models, (d) asymmetric bidders, (e) asymmetry in both preferences and private values.

5.1 Beta Distribution

An immediate application for our results would be the identification of auctions when values are drawn from Beta distributions. With a slight abuse of notation, the Beta distribution $\mathcal{B}(\alpha, \beta)$ is a family of continuous distributions defined on $[0, 1]$ and parameterized by two positive shape parameters $\alpha, \beta > 0$. Its density function is

$$f(v; \alpha, \beta) = \frac{v^{\alpha-1}(1-v)^{\beta-1}}{\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx},$$

if $v \in [0, 1]$, and $f(v; \alpha, \beta) = 0$ otherwise. It is a flexible distribution that has bounded support. In fact, any continuous distribution function on $[0, 1]$ can be approximated by a convex combination of Beta distributions (see Diaconis and Ylvisaker (1985)).

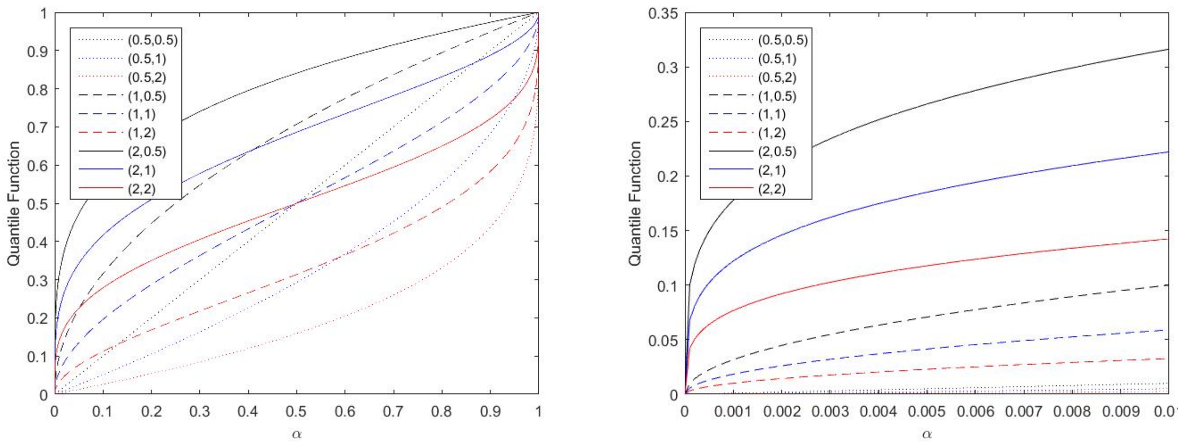
It is known that this family of distributions cannot be ranked by FOSD relations. On the other hand, RSD relations lead to a lexicographical ordering: first we rank α; the larger the α, the higher ranked the distribution; in case the α are the same, we rank β; the larger the β, the lower ranked the distribution.[10] Figure 2 shows the quantile functions of Beta distributions on $[0, 1]$ and $[0, 0.01]$ when $\alpha, \beta \in \{0.5, 1, 2\}$. While these functions cannot be ordered on $[0, 1]$, they do in a neighborhood of 0.

In summary, any set of distinct Beta distributions satisfies Assumption 4.[11] There-

[10]In fact, all Beta distributions split at $\alpha_\dagger = 0$. Consider two different Beta distributions defined by (α_1, β_1) and (α_2, β_2), respectively. It is easy to show that if $v \to 0$, then the difference $\log f_2(v) - \log f_1(v)$ converges to (1) $-\infty$ if $\alpha_1 < \alpha_2$, (2) $+\infty$ if $\alpha_1 > \alpha_2$, (3) a positive number if $\alpha_1 = \alpha_2$ and $\beta_1 < \beta_2$, and (4) a negative number if $\alpha_1 = \alpha_2$ and $\beta_1 > \beta_2$.

[11]Consider two different Beta distributions parameterized by (α_1, β_1) and (α_2, β_2), respectively. Note that if $f_1 \leq f_2, \forall a \in [0, x']$, then $F_1 \leq F_2, \forall a \in [0, x']$. Thus, we only need to show that there exists a point x such that the difference between the two PDFs does not change sign on $(0, x')$. We prove by contradiction. Suppose that for any x', we can find a change in sign on $(0, x')$ and thus a point $x \in (0, x')$ such that $x^{\alpha_1-1}(1-x)^{\beta_1-1}/B_1 = x^{\alpha_2-1}(1-x)^{\beta_2-1}/B_2$, where $B_t = \int_0^1 x^{\alpha_t-1}(1-x)^{\beta_t-1}dx$. Taking the logarithm

Figure 2: The Quantile Functions of Beta Distributions: $\alpha, \beta \in \{0.5, 1, 2\}$



(a) Probability $\alpha \in [0, 1]$

(b) Probability $\alpha \in [0, 0.01]$

fore, under Assumption 2, if $I \geq 3$ and the values are *i.i.d.* draws from Beta distributions $F(\cdot; \alpha_k, \beta_k)$ conditioning on $k \in \{1, \ldots, K\}$, the symmetric first-price auction model with discrete unobserved heterogeneity and risk neutrality is identified.

5.2 Guerre, Perrigne, and Vuong (2009): Risk Aversion

Theorem 1 says that the ordered trees are the same for the induced bid distributions regardless of the utility function and the number of bidders. This result is useful to identify auction models with discrete unobserved heterogeneity k and risk aversion (i.e., $U''(\cdot) < 0$) in the Guerre, Perrigne, and Vuong (2009) framework. In particular, we consider two sets of auctions with unobserved heterogeneity k, which have I_1 and I_2 bidders, respectively. We observe the joint distribution of three bids (b_i, b_j, b_ℓ) conditioning on $I = I_1, I_2$. We consider the identification of unobserved heterogeneity distributions $p_{k,I}$, utility function $U(\cdot)$, as well as component value distributions $F_{k,I}(\cdot)$. If bidders are risk averse, we need to deal with the

gives

$$T(x) = (\alpha_2 - \alpha_1) \log x + (\beta_2 - \beta_1) \log(1 - x) - \log(B_2/B_1) = 0.$$

Note that for any natural number M, we can find a point x_M such that $x_M \in (0, 1/2^M)$ and $T(x_M) = 0$. Consider two cases: (1) if $\alpha_2 = \alpha_1$ or $\beta_2 = \beta_1$, $T(\cdot)$ can only have one zero point, a contradiction. (2) if $\alpha_2 \neq \alpha_1$ and $\beta_2 \neq \beta_1$. When $M \to +\infty$, we have $x_M \to 0, \log x_M \to -\infty, \log(1 - x_M) \to 0$. Therefore, for an arbitrary $\epsilon > 0$, $\|T(x_M)\| > \epsilon$ when M is large enough, a contradiction.

unknown utility function $U(\cdot)$. We follow Guerre, Perrigne, and Vuong (2009) to exploit the exogenous participation restriction.

Assumption 5. $F_{k,I_1}(\cdot) = F_{k,I_2}(\cdot) = F_k(\cdot)$.

Note that we allow p_{k,I_1} and p_{k,I_2} to differ. Conditioning on the number of bidders $I \in \{I_1, I_2\}$, we can apply Theorem 2 to obtain unobserved heterogeneity distributions $p_{k,I}$, component bid distributions $G_{k,I}(\cdot)$ and corresponding bid quantile functions $b_{k,I}(\cdot)$. Identification of $\lambda(\cdot)$ is achieved if we can match the two bid quantile functions that are derived from the same unobserved heterogeneity k but with different numbers of bidders, say $3 \leq I_1 < I_2$.

The difficulty lies in how to match the bid quantile functions in the two lists. The reason is that with the same unobserved heterogeneity, the resulting component bid distributions are different due to different competition levels I_1 and I_2. If bidders are risk neutral, these two bid distributions are related through a known mapping due to exogenous participation[12]:

$$b_{k,I_2}(\alpha) = \frac{I_2 - 1}{I_1 - 1}\Big[b_{k,I_1}(\alpha) + (I_1 - I_2)\alpha^{1-I_2}\int_0^\alpha b_{k,I_1}(x)x^{I_2-2}dx\Big]. \tag{3}$$

With risk averse bidders, it is unclear how to make a similar connection before identifying bidders' utility function. In other words, we are not sure whether the difference between two bid quantile functions is due to different competition levels or unobserved heterogeneity.

We now show how to deal with this issue using our intermediate results. While we are not able to match bid distributions through a known functional, the corresponding bid quantile functions share the same ordered tree, which is inherited from the same set of value quantile functions and unaffected by the number of bidders.

Applying Theorem 2, we first identify the component bid quantile functions $G_{k,I}$ conditioning on the number of bidders $I = I_1, I_2$, respectively. This gives two ordered trees, $\mathcal{P}^{I_1}$ and $\mathcal{P}^{I_2}$, for each of the two samples, respectively. The two trees are identical. Second, we

[12]See Appendix A.5 for its proof.

apply Theorem 1 to match these bid quantile functions into pairs. The idea is that two bid quantile functions are generated by the same value quantile function if they are the same leaves on two identical ordered trees. Finally, we apply Guerre, Perrigne, and Vuong (2009) to identify risk aversion by exploiting the compatibility condition:

$$b_{k,I_1}(\alpha) + \lambda^{-1}\Big(\frac{1}{I_1 - 1}\alpha b'_{k,I_1}(\alpha)\Big) = b_{k,I_2}(\alpha) + \lambda^{-1}\Big(\frac{1}{I_2 - 1}\alpha b'_{k,I_2}(\alpha)\Big).$$

Therefore, the model is identified with two sets of auctions under different competition levels.

In summary, under Assumptions 2, 4, and 5, the symmetric first-price auction model with discrete unobserved heterogeneity and risk aversion is identified if $I_2 > I_1 \geq 3$. In a related paper, Grundl and Zhu (2018) show that the deconvolution method applies to nonparametric utility functions if unobserved heterogeneity enters values additively, but only to CRRA utility functions if it enters multiplicatively. While the Hu, McAdams, and Shum (2013) approach requires more bids than the deconvolution method, our results say that their misclassification approach is robust to the utility function. The intuition comes from the fact that the independence of bids can be inherited from independent values, while the multiplicative separability of bids may not necessarily be inherited from multiplicatively separable values.

Discussion

Note that Theorem 2 identifies $2K$ functions (i.e., K component bid distributions for each I), but the number of unknown primitive functions is $1+K$ (i.e., the common utility function $U(\cdot)$ and the value distributions $F_k(\cdot)$). Obviously, this model is overidentified. An interesting extension is to allow the utility function be state-specific, say $U_k(\cdot)$. The model becomes more general and its primitives become $\{U_k(\cdot), F_k(\cdot), p_{k,I_1}, p_{k,I_2}\}$.

To identify the new model, complications arise due to the interaction of $U_k(\cdot)$ and $F_k(\cdot)$. Obtaining comparative statics is difficult using the usual notion of stochastic dominance.

Furthermore, since $U_k(\cdot)$ and $F_k(\cdot)$ jointly determine $b_k(\cdot)$, the same RSD ordering may not pass from value quantile functions to bid quantile functions.

Despite these challenges, the notion of RSD is still useful in terms of identifying ordering because it only requires comparing function locally. One possible identification strategy is to use a point where the interaction of $U_k(\cdot)$ and $F_k(\cdot)$ disappears. In particular, we focus on probability $\alpha = 0$ and show that $v_k'(0) = \frac{I}{I-1} b_k'(0)$ if the coefficient of absolute risk aversion at 0, i.e., $U_k''(0)/U_k'(0)$, is finite, where $I = I_1, I_2$ and $k \in \mathcal{K}$. Therefore, the same RSD ordering passes from value quantile functions to bid quantile functions. In summary, the notion of RSD leads to a simple identification assumption.[13]

Assumption 6. *$U_k''(0)/U_k'(0)$ is finite for all k, and $v_1'(0) < v_2'(0) < \cdots < v_K'(0)$.*

The first part of Assumption 6 excludes a special case of hyperbolic absolute risk aversion utility functions, the CRRA utility function. In fact, $U(x) = x^\theta$ implies that $\lim_{x\downarrow 0} -U''(x)/U'(x) = \lim_{x\downarrow 0}(1-\theta)/x = +\infty$, where $\theta \in (0,1)$. The second part says that the splitting point is 0 for any two value quantile functions $v_k(\cdot)$ and $v_{k'}(\cdot)$. Many families of distribution functions with bounded supports satisfy this condition. For instance, the Beta distribution, the uniform distribution, and the two-sided power distribution.

In summary, under Assumptions 2, 4, 5, and 6, the symmetric first-price auction model with discrete unobserved heterogeneity and state-specific risk aversion is identified if $I_2 > I_1 \geq 3$.

5.3 Gentry and Li (2014): Affiliated-Signal Model

Gentry and Li (2014) study identification of the Affiliated-Signal model under risk neutrality,

[13] Equation (1) implies that

$$v'(\alpha) = b'(\alpha) + \frac{1}{I-1} \times \lambda^{-1\prime}\left(\frac{1}{I-1}\alpha b'(\alpha)\right) \times [\alpha b''(\alpha) + b'(\alpha)],$$

where we omit the subscript k for simplicity. Therefore, $v'(0) = [1 + \frac{1}{I-1}\lambda^{-1\prime}(0)]b'(0) = \frac{I}{I-1}b'(0)$. In fact, note that $\lambda(\cdot) = U(\cdot)/U'(\cdot)$ and we assume $U(0) = 0, U'(\cdot) > 0, U''(\cdot) \leq 0$. Thus, $\lambda(0) = 0$. Note that $\lambda'(\cdot) = \frac{(U'(\cdot))^2 - U(\cdot)U''(\cdot)}{(U'(\cdot))^2} = 1 - \lambda(\cdot)U''(\cdot)/U'(\cdot)$. Thus, $\lambda'(0) = (U'(0))^2/(U'(0))^2 = 1$ if $U''(0)/U'(0)$ is finite. In summary, $\lambda^{-1\prime}(0) = 1/\lambda'(0) = 1$.

in which potential bidders have private values v, observe imperfect signals s of their true values prior to entry, choose whether to undertake a costly entry process to learn their exact values and submit bids. Let $F(v, s|I, z)$ be the joint distribution of private value and signal, conditioning on the number of bidders I and a continuous cost shifter z, and let $c(I, z)$ be the entry cost function. With two excludable variations, $F(v, s|I, z) = F(v, s)$ and $c(I, z) = c(z)$, Gentry and Li (2014) show that conditional value distribution $F(\cdot|s)$ and entry cost function $c(z)$ are point-identified when z is continuous.[14]

They generalize their results to accommodate unobserved heterogeneity k by introducing the following assumptions: (1) conditional independence: $(v_i, s_i) \perp (v_j, s_j)|k$; (2) conditional excludability: $F(v, s|I, z, k) = F(v, s|k)$ and $c(I, z, k) = c(z; k)$; and (3):

Assumption 7 (Gentry and Li (2014))**.** *1. Stochastic ordering:* $F(v|s, k') \leq F(v|s, k)$ *if* $k' \geq k$*; 2. Entry ordering: for all* (I, z)*, entry threshold* $s^*(I, z; k)$ *is decreasing in* k*.*

While Assumptions (1) and (2) are standard and both stochastic ordering and entry ordering are intuitive, the orderings are assumed to ensure that the CDF of "realized bids" G_w^* can be indexed with respect to k.

$$G_w^*(W|I, z; k) = \begin{cases} s^*(I, z; k) + [1 - s^*(I, z; k)]G_b^*(b|I, z; k), \text{if } W > 0 \\ s^*(I, z; k), \text{if } W = 0 \end{cases},$$

where $W_i = B_i$ if bidder i enters and $W_i = 0$ otherwise; $s^*(I, z; k)$ is the entry threshold. The authors remark that the mixed random variables $(W_1, \ldots, W_I)$ are conditionally independent given k and stochastically increasing in k under Assumption 7.

The identification proceeds as follows: Given $I \geq 3$ and z, applying the Hu, McAdams, and Shum (2013) results identifies $G_w^*(W_i|I, z; k)$ and $p_k(I, z)$ up to a permutation from joint distribution $G_W(W_1, \ldots, W_I|I, z)$. Stochastic ordering of $G_w^*(W_i|I, z; k)$ ensures a unique mapping from the anonymous distributions to the elements of $\mathcal{K}$ for every pair (I, z).

[14] They have interesting results on partial identification with incomplete variation, which we omit here.

In fact, monotonicity of the entry threshold is sufficient for ordering unobserved heterogeneity conditioning on every pair (I, z). To see this, we construct an ordered tree using the corresponding quantile function of $\{G^*_w(W_i|I, z; k)\}_{k=1,\ldots,K}$. Since the quantile function remains at zero on $[0, s^*(I, z; k)]$ and jumps to a positive number thereafter, the RSD ordering of $G^*_w(W_i|I, z; k)$ is increasing with respect to k. Therefore, allowing for unobserved heterogeneity, the Gentry and Li (2014) results hold under a smaller set of assumptions: (1) conditional independence, (2) conditional excludability, and (3) entry ordering.

Discussion

In general, by introducing the covariate z, we have a problem of identifiability of finite mixture models with covariates, which suffers from a "label switching" problem. To deal with this, further assumptions are usually made. For instance, to identify auction models with asymmetric anonymous bidders and covariates, Lamy (2012) adopts a similar strategy by assuming that any two bid distributions are either equal or can be strictly ordered according to FOSD. That is, either $G(\cdot|z, k)$ and $G(\cdot|z, k')$ are the same for any z or $G(b|z, k) < G(b|z, k')$ for any b and any z. In view of our results, identification can be achieved by assuming that there exists a known functional $\mathcal{M}$ that yields the same ordering when applied to the set of bid functions $\{G(\cdot|z, k)\}_{k=1,\ldots,K}$ for every z.

For instance, consider symmetric first-price auctions with auction-specific covariates z and unobserved heterogeneity k, where private values are *i.i.d.* draws from the Beta distribution with shape parameters $(\alpha(z, k), \beta(z, k))$. The analyst observes repeated measurements, i.e. three bids per auction (b_1, b_2, b_3), which has joint cumulative distribution

$$G(b_1, b_2, b_3|z) = \sum_{k=1}^{K} p_{k|z} \times \Pi_{i=1}^{3} G_{\mathcal{B}}\big(b_i; \alpha(z, k), \beta(z, k)\big),$$

where $p_{k|z}$ is the conditional probability of unobserved heterogeneity, $(\alpha(z, k), \beta(z, k))$ are the shape parameters conditioning on (z, k), and $G_{\mathcal{B}}(\cdot; \alpha(z, k), \beta(z, k))$ is the corresponding bid distribution. In view of our results in Subsection 5.1, we provide a sufficient condition for nonparametric identification of the model primitives $\{\alpha(\cdot,\cdot), \beta(\cdot,\cdot), p_{k|z}\}$.[15]

In summary, under Assumption 2, if $I \geq 3$, $\alpha(z,k)$ is strictly monotone in k, and the values are *i.i.d.* draws from Beta distributions $F(\cdot; \alpha(z,k), \beta(z,k))$ conditioning on (z,k), then the symmetric first-price auction model with auction-specific covariates z, discrete unobserved heterogeneity k, and bidder risk neutrality is identified.

5.4 An (2017): Asymmetric Bidders

An (2017) studies identification of first-price auction models where bidders' values are asymmetrically distributed.[16] He considers a model with I risk neutral bidders whose private values are independent draws from $F_1(\cdot), \ldots, F_K(\cdot)$, which are CDFs with identical support $[\underline{v}, \overline{v}]$. Each bidder is one of K types. Let τ_i denote bidder i's type known to all bidders but unknown to the econometrician. Bidder i's bidding strategy is a mapping from his/her value to his/her bid, i.e., $s_i(\cdot,\cdot) : [\underline{v}, \overline{v}] \times \{1, \ldots, K\} \to [\underline{v}, \overline{v}]$. An (2017) assumes that each bidder participates in several auctions and that bidder type is invariant, so the econometrician observes the bidders' identity and joint distribution $G(b_\ell, b_{\ell'}, b_{\ell''})$.

Applying the results in Hu (2008), he identifies the type-specific bid distribution $G_k(\cdot)$ and then value distribution $F_k(\cdot)$ under Assumption 3. In view of our results, his results hold under a weaker condition, Assumption 4.

5.5 Multidimensional Unobserved Heterogeneity: Asymmetry in Both Preferences and Private Values

Let us now generalize An (2017)'s results to allow for asymmetry in both preferences and private values. That is, bidders differ not only in their value distributions but also in their

[15]While we constrain ourselves to discrete unobserved heterogeneity in this paper, RSD ordering can be applied to the continuous case. See, e.g., Hu and Schennach (2008). We leave this for future research.

[16]An (2017) also studies identification of the first-price auction model with non-equilibrium beliefs. We omit the discussion here because ordering follows from model restrictions.

utility functions. For simplicity, we assume bidders have CRRA utility function $U(x) = x^\theta$, where risk aversion parameter $\theta \in \{\theta_H, \theta_L\}$. Assume $0 < \theta_L < \theta_H \leq 1$ so that type-θ_L bidders are more risk averse than type-θ_H bidders. Moreover, we assume that the econometrician knows the values of θ_L and θ_H for simplicity of exposition.[17]

Each bidder is also one of K types whose private values are independent draws from $F_1(\cdot), \dots, F_K(\cdot)$. Following Guerre, Perrigne, and Vuong (2000) and Flambard and Perrigne (2006), the FOC can be rewritten as (see also Appendix C)

$$s_i^{-1}(b) = b + \frac{\theta_i}{\sum_{j \neq i} g_j(b)/G_j(b)}, \tag{4}$$

where $g_i(\cdot)$ and $G_i(\cdot)$ are the bid density and distribution functions of bidder $i = 1, \dots, I$. Since $F_i(\cdot) = G_i(s_i(\cdot))$, identification of $s_i^{-1}(\cdot)$ and $G_i(\cdot)$ implies identification of $F_i(\cdot)$.

Note that bidder i is of type $k \in \{1, \dots, K\}$ and $\theta \in \{\theta_H, \theta_L\}$. While the bidders' two types θ and k are common knowledge among bidders, they are unknown to the econometrician, which leads to the problem of multidimensional unobserved heterogeneity.[18] The model primitives are the component value quantile functions $\{v_k(\cdot)\}_{k\in\{1,\dots,K\}}$ and the proportions of each bidder type combination $\{p_{k\theta}\}_{k\in\{1,\dots,K\},\theta\in\{\theta_H,\theta_L\}}$.

We study the identification of this model in several steps. Following An (2017), we assume that each bidder participates in at least three independent auctions and that bidder type is invariant throughout. First, let us relabel bidder type as a combination of value distribution type k and risk aversion level θ, i.e. $\tau = (k, \theta)$. There are $2K$ types of bidders so $\tau = 1, \dots, 2K$. Let $G_{k\theta}(\cdot)$ represent the bid distribution function generated from bidders with value quantile function $v_k(\cdot)$ and risk aversion level θ. Second, applying Theorem 2, we can identify component bid quantile functions $b_\tau(\cdot)$ up to a permutation and the probability of type τ. Obviously, identification of the value distribution functions relies on correctly

[17] To identify these two parameters, we can introduce exogenous variation in competition level I as in Guerre, Perrigne, and Vuong (2009); see Subsection 5.2.

[18] Identification of multidimensional unobserved heterogeneity is also studied by Aguirregabiria and Mira (2018) for static games of incomplete information and Luo, Xiao, and Xiao (2018) for dynamic games.

assigning θ to the component bid quantile functions. Moreover, identification of $p_{k\theta}$ requires correctly assigning each τ to a pair (k, θ). However, identification up to a permutation does not provide the correspondence between τ and (k, θ).

Now we describe how to apply our results toward assigning θ to the list of "anonymous" bid quantile functions, which involves iterating the following two steps.

Step 1: We identify the lowest-ranked component bid distribution function and its corresponding value distribution function, say $F_{k_1}(\cdot)$. For any $k \in \mathcal{K}$, Appendix C shows that the bid quantile function of (k, θ_L)-type bidders ranks higher than that of (k, θ_H)-type bidders in our ordered tree constructed from $\{b_\tau(\cdot)\}_{\tau \in \{1,\dots,2K\}}$. Therefore, the lowest ranked bid quantile function in our ordered tree corresponds to the least risk averse bidders, i.e., $\theta = \theta_H$. Denote its distribution function as $G_{k_1\theta_H}(\cdot)$. Given this, we can identify the inverse bidding strategy, say $s^{-1}_{k_1\theta_H}(\cdot)$, and its corresponding value distribution function, say $F_{k_1}(\cdot)$, via Equation (4) with $\theta_i = \theta_H$.

Step 2: We identify the other component bid distribution function corresponding to the same value distribution function $F_{k_1}(\cdot)$ but for a different risk aversion level. In particular, Appendix C shows that

$$\frac{d}{db} \log F_{k_1}(s^{-1}_{k_1\theta_H}(b)) - \frac{d}{db} \log F_{k_1}(s^{-1}_{k_1\theta_L}(b)) = \frac{\theta_L}{s^{-1}_{k_1\theta_L}(b) - b} - \frac{\theta_H}{s^{-1}_{k_1\theta_H}(b) - b}, \tag{5}$$

where $s^{-1}_{k_1\theta_L}(\underline{v}) = \underline{v}$ and $s^{-1}_{k_1\theta_L}(\overline{b}) = \overline{v}$. We can show that this two-point boundary value problem identifies $s^{-1}_{k_1\theta_L}(\cdot)$, which makes use of the following simple lemma.

Lemma 5. *Let h be a differentiable function on $[\underline{x}, \overline{x}]$ for which (i) $h(\overline{x}) = 0$, (ii) $h(x) > 0$ implies that $h'(x) \geq 0$ and (iii) $h(x) < 0$ implies that $h'(x) \leq 0$. Then $h(x) = 0$ for all $x \in [\underline{x}, \overline{x}]$.*

Lemma 6. *The two-point boundary value problem (5) has a unique solution $s^{-1}_{k_1\theta_L}(\cdot)$.*

Finally, rewriting Equation (4) gives

$$\frac{g_{k_1\theta_L}(b)}{G_{k_1\theta_L}(b)} = \sum_{\imath=1}^{I}\frac{g_\imath(b)}{G_\imath(b)} - \frac{\theta_L}{s^{-1}_{k_1\theta_L}(b) - b},$$

which identifies $G_{k_1\theta_L}(\cdot)$ as follows:

$$G_{k_1\theta_L}(b) = \exp\Big\{-\int_b^{\bar{b}}\Big[\sum_{\imath=1}^{I}\frac{g_\imath(x)}{G_\imath(x)} - \frac{\theta_L}{s^{-1}_{k_1\theta_L}(x) - x}\Big]dx\Big\} = \Pi_{\imath=1}^{I}G_\imath(b)\cdot\exp\Big\{\int_b^{\bar{b}}\frac{\theta_L}{s^{-1}_{k_1\theta_L}(x) - x}dx\Big\},$$

which is derived from the value distribution function $F_{k_1}(\cdot)$ and risk aversion level θ_L.

Dropping these two bid quantile functions from the list leaves us with remaining $2(K-1)$ "anonymous" bid quantile functions. We again construct an ordered tree and identify the value quantile function that corresponds to the lowest ranked bid quantile function, as well as the bid quantile function of more risk averse bidders who have the same value distribution. We can repeat this process until each τ has been assigned a value of θ and a value of k.

In summary, the first-price auction model with unobserved asymmetry in both preferences and private values is identified with the same data requirements as in An (2017).

6 Empirical Application

6.1 Data

In this section, we apply our results to an empirical analysis of the United States Forest Service (USFS) timber auctions. See, e.g., Baldwin, Marshall, and Richard (1997), Haile (2001), and Haile and Tamer (2003). In particular, we analyze the sealed-bid auction data from 1982 to 1990, which are constructed from the data available on Philip A. Haile's website. Following Haile, Hong, and Shum (2003), we consider only scaled sales in Forest Service Regions 1 and 5 to minimize the significance of subcontracting/resale and thus common value. Moreover, we drop sales that are set aside for small businesses and salvage sales. In summary, we focus on auctions that are most likely to satisfy the independent private value assumption. For simplicity, we focus on the 159 auctions with three bidders.

Table 1 provides summary statistics on a list of auction-specific observables: winning bid, size of tract (in acres), estimated volume of timber (in MBF), appraisal value (per MBF), estimated selling value (per MBF), estimated harvesting cost (per MBF), estimated manufacturing cost (per MBF), species concentration index (HHI) and Forest Service region.[19]

Table 1: Summary Statistics

Variable	Mean	Std. Dev.	Min	Max
winning bid	95688.19	148656.60	2848.18	1112966
acres	499.35	791.24	4	7000
vol_sum	1776.11	2493.26	52	12900
AppValue_avg	29.49	26.95	0.50	162.69
SellValue_avg	349.68	68.22	154.91	532.57
LogCost_avg	134.81	28.09	64.81	259.44
MfgCost_avg	169.11	33.87	3.71	248.62
HHI	0.5788	0.2447	0.1722	1
D5	0.4654	0.5004	0	1

Note: D5=1 if the Forest Service region is 5, =0 otherwise.

Before implementing our methodology, we apply the Haile, Hong, and Shum (2003) method to homogenize the bids. Table 2 displays the regression results. Regression (3) includes all control variables as well as year dummies. All estimated coefficients have the expected signs. We calculate homogenized bids as the exponential of the differences between the logarithm of the original total bids and the demeaned fitted values of Regression (3).

6.2 Estimation Results

In view of the small sample size, we estimate a parametric auction model with risk neutral bidders. In particular, we propose the family of two-sided power distributions introduced into the statistic literature by Van Dorp and Kotz (2002). With a location parameter θ

[19] We construct the variables following Liu and Luo (2017). All dollar values are nominal and all volume values are in thousand board feet (MBF) of timber.

Table 2: Regression Results

VARIABLES	(1)	(2)	(3)
log_acres		-0.0188	-0.0330
		(0.0252)	(0.0236)
log_vol_sum	1.014***	1.032***	1.058***
	(0.0213)	(0.0289)	(0.0293)
log_AppValue_avg	0.595***	0.456***	0.458***
	(0.0343)	(0.0470)	(0.0446)
log_SellValue_avg		1.166***	1.242***
		(0.201)	(0.209)
log_LogCost_avg		-0.437***	-0.421***
		(0.133)	(0.131)
log_MfgCost_avg		-0.205***	-0.268***
		(0.0641)	(0.0645)
HHI		-0.0973	-0.0504
		(0.103)	(0.0977)
Observations	477	477	477
R-squared	0.841	0.853	0.866
Year FE			YES

Robust standard errors in parentheses
*** p<0.01, ** p<0.05, * p<0.1

and a shape parameter γ, the two-sided power distribution has two important advantages in the context of auctions. First, this family of distributions is similar in flexibility to the Beta distribution and includes uniform distributions, standard power distributions, and triangular distributions. Moreover, similar to the Beta distribution, this family of distributions can also approximate any continuous distribution function on $[0, 1]$ arbitrarily close by a convex combination of two-sided power distribution functions (see Bornkamp and Ickstadt (2009)). Second, it is easy to solve auction models where bidder values are drawn from two-sided power distributions.

The two-sided power distribution function is given by

$$F(x;\theta,\gamma) = \begin{cases} \theta(\frac{x}{\theta})^{\gamma} & \text{if } x \in [0,\theta] \\ 1-(1-\theta)(\frac{1-x}{1-\theta})^{\gamma} & \text{if } x \in (\theta,1] \end{cases},$$

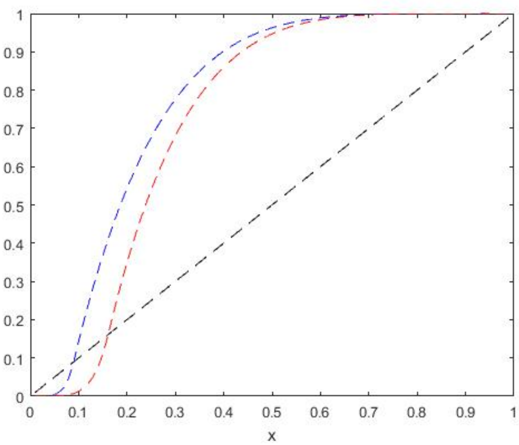
where location parameter $\theta \in [0, 1]$ and shape parameter $\gamma > 0$. Since the two-sided power distribution is supported on range $[0,1]$, we alter the location and scale of the distribution by introducing a linear transformation using an additional location parameter κ_0 and a scale parameter κ_1. In particular, we assume that the values satisfy $v_i = \kappa_0 + \kappa_1 x_i$, where $i \in \{1,2,3\}$, and x_i are *i.i.d.* draws from the two-sided power distribution conditioning on the auction-specific state (γ, θ).

For simplicity, assume that each of the state parameters (γ, θ) has two possible values, i.e., $\gamma \in \{\gamma_L, \gamma_H\}$ and $\theta \in \{\theta_L, \theta_H\}$, where $\gamma_L < \gamma_H$ and $\theta_L < \theta_H$. Two-dimensional unobserved heterogeneity arises because the analyst does not observe (γ, θ). The primitives are the location and scale parameters (κ_0, κ_1), the location and shape parameters $\{\gamma_L, \gamma_H, \theta_L, \theta_H\}$ and the probability distribution of the states $\{p_{LL}, p_{LH}, p_{HL}, p_{HH}\}$, where the two subscripts represent the levels of γ and θ, respectively.

We estimate the model via maximum likelihood.[20] See, e.g., Donald and Paarsch (1993), Hirano and Porter (2003), and Chernozhukov and Hong (2004). All dollar values are reported in thousands. κ_0 is estimated to be the minimum homogenized bid, i.e. 2.63. The estimated scale parameter is $\widehat{\kappa}_1 = 187.59$. The estimated location and shape parameters are $\widehat{\gamma}_L = 1.00, \widehat{\gamma}_H = 5.34, \widehat{\theta}_L = 0.0896$, and $\widehat{\theta}_H = 0.1610$. The mixing weights are $\widehat{p}_{LL} = 0.2612, \widehat{p}_{LH} = 0.2365, \widehat{p}_{HL} = 0.2459$, and $\widehat{p}_{HH} = 0.2564$. Note that the two-sided power distribution becomes a uniform distribution when $\gamma = 1$. Thus, assuming that our model is correctly specified, our results imply that the bids we observe are most likely from a mixture of a uniform distribution and two two-sided power distributions with parameters $(\widehat{\gamma}_H, \widehat{\theta}_L)$ and $(\widehat{\gamma}_H, \widehat{\theta}_H)$, respectively. The mixing weights are 49.77%, 24.59%, and 25.64%, respectively. Figure 3 shows the three component value distribution functions.

[20] We use the Nelder-Mead Simplex Method for maximizing the objective function. Our estimation converges to the same solution for all attempted starting values.

Figure 3: Estimated Component Value Distributions



7 Conclusion

In this paper, we introduce the concept of restricted stochastic dominance in auctions. We show that the same ordering passes from value quantile functions to bid quantile functions. Relying on these results, we define an RSD ordering for a set of distinct quantile functions and show that this ordering is preserved by the mapping from value quantile functions to bid quantile functions in symmetric first-price auctions. This property is used to generalize Hu, McAdams, and Shum (2013) by allowing for nonseparable unobserved heterogeneity under an RSD condition. This condition is more general than the FOSD condition and allows for all analytic functions as well as some non-analytic ones. Therefore, our results further expand the breadth of applicability of the misclassification approach. Finally, we apply our results to the identification of various auction models with unobserved heterogeneity.

While we focus on auction models in this paper, RSD ordering can be useful in other applications of measurement error models. See Hu (2017) for a recent survey on applications in empirical industrial organization and labor economics. Moreover, although we only study discrete unobserved heterogeneity in this paper, RSD ordering can also be applied to the continuous case. See, e.g., Hu and Schennach (2008). We leave this for future research.

Appendices

A Proofs

A.1 Proof of Lemma 1

Proof. Consider an arbitrary $\epsilon > 0$. First, consider $v_1(\cdot)$ and $v_2(\cdot)$. There exists $x_{12} \in [\alpha_\dagger, \alpha_\dagger + \epsilon)$ such that $\forall \alpha \in [0, x_{12}], v_1(\alpha) \leq v_2(\alpha), \& \exists \alpha_{12} \in (\alpha_\dagger, x_{12}], v_1(\alpha_{12}) < v_2(\alpha_{12})$. Second, consider $v_2(\cdot)$ and $v_3(\cdot)$. There exists $x_{23} \in [\alpha_\dagger, \alpha_{12})$ such that $\forall \alpha \in [0, x_{23}], v_2(\alpha) \leq v_3(\alpha), \& \exists \alpha_{23} \in (\alpha_\dagger, x_{23}], v_2(\alpha_{23}) < v_3(\alpha_{23})$. Note that $0 \leq \alpha_\dagger < \alpha_{23} \leq x_{23} < \alpha_{12} \leq x_{12} < \alpha_\dagger + \epsilon$. In summary, $\alpha_{23} \in [0, x_{23}] \subset [0, x_{12}]$ and $v_1(\alpha_{23}) \leq v_2(\alpha_{23}) < v_3(\alpha_{23})$, where the first inequality follows $\forall \alpha \in [0, x_{12}], v_1(\alpha) \leq v_2(\alpha)$. Moreover, $\forall \alpha \in [0, x_{23}], v_1(\alpha) \leq v_2(\alpha) \leq v_3(\alpha)$. Therefore, $v_1(\cdot) \underset{\alpha_\dagger}{\prec} v_3(\cdot)$. □

A.2 Proof of Lemma 2

Proof. Consider an arbitrary $\epsilon > 0$. We consider three exhaustive and exclusive cases: (1) If $\alpha_\dagger = \alpha_{\dagger\dagger}$, we obtain $v_1(\cdot) \underset{\alpha_\dagger}{\prec} v_3(\cdot)$ because Lemma 1 applies. (2) If $\alpha_\dagger < \alpha_{\dagger\dagger}$, $v_2(\cdot) \underset{\alpha_{\dagger\dagger}}{\prec} v_3(\cdot)$ implies that there exists $x_{23} \in [\alpha_{\dagger\dagger}, \alpha_{\dagger\dagger} + \epsilon)$ such that $\forall \alpha \in [0, x_{23}], v_2(\alpha) \leq v_3(\alpha), \& \exists \alpha_{23} \in (\alpha_{\dagger\dagger}, x_{23}], v_2(\alpha_{23}) < v_3(\alpha_{23})$. Moreover, $v_1(\cdot) \underset{\alpha_\dagger}{\prec} v_2(\cdot)$ implies that there exists $x_{12} \in [\alpha_\dagger, \min\{\alpha_{23}, \alpha_\dagger + \epsilon\})$ such that $\forall \alpha \in [0, x_{12}], v_1(\alpha) \leq v_2(\alpha), \& \exists \alpha_{12} \in (\alpha_\dagger, x_{12}], v_1(\alpha_{12}) < v_2(\alpha_{12})$. Note that $0 \leq \alpha_\dagger < \alpha_{12} \leq x_{12} < \alpha_{23} \leq x_{23} < \alpha_{\dagger\dagger} + \epsilon$. In summary, $\alpha_{12} \in [0, x_{12}] \subset [0, x_{23}]$ and $v_1(\alpha_{12}) < v_2(\alpha_{12}) \leq v_3(\alpha_{12})$, where the second inequality follows $\forall \alpha \in [0, x_{23}], v_2(\alpha) \leq v_3(\alpha)$. Moreover, $\forall \alpha \in [0, x_{12}], v_1(\alpha) \leq v_2(\alpha) \leq v_3(\alpha)$. Therefore, $v_1(\cdot) \underset{\alpha_\dagger}{\prec} v_3(\cdot)$. (3) If $\alpha_\dagger > \alpha_{\dagger\dagger}$, we obtain $v_1(\cdot) \underset{\alpha_{\dagger\dagger}}{\prec} v_3(\cdot)$ following similar reasoning as in (2). □

A.3 Proof of Lemma 3

Proof. We prove by contradiction. Suppose $b_1(\alpha_R) > b_2(\alpha_R)$ for some $\alpha_R \in [0, x]$. Let $\alpha_L = \inf\{x' \in (0, \alpha_R] | b_1(\alpha) > b_2(\alpha), \forall \alpha \in (x', \alpha_R]\}$. From the continuity of $b_k(\cdot)$s, we have $b_1(\alpha_L) = b_2(\alpha_L)$. On the other hand, since $b_1(\alpha) > b_2(\alpha), \forall \alpha \in (\alpha_L, \alpha_R]$, we have $b_1'(\alpha) < b_2'(\alpha), \forall \alpha \in (\alpha_L, \alpha_R]$. In fact, since $v_1(\alpha) \le v_2(\alpha), \forall \alpha \in [0, x]$, we have $v_1(\alpha) - b_1(\alpha) < v_2(\alpha) - b_2(\alpha), \forall \alpha \in (\alpha_L, \alpha_R]$. Thus, monotonicity of $\lambda(\cdot)$ implies that

$$\frac{I-1}{\alpha}\lambda(v_1(\alpha) - b_1(\alpha)) < \frac{I-1}{\alpha}\lambda(v_2(\alpha) - b_2(\alpha)),$$

which implies that $b_1'(\alpha) < b_2'(\alpha), \forall \alpha \in (\alpha_L, \alpha_R]$ due to Equation (2). Therefore, $b_1(\alpha_R) = b_1(\alpha_L) + \int_{\alpha_L}^{\alpha_R} b_1'(x)dx < b_2(\alpha_L) + \int_{\alpha_L}^{\alpha_R} b_2'(x)dx = b_2(\alpha_R)$, a contradiction. □

A.4 Proof of Lemma 4

Proof. Let $\alpha_L = \sup\{x' \in [0, \alpha_*) | v_1(\alpha) = v_2(\alpha), \forall \alpha \in [0, x']\}$. By continuity of $v_k(\cdot)$, $\alpha_L < \alpha_*$ and $v_1(\alpha_L) = v_2(\alpha_L)$. Lemma 3 implies that $b_1(\alpha_L) = b_2(\alpha_L)$ as well. By definition of α_L, for any $\epsilon > 0$, there exists a probability $\alpha \in (\alpha_L, \alpha_L + \epsilon)$ such that $v_1(\alpha) < v_2(\alpha)$.

Consider a probability $\alpha \in (\alpha_L, x]$. Equation (2) implies that

$$\begin{aligned} \alpha[b_2'(\alpha) - b_1'(\alpha)] &= (I-1)[\lambda(v_2(\alpha) - b_2(\alpha)) - \lambda(v_1(\alpha) - b_1(\alpha))] \\ &= (I-1)\lambda'(\widetilde{R(\alpha)})[(v_2(\alpha) - v_1(\alpha)) - (b_2(\alpha) - b_1(\alpha))], \end{aligned}$$

where the second equation follows the mean value theorem and $\widetilde{R(\alpha)}$ is a value between $v_1(\alpha) - b_1(\alpha)$ and $v_2(\alpha) - b_2(\alpha)$. Denote $y(\alpha) = b_2(\alpha) - b_1(\alpha)$, $q(\alpha) = -(I-1)\lambda'(\widetilde{R(\alpha)})/\alpha$, $p(\alpha) = (I-1)(v_2(\alpha) - v_1(\alpha))\lambda'(\widetilde{R(\alpha)})/\alpha$. We obtain $y'(\alpha) = q(\alpha)y(\alpha) + p(\alpha)$ of which solutions take the form

$$y(\alpha) = ce^{Q(\alpha)} + e^{Q(\alpha)} \int_{\alpha_L}^{\alpha} e^{-Q(x)} p(x)dx,$$

where $Q(\alpha) = \int_{\alpha_L}^{\alpha} q(x)dx$ and c is a constant. The boundary condition $y(\alpha_L) = b_2(\alpha_L) - b_1(\alpha_L) = 0$ implies that $c = 0$. Since $\lambda'(\cdot) = (U(\cdot)/U'(\cdot))' = 1 - U(\cdot)U''(\cdot)/(U'(\cdot))^2 \ge 1$, $v_2(\alpha) - v_1(\alpha) \ge 0, \forall \alpha \in (\alpha_L, \alpha]$ and $v_2(\alpha') - v_1(\alpha') > 0$ for some $\alpha' \in (\alpha_L, \alpha)$, we have $p(\alpha) \ge 0, \forall \alpha \in (\alpha_L, \alpha]$ and $p(\alpha') > 0$ for some $\alpha' \in (\alpha_L, \alpha)$. Therefore, $y(\alpha) =$

$\int_{\alpha_L}^{\alpha} e^{Q(\alpha)-Q(x)}p(x)dx > 0$. In other words, $b_2(\alpha) > b_1(\alpha), \forall\alpha \in (\alpha_L, x]$. □

A.5 Proof of Equation 3

Proof. For simplicity, consider homogeneous auctions of I bidders. First, the bidder value quantile function satisfies the FOC

$$v(\alpha) = b_I(\alpha) + \frac{1}{I-1}\alpha b_I'(\alpha).$$

Let $y(\alpha) = b(\alpha), q(\alpha) = -(I-1)/\alpha, p(\alpha) = (I-1)v(\alpha)/\alpha$. We obtain $y'(\alpha) = q(\alpha)y(\alpha)+p(\alpha)$ of which solutions take the form

$$y(\alpha) = ce^{Q(\alpha)} + e^{Q(\alpha)} \int_0^{\alpha} e^{-Q(x)}p(x)dx,$$

where $Q(\alpha) = \int_0^{\alpha} q(x)dx$ and c is a constant. Note that $Q(\alpha) - Q(x) = \int_x^{\alpha} q(z)dz = \ln(x/\alpha)^{I-1}$. The boundary condition $y(0) = \underline{v}$ implies that $c = 0$. Rearranging terms leads to an explicit mapping from value quantile function to bid quantile function

$$b_I(\alpha) = (I-1)\alpha^{1-I} \int_0^{\alpha} v(x)x^{I-2}dx.$$

See also Gimenes and Guerre (2016) and Liu and Luo (2017). Therefore, under different auction competition levels, the two bid quantile functions would satisfy the following condition

$$b_{I_2}(\alpha) = (I_2-1)\alpha^{1-I_2} \int_0^{\alpha} [b_{I_1}(x) + \frac{1}{I_1-1}xb_{I_1}'(x)]x^{I_2-2}dx.$$

Integration by parts gives

$$b_{I_2}(\alpha) = \frac{I_2-1}{I_1-1}\Big[b_{I_1}(\alpha) + (I_1-I_2)\alpha^{1-I_2} \int_0^{\alpha} b_{I_1}(x)x^{I_2-2}dx\Big].$$

Moreover, it is also known that $b_{I_1}(\cdot) < b_{I_2}(\cdot)$ on $(0,1]$ if $I_1 < I_2$, and $b_{I_1}(0) = b_{I_2}(0) = \underline{v}$. See Guerre, Perrigne, and Vuong (2009). $\square$

A.6 Proof of Lemma 5

Proof. We prove by contradiction. Without loss of generality, suppose that there exists a point $x_* \in [\underline{x}, \overline{x})$ such that $h(x_*) > 0$. Let $x_L = \inf\{x \in (x_*, \overline{x}] | h(x) = 0\}$. From the continuity of $h(\cdot)$, we have $h(x_L) = 0$. Moreover, $h(x) > 0, \forall x \in (x_*, x_L)$. Otherwise we have a contradiction in the definition of x_L in either of the two exhaustive cases: (1) $h(x_1) = 0$ for some $x_1 \in (x_*, x_L)$; (2) $h(x_2) < 0$ for some $x_2 \in (x_*, x_L)$. In this case, by the intermediate value theorem, there exists another $x_3 \in (x_*, x_2)$ such that $h(x_3) = 0$.

Since $h(x) > 0, \forall x \in (x_*, x_L)$, condition (ii) implies that $h'(x) \geq 0, \forall x \in (x_*, x_L)$. However, by the mean value theorem, there exists a point $x \in (x_*, x_L)$ such that $h'(x) = \frac{h(x_L) - h(x_*)}{x_L - x_*} < 0$, which contradicts $h'(x) \geq 0, \forall x \in (x_*, x_L)$. $\square$

Proof of Lemma 6

Proof. Consider two solutions $\xi(\cdot)$ and $\widetilde{\xi}(\cdot)$. Denote $h(x) = \log F_{k_1}(\widetilde{\xi}(\cdot)) - \log F_{k_1}(\xi(\cdot))$. Thus, $h(\overline{b}) = 0$ and $\frac{d}{db} \log F_{k_1}(\xi(b)) + \frac{\theta_L}{\xi(b) - b} = \frac{d}{db} \log F_{k_1}(\widetilde{\xi}(b)) + \frac{\theta_L}{\widetilde{\xi}(b) - b}$, which implies that

$$\frac{d}{db} \log F_{k_1}(\widetilde{\xi}(b))/F_{k_1}(\xi(b)) = \frac{\theta_L(\widetilde{\xi}(b) - \xi(b))}{(\xi(b) - b)(\widetilde{\xi}(b) - b)}.$$

Thus, (i) if $h(b) > 0$, then $\widetilde{\xi}(b) > \xi(b)$ and also $h'(b) = \frac{d}{db} \log F_{k_1}(\widetilde{\xi}(b))/F_{k_1}(\xi(b)) > 0$ and (ii) if $h(b) < 0$, then $\widetilde{\xi}(b) < \xi(b)$ and also $h'(b) = \frac{d}{db} \log F_{k_1}(\widetilde{\xi}(b))/F_{k_1}(\xi(b)) < 0$. Lemma 5 implies that $h(b) = 0$ for all $b \in [\underline{b}, \overline{b}]$. In other words, $\widetilde{\xi}(\cdot) = \xi(\cdot)$. $\square$

B Sufficient Conditions for Assumptions 2 and 4

We now introduce a sufficient condition for Assumption 2 rank condition.

Assumption 8. *The component bid distribution functions are linearly independent.*

Proof. Note that $L_{D_j,D_i} = L_{D_j|K} \times D_K \times L^T_{D_i|K}$, where $D_K = diag\{Pr\{K = k\}_{k\in\{1,\ldots,K\}}\}$ has rank K. We only need to show that both $L_{D_j|K}$ and $L_{D_i|K}$ have rank K.

For an arbitrary list of cutoff points $x_1 < \ldots < x_\imath$,

$$L_{D_j|K} = \begin{pmatrix} F_1(x_1) - 0 & F_2(x_1) - 0 & \ldots & F_K(x_1) - 0 \\ F_1(x_2) - F_1(x_1) & F_2(x_2) - F_2(x_1) & \ldots & F_K(x_2) - F_K(x_1) \\ \ldots & \ldots & \ldots & \ldots \\ F_1(x_\imath) - F_1(x_{\imath-1}) & F_2(x_\imath) - F_2(x_{\imath-1}) & \ldots & F_K(x_\imath) - F_K(x_{\imath-1}) \\ 1 - F_1(x_\imath) & 1 - F_2(x_\imath) & \ldots & 1 - F_K(x_\imath) \end{pmatrix}.$$

By replacing each row by the sum of all the rows that proceed it and then reordering rows, we obtain the following matrix which has the same rank as $L_{D_j|K}$:

$$A_\imath = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ F_1(x_1) & F_2(x_1) & \ldots & F_K(x_1) \\ F_1(x_2) & F_2(x_2) & \ldots & F_K(x_2) \\ \ldots & \ldots & \ldots & \ldots \\ F_1(x_\imath) & F_2(x_\imath) & \ldots & F_K(x_\imath) \end{pmatrix}.$$

For notational convenience, we look for a list of cutoff points such that $A_\imath$ has rank K.

The proof has two steps. In the first step, following Allman, Matias, and Rhodes (2009), we show there exists a list of cutoff points such that $A_\imath$ is of rank K by adding well-chosen points to an arbitrary list of cutoffs. Take an arbitrary list of cutoff points $x_1 < x_2 < \ldots < x_\imath$, where $\imath \geq K - 1$. If $A_\imath$ is of rank K, we move directly to the second step of the proof. If $A_\imath$ is not of rank K, we can find a nonzero vector $\beta \in R^K$ such that $A_\imath\beta = 0$. Since the distributions are linearly independent, we can find $x_{\imath+1}$ such that $\sum_{k=1}^K F_k(x_{\imath+1})\beta_k \neq 0$. Let

$$A_{\imath+1} = \begin{pmatrix} & A_\imath & & \\ F_1(x_{\imath+1}) & F_2(x_{\imath+1}) & \ldots & F_K(x_{\imath+1}) \end{pmatrix}.$$

We now show that $A_{\imath+1}$ has a larger rank than $A_\imath$. If not, they are of the same rank, which implies that the new row is a linear combination of the original rows. That is, $\forall k, F_k(x_{\imath+1}) = \sum_{\ell=0}^{\imath} \gamma_\ell F_k(x_\ell)$ for some $(\gamma_0, \ldots, \gamma_\imath)'$, where $F_k(x_0) = 1$. Since $\sum_{k=1}^{K} F_k(x_{\imath+1})\beta_k \neq 0$, we have $\sum_{k=1}^{K}[\sum_{\ell=0}^{\imath} \gamma_\ell F_k(x_\ell)]\beta_k \neq 0$. On the other hand, $A_\imath \beta = 0$ implies that $0 = \sum_{\ell=0}^{\imath} \gamma_\ell[\sum_{k=1}^{K} F_k(x_\ell)\beta_k] = \sum_{k=1}^{K}[\sum_{\ell=0}^{\imath} \gamma_\ell F_k(x_\ell)]\beta_k$, a contradiction.

We can continue adding new rows in this manner until the rank increases to K. In the second step, based on the list of cutoffs, we can find exactly $K-1$ cutoff points $\{x_1, \ldots, x_{K-1}\}$ such that A_K is of rank K by applying Lemma 2 from Xiao (2018). □

Assumption 8 has been used in An (2017) for identifying the number of types K. Following his paper, one could also describe the linear independence condition in terms of primitives: the functions $\{F_1(s_1^{-1}(\cdot)), \ldots, F_K(s_K^{-1}(\cdot))\}$ are linearly independent, where $s_k^{-1}(\cdot)$ is the inverse bidding strategy in state-k auctions defined by the model primitives $\{U(\cdot), F_k(\cdot)\}$.

Recall that the FOSD condition is sufficient for both the UH monotonicity condition and the UH full-rank condition. See Hu, McAdams, and Shum (2013). A key step in their proof is showing that the maximum bid is increasing in state k, which implies that the component bid distribution functions are linearly independent.[21]

We now describe a sufficient condition for Assumption 4.

Assumption 9. $\{v_k(\cdot)\}_{k=1,\ldots,K}$ *are analytic.*

Proof. Suppose $v_k(\cdot)$ and $v_{k'}(\cdot)$ are analytic and Assumption 4 is not true. Then, for any $x \in (0, 1]$, we can find a switch in sign for the function $v_k(\cdot) - v_{k'}(\cdot)$ on $[0, x]$. By the intermediate value theorem, there exists an $\alpha_{(1)}$ on $(0, 1]$ such that $v_k(\alpha_{(1)}) = v_{k'}(\alpha_{(1)})$. For the same reason, we can find an $\alpha_{(2)}$ on $(0, \min\{1/2, \alpha_{(1)}\}]$ such that $v_k(\alpha_{(2)}) = v_{k'}(\alpha_{(2)})$. Repeating this procedure yields a converging sequence $\{\alpha_{(m)}\}_{m=1,2,\ldots}$, where $\alpha_{(m)} \in (0, \min\{1/2^{m-1}, \alpha_{(m-1)}]$, such that $v_k(\alpha_{(m)}) = v_{k'}(\alpha_{(m)})$, and $\lim_{m\to\infty} \alpha_{(m)} = 0$. Note that $v_k(0) - v_{k'}(0) = 0$.

[21] In fact, if $\sum_{k=1}^{K} \gamma_k G_k(b) = 0$, then $\gamma_k = 0, \forall k \in \mathcal{K}$. To see this, consider an arbitrary $b \in (\bar{b}_{K-1}, \bar{b}_K)$. Since $\sum_{k=1}^{K} \gamma_k G_k(b) = \gamma_K G_K(b) = 0$, we have $\gamma_K = 0$. Similarly, we can show that $\gamma_k = 0$ for all $k = K-1, \ldots, 1$ by considering a bid b in $(\bar{b}_{K-2}, \bar{b}_{K-1}), \ldots, (\bar{b}_1, \bar{b}_2)$ sequentially.

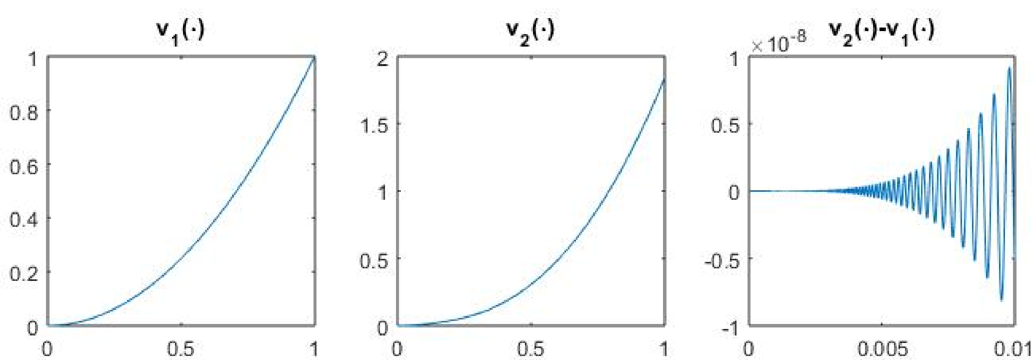
Therefore, the difference must be 0 on $(0, 1]$ by the principle of permanence, meaning that $v_k(\alpha) = v_{k'}(\alpha), \forall \alpha \in [0, 1]$, a contradiction. □

An example of a non-analytic function is $f(x) = x^4 \sin(1/x)$, where $x \in [0, 1]$. See Figure 4 for its graph on $[0, 0.01]$. In fact, for any $x \in (0, 1]$, we can always find a change of sign in $(0, x]$. If this function were the difference of the two value quantile functions, it is obvious that while it would be continuously differentiable, there would exist no splitting point. For instance, let $v_1(\alpha) = \alpha^2$ and $v_2(\alpha) = \alpha^2 + \alpha^4 \sin(1/\alpha)$. Figure 4 shows that $v_1(\cdot)$ and $v_2(\cdot)$ are strictly increasing and hence valid quantile functions but they violate Assumption 4.

Note that analytic functions are infinitely differentiable and include all elementary functions such as polynomial functions, exponential functions, and trigonometric functions. Moreover, the Weierstrass Approximation Theorem says that any continuous function on a bounded interval can be uniformly approximated by polynomial functions, which are analytic. For assuming analytic functions achieve identification in various econometrics contexts, see, e.g., Fox, Kim, Ryan, and Bajari (2012) for a random coefficients logit model, Hernández, Quint, and Turansick (2018) for English auctions, Fox and Gandhi (2016) for a multinomial choice model, and Aryal (2016) for an adverse selection model.

Nevertheless, Assumption 4 does allow for some non-analytic functions. For instance, $v_1(\alpha) = \alpha, v_2(\alpha) = \alpha + \exp(-1/\alpha)$, if $\alpha \in (0, 1]$ and $v_1(0) = v_2(0) = 0$.

Figure 4: A non-analytic function: $v_2(\alpha) - v_1(\alpha) = \alpha^4 \sin(1/\alpha)$



C Restricted Stochastic Dominance in Asymmetric Auctions

Asymmetry among bidders can arise from either different value distributions $F_i(\cdot)$ and/or different utility functions $U_i(\cdot)$. In an independent private value first-price auction, bidder i maximizes the payoff

$$U_i(v-b)\cdot\Pi_{j\neq i}F_j(s_j^{-1}(b)),$$

where $\Pi_{j\neq i}F_j(s_j^{-1}(b))$ is the probability of winning. After some algebra, the FOC can be rewritten as

$$\frac{1}{\lambda_i(v-b)}=\sum_{j\neq i}\frac{f_j(s_j^{-1}(b))}{F_j(s_j^{-1}(b))}\frac{1}{s_j'(s_j^{-1}(b))}.$$

For notation simplicity, we first consider RSD ordering in auctions with asymmetry in private values and then in auctions with asymmetry in preferences.

Asymmetry in Private Values

Consider the asymmetric IPV model with risk neutral bidders. Assume that the I bidders have potentially different value distributions $F_k(\cdot)$ with support $[\underline{v},\overline{v}]$, where $k=1,\ldots,I$. Lebrun (1999) and Maskin and Riley (2000) show that the equilibrium is characterized by

$$\frac{d}{db}\xi_k(b)=\frac{F_k(\xi_k(b))}{(I-1)f_k(\xi_k(b))}\left\{\sum_{\ell=1}^{I}\frac{1}{\xi_\ell(b)-b}-\frac{I-1}{\xi_k(b)-b}\right\},$$

along with the boundary conditions: $\xi_k(\overline{b})=\overline{v}$ and $\xi_k(\underline{v})=\underline{v}$, where $\xi_k(\cdot)=s_k^{-1}(\cdot)$ is the inverse bidding strategy. We remark that the theory predicts bidders' bids share the same support. In contrast to symmetric auctions, we focus on the distribution functions instead of the quantile functions. Moreover, we treat the graph of $F_k(\cdot)$ as directioned while letting $(\overline{v},1)$ and $(\underline{v},0)$ be our starting point and ending point, respectively.

Suppose $F_i(\cdot)\geq F_j(\cdot)$ in $[z,\overline{v}]$, where $i\neq j\in\{1,\ldots,I\}$. We now show that the same RSD relation passes to the bid distributions, i.e. $G_i(\cdot)\geq G_j(\cdot)$ on $[s_i(z),\overline{b}]$. In general, no closed-form solution exists, which precludes direct comparisons.

First, note that $\frac{d}{db}\log F_k(\xi_k(b)) = \frac{f_k(\xi_k(b))}{F_k(\xi_k(b))}\left[\frac{d}{db}\xi_k(b)\right]$. The FOC becomes

$$\frac{d}{db}\log F_k(\xi_k(b)) = \frac{1}{I-1}\sum_{\ell=1}^{I}\frac{1}{\xi_\ell(b)-b} - \frac{1}{\xi_k(b)-b},$$

which implies that

$$\frac{d}{db}\log F_j(\xi_j(b)) - \frac{d}{db}\log F_i(\xi_i(b)) = \frac{1}{\xi_i(b)-b} - \frac{1}{\xi_j(b)-b}.$$

Substituting $b = s_i(v)$ and changing the differentiation to v gives

$$\frac{d}{dv}\log F_j(\xi_j(s_i(v))) = \frac{d}{dv}\log F_i(v) + s_i'(v)\Big[\frac{1}{v-s_i(v)} - \frac{1}{\xi_j(s_i(v))-s_i(v)}\Big]. \tag{6}$$

Second, $F_j(F_j^{-1}(F_i(v))) = F_i(v)$ implies that

$$\frac{d}{dv}\log F_j(F_j^{-1}(F_i(v))) = \frac{d}{dv}\log F_i(v). \tag{7}$$

Note that: 1) $\log F_j(F_j^{-1}(F_i(\overline{v}))) = \log F_i(\overline{v}) = 0$ and $\log F_j(\xi_j(s_i(\overline{v}))) = 0$; 2) if $\log F_j(\xi_j(s_i(v))) = \log F_j(F_j^{-1}(F_i(v)))$ in $[v_\dagger, \overline{v}]$, we have $\xi_j(s_i(v)) = F_j^{-1}(F_i(v)) \geq v$ because $F_i(v) \geq F_j(v)$. Thus, $\left[\frac{1}{v-s_i(v)} - \frac{1}{\xi_j(s_i(v))-s_i(v)}\right] = \frac{\xi_j(s_i(v))-v}{(v-s_i(v))(\xi_j(s_i(v))-s_i(v))} \geq 0$. Equations (6) and (7) imply that $\frac{d}{dv}\log F_j(\xi_j(s_i(v))) \geq \frac{d}{dv}\log F_i(v) = \frac{d}{dv}\log F_j(F_j^{-1}(F_i(v)))$ in $[z, \overline{v}]$. Finally, 1) and 2) imply that $\log F_j(\xi_j(s_i(v))) \leq \log F_j(F_j^{-1}(F_i(v)))$, which implies that

$$\xi_j(s_i(v)) \leq F_j^{-1}(F_i(v)), \forall v \in [z, \overline{v}].$$

See, e.g., Milgrom and Weber (1982) and Lebrun (1999). Replacing $v = \xi_i(b)$ gives $F_j(\xi_j(b)) \leq F_i(\xi_i(b))$. That is, $G_i(\cdot) \geq G_j(\cdot)$ in $[s_i(z), \overline{b}]$.

Asymmetry in Preferences

Alternatively, we could assume that bidders are symmetric in private values but asymmetric in preferences, i.e., degrees of risk aversion. Again, we consider the case in which bidders have CRRA utility functions. The equilibrium is characterized by

$$\frac{d}{db}\xi_i(b) = \frac{F(\xi_i(b))}{(I-1)f(\xi_i(b))}\left\{\sum_{\imath=1}^{I}\frac{\theta_\imath}{\xi_\imath(b)-b} - \frac{(I-1)\theta_i}{\xi_i(b)-b}\right\},$$

along with the boundary conditions: $\xi_i(\overline{b}) = \overline{v}$ and $\xi_i(\underline{v}) = \underline{v}$, where $\xi_i(\cdot) = s_i^{-1}(\cdot)$ is the inverse bidding strategy of bidder i. $F(\cdot)$ is the common value distribution and $\theta_i \in (0,1]$ is bidder i's CRRA parameter (i.e., $U_i(x) = x^{\theta_i}$).

Consider two bidders i and j such that $\theta_i < \theta_j$. By similar logic, we obtain

$$\frac{d}{db}\log F(\xi_j(b)) - \frac{d}{db}\log F(\xi_i(b)) = \frac{\theta_i}{\xi_i(b)-b} - \frac{\theta_j}{\xi_j(b)-b}.$$

Substituting $b = s_i(v)$ and changing the differentiation to v gives

$$\frac{d}{dv}\log F(\xi_j(s_i(v))) = \frac{d}{dv}\log F(v) + s_i'(v)\Big[\frac{\theta_i}{v-s_i(v)} - \frac{\theta_j}{\xi_j(s_i(v))-s_i(v)}\Big].$$

Note that: 1) $\log F(\overline{v}) = 0$ and $\log F(\xi_j(s_i(\overline{v}))) = 0$; 2) if $\log F(\xi_j(s_i(v))) = \log F(v)$, we have $\xi_j(s_i(v)) = v$ and then

$$\frac{\theta_i}{v-s_i(v)} - \frac{\theta_j}{\xi_j(s_i(v))-s_i(v)} = \frac{\theta_i-\theta_j}{v-s_i(v)} < 0,$$

which implies that $\frac{d}{dv}\log F(\xi_j(s_i(v))) < \frac{d}{dv}\log F(v)$. In summary, 1) and 2) imply that $\log F(\xi_j(s_i(v))) > \log F(v)$, which implies that $s_i(v) > s_j(v), \forall v \in (\underline{v}, \overline{v})$ and $G_j(b) > G_i(b), \forall b \in (\underline{b}, \overline{b})$. In other words, the more risk averse the bidder is, the higher he/she bids.

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