Shopping Cost and Multi-Seller Agglomeration (very preliminary - please do not circulate)

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Abstract

We provide a theory of seller agglomeration from the perspective of standard multiproduct pricing. When consumers face a shopping cost (e.g., due to transportation) to access a group of sellers (e.g., those clustering in a downtown shopping area or a mall) who face otherwise independent demands, complementarities are created across all these sellers, and their equilibrium prices are too high from a joint-profit maximization perspective. Increasing the number of sellers raises all existing sellers' profits and generates an endogenous positive network externality that naturally gives rise to multi-seller agglomeration. Conversely, a negative shopping cost (e.g., free parking, free gifts and other amenities provided by a shopping mall) creates substitutability across sellers, resulting in equilibrium prices that are too low. With a negative shopping cost, increasing the number of sellers reduces seller profits and generates a negative network externality.

Key Words: shopping cost, agglomeration, multiproduct pricing

JEL **Classification**: D47, L11, L12, L81.

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1 Introduction

Shoppers often face a shopping cost to access sellers. For instance, it takes time and transportation cost to get to a downtown shopping area where the brick-and-motar shops cluster. In the context of online shopping, it also takes time and effort to set up customer accounts and/or online payment accounts. In this article, we show that such shopping costs can naturally give rise to multi-seller agglomeration.

1. Pricing externalities due to shopping costs

Because these shopping costs usually do not vary with what a consumer actually buys on a shopping trip, it links the demands for the different products that may be purchased. In particular, if without the shopping cost the products that different sellers provide are neither complements nor substitutes to any individual consumer - in which case we will simply call them independent - the existence of a positive shopping cost will actually create *complementarities* on the aggregate level across all these products. Consider any two different sellers' products. For marginal shoppers who buy both products (with or without other products) and obtain just enough surplus to cover the shopping cost, a higher price of either product will dissuade them from taking the shopping trip altogether, and thereby they also forego the other product. This implies that in equilibrium the prices set by individual sellers are too high from a joint-profit maximization perspective. This therefore represents a prisoner's dilemma game where the equilibrium profit for each seller is lower than in the "collusive" outcome.

In some other circumstances there may exist a negative net shopping cost. For instance, it is quite common for shopping malls or online shopping platforms to offer free parking/shipping services, gift vouchers, shopping credit and/or other amenities to "offset" the otherwise positive shopping costs that consumers may incur. In contrast to positive shopping costs, a negative shopping cost actually creates *substitutability* - on the aggregate level - across otherwise independent products. This will in turn result in equilibrium prices that are lower than the collusive level (which maximizes joint profit of all sellers).

Therefore, in the presence of positive or negative shopping costs, there exist endogenous *pricing externalities* amongst multiple sellers - even if their products are otherwise independent to any individual consumers. We provide a general framework to study these externalities among any number of such independent sellers.

2. Network externalities due to shopping costs

We show that shopping costs also generate endogenous *network externalities* amongst sellers, through analyzing how pricing externalities change with the number of sellers. In particular, when there exists a positive shopping cost, complementarity implies that increasing the number of sellers raises all existing sellers' profits, and thus generates a positive network externality. Therefore, positive shopping costs can naturally induce multi-seller agglomeration. With a negative shopping cost, on the other hand, substitutability implies that adding new sellers lowers existing sellers' profits, and generates a negative network externality.

For the purpose of our analysis, it is easier to understand the impact of an additional seller on the existing system of say n sellers by considering the new seller's price to be infinitely high before she joins the market. The new seller "begins" to participate in the market as soon as she lowers her price from infinity to a level where some consumers begin to purchase from her. In the presence of positive shopping costs, because all products are complements on the aggregate level, lowering the new seller's price will raise the demands - and profits - of all existing sellers. This induces these sellers to raise their own prices under standard conditions. When a new equilibrium is reached in the new market with $n+1$ sellers, prices will therefore be higher than when there were just n sellers. Because all existing sellers' profits rise in the new market equilibrium, they benefit from a positive network effect as the number of sellers increases.

With a negative shopping cost, all independent products become aggregate substitutes. When a new seller enters the market by lowering her price from infinity to a level where some consumers start to buy, demands for all existing sellers' products decrease as a result. This induces them to lower prices, and therefore the new equilibrium prices with $n+1$ sellers are lower than before. Because all the existing sellers obtain lower profits in the new equilibrium, they suffer from a negative network externality when the number of sellers increases.

We also study how entry of new sellers affect the collusive price (which maximize all sellers' joint profit), under positive or negative shopping costs. These findings have practical implications for market designers, platform businesses and regulators, especially when it is feasible to manipulate consumers' shopping costs, e.g., through charges or subsidies (which endogenize the shopping costs).

Review of Related Literature (to be completed)

This research is related to three stands of literature.

1. Shopping cost

Chen and Rey (2012), AER, "Loss Leading as an Exploitative Practice"

2. Multiproduct two-part tariff

Armstrong (1999), REStud, "Price Discrimination by a Many-Product Firm"

Calem and Spulber (1984), IJIO, "Multiproduct Two Part Tariffs"

3. Multiproduct demand system

Manelli, A. M., D. Vincent, (2006), JET

2 Modelling Framework

There is a group of $n \in \mathbb{N}$ sellers who each sell one product. Denote $N \equiv \{1, 2, ..., n\}$ the set of all sellers, and let $j \in N$ denote a seller and the product she[1] provides, at marginal cost $c_j \geq 0$. Sellers are forbidden to coordinate in any way, such that each seller can only individually choose a price $p_j \in \mathbb{R}$ for her product.

There is a continuum of consumers of mass 1, each of whom is represented by an n-dimensional real-valued vector $\mathbf{x} \equiv (x_1, ..., x_n)$, where x_j is his private valuation of (or the gross utility he derives from) product j. Each consumer demands 0 or 1 unit of each product. Each consumer faces an exogenous shopping cost $m \in \mathbb{R}$ (which may be negative) to purchase from any seller(s).

The net utility a consumer derives from purchasing a set of products is simply the sum of his valuations for these products, minus all the prices he pays and the shopping cost incurred. Therefore consumer valuation is *additive* and there exists no intrinsic complementarity or substitutability between different products in individual consumption. However, as we will show shortly, different products may still exhibit complementarity or substitutability on the *aggregate* level as a result of the shopping cost. No purchase results in zero utility.

Consumers are heterogeneous such that $\mathbf{x}$ varies across them following a joint distribution F with density f. The sellers know this distribution but not the exact value of each consumer's type. The support of f is assumed to be bounded and is normalized to $[0, 1]^n \equiv I^n$.

Assumption 1 *f is atomless, and $f(\mathbf{x}) > 0$ if and only if $\mathbf{x} \in I^n$.*[2]

We assume $c_j \in [0, 1)$ such that it is possible for each seller to make some profit.

[1] In this article, "she" refers to a seller, and "he" refers to a consumer.

[2] Using a unit hypercube as support simplifies expressions but is not crucial. Our main results still hold when we extend the support of f to a weakly convex and bounded subset of $\mathbb{R}^{+n}$ with full dimension.

Timing Consumers each maximize their own utility, and sellers each maximize their own profit. The timing of the game that they all play is as follows.

Stage 1. The sellers simultaneously announce their prices. (It is worth noting that, real-life shops can usually freely change their prices at any time, which indicates a lack of commitment power. Therefore we model the pricing stage as a static game.)

Stage 2. Consumers observe prices and decide whether to incur the shopping cost to visit (a subset of) the sellers. (When there is a negative shopping cost, we assume a consumer can only receive the benefit if he purchases something.)

Stage 3. Consumers choose which product(s) to purchase, if any.

Our analysis follows backward induction. We begin by characterizing consumers' demand, given the prices set by sellers. Then we find each seller's optimal price in response to one another, and characterize properties of these "best responses". We then compare equilibrium prices to collusive prices that maximize joint profit of all sellers. Finally, we analyze the impact of raising n on equilibrium and collusive prices, and profits.

Consumer Choice What combination, or bundle, of products each consumer purchases (if any) depends on his type $\mathbf{x}$ and the total final cost he perceives for all possible bundles. Any bundle, denoted by J, is a subset of N, the full bundle of all n products, i.e. $J \subset N$. The empty bundle is $\varnothing \subset N$.

Denote $\mathbf{p} \equiv \{p_j\}_{j \in N}$ the vector of prices charged by all sellers. A general profile of the shopping cost and all sellers' prices is denoted $(m, \mathbf{p})$. Here we highlight shopping cost m along with the seller prices because m is also perceived by consumers as part of the total final cost of their purchase. Given $(m, \mathbf{p})$, the total final cost that a consumer who chooses bundle $J \subset N$ incurs, denoted q_J, is given by

$$q_J = \begin{cases} \sum_{j \in J} p_j + m & , \text{ if } J \neq \varnothing; \\ 0 & , \text{ if } J = \varnothing; \end{cases} \tag{1}$$

which essentially defines a multiproduct two-part tariff. The shopping cost m applies if you buy anything, whereas each seller's price applies only if you buy from that seller. m therefore adds to the final prices of *all* non-empty bundles by the same amount.

Given such a two-part tariff $(m, \mathbf{p})$, consumer $\mathbf{x} = (x_1, ..., x_n)$ chooses bundle

$J \subset N$, $J \neq \varnothing$, if and only if

$$\sum_{j \in J}(x_j - p_j) - m \geq \max\{0, \sum_{k \in K}(x_k - p_k) - m; \forall K \subset N, K \neq \varnothing\}, \tag{2}$$

which guarantees that J provides the highest surplus. Otherwise, he chooses the empty bundle. Denote $A_J(m, \mathbf{p})$ the set of all the consumers who choose the non-empty bundle J, also called the demand segment of J, and we must have

$$A_J(m, \mathbf{p}) = \{\mathbf{x} \in I^n | \mathbf{x} \text{ satisfies (2)}\}. \tag{3}$$

Whenever indifferent, we assume that a consumer chooses the largest bundle, or randomizes with equal probabilities among equal-sized bundles.[3] For any $A \subset I^n$, we denote the probability measure of A as $\Pr[\mathbf{x} \in A] = \int_A f(\mathbf{x}) d\mathbf{x}$.

Demand

Definition 1 (Individual Seller's Demand) *Given $(m, \mathbf{p})$, the demand segment of seller $j \in N$ is the set of all consumers who buy bundles that contain product j, denoted*

$$\mathcal{B}_j(m, \mathbf{p}) \equiv \bigcup\nolimits_{J \ni j} A_J(m, \mathbf{p}), \tag{4}$$

*and the **demand for seller** $j \in N$ is the probability measure of $\mathcal{B}_j(m, \mathbf{p})$, denoted*

$$D_j(m, \mathbf{p}) \equiv \int_{\mathcal{B}_j(m, \mathbf{p})} f(\mathbf{x}) d\mathbf{x}. \tag{5}$$

Note that $\mathcal{B}_j(m, \mathbf{p})$ represents all consumers who buy product j, no matter if they also buy other products. Seller j's demand segment is denoted by calligraphic $\mathcal{B}_j$ for distinction from the demand segment of the single-product bundle $A_j(m, \mathbf{p})$. In fact, by (4), $\mathcal{B}_j(m, \mathbf{p}) \supset A_j(m, \mathbf{p})$.

Definition 2 (Total Demand) *Given $(m, \mathbf{p})$, denote the set of all consumers who purchases from any seller(s)*

$$\mathcal{B}_0(m, \mathbf{p}) \equiv \bigcup\nolimits_{j \in N} \mathcal{B}_j(m, \mathbf{p}) = \bigcup\nolimits_{J \neq \varnothing} A_J(m, \mathbf{p}) = I^n \backslash A_{\varnothing}(m, \mathbf{p}), \tag{6}$$

[3] From (3) we know that all demand segments are *closed*. The intersection of different demand segments defines their "boundary", i.e. the set of indifferent consumers. As density f has no atoms, consumers on any boundary have zero mass and therefore do not pose a problem for demand measurement based on f to be defined shortly.

and ***the total demand*** *for all sellers is the probability measure of* $\mathcal{B}_0(m, \mathbf{p})$*, denoted*

$$D_0(m, \mathbf{p}) \equiv \int_{\mathcal{B}_0(m,\mathbf{p})} f(\mathbf{x}) d\mathbf{x} \tag{7}$$

Lemma 1 *Given* $(m, \mathbf{p})$*, for any* $j \in N$*,* $D_j(m, \mathbf{p})$ *and* $D_0(m, \mathbf{p})$ *exist.*

For expository simplicity, all omitted proofs are provided in the Appendix, as are the details of the demand functions.

Given $(m, \mathbf{p})$, the maximized aggregate consumer surplus is denoted

$$V(m, \mathbf{p}) \equiv \mathbf{E}_{\mathbf{x}}[\max\{0, \sum_{k \in K}(x_k - p_k) - m, K \subset N, K \neq \varnothing\}] \tag{8}$$

Assuming $V(\cdot)$ is twice differentiable, and using the demand functions and the aggregate consumer surplus, by an envelope argument we have, for any $j \in N$,

$$D_j = -\frac{\partial V}{\partial p_j}, \text{ and } D_0 = -\frac{\partial V}{\partial m},$$

and therefore the next result follows.

Lemma 2 (Pricing Externality) *When* $n \geq 2$*, given* $(m, \mathbf{p})$*, suppose there exist* $j, k \in N$*, such that* $j \neq k$*,* $D_j > 0$*, and* $D_k > 0$*. Then we have*

i) $\frac{\partial D_j}{\partial p_k} = \frac{\partial D_k}{\partial p_j}$, $\frac{\partial D_j}{\partial p_j} < \frac{\partial D_j}{\partial m} = \frac{\partial D_0}{\partial p_j} < 0$, $\left|\frac{\partial D_j}{\partial p_k}\right| < \left|\frac{\partial D_j}{\partial p_j}\right|$*, and* $\frac{\partial D_0}{\partial m} < \frac{\partial D_j}{\partial m}$*;*

ii) When $m > 0$*,* $\frac{\partial D_j}{\partial p_k} < 0$ *(i.e. a positive shopping cost creates complementarity);*

iii) When $m < 0$*,* $\frac{\partial D_j}{\partial p_k} > 0$ *(i.e. a negative shopping cost creates substitutability).*

Point i) follows from the demand definitions and the Slutsky symmetry of the aggregate consumer surplus V in (8).

Points ii) and iii) show that shopping costs can make the products of any two sellers complements or substitutes on the aggregate level, even though each individual consumer has additive valuation of any combination of products. The intuition of these results are as follows.

A positive shopping cost precludes marginal consumers of all product bundles from visiting the host, but does not affect how consumers choose *among* products. It creates complementarity between any two products, because of marginal multi-seller consumers who buy both products (with or without other products) and obtain just enough surplus to justify their visit to the host (i.e. their participation constraints are binding). For these consumers, a higher price of either product will dissuade them from going shopping altogether, and by doing this they also forego the other product.

A negative shopping cost (or a shopping subsidy), on the other hand, induces some marginal consumers to shop just to buy from a single seller so that they qualify for the subsidy. It creates substitutability between any two sellers, because they now compete for those single-seller consumers who are lured in by the subsidy but are indifferent between their products. Therefore a higher price of either product will send such consumers straight to the "competitor" next door.

However, under a shopping subsidy the complementarity described previously no longer exists. When the price of a product increases, no multi-seller consumers who are indifferent between buying and not buying this product will leave altogether, because their participation is "cushioned" by the subsidy. They simply stop buying this now more expensive product while keeping whichever other product(s) they have already bought.

Nor does the substitutability under a shopping subsidy described previously exist under a positive shopping cost. When there is a positive cost to shop, if a consumer is indifferent between buying two stand-alone products, then buying a bundle of both will be a even better choice. This is because each consumer only needs to incur the positive shopping cost *once*. If he is willing to incur this cost for either single product, buying both is strictly better because he will save one such cost. Even if neither product alone is good enough to attract him to go shopping, it is still possible that buying both will be worthwhile due to a "de facto discount" (equal to the positive shopping cost) enjoyed by multi-seller consumers. Either way, indifference between any two separate products will not be a binding constraint on his choice among different product bundles, and therefore there is no substitutability under a positive shopping cost.

3 Individual Seller Pricing

Because all sellers set their prices at the same time, seller j chooses her optimal price p_j^* taking the prices of all other sellers, denoted $\mathbf{p}_{-j} \equiv \{p_k\}_{k\in N, k\neq j}$, as given. Given $(m, \mathbf{p}_{-j})$, denote seller j's profit

$$\pi_j(m, p_j, \mathbf{p}_{-j}) = (p_j - c_j) \cdot D_j(m, p_j, \mathbf{p}_{-j}), \tag{9}$$

and assume $\pi_j(m, p_j, \mathbf{p}_{-j})$ is concave in p_j such that her optimal price p_j^* is given by the first-order condition:

$$D_j(m, p_j^*, \mathbf{p}_{-j}) + (p_j^* - c_j) \cdot \frac{\partial}{\partial p_j} D_j(m, p_j^*, \mathbf{p}_{-j}) = 0. \tag{10}$$

Assumption 2 *For any seller* $j \in N$ such that $D_j > 0$, $D_j(m, \mathbf{p})$ *is logconcave in* p_j*, i.e.,* $\frac{\partial^2}{\partial p_j^2} D_j(m, \mathbf{p}) \cdot D_j(m, \mathbf{p}) < (\frac{\partial}{\partial p_j} D_j(m, \mathbf{p}))^2$.

Lemma 3 $\pi_j(m, \mathbf{p})$ *is concave in* p_j *under Assumption 2.*

Denote

$$\epsilon_j(m, \mathbf{p}) \equiv -\frac{\partial}{\partial p_j} D_j(m, \mathbf{p}) \cdot \frac{p_j}{D_j(m, \mathbf{p})}, \text{ seller } j\text{'s price elasticity of demand, and}$$
$$\sigma_j(m, \mathbf{p}) \equiv -\frac{\partial}{\partial p_j} D_j(m, \mathbf{p}) \cdot \frac{1}{D_j(m, \mathbf{p})} = \frac{\epsilon_j}{p_j} (> 0), \text{ seller } j\text{'s ``semi-elasticity'' of demand.}$$

Note that the semi-elasticity of demand is equal to the proportion of a seller's marginal consumers in her demand.[4] Then we have the following familiar Lerner (1934) formula.

Proposition 1 *Given* m *and* $\mathbf{p}_{-j} = \{p_k\}_{k \in N, k \neq j}$*, seller* j*'s optimal price* p_j^* *satisfies*

$$\frac{p_j^* - c_j}{p_j^*} = \frac{1}{\epsilon_j(m, p_j^*, \mathbf{p}_{-j})}, \text{ or equivalently, } p_j^* = c_j + \frac{1}{\sigma_j(m, p_j^*, \mathbf{p}_{-j})}. \tag{11}$$

Therefore, each seller marks up on her cost according to her (semi-)elasticity. In equilibrium, all sellers set prices according to (11), and we add superscript $*$ to the relevant (semi-)elasticities to denote their equilibrium values. Clearly $p_j^* > c_j \geq 0$, for any $j \in N$, i.e., $\mathbf{p}^* >> \mathbf{c}(> \mathbf{0})$. Without loss of generality, from now on we focus on such profitable individual prices set by sellers.

Lemma 4 *Given* $(m, \mathbf{p})$*, Assumption 2 implies that* $p_j - c_j - \frac{1}{\sigma_j(m, \mathbf{p})}$ *is increasing in* p_j*, which in turn implies that* $p_j - c_j > \frac{1}{\sigma_j(m, \mathbf{p})}$ *if and only if* $p_j > p_j^*$.

Assumption 3 Given $(m, \mathbf{p})$, for any $j, k \in N$, such that $j \neq k$, $D_j > 0$, and $D_k > 0$, $\frac{\partial^2}{\partial p_j^2} D_k(m, \mathbf{p}) < 0$, i.e., $D_k(m, \mathbf{p})$ is concave in p_j.

[4] The term and definition of "semi-elasticity" here follow Rochet and Tirole (2006). In typical single-product demand systems, the semi-elasticity as a function of price can be shown to equal the hazard rate.

Lemma 5 *Under Assumption 3,* $\frac{\partial}{\partial p_k}\sigma_j(m,\mathbf{p}) > 0$ *if* $m > 0$.

First-order condition (10) defines seller j's best response to $\mathbf{p}_{-j}$ (given m) as an implicit function $p_j^*(m,\mathbf{p}_{-j})$. We have the following result about its property.

Proposition 2 (Pass-through Properties) *i)* $\frac{\partial p_j^*}{\partial m} < 0$ *if* $\frac{\partial^2 D_j}{\partial p_j^2} \leq 0$ *and* $\frac{\partial^2 D_0}{\partial p_j^2} \leq 0$*; ii) For* $k \neq j$, $\frac{\partial p_j^*}{\partial p_k} < 0$ *if* $\frac{\partial D_j}{\partial p_k} < 0$, $\frac{\partial^2 D_j}{\partial p_j^2} \leq 0$ *and* $\frac{\partial^2 D_j}{\partial p_j \partial p_k} \leq 0$.

Part i) shows that a seller may respond to an increase in shopping cost with a lower price under some conditions, which indicates that these prices are strategic substitutes. The conditions $\frac{\partial^2 D_j}{\partial p_j^2} \leq 0$ and $\frac{\partial^2 D_0}{\partial p_j^2} \leq 0$ require that the demand of seller j and the total demand are both weakly concave in the seller's price. They hold, for instance, when f is the uniform distribution, where $\frac{\partial^2 D_j}{\partial p_j^2} = 0$ and $\frac{\partial^2 D_0}{\partial p_j^2} < 0$.

Part ii) shows that different sellers' prices are also strategic substitutes under some conditions. Condition $\frac{\partial D_j}{\partial p_k} < 0$ requires that j and k are complements, and this happens if and only if $m > 0$ according to Lemma 2. Condition $\frac{\partial^2 D_j}{\partial p_j \partial p_k} \leq 0$ requires that seller j's demand is submodular in its own price and another seller k's price, which also holds when f is the uniform distribution, where $\frac{\partial^2 D_j}{\partial p_j \partial p_k} = 0$.

4 Joint Profit Maximization[5]

Given sellers' prices $\mathbf{p}$, denote $\pi(m,\mathbf{p})$ the total profits of all sellers:

$$\pi(m,\mathbf{p}) = \sum_{k\in N} \pi_k = \sum_{k\in N} (p_k - c_k) D_k(m,\mathbf{p}). \tag{12}$$

Given $\mathbf{p}_{-j}$, denote $\hat{p}_j$ the *collusive price* of seller j, which maximizes $\pi(m,\mathbf{p})$.

Lemma 6 *Under Assumption 3,* $\pi(m,\mathbf{p})$ *is concave in each seller's price.*

Thus $\hat{p}_j$ must satisfy the following first-order condition of (12):

$$\frac{\partial}{\partial p_j}\pi_j(m,\hat{p}_j,\mathbf{p}_{-j}) + \sum_{k\in N, k\neq j} (\hat{p}_k - c_k) \cdot \frac{\partial}{\partial p_j} D_k(m,\hat{p}_j,\mathbf{p}_{-j}) = 0. \tag{13}$$

[5] The analysis in this section also applies in the case when a monopolist seller of all n products chooses the optimal prices for stand-alone products in order to maximize total profits.

Given $(m, \mathbf{p})$, define

$$\sigma_{j\times k}(m, \mathbf{p}) \equiv -\frac{\partial}{\partial p_k} D_j(m, \mathbf{p}) \cdot \frac{1}{D_j(m, \mathbf{p})}$$

the cross-semi-elasticity of seller j's demand with respect to p_k.

Denote $\hat{\mathbf{p}} \equiv \{\hat{p}_j\}_{j\in N}$ the collusive price vector, and use $\frac{\partial \pi_j}{\partial p_j} = D_j + (\hat{p}_j - c_j) \cdot \frac{\partial D_j}{\partial p_j}$ and the Slutsky symmetry $\frac{\partial D_k}{\partial p_j} = \frac{\partial D_j}{\partial p_k}$ in (13), we have the following result.

Proposition 3 (Collusive Price) *When $n \geq 2$, for any $j \in N$, given $\mathbf{p}_{-j}$, seller j's collusive price is given by*

$$\hat{p}_j = c_j + \frac{1}{\sigma_j(m, \hat{p}_j, \mathbf{p}_{-j})}[1 - \sum_{k\in N, k\neq j} (p_k - c_k) \cdot \sigma_{j\times k}(m, \hat{p}_j, \mathbf{p}_{-j})]. \tag{14}$$

Proposition 3 shows that the collusive prices $\hat{\mathbf{p}} = \{\hat{p}_j\}_{j\in N}$ are jointly determined by each seller's own-price semielasticity and the cross-price semielasticities between each pair of two sellers, according to the system of n equations in the format of (14).

By parts *ii*) and *iii*) of Lemma 2, we know $\sigma_{j\times k}$ has the same sign as m, and a comparison between (11) and (14) therefore immediately implies the following conclusion.

Proposition 4 (Pricing Externality) *Under Assumptions 2 and 3, when $n \geq 2$, for any seller $j \in N$,*

$$\hat{p}_j < p_j^* \text{ if and only if } m > 0.$$

With $m > 0$, each seller's collusive price (which maximizes joint profit) is lower than her equilibrium price (which maximizes her own profit). With $m < 0$, the reverse is true. Clearly, $\hat{p}_j = p_j^*$ when $m = 0$.

Lemma 7 *When $n \geq 2$, given $(m, \mathbf{p})$, suppose there exist at least two sellers j and k with positive demand. Then we have*

$$|\sigma_{j\times k}| < \sigma_j.$$

This property is quite intuitive and directly implied by part *i*) of Lemma 2. Whenever there are multiple sellers, the impact (in percentage) on a seller's demand due to a change of her own price will always be larger than that due to an equal change in another seller's price.

Proposition 5 (Total Markup) *At the collusive prices* $\hat{\mathbf{p}} = \{\hat{p}_j\}_{j\in N}$, *for any* $j \in N$, *we have*

$$\sum_{k\in N}(\hat{p}_k - c_k) > \frac{1}{\sigma_j(m, \hat{\mathbf{p}})}.$$

Proof. Fixing $\mathbf{p}_{-j} = \hat{\mathbf{p}}_{-j}$, then $p_j^* = c_j + \frac{1}{\sigma_j(m, p_j^*, \mathbf{p}_{-j}^*)}$, and by Lemma 7, we have

$$\begin{aligned}
\hat{p}_j &= c_j + \frac{1}{\sigma_j(m, \hat{\mathbf{p}})}[1 - \sum_{k\in N, k\neq j}(\hat{p}_k - c_k)\cdot \sigma_{j\times k}(m, \hat{\mathbf{p}})] \\
&> c_j + \frac{1}{\sigma_j(m, \hat{\mathbf{p}})}[1 - \sigma_j(m, \hat{\mathbf{p}})\cdot \sum_{k\in N, k\neq j}(\hat{p}_k - c_k)] \\
&= c_j + \frac{1}{\sigma_j(m, \hat{\mathbf{p}})} - \sum_{k\in N, k\neq j}(\hat{p}_k - c_k) \\
&> c_j + \frac{1}{\sigma_j(m, \mathbf{p}^*)} - \sum_{k\in N, k\neq j}(\hat{p}_k - c_k).
\end{aligned}$$

■

When colluding, although all sellers lower their prices, the sum of all their price-cost markups is still larger than any individual seller's optimal markup.

4.1 Special Case with Symmetric Sellers

Notice that (14) also implies that the collusive prices depend on the number of sellers n. To see this more clearly, consider temporarily a special case where all sellers are *symmetric* in the sense that $c_j = c_1$ for all $j \in N$, and that for any consumer type $\mathbf{x}$ the distribution $f(\mathbf{x})$ is invariant in any permutation of $\mathbf{x}$'s elements. Now focus on the symmetric pricing equilibrium where all sellers choose the same collusive price, denoted $\hat{p}^{(n)}$, to maximize joint profit. The superscript (n) is used to emphasize the number of sellers in the system. Denote $\sigma(m, \hat{p}^{(n)})$ each seller's own-price semielasticity and $\sigma_\times(m, \hat{p}^{(n)})$ her cross-price semielasticity (with respect to any other seller's price). Then (14) implies

Corollary 1 *When sellers are symmetric, their symmetric collusive price is given by*

$$\hat{p}^{(n)} = c_1 + \frac{1}{\sigma(m, \hat{p}^{(n)}) + (n-1)\cdot \sigma_\times(m, \hat{p}^{(n)})}. \tag{15}$$

(15) shows that $\hat{p}^{(n)}$ directly depends on n. Because Lemma 7 implies that $\sigma(m, \hat{p}^{(n)}) > \sigma_\times(m, \hat{p}^{(n)})$, (15) also implies that

$$\frac{1}{n \cdot \sigma(m, \hat{p}^{(n)})} < \hat{p}^{(n)} - c_1 < \frac{1}{n \cdot \sigma_\times(m, \hat{p}^{(n)})}.$$

Moreover, we have the following property.

Lemma 8 *When sellers are symmetric, and they all set the same price* $p \in (c_1, 1)$*, we have*

i) $\sigma_\times(m, p)$ *decreases exponentially in* n *when* $m > 0$*; and*

ii) $|\sigma_\times(m, p)|$ *decreases exponentially in* n *when* $m < 0$*.*

Proof. See Appendix. ■

To understand this result, recall the original definition of $\sigma_{j \times k}$. First consider $m > 0$, which implies that different products are complements by point ii) of Lemma 2. In this case $\sigma_{j \times k}$ is positive and measures, amongst all of seller j's consumers, the proportion of the marginal ones who stop buying from seller j when another seller k raises her price by one unit. Clearly, for any consumers who will still buy something after k's price goes up, this price change will not change which other product(s) (besides k) they will purchase, because these decisions are independent once the positive shopping cost is incurred. Therefore, k's price change can only affect some consumers' participation decisions, which means that any marginal consumers who stop buying from seller j actually also stop buying anything else, if any, and leave the market. This in turn implies that these marginal consumers' valuations for any other sellers' products (besides j) must be relatively low, such that they were either not buying any other products, or ready to stop buying them once p_k was one-unit higher.

As the number of sellers increases, the probability for seller j's consumers to have a low valuation for all other sellers' products decreases generally (and in the symmetric case, exponentially), but seller j's total demand does not - the proportion of marginal consumers measured by $\sigma_{j \times k}$ therefore decreases in n.

Now consider the case when $m < 0$, which implies that different products are substitutes. In this case $\sigma_{j \times k}$ is negative, and $|\sigma_{j \times k}|$ measures, amongst all of seller j's consumers, the proportion of the marginal ones who start buying from her when another seller k raises her price by one unit. Similar to the previous explanations provided for part iii) of Lemma 2, all the marginal consumers measured by $\sigma_{j \times k}$ must be single-seller purchasers who are substituting product j for k after k's price rises, because now j appears to be the more attractive product to purchase in order to qualify for the negative shopping cost. Like in the case when $m > 0$, these marginal consumers must have relatively low valuations for all other products, because otherwise they would have also purchased them. Therefore $|\sigma_{j\times k}|$ also decreases in n.

In the symmetric case, for $m > 0$ and n large enough, $\sigma_\times$ decreases in n faster than $(n-1)$ increases, which implies that $(n-1)\sigma_\times$ will decrease in n. Similarly, when $m < 0$, $(n-1)\sigma_\times$ will increase in n for large enough n. Therefore we have the following result.

Lemma 9 *When sellers are symmetric, and they all set the same collusive price $\hat{p}^{(n)}$, there exists an integer $\hat{n} \geq 1$ such that $\hat{p}^{(n)}$ strictly increases (resp. decreases) in n for all $n > \hat{n}$ when $m > 0$ (resp. $m < 0$).*

4.2 Special Case with Uniform Distribution

Suppose now consumers' valuations follow the n-dimensional uniform distribution on $[0,1]^n$, which is a special case of the symmetric distribution. Focus on the symmetric price of all products, denoted p. When $0 < p < p + m < 1$, it can be shown that

$$
\begin{aligned}
D_1^{(n)}(m,p) &= \sum_{i=1}^{n} [(1-p)^i - \frac{1}{i!} \cdot m^i] \cdot p^{n-i} \cdot \binom{n}{i}, \text{ and} \\
-\frac{\partial}{\partial p_2} D_1^{(n)}(m,p) &= \sum_{i=1}^{n-1} \frac{1}{i!} \cdot m^i \cdot p^{n-i-1} \cdot \binom{n-2}{n-i-1},
\end{aligned}
$$

where the superscript (n) is used to emphasize the number of sellers, and $\binom{n}{i}$ represents the number of i-combinations out of n elements in total. Because $-\frac{\partial D_1}{\partial p_2}$ is a homogeneous polynomial of degree $(n-1)$ in m and p, which are both strictly between 0 and 1, and the coefficient of each term is decreasing ([check]), $-\frac{\partial D_1}{\partial p_2}$ decreases in n exponentially.

5 Endogenous Network Externalities

What happens to equilibrium and collusive prices when new sellers join the market?

Because the formulas and equations in our general framework do not explicitly depend on n, it is difficult to conduct a standard comparative statics analysis by treating n as a parameter. However we develop an alternative method that can show the impact of adding more sellers to the market.

Consider all potential new sellers who, before entering the market, have kept their prices high enough so that there is no demand. Given the normalized support of consumer valuations, prices that are well above 1 suffice for this purpose in our model. When such a new seller enters the market, the impact to all other sellers is as if this new seller has lowered her price below 1 so that consumers begin purchasing from her.

When $m > 0$ (respectively, $m < 0$), because the new seller's product will be a complement (respectively, substitute) to all other existing products, lowering its price means demands for all other products increase (respectively, decrease). Therefore we are able to analyze the impact of a new seller on existing sellers through the pricing externalities we have identified previously in Lemma 2.

5.0.1 Equilibrium Price

According to (11), the only channel that the addition of a new seller can change an existing seller's equilibrium price must be through changing the latter's semi-elasticity:

$$\sigma_j(m, \mathbf{p}) = -\frac{\partial}{\partial p_j} D_j(m, \mathbf{p}) \cdot \frac{1}{D_j(m, \mathbf{p})},$$

which depends on both the demand $D_j(m, \mathbf{p})$ and its derivative $\frac{\partial D_j}{\partial p_j}$. Lemma 2 also clearly tells us that $\frac{\partial D_j}{\partial p_k}$ and m have opposite signs. Therefore the crucial determinant of how σ_j changes with some other seller's price p_k is the second-order derivative $\frac{\partial^2 D_j}{\partial p_j \partial p_k} = \frac{\partial^2 D_k}{\partial p_j^2}$ (by the Slutsky symmetry).

Denote all the variables when there are n sellers with a superscript (n), and call the new seller $n+1$. Also denote (with some abuse of notation) the full bundle with $n+1$ sellers $N^{(n+1)} \equiv N \cup \{n+1\}$.

Proposition 6 *Under Assumption 3, when a new seller $n+1$ joins the market, for any seller $j \in N^{(n+1)}$, we have $p_j^{(n+1)*} > p_j^{(n)*}$ if $m > 0$.*

Proof. Assumption 3 also holds for the new seller, we have $\frac{\partial^2}{\partial p_j^2} D_{n+1}(m, \mathbf{p}^{(n)*}, p_{n+1}) \leq 0$. By Lemma 5, $\frac{\partial^2 D_{n+1}}{\partial p_j^2} \leq 0$ implies $\frac{\partial \sigma_j}{\partial p_{n+1}} > 0$. Therefore when p_{n+1} lowers from above 1 to $p_{n+1}^{(n+1)*}$, say, p_j^* must rise according to (11). ■

This is immediately implied by (11) and Lemma 5.

Proposition 7 (Network Externality) *Under Assumption 3, when a new seller $n+1$ joins the market, for any existing seller $j \in N$, we have $\pi_j^{(n+1)*} > \pi_j^{(n)*}$ if and only if $m > 0$.*

Proof. Assumption 3 also holds for the new seller, we have $\frac{\partial^2}{\partial p_j^2} D_{n+1}(m, \mathbf{p}^{(n)*}, p_{n+1}) \leq 0$. For any $j \in N$, whenever $p_{n+1} < 1$, we have $\frac{\partial}{\partial p_{n+1}} \pi_j(m, \mathbf{p}^{(n)*}, p_{n+1}) = (p_j^{(n)*} - c_j) \cdot \frac{\partial}{\partial p_{n+1}} D_j(m, \mathbf{p}^{(n)*}, p_{n+1}) < 0$ if and only if $m > 0$, according to parts *ii)* and *iii)* of Lemma 2. Therefore when p_{n+1} lowers from above 1 to $p_{n+1}^{(n+1)*}$, say, π_j rises if and only if $m > 0$. ■

5.0.2 Collusive Price

Simulation results Under uniform distributions $F\tilde{}U[0,1]^n$, we have found the following numerical results. For simplicity we have set $c_j = 0$ for any $j \in N$.[6]

Table 1. Simulation Results under a Positive Shopping Cost ($m = 0.1$)

n	$p_j^{(n)*}$	$\hat{p}_j^{(n)}$	$p_j^{(n)*} - \hat{p}_j^{(n)}$	$\pi_j^{(n)*}$	$\hat{\pi}_j^{(n)}$	$\hat{\pi}_j^{(n)} - \pi_j^{(n)*}$
1	0.45	0.45	0	0.2025	0.2025	0
2	0.4738	0.4523	0.0215	0.2245	0.2250	0.00051
3	0.4857	0.4631	0.0226	0.2359	0.2365	0.00059

Under a positive shopping cost, both the equilibrium and collusive prices rise as n does. The equilibrium price rises at a faster rate than its collusive counterpart, leaving their gap also rising as n does. Each seller's profit also rises in each scenario, and the gap between the collusive profit and equilibrium profit also expands as n increases.

Table 2. Simulation Results under a Negative Shopping Cost ($m = -0.1$)

n	$p_j^{(n)*}$	$\hat{p}_j^{(n)}$	$\hat{p}_j^{(n)} - p_j^{(n)*}$	$\pi_j^{(n)*}$	$\hat{\pi}_j^{(n)}$	$\hat{\pi}_j^{(n)} - \pi_j^{(n)*}$
1	0.55	0.55	0	0.3025	0.3025	0
2	0.5237	0.5528	0.0291	0.2742	0.2750	0.00076
3	0.5104	0.5379	0.0276	0.2605	0.2611	0.00065

Under a negative shopping cost (Table 2), note that we have $\hat{p}_j^{(n)} > p_j^{(n)*}$, unlike in the case with a positive shopping cost where the reverse is true. The equilibrium

[6] All the calculations were done in WolframAlpha and Excel.

price drops consistently as n increases. However, the collusive price rises when the number of sellers increases from 1 to 2, but drops when it further increases to 3. The gap between the collusive and equilibrium prices first rises and then drops as n increases from 1 to 3.

Both the collusive and equilibrium profits of each seller consistently decrease throughout, but their gap first rises and then drops.

6 Welfare Analysis [to be completed]

7 Conclusion [to be completed]

8 Appendix - Proofs [Incomplete]

Some new notations In these proofs, it is sometimes more convenient to use general price schedules, denoted $\mathbf{P} \equiv \{p_J\}_{J \subset N}$, where $p_J \in \mathbb{R}$ for any $J \subset N$. We assume $p_\varnothing = 0$ in all price schedules without loss of generality.

For expository simplicity, in the following characterization we only consider price schedules where $p_j \geq 0$ for all $j \in N$. A negative price proves unprofitable in seller j's maximization problem.

The bundle that only contains product j is denoted $\{j\}$, and simplified to j when it does not cause confusion. Denote $J^C \equiv N \backslash J$ the complementary bundle of J. j^C simply means $\{j\}^C$. For any bundle $J \subset N$, denote $|J|$ its bundle size - the number of products in it. Denote $\mathbf{P} \equiv \{p_J\}_{J \subset N}$ a *general price schedule*, where each p_J is the final price for a consumer who purchases bundle J. A consumer *allocation* given $\mathbf{P}$ is the profile of demand segments of all bundles induced by $\mathbf{P}$, denoted $\{A_J(\mathbf{P})\}_{J \subset N}$. A price schedule $\mathbf{P} \equiv \{p_J\}_{J \subset N}$ is called *additive* if $p_\varnothing = 0$ and $p_J = \sum_{j \in J} p_j$ for any non-empty $J \subset N$. An additive price schedule can also be simply written as $\{p_j\}_{j \in N}$. From (1), we know the two-part tariff $(m, \mathbf{p})$ is additive if and only if $m = 0$. In the following proofs we often use $\mathbf{Q} \equiv \{q_J\}_{J \subset N}$ to represent the two-part tariff $(m, \mathbf{p})$ in (1) as a price schedule.

Lemma 1 By (1) through (7), the demand functions are:

$$D_j(m, \mathbf{p}) = \begin{cases} \int_{p_j}^{1} f_j(x_j) \cdot \Pr[\sum_{i \in N} \max(x_i - p_i, 0) \geq m |\, x_j] dx_j, & \text{if } m \geq 0; \\ 1 - F_j(p_j) + \int_{p_j+m}^{p_j} f_j(x_j) \cdot \Pr[x_j - p_j \geq \max(x_i - p_i, \forall i \neq j) |\, x_j] dx_j, & \text{if } m < 0. \end{cases} \tag{16}$$

$$D_0(m, \mathbf{p}) = \begin{cases} \int_0^{p_j} f_j(x_j) \cdot \Pr[\sum_{i \in N, i \neq j} \max(x_i - p_i, 0) \geq m |\, x_j] dx_j + D_j(m, \mathbf{p}), & \text{if } m \geq 0; \\ 1 - \int_0^{p_1+m} \int_0^{p_2+m} \cdots \int_0^{p_n+m} f(\mathbf{x}) dx_n \cdots dx_2 dx_1, & \text{if } m < 0. \end{cases} \tag{17}$$

Assumption 1 implies that D_0 and D_j are differentiable with respect to m, p_j and p_k, for any $j, k \in N$. In particular, we have

$$\begin{aligned} \lim_{m \to 0^+} \frac{\partial}{\partial m} D_j(m, \mathbf{p}) &= \lim_{m \to 0^-} \frac{\partial}{\partial m} D_j(m, \mathbf{p}) = -f_j(p_j) \cdot \Pr[x_i < p_i, \forall i \neq j |\, x_j = p_j], \text{ and} \\ \lim_{m \to 0^+} \frac{\partial}{\partial m} D_0(m, \mathbf{p}) &= \lim_{m \to 0^-} \frac{\partial}{\partial m} D_0(m, \mathbf{p}) = -\sum_{j \in N} f_j(p_j) \cdot \Pr[x_i < p_i, \forall i \neq j |\, x_j = p_j]. \blacksquare \end{aligned}$$

Lemma 2 Note that in the following proof we focus on the domains of $(m, \mathbf{p})$ where $p_j, p_k, p_j + m, p_k + m \in [0, 1)$ such that $D_j > 0$ and $D_k > 0$.

i) This is implied directly by the symmetry of second-order derivatives of V in (8).

ii) When $m > 0$, $\frac{\partial D_j}{\partial p_k} = \int_{p_j}^{1} f_j(x_j) \cdot \frac{\partial}{\partial p_k} \Pr[\sum_{i \in N} \max(x_i - p_i, 0) \geq m |\, x_j] dx_j$. Given x_j, the event $\sum_{i \in N} \max(x_i - p_i, 0) \geq m$ is affected (in a set-theoretic sense) by p_k only through the term $\max(x_k - p_k, 0)$ which clearly decreases in p_k. Therefore, $\frac{\partial}{\partial p_k} \Pr[\sum_{i \in N} \max(x_i - p_i, 0) \geq m |\, x_j] < 0$, which implies $\frac{\partial D_j}{\partial p_k} < 0$.

iii) When $m < 0$, $\frac{\partial D_j}{\partial p_k} = \int_{p_j+m}^{p_j} f_j(x_j) \cdot \frac{\partial}{\partial p_k} \Pr[x_j - p_j \geq \max(x_i - p_i, \forall i \neq j) |\, x_j] dx_j$. Given x_j, the event $x_j - p_j \geq \max(x_i - p_i, \forall i \neq j)$ is affected by p_k only through the term $\max(x_i - p_i, \forall i \neq j)$ which clearly decreases in p_k (because $k \neq j$). Therefore, $\frac{\partial}{\partial p_k} \Pr[x_j - p_j \geq \max(x_i - p_i, \forall i \neq j) |\, x_j] > 0$, which implies $\frac{\partial D_j}{\partial p_k} > 0$.■

Lemma 3

Proposition 1 Apply $\epsilon_j \equiv -\frac{\partial D_j}{\partial p_j} \cdot \frac{p_j}{D_j}$ in (10) and we immediately get (11).■

Lemma 4 Given $(m, \mathbf{p}_{-j}, p_j)$,

$$\frac{\partial}{\partial p_j}(p_j - c_j - \frac{1}{\sigma_j(m, \mathbf{p}_{-j}, p_j)}) = 1 - \frac{1}{\sigma_j^2} \cdot \frac{1}{D_j^2} \cdot [\frac{\partial^2 D_j}{\partial p_j^2} \cdot D_j - (\frac{\partial D_j}{\partial p_j})^2] > 0,$$

because D_j is logconcave in p_j by Assumption 2.

Given any $\mathbf{p}_{-j}$, by (11) we know $p_j^* - c_j - \frac{1}{\sigma_j(m, \mathbf{p}_{-j}, p_j^*)} = 0$, which in turn implies that $p_j - c_j - \frac{1}{\sigma_j(m, \mathbf{p}_{-j}, p_j)} > 0$ if and only if $p_j > p_j^*$.■

Lemma 5

Proposition 2 In condition (10), let $p_j^* = p_j^*(m, \mathbf{p}_{-j})$ and take partial derivative with respect to m on both sides, we have

$$\frac{\partial D_j}{\partial m} + \frac{\partial D_j}{\partial p_j}\frac{\partial p_j^*}{\partial m} + \frac{\partial D_j}{\partial p_j}\frac{\partial p_j^*}{\partial m} + (p_j^* - c_j)\frac{\partial^2 D_j}{\partial p_j^2}\frac{\partial p_j^*}{\partial m} + (p_j^* - c_j)\frac{\partial^2 D_j}{\partial p_j \partial m} = 0$$

$$\frac{\partial p_j^*}{\partial m} = -\frac{(p_j^* - c_j)\frac{\partial^2 D_j}{\partial p_j \partial m} + \frac{\partial D_j}{\partial m}}{2 \times \frac{\partial D_j}{\partial p_j} + (p_j^* - c_j)\frac{\partial^2 D_j}{\partial p_j^2}}$$

By Lemma 2 we know $\frac{\partial D_j}{\partial m} < 0$, $\frac{\partial D_j}{\partial p_j} < 0$. By Slutsky symmetry of V defined in (8), we have $\frac{\partial^2 D_j}{\partial p_j \partial m} = \frac{\partial^2 D_0}{\partial p_j^2}$. Therefore $\frac{\partial p_j^*}{\partial m} < 0$ when $\frac{\partial^2 D_j}{\partial p_j^2} \leq 0$ and $\frac{\partial^2 D_0}{\partial p_j^2} \leq 0$.

In condition (10), for $k \neq j$, take partial derivative with respect to p_k on both sides, we have

$$\frac{\partial p_j^*}{\partial p_k} = -\frac{(p_j^* - c_j)\frac{\partial^2 D_j}{\partial p_j \partial p_k} + \frac{\partial D_j}{\partial p_k}}{2 \times \frac{\partial D_j}{\partial p_j} + (p_j^* - c_j)\frac{\partial^2 D_j}{\partial p_j^2}}$$

Therefore $\frac{\partial p_j^*}{\partial p_k} < 0$ when $\frac{\partial D_j}{\partial p_k} < 0$, $\frac{\partial^2 D_j}{\partial p_j^2} \leq 0$ and $\frac{\partial^2 D_j}{\partial p_j \partial p_k} \leq 0$.■

Lemma 6

Proposition 3

Proposition 4 Suppose $m > 0$. Let $\mathbf{p}_{-j} = \hat{\mathbf{p}}_{-j}$ in (14), which holds for any $\mathbf{p}_{-j}$. Because $\sigma_{j\times k} > 0$ iff $m > 0$ by parts *ii*) and *iii*) of Lemma 2, and $\hat{p}_k > c_k$ for any k, we have $\hat{p}_j < c_j + \frac{1}{\sigma_j(m, \hat{p}_j, \hat{\mathbf{p}}_{-j})}$.

Lemma 7

Corollary 1

Lemma 8 Suppose $n \geq 3$. Assume all x_j's follow i.i.d. distribution, denoted $F_1(x)$ with density $f_1(x)$. That is, the joint density $f(\mathbf{x}) = \prod_{j=1}^n f_1(x_j)$. This is therefore a special case of the symmetric distribution. We first characterize $-\frac{\partial D_1}{\partial p_2}$ for the case when $n = 3$ as shown in Figure 2. From Figure 2 we know

$$\begin{aligned} -\frac{\partial D_1}{\partial p_2} &= \int_{p_1}^{p_1+m} \int_0^{p_3} f(x_1, p_1 + p_2 + m - x_1, x_3) dx_3 dx_1 \\ &+ \int_{p_1}^{p_1+m} \int_{p_3}^{p_3+m} f(x_1, p_1 + p_2 + p_3 + m - x_1 - x_3, x_3) dx_3 dx_1 \end{aligned}$$

Note that the first term on the RHS (which is the measure of the green rectangle in Figure 2) satisfies:

$$\begin{aligned}
& \int_{p_1}^{p_1+m}\int_0^{p_3} f(x_1, p_1+p_2+m-x_1, x_3)dx_3dx_1 \\
< & \int_{p_1}^{p_1+m}\int_{p_2}^{p_2+m}\int_0^{p_3} f(x_1,x_2,x_3)dx_3dx_2dx_1 \\
= & [F_1(p_1+m)-F_1(p_1)][F_1(p_2+m)-F_1(p_2)]F_1(p_3),
\end{aligned}$$

and the second term on the RHS (which is the measure of the red triangle in Figure 2) satisfies:

$$\begin{aligned}
& \int_{p_1}^{p_1+m}\int_{p_3}^{p_3+m} f(x_1, p_1+p_2+p_3+m-x_1-x_3, x_3)dx_3dx_1 \\
< & \int_{p_1}^{p_1+m}\int_{p_2}^{p_2+m}\int_{p_3}^{p_3+m} f(x_1,x_2,x_3)dx_3dx_2dx_1 \\
= & [F_1(p_1+m)-F_1(p_1)][F_1(p_2+m)-F_1(p_2)][F_1(p_3+m)-F_1(p_3)].
\end{aligned}$$

Therefore we have

$$-\frac{\partial D_1}{\partial p_2} < [F_1(p_1+m)-F_1(p_1)][F_1(p_2+m)-F_1(p_2)]F_1(p_3+m).$$

When prices are also symmetric, i.e., $p_i = p$ for all i, we have

$$-\frac{\partial D_1}{\partial p_2} < [F_1(p+m)-F_1(p)]^2 F_1(p+m).$$

More generally, for $n \geq 3$, we have

$$0 < -\frac{\partial D_j}{\partial p_k} < [F_1(p+m)-F_1(p)]^2 \cdot [F_1(p+m)]^{n-2}.$$

Therefore given (m, p), as long as $0 < p < p+m < 1$, we know $-\frac{\partial}{\partial p_k}D_j(m,p)$ decreases exponentially in n. Because $D_j(m,p) > 0$ and increases in n, we know $\sigma_{j\times k}(m,p) = \frac{-\frac{\partial}{\partial p_k}D_j(m,p)}{D_j(m,p)}$ decreases exponentially in n.

Note from Figure 2 that

$$1 - F_1(p) > D_1 > 1 - F_1(p) - [F_1(p+m)]^2 \cdot [F_1(p+m) - F_1(p)],$$

and more generally, for $n \geq 3$,

$$1 - F_1(p) > D_j > 1 - F_1(p) - [F_1(p+m)]^{n-1} \cdot [F_1(p+m) - F_1(p)],$$

which implies

$$\lim_{n\to\infty} D_j(m,p) = 1 - F_1(p).$$

Moreover, we have

$$0 \leq \lim_{n\to\infty}(-\frac{\partial}{\partial p_k} D_j(m,p)) \leq \lim_{n\to\infty}[F_1(p+m) - F_1(p)]^2 \cdot [F_1(p+m)]^{n-2} = 0.$$

And therefore we have

$$\lim_{n\to\infty} \sigma_{j\times k}(m,p) = \lim_{n\to\infty}(\frac{-\frac{\partial}{\partial p_k} D_j(m,p)}{D_j(m,p)}) = 0.\blacksquare$$

Lemma 9

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