Recursive Nash-in-Nash Bargaining Solution

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Abstract. We propose a new bargaining solution for interdependent bilateral negotiations, which we call the recursive Nash-in-Nash bargaining solution. The main difference of this bargaining framework with the standard Nash-in-Nash is in the treatment of the disagreement point, which we assume is the bargaining payoffs given that all other negotiations happen with recognition of this disagreement. This bargaining framework is mostly motivated by the observations that renegotiations and contingent contracts are not captured by the standard Nash-in-Nash and that the predictions of standard Nash-in-Nash are counter-intuitive in some simple examples. We show that under some assumptions, the recursive Nash-in-Nash bargaining solution is the same as the Shapley value or the more general Myerson value for the corresponding cooperative game.

I. Introduction

The Nash bargaining model (Nash (1950)) has been widely used as a framework for analyzing markets involving bargaining in a wide range of applications.[2] For example, courts have relied on it to predict merger effects on competition, such as in *FTC v. ProMedica*;[3] the Federal Communications Commission applied a Nash bargaining model in the analysis of the Comcast and NBC Universal merger (FCC 2011); in the determination of royalty rates and terms for sound recordings by satellite radio before the Copyright Royalty Judges, economic experts of both the copyright user (Sirius XM) and the copyright owner (SoundExchange) used the Nash bargaining solution to analyze music royalty rates (Willig (2016) and Farrell (2017)).

Nash's seminal paper considers a two-person game, but many economic problems involve multiple interdependent pairs of bilateral bargaining, such as buyer-seller networks and wage negotiations between a firm and its individual workers. One commonly used extension of the Nash bargaining solution to the problem of bilateral bargaining between multiple pairs is the "Nash equilibrium in

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[2] In many applications, the Nash bargaining solution is used to identify the division of surplus assuming gains from trade do not vary over bargaining terms. In such applications, it would probably be more straightforward to describe the outcome as involving splitting the surplus in a constant proportion. In such situations the only operative axioms of the four axioms characterizing the Nash bargaining solution as summarized in the literature (see, for example, Osborne and Rubinstein (1990), Chapter 2) is the symmetry and efficiency axioms. Despite this, we will follow the practice in the applied literature and refer to these as involving a Nash bargaining solution.

[3] *FTC v. ProMedica Health Sys., Inc.*, 2011-1 Trade Cas. (CCH) ¶ 77,395 (N.D. Ohio Mar. 29, 2011).

Nash bargains”, or “Nash-in-Nash” bargaining (Collard-Wexler, Gowrisankaran, and Lee (2018)). It was first proposed by Horn and Wolinsky (1988) to study horizontal mergers given a buyer-seller network and has been applied in various economic environments (see, for example, Crawford and Yurukoglu (2012), Gowrisankaran, Nevo, and Town (2015), Ho and Lee (2017), and Crawford, Lee, Whinston and Yurukoglu (2018)).

The Nash-in-Nash framework has also been used in numerous practical applications. The implication of a merger between two competing hospital systems on the prices that the hospitals negotiate with insurance companies is often analyzed using a Nash-in-Nash bargaining model.[4] In a proceeding before the US Copyright Royalty Board, a Nash-in-Nash bargaining model was used to illustrate the effects to music services such as Sirius XM to induce competition between copyright owners by changing the frequency that music is played in response to differences in royalty rates.[5] These models are increasingly becoming an important tool for applications involving markets where prices are determined through bargaining.

The Nash-in-Nash bargaining solution is defined as the set of bilateral bargaining outcomes that are consistent with all of the other bilateral bargaining outcomes assuming that in the case of a bilateral disagreement (out of equilibrium) all of the other bargaining outcomes remain at the proposed equilibrium levels. That is, each agent believes that all of the other pairs of bargaining outcomes remain as predicted by the equilibrium even if there is an out of equilibrium breakdown in bargaining among one pair of agents. This may be a good approximation of some real world bargaining outcomes in certain situations, but in other situations, agreements of interdependent bargaining pairs may change if the negotiation of one pair breaks down unexpectedly. We propose another extension of the Nash bargaining solution to the multiple bilateral bargaining problems to treat these situations, which we call “recursive Nash-in-Nash” bargaining solution.[6]

The recursive Nash-in-Nash bargaining solution is similar to the standard Nash-in-Nash in that each bargaining pair’s payoffs are their Nash bargaining solution assuming that all other negotiating pairs reach agreements and that they all split surplus bilaterally according to the Nash bargaining solutions. The difference is in the disagreement payoffs. In the recursive Nash-in-Nash framework, a bargaining

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4 Federal Trade Commission and Commonwealth of Pennsylvania v. Penn State Hershey Medical Center and Pinnacle Health System, No. 16-2365 (3d Cir. 2016); Federal Trade Commission and State of Illinois v. Advocate Health Care Network, et al., No. 15 C 11473 (7th Cir. 2016).

5 Written Rebuttal Testimony of Joseph Farrell, in re “Determination of Royalty Rates and Terms for Transmission of Sound Recordings by Satellite Radio and ‘Preexisting’ Subscription Services (SDARS III),” Docket No. 16-CRB-0001 SR/PSSR (2018-2022).

6 A similar bargaining solution is independently defined by Froeb, Mares and Tschantz. That work came to the authors’ attention after early drafts of this paper.

pair's disagreement payoffs are determined assuming that all other bargaining pairs' agreements are negotiated expecting that disagreement has been reached by this bargaining pair.

The recursive Nash-in-Nash bargaining solution is likely a better description of outcomes than the standard Nash-in-Nash outcome in certain applications. For example, consider a manufacturer who is bargaining with two retailers. Suppose the manufacturer has the same fixed cost whether it reaches agreements with one or two retailers and the total surplus increases in the number of agreements. In Nash-in-Nash, the retailers' payoffs do not depend on the size of the fixed cost. That is, the fixed cost is not shared between the manufacturer and the retailers. This is because in Nash-in-Nash, the manufacturer gets the same payment from the other retailer and incurs the same fixed cost if it does not reach an agreement with one retailer. However, realizing that it cannot reach agreement with one retailer, the manufacturer may seek to renegotiate the payment from the other retailer to help cover the fixed cost or alternatively the manufacturer might negotiate contingent contracts. As shone in Example 1 below, the supplier's fixed cost affect the price negotiated with the retailers in the recursive Nash-in-Nash bargaining framework but not in the standard Nash-in-Nash framework.

Predictions relating to the covering of fixed costs can be important in some regulatory contexts where regulators are legislatively directed to set a price that would prevail in some hypothetically competitive market. The authors' analysis of Sirius XM's hypothetical bargaining with copyright owners and whether that resulted in the copyright owners in some sense sharing the cost of Sirius XM's satellites was the original motivation for this research.[7]

Raskovich (2003) has a similar example regarding how fixed costs are shared between a supplier and multiple buyers. His model setup has two stages: after a simultaneous bilateral bargaining stage, the supplier can declare bankruptcy and does not implement any contract if the payments it received in the first stage is not enough to cover its total costs. This assumption makes the out of equilibrium disagreement involving "pivotal buyers" similar to the recursive Nash-in-Nash in the sense that disagreement causes a change in the outcome in the other bilateral negotiations while disagreement with non-pivotal buyers is similar to the standard Nash-in-Nash model as it does not affect other negotiated outcomes. Consequently, the supplier's fixed costs are shared only with "pivotal buyers," in contrast with the recursive Nash-in-Nash where the supplier's fixed costs would be shared with all buyers and with the standard Nash-in-Nash where they would not be shared with any buyer.

Consider another example involving a manufacturer who is bargaining with two suppliers with complementary inputs such that if there is agreement with only one supplier there are no gains from trade, but there are gains if the manufacturer is able to complete agreements with both suppliers

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7 In *re* DETERMINATION OF ROYALTY RATES AND TERMS FOR TRANSMISSION OF SOUND RECORDING BY SATELLITE RADIO AND "PREEXISTING" SUBSCRIPTION SERVICES (SDARS III), Docket No. 16-CRB-0001-SR/PSSR (2018-2022).

because it needs the inputs from both of them to produce. In such a situation, the manufacturer might seek to make agreement with each supplier contingent on agreement by the other supplier. Such contingent contracts would be inconsistent with the standard Nash-in-Nash solution. This is illustrated in Example 2 below.

The third motivating example considers a firm negotiating wages with two workers. This example shows that whether revenues are collected by the workers or by the firm affects the standard Nash-in-Nash payoffs but not the recursive Nash-in-Nash payoffs. Therefore, in situations where which negotiating party collects the revenue does not matter as long as the total values created by bargaining groups are the same, the recursive Nash-in-Nash may be a better framework than the standard Nash-in-Nash.

After defining the recursive Nash-in-Nash bargaining solution, we show that it is the same as the Shapley value of a corresponding game in characteristic function form if there are no externalities across different bargaining groups (externalities within the same bargaining groups are allowed). Moreover the recursive Nash-in-Nash bargaining solution is the same as the Myerson value of a corresponding game in partition function form (Myerson (1977b)) if there are externalities across different groups of bargaining agents.[8] These results have the potential to simplify the quantification of outcomes using these models for applied work.

Literature has recognized that different outside options should be applied in different real world environments and that the Nash-in-Nash bargaining solution may not describe the surplus division in some real-world settings satisfactorily. Horn and Wolinsky (1988) mention that a proper specification of the outside options is not an obvious matter. In the setting where one supplier bargains with two producers who needs the inputs of the supplier to produce, they note that the disagreement point of the negotiation between the supplier and a producer can be either that the other producer operate at the anticipated equilibrium level or at the monopoly level. In the setting of "narrow network" health insurance plans, Ho and Lee (2018) define a "Nash-in-Nash with Threat of Replacement" bargaining solution, where at the disagreement point, an insurer can trade with a hospital not in its equilibrium network. As discussed above, Raskovich (2003) proposes a two-stage game with bankruptcy contingency to study contracting problems in markets with high fixed costs and in particular, the FCC's horizontal ownership limits on cable system operators. Collard-Wexler, Gowrisankaran and Lee (2018) explain that Nash-in-Nash solutions may not emerge if there are renegotiations upon disagreement, agreements have contingencies, or there are large complementarities on one side of the

8 The Myerson value sometimes refers to the value derived in Myerson (1977a) (e.g. Navarro and Perea 2013), where he generalizes the Shapley value to games with a network structure; it sometimes refers to the value derived in Myerson (1977b) (e.g. Navarro (2007)), where he generalizes the Shapley value to games with externalities across coalitions. We use the Myerson value to refer to Myerson (1977b) since the value derived in Myerson (1977a) can be written as the Shapley value of a corresponding game whereas the value derived in Myerson (1977b) generally cannot.

buyer-seller network. Our framework is most similar to Stole and Zwiebel (1996)'s "stable outcome profiles of bargaining," which they use to study intra-firm wage bargaining and show that the payoffs are the same as the Shapley value. In their model, all negotiations involve the firm, which is needed to create gains from trade. The recursive Nash-in-Nash can be viewed as a generalization of their stable outcome profiles to settings with multiple buyers and sellers and settings with externalities across bargaining groups.

The standard Nash-in-Nash bargaining solution is usually different from the Shapley value. However, as we show in this paper, the recursive Nash-in-Nash bargaining solution is the same as the Shapley value or the more general Myerson value of a corresponding cooperative game under some assumptions. This result is related to the observations made in Myerson (1977a), Jackson and Wolinsky (1996) and Navarro (2007) that the unique allocation rule that satisfies the "component balance" condition (the value generated by a group is allocated within the group) and the "fairness" condition (Eq. 6 in Myerson (1977a)) is the Shapley value or the more general Myerson value. The recursive definition of our bargaining solution makes the equal split of bargaining surplus condition the same as the fairness condition in these allocation rules, which leads to the equivalence between the recursive Nash-in-Nash solution and the Shapley value and the Myerson value. In addition to the recursive nature of our bargaining solution definition, a difference between the recursive Nash-in-Nash bargaining solution and these allocation rules is individual rationality. That is, everyone's gains from trade from a bargaining agreement needs to be non-negative. Our proof of the uniqueness of the bargaining solution is also different from the methods of proofs used in these papers.

Stole and Zweibel (1996), Navarro and Perea (2013), Bruegemann, Gautier, and Menzio (2016), de Fontenay and Gans (2014) construct extensive form bargaining games such that the equilibrium payoffs are the same as the Shapley value or the more general Myerson value. In Stole and Zweibel (1996), Navarro and Perea (2013), and Bruegemann, Gautier, and Menzio (2016), if bargaining breaks down between one pair of negotiating agents, all other bargaining pairs renegotiate assuming that this pair has reached disagreement. In de Fontenay and Gans (2014), agents do not renegotiate, instead they negotiate over contracts contingent on the agreements of other negotiating pairs. The recursive Nash-in-Nash bargaining solution can be viewed as a reduced form version of these extensive form bargaining games.

The paper is organized as follows. Section 2 provides three motivating examples of the recursive Nash-in-Nash. Section 3 has the model and the definition of the recursive Nash-in-Nash bargaining solution. Section 4 shows the equivalence between recursive Nash-in-Nash bargaining solution and the Shapley value and the Myerson value under certain conditions. Section 5 discusses potential applications. All the proofs are in the Appendix.

II. Motivating Examples

II.A. Example 1

A manufacturer negotiates with two symmetric retailers 1 and 2. Assume they all have equal bargaining power in the sense that each bargaining pair splits the gains from trade equally. The manufacturer incurs the same fixed cost K as long as any agreement is reached and has no marginal cost. If both agreements are reached, each retailer gets revenue R. If only one agreement is reached, the retailer with the agreement gets revenue R' and the other retailer gets no revenue. Assume $2R > R' > K$ and that retailers have no cost other than the payments to the manufacturer.

Assume that the manufacturer reaches agreements with both retailers and let t_1 and t_2 be the payments from retailers 1 and 2 to the manufacturer, respectively. In the Nash-in-Nash framework, t_1 and t_2 satisfy the following equal split of surplus conditions:

$$(t_1 + t_2 - K) - (t_2 - K) = (R - t_1) - 0,$$

$$(t_1 + t_2 - K) - (t_1 - K) = (R - t_2) - 0.$$

The left hand side of the above equations is the manufacturer's gains from trade and the right hand side of the above equations is retailer 1's and 2's gains from trade, respectively. Solving these two equations give $t_1 = t_2 = R/2$. Therefore, the Nash-in-Nash bargaining solution in this example is that each retailer pays the manufacturer $R/2$, keeps half of its revenue $R/2$ and the manufacturer gets $R - K$. Note that the retailers' profits and payments do not depend on K. Note that this is qualitatively different from the bargaining outcome when there were only one retailer, where the Nash bargaining solution predicts that the retailer and the manufacturer split the total gains from trade $R' + K$ and thus the retailer shares the manufacturer's fixed cost.

The disagreement point of a bilateral negotiation in the Nash-in-Nash framework assumes that all other bilateral agreements remain the same as if this bilateral agreement is reached. Therefore, the manufacturer's disagreement payoff when bargaining with retailer i is $t_j - K$. That is, if the manufacturer and retailer i do not reach an agreement, the manufacturer still gets the same payment t_j from retailer j and incurs the same fixed cost K. Therefore, the bilateral gains from trade between the manufacturer and retailer i in the Nash-in-Nash framework is simply the revenue of retailer i. It does not depend on the fixed cost and retailer j's payment. The disagreement point assumption of the Nash-in-Nash framework leads to the prediction that fixed costs in cases like this are not shared if the manufacturer supplies more than one retailer.

In contrast, in the recursive Nash-in-Nash bargaining framework defined in the next section, the disagreement point of the bargaining between the manufacturer and retailer i assumes that the other negotiation happens expecting the disagreement of this negotiation. That is, if bargaining breaks down between the manufacturer and retailer i, they get the payoffs the same as what would occur under the Nash bargaining solution assuming there is no agreement with retailer i and agreement can only be reached between the manufacturer and retailer j. Therefore, the manufacturer's disagreement payoff is $(R' - K)/2$ and a retailer's disagreement payoff is 0.

With these different disagreement payoffs, an equal split of the surplus implies

$$(t_1 + t_2 - K) - \frac{R' - K}{2} = (R - t_1) - 0,$$

$$(t_1 + t_2 - K) - \frac{R' - K}{2} = (R - t_2) - 0.$$

Again the left hand side of the above equations is the manufacturer's gains from trade and the right hand side of the above equations is retailer 1's and 2's gains from trade, respectively.

Solving these two equations gives $t_1 = t_2 = (2R + R' + K)/6$. Therefore, the recursive Nash-in-Nash bargaining payoffs are $R - t_i = (4R - R' - K)/6$ for each retailer and $t_1 + t_2 - K = (2R + R' - 2K)/3$ for the manufacturer. Different from the Nash-in-Nash solution, the retailers' payoffs depend on the fixed cost K. That is, in the recursive Nash-in-Nash framework, fixed costs in cases like this are shared. Moreover, different from the Nash-in-Nash solution, the recursive Nash-in-Nash payoffs here depend on a retailer's revenue when the other retailer does not reach an agreement, which seems to better reflect a retailer's marginal contribution. This feature is similar to the observation made in Stole and Zwiebel (1996) where the firm's equilibrium profit and the employees' wages depend on both the marginal and infra-marginal outputs. It is easy to check that the recursive Nash-in-Nash payoffs in this example are the same as the Shapley values whereas the Nash-in-Nash payoffs are not.

II.B. Example 2

A manufacturer needs inputs from two suppliers (1 and 2) to produce. If the manufacturer has only inputs from one supplier, its revenue is zero. If it has both inputs, its revenue is $R > 0$. Suppose the suppliers have no costs and do not have revenues other than the payments from the manufacturer.

Suppose the manufacturer reaches agreements with both suppliers and let t_1 and t_2 be the payment from the manufacturer to supplier 1 and 2, respectively. In Nash-in Nash, if the manufacturer disagrees with supplier i, it still pays t_j to supplier j although its revenue is zero. Therefore, in the Nash-in-Nash framework, t_1 and t_2 satisfy the following equations:

$$(R - t_1 - t_2) - (-t_2) = t_1 - 0,$$

$$(R - t_1 - t_2) - (-t_1) = t_2 - 0,$$

which are the equal surplus split conditions for the bargaining between the manufacturer and supplier 1 and supplier 2, respectively. The left hand side of the above equations is the manufacturer's gains from trade and the right hand side of the above equations is supplier 1's and 2's gains from trade, respectively.

Solving these two equations gives $t_1 = t_2 = R/2$. Therefore, the Nash-in-Nash bargaining solution in this example is that each supplier gets $R/2$ whereas the manufacturer gets zero profit. Here all the agents are assumed to have equal bargaining power and the manufacturer makes positive contribution to the joint surplus, but in the Nash-in-Nash solution of this example the manufacture gets zero profit. Moreover, note that if the manufacture only needs one suppliers' inputs to produce revenue R, then the Nash bargaining solution predicts a strictly positive payoff, $R/2$. Therefore, as in Example 1, the Nash-in-Nash solution may not be a good extension of the Nash bargaining solution in some multilateral situations.

In the recursive Nash-in-Nash bargaining framework, if bargaining breaks down between the manufacturer and one of the suppliers, they get the payoffs the same as what would occur under the Nash bargaining solution assuming there is no agreement between them and bargaining only happens between the manufacturer and the other supplier, which are zero in this example since the total surplus is zero with only one supplier. The recursive Nash-in-Nash payments satisfy the following equations:

$$(R - t_1 - t_2) - 0 = t_1 - 0,$$

$$(R - t_1 - t_2) - 0 = t_2 - 0.$$

which again are the equal surplus split conditions for the bargaining between the manufacturer and supplier 1 and supplier 2, respectively.

Solving these two equations gives $t_1 = t_2 = R/3$. Therefore, the recursive Nash-in-Nash solution in this example gives the manufacturer a strictly positive payoff, $R/3$. Again, the recursive Nash-in-Nash payoffs is the same as the Shapley values in this example.

II.C. Example 3

A firm negotiates wages with two symmetric workers (1 and 2). If the firm hires only one worker, the worker collects revenue R'. If the firm hires both workers, then each of them collects revenue R. Assume $2R > R'$ and there are no costs. Note that this example is the same as Example 1 if $K = 0$. Therefore, the recursive Nash-in-Nash payoffs for the firm and each worker are $(2R + R')/3$ and $(4R - R')/6$, respectively and the standard Nash-in-Nash payoffs for the firm and each worker are R and $R/2$, respectively.

Now suppose that the workers do not directly collect any revenue and that the firm collects revenue R' if it hires one worker and $2R$ if it hires both workers. It is easy to show that the recursive Nash-in-Nash payoffs remain the same as above, but the standard Nash-in-Nash payoffs for the firm and each worker becomes R' and $R - R'/2$, respectively.

This example illustrates that the recursive Nash-in-Nash bargaining solution does not depend on which agent directly collects the revenue but the standard Nash-in-Nash does.

III. Definition of the Recursive Nash-in-Nash Bargaining Solution

Let $N = \{1, \ldots, n\}$ denote the set of agents and let g denote the set of agent pairs who bargain with each other, where g is an undirected network such that if agents $i \in N$ and $j \in N$ negotiate with each other, then the unordered pair ij is in g. The pairwise negotiations are over lump-sum transfers.

Following Collard-Wexler, Gowrisankaran, and Lee (2018), we assume that each agent's profits without transfers (called “gross profits” henceforth) do not depend on the transfers negotiated without him but they can depend on whether agreements are reached in those negotiations. As in Collard-Wexler, Gowrisankaran, and Lee (2018), we take gross profits at all the subsets of bilateral negotiations in g as primitives of the game. Let $\pi: \{g' | g' \subseteq g\} \rightarrow R^{|N|}$ be the gross profit function. It is a vector-valued function whose ith entry $\pi_i(g')$ is agent i's gross profit if the set of agreements reached is $g' \subseteq g$. In practice, agents often negotiate over “actions” that affect their gross profits, such as how much investment to make and the characteristics of the goods being traded. $\pi(g')$ can be viewed as the gross profits when all the agreements in g' include bilaterally efficient actions. We abstract away from how the actions are determined and focus on the surplus division.

The following notations mostly follow Jackson (2010) and Navarro (2007). A *network* in this game is a set of bilateral negotiations. We use $g \backslash ij$ to indicate the subnetwork of a network g such that it is identical to g except that $ij \in g$ is removed. We use $|\cdot|$ to denote the size of a set. For example, $|g|$ denotes the number of negotiating pairs in g.

A *path* in a network g between agents i and j is a sequence of agents $i_1, \dots, i_K$ such that $i_k i_{k+1} \in g$ for each $k \in \{1, \dots, K-1\}$, with $i_1 = i$ and $i_K = j$, and such that each agent in the sequence $i_1, \dots, i_K$ is distinct. i and j are *connected* in a network g if there is a path between them in g. i is *connected* in g if $\exists j \in N$ such that i and j are connected in g. A network is *connected* if there is a path between any two agents of the network.

A subset $S \subseteq N$ is called a *coalition*. A coalition S is a *component* of N in g if: (1) any two agents in S are connected in g, (2) any two agents such that only one of them is in S are not connected. A component in our setting is a bargaining group. Agents in different bargaining groups do not bargain with each other and all of an agent's bargaining partners are in the same bargaining group as this agent. Let Q_g be the set of N's components in g. Note that Q_g is a partition of N.

Definition. *Given a set of agents N, a set of bilateral negotiations g and a gross profit function $\pi: \{g' | g' \subseteq g\} \to R^{|N|}$, the recursive Nash-in-Nash bargaining solution is a vector of payoffs $U^g \in R^{|N|}$ such that*

$$\sum_{i \in S} U_i^g = \sum_{i \in S} \pi_i(g), \forall S \in Q_g, \qquad (Component\ balance)$$

$$U_i^g - U_i^{g \setminus ij} = U_j^g - U_j^{g \setminus ij}, \forall ij \in g, \qquad (Fairness)$$

$$U_i^g \geq U_i^{g \setminus ij}, U_j^g \geq U_j^{g \setminus ij}, \forall ij \in g, \qquad (Individual\ rationality)$$

where $U^{g \setminus ij} = \pi(\emptyset)$ if $|g| = 1$ and otherwise $U^{g \setminus ij}$ is the recursive Nash-in-Nash bargaining solution given agents N, bilateral negotiations $g \setminus ij$ and the gross profit function π.

The component balance condition says that the total payoff of a bargaining group is the sum of its members' gross profits. That is, there are no net transfers across different bargaining groups. The fairness condition says that every pair of negotiating agents splits the gains from trade equally. The individual rationality condition says that each agent's payoff in an agreement is at least as high as his disagreement payoff. When $|g| = 1$, the recursive Nash-in-Nash solution is the same as the Nash bargaining solution.

The recursive Nash-in-Nash is defined recursively and the disagreement payoffs $U_i^{g \setminus ij}$ and $U_j^{g \setminus ij}$ in the fairness and individual rationality conditions are the recursive Nash-in-Nash bargaining payoffs when ij is removed from the set of negotiations in g assuming all the other contracts are "renegotiated" without this pair's agreement. This is the key difference between recursive Nash-in-

Nash and Nash-in-Nash, where $U_i^{g\setminus ij}$ and $U_j^{g\setminus ij}$ in the fairness and individual rationality conditions would be replaced by i's and j's payoffs assuming all the other contracts remain the same as if i and j have reached an agreement.

The recursive Nash-in-Nash bargaining solution can be solved by "backward" induction starting from one pair of negotiating agents in g and increasing the number of negotiating agent pairs one by one. First, solve for $U^{g'} \in R^{|N|}$ for all $g' \subseteq g$ with $|g'| = 1$. For the only pair of negotiating agents, their payoffs are the same as their Nash bargaining payoffs, and for the other agents, their payoffs are the same as their gross profits given g' by the component balance condition. Then we can solve for the bargaining solution for all $g' \subseteq g$ with $|g'| = 2$ using the previous bargaining solutions as disagreement payoffs. Similarly, after solving for $U^{g'}$ for all $g' \subseteq g$ with $|g'| = m - 1$, one can solve for the bargaining solutions for all $g' \subseteq g$ with $|g'| = m$ until $g' = g$.

Consider the example shown in Figure 1 for an illustration of the "backward" induction. There are two pairs of negotiations in g: 1 and 3, and 2 and 3. If 1 and 3 disagree, then the set of negotiations becomes g_1 and if 2 and 3 disagree, the set of negotiations becomes g_2. One can first solve the recursive Nash-in-Nash bargaining solutions for the negotiation sets g_1 ($U_i^{g_1}, i = 1, 2, 3$) and g_2 ($U_i^{g_2}, i = 1, 2, 3$). Then the recursive Nash-in-Nash solution for the negotiations in g ($U_i^g, i = 1,2,3$) can be solved using the solutions for g_1 and g_2 as disagreement payoffs in the fairness and individual rationality conditions. For example, the disagreement payoffs for the negotiation between 1 and 3 in g are $U_1^{g\setminus 13} = U_1^{g_1}$ and $U_3^{g\setminus 13} = U_3^{g_1}$.

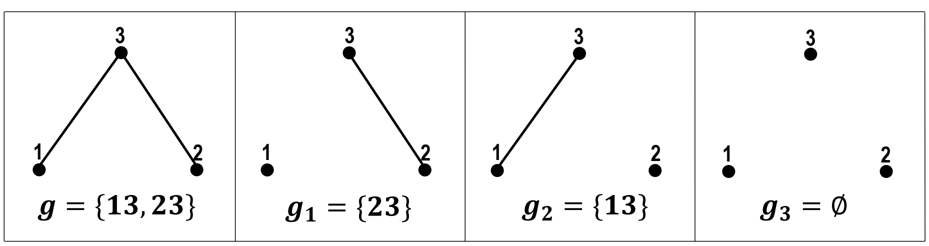

Figure 1: An example of a set of bilateral negotiations g and its subsets g_1, g_2 and g_3.

IV. The Recursive Nash-in-Nash Bargaining Solution, Shapley Value and Myerson Value

In this section, we show that under certain assumptions the recursive Nash-in-Nash bargaining solution is the same as the Shapely value or the more general Myerson value (defined in Myerson (1977b)) for the corresponding cooperative game. As we will see in Proposition 1, which links the recursive Nash-in-Nash to the Shapley value, the value of a coalition for the corresponding game in characteristic function form is the coalition's total gross profit assuming that only the subset of negotiations within this coalition reach agreements; and as we will see in Proposition 2, which links the recursive Nash-in-Nash to the Myerson value, the value of a coalition given a partition of all the agents into coalitions is the sum of this coalition's total gross profit assuming only the subset of negotiations within the coalitions specified by the partition reach agreements.

Define the restriction of a network g to a coalition S as $g|_S = \{ij \in g: i \in S \text{ and } j \in S\}$.

Assumption 1. (No externality across components) $\pi_i(g') = \pi_i(g'|_{S_i^{g'}})$, $\forall g' \subseteq g, \forall i \in N$, where $S_i^{g'}$ denotes the component of N in g' that i belongs to.

This assumption is used in Proposition 1 but not in Proposition 2. It says that any agent's gross profit only depends on the set of agreements made in his bargaining group. In other words, the agreements made in other bargaining groups do not have externalities on him. For example, in Figure 1, this assumption implies that $\pi_1(g_1) = \pi_1(g_3)$ and $\pi_2(g_2) = \pi_2(g_3)$. Jackson and Wolinsky (1996) and Navarro and Perea (2013) use similar assumptions. In addition, games with a "veto" player (such that any bargaining groups without this player do not have gains from trade) such as the intra-firm wage bargaining game studied in Stole and Zwiebel (1996) satisfy this assumption.

Assumption 2. (Monotonicity) $\forall g' \subseteq g, \forall ij \in g', \sum_{k \in N} \pi_k(g') \geq \sum_{k \in N} \pi_k(g' \backslash ij)$.

This assumption says that for any subset of agreements, the agents' total gross profit is weakly higher with these agreements than with one less agreement. For example, in Figure 1, this assumption implies that the three agents' total gross profit given g is weakly higher than that given g_1 or g_2, which in turn is weakly higher than that given g_3. This assumption is used in Proposition 1 to guarantee positive gains from trade.

Proposition 1. *Given a set of agents N, a set of bilateral negotiations g and a gross profit function* $\pi: \{g' | g' \subseteq g\} \to R^{|N|}$ that *satisfies Assumptions 1 and 2, the recursive Nash-in-Nash bargaining solution exists and it is the same as the Shapley values of the cooperative game (N, v^g) in characteristic function form, where $v^g(S) = \sum_{i \in S} \pi_i(g|_S), \forall S \subseteq N$.*

The cooperative game (N, v^g) can be interpreted as follows. We want to allocate to each agent in N the total value of the grand coalition, which is the total gross profit of all agents when all the agreements in g are reached, i.e. $v^g(N) = \sum_{i \in N} \pi_i(g)$. The value of each coalition S is the total gross profit of its members when only the subset of agreements within S, $g|_S$, is reached. Proposition 1 says that the Shapely values of this game are the same as the recursive Nash-in-Nash bargaining solution given Assumptions 1 and 2.

In some real world situations, Assumption 1 is not satisfied and the recursive Nash-in-Nash is not the same as the Shapley value. Consider the following example adapted from Myerson (1977b). Assume the set of negotiations is g_1 in Figure 1. It has only one subset, $g_3 = \emptyset$. If $\pi_1(g_1) \neq \pi_1(g_3)$, then Assumption 1 is not satisfied. This may happen when agents 1 and 2 are competing retailers and agent 3 is a supplier, and the agreement between retailer 2 and the supplier has negative externality on retailer 1. The recursive Nash-in-Nash bargaining solution given g_1 is different from the Shapley values for the corresponding game $(\{1,2,3\}, v^{g_1})$. This can be seen by comparing agent 1's payoffs. In the recursive Nash-in-Nash bargaining solution, agent 1's payoff is simply $\pi_1(g_1)$ by the component balance condition, whereas 1's Shapley value depends on both $\pi_1(g_1)$ and $\pi_1(g_3)$ since it is the average of 1's marginal contributions over all possible orders by which the agents arrive at a hypothetical market.[9]

In these situations with externalities across bargaining groups, the recursive Nash-in-Nash bargaining solution is the same as the more general Myerson value for a corresponding game in partition function form if the Myerson values satisfy the individual rationality condition. The following definitions largely follow those in Myerson (1977b).

Given a set of agents N, let PT be the set of partitions of N and let ECL be the set of *embedded coalitions*, that is the set of coalitions together with specifications as to how the other agents are assigned.[10] Formally:

$$ECL = \{(S, Q) | S \in Q \in PT\}.$$ [11]

A game in partition function form is defined as a vector $w \in R^{|ECL|}$, where $w_{S,Q}$ (the (S, Q)-component of w) is interpreted as the wealth, measured in units of transferable utility, which coalition S would have created if all the agents are aligned into coalitions of partition Q.

Given a network g and a partition Q, let $g|Q$ be the subnetwork of g such that all negotiations in g between agents in different coalitions in Q are removed from network g. That is, $g|Q = \cup_{T \in Q} g|_T$. (Note that in our notation $g|A$ and $g|_A$ are not the same.)

Proposition 2. *Given a set of agents N, a network of potential bilateral negotiations g and a gross profit function $\pi: \{g' | g' \subseteq g\} \to R^{|N|}$, if the recursive Nash-in-Nash solution exists then it is the same*

9 The example in Myerson 1977b also considers other definitions of v^{g_1} since the value of coalition {1} depends on whether 2 and 3 form a coalition. The Shapley values of these other games do not satisfy the component balance condition either.

10 For example, if $N = \{1,2,3\}$, then for N, $PT = \{\{\{1\},\{2\},\{3\}\}, \{\{1,2\},\{3\}\}, \{\{1\},\{2,3\}\}, \{\{1,3\},\{2\}\}, \{\{1,2,3\}\}\}$

11 For the example in the previous footnote, $ECL = \{(\{1\}, \{\{1\},\{2\},\{3\}\}), (\{2\}, \{\{1\},\{2\},\{3\}\}), (\{3\}, \{\{1\},\{2\},\{3\}\}), (\{1,2\}, \{\{1,2\},\{3\}\}), (\{3\}, \{\{1,2\},\{3\}\}), (\{1\}, \{\{1\},\{2,3\}\}), (\{2,3\}, \{\{1\},\{2,3\}\}), (\{1,3\}, \{\{1,3\},\{2\}\}), (\{2\}, \{\{1,3\},\{2\}\}), (\{1,2,3\}, \{\{1,2,3\}\})\}$.

as the Myerson values of the corresponding cooperative game in partition function form w^g*, where* $w^g \in R^{|ECL|}$ *and* $w^g_{S,Q} = \sum_{i \in S} \pi_i(g|Q), \forall (S,Q) \in ECL.$

Similar to the game (N, v^g), the cooperative game w^g can be interpreted as follows. We want to allocate to each agent in N the total value of the grand coalition, which is the total gross profit of all agents when all the agreements in g are reached, i.e. $w^g_{N,\{N\}} = \sum_{i \in N} \pi_i(g)$. The value of a coalition S given partition Q is the total gross profit of its members when only the subset of agreements "divided" by Q, $g|Q$, is reached. Proposition 2 says that the Myerson values of this game are the same as the recursive Nash-in-Nash bargaining solution it exists.

Since the Myerson value was proposed as a solution to cooperative games, it does not necessarily satisfy the individual rationality condition typically assumed for bargaining solutions. The numeric example below shows a parameter range of the primitives where the Myerson values satisfy individual rationality condition in the definition of the recursive Nash-in-Nash solution and a parameter range where they do not. In the latter case, the recursive Nash-in-Nash solution does not exist given the set of negotiations g, suggesting that g is not the set of equilibrium agreements. As in in Collard-Wexler, Gowrisankaran and Lee (2018) and in most of the applied work using Nash-in-Nash, we have assumed the set of negotiations g is exogenous. If recursive Nash-in-Nash is an appropriate framework for a specific bargaining problem and g is the observed set of agreements, then presumably the recursive Nash-in-Nash solution will exist.

Consider a set of bilateral negotiations between two manufacturers and two retailers as depicted by network g in Figure 2: manufacturer 1 negotiates with both retailer 2 and retailer 4 and manufacturer 3 only negotiates with retailer 4. $g_1, \ldots g_7$ are subsets of g. Every player's gross profit for each set of agreements is shown in the parenthesis: the gross profits of the manufacturers are zero in all cases; the gross profit of a retailer is 0 if it does not reach agreements with any manufacturer; when there is only one agreement (g_4, g_5 and g_6), the gross profit of the retailer with agreement is 1; when there is more than one agreement, the retailers' gross profits may be higher or lower than 1 (g, g_1, g_2 and g_3). In particular, the agreement between 3 and 4 may exert externality on 1 and 2 if player 2's gross profit in g_1 is different from that in g_4, i.e. $x \neq 1$.

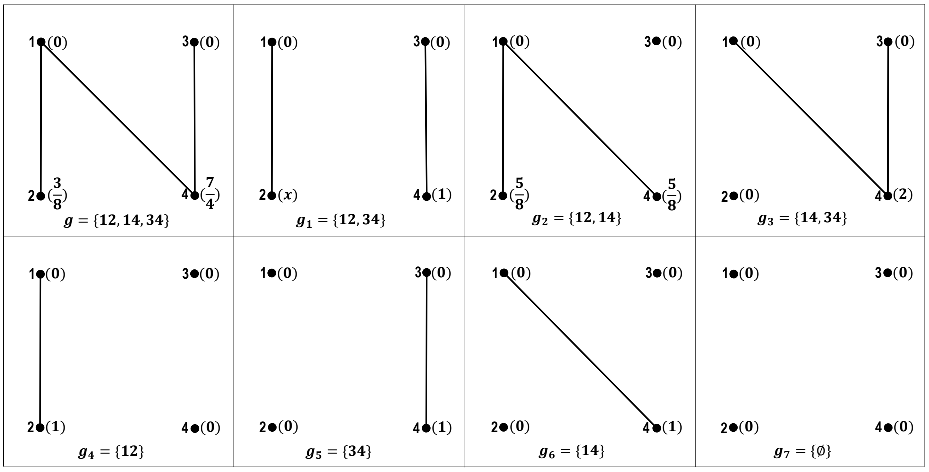

Figure 2. An example of a set of bilateral negotiations g and its subsets $g_1, \dots, g_7$. Numbers in the parentheses are players' gross profits.

Player 2's Myerson value for game w^g is $(8x-1)/32$. If $0 \leq x < 1/8$, then player 2's Myerson value is negative and thus does not satisfy the individual rationality condition in recursive Nash-in-Nash (since 2's payoff is zero if it does not reach an agreement with 1) although the monotonicity assumption is satisfied. In this case, the recursive Nash-in-Nash solution for negotiation set g does not exist since players 1 and 2 are better off if they do not trade given the agreements between 1 and 4, and 3 and 4.[12] If $\frac{1}{8} \leq x \leq 1\frac{1}{8}$, then the monotonicity assumption is satisfied and the Myerson values for game w^g are the same as the recursive Nash-in-Nash bargaining solution (if $x = 1$, then there are no externalities across components and the Myerson values for game w^g are the same as the Shapley values for game (N, v^g)).

V. Potential Applications

In this section, we consider some potential applications of the recursive Nash-in-Nash bargaining solution in regulation and litigation settings.

The recursive Nash-in-Nash bargaining solution is likely better suited in markets where trading partners renegotiate when the negotiation of a pair unexpectedly breaks down or contracts are contingent on whether trades happen in other negotiations. For example, if a virtual MVPD entrant

[12] Literature has recognized that the Myerson value sometimes gives unintuitive results. For example, De Clippel and Serrano (2008) points out that the Myerson value does not satisfy a monotonicity in marginal contributions. The following example is the second partition function in their Example 6 using our notations. $N = \{1,2,3\}$. $w_{N,\{N\}} = w_{\{1\},\{\{1\},\{2\},\{3\}\}} = 1$, $w_{S,Q} = 0$ for all other $(S,Q) \in ECL$. The Myerson values of this game is $(0,1/2,1/2)$. Agent 1's Myerson value is counterintuitive because it creates strictly positive value in some partitions and zero in the rest, so one would expect its Myerson value to be strictly positive.

cannot launch its service without content with some major programmers, it may condition its payments to a programmer on successful negotiations with all the necessary programmers. Similarly, if an insurer cannot launch a new health insurance plan profitably without two popular hospitals, it may renegotiate with one of them if the other negotiation unexpectedly breaks down instead of launching the plan.

Using the appropriate bargaining solution is not merely a theoretical issue; instead, it can affect conclusions regarding to whether a horizontal or vertical merger is anticompetitive or procompetitive. Antitrust agencies often weigh the potential harm and benefits of a merger.[13] Recursive Nash-in-Nash and standard Nash-in-Nash would generally predict different harm. Consider the example in Nevo (2014). A distributor negotiates with two symmetric providers. The distributor's profit is $120 if its bundle includes both providers, $100 if its bundle includes only one provider. Both standard Nash-in-Nash and recursive Nash-in-Nash solutions predict that the distributor payment is $60 after the merger since both bargaining solutions are the same as Nash bargaining solution when there is only one negotiation. However, Nash-in-Nash predicts that the merger increases the distributor's total payments by $40 whereas the payment increase predicted by the recursive Nash-in-Nash is only a third of that. Assuming that the merger specific efficiencies is $30, then Nash-in-Nash would predict that the merger is harmful whereas recursive Nash-in-Nash would predict the opposite.

In addition to merger reviews, recursive Nash-in-Nash can also be applied in some regulatory contexts where regulators are legislatively directed to set a price that would prevail in some hypothetically competitive market. As shown in Example 1, recursive Nash-in-Nash and Nash-in-Nash may have very different predictions on how fixed costs are shared. If empirical evidence of fixed-cost sharing is more consistent with recursive Nash-in-Nash instead of Nash-in-Nash in some markets, then recursive Nash-in-Nash is probably the more natural framework to use in these settings.

VI. References

Bruegemann, Björn, Peter Gautier, and Guido Menzio (2016). "Intra Firm Bargaining and Shapley Values." In 2016 Meeting Papers (No. 389). Society for Economic Dynamics.

Collard-Wexler, Allan, Gautam Gowrisankaran, and Robin S. Lee (2018). "'Nash-in-Nash' Bargaining: A Microfoundation for Applied Work," forthcoming, *Journal of Political Economy*.

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[13] *See*, e.g., Federal Trade Commission and U.S. Department of Justice, "Commentary on the Horizontal Merger Guidelines," 29-31, 2010, *available at https://www.ftc.gov/sites/default/files/attachments/merger-review/100819hmg.pdfhttps://www.justice.gov/atr/file/801216/download*

Crawford, Gregory S., Robin S. Lee, Michael D. Whinston, and Ali Yurukoglu (2018). "The welfare effects of vertical integration in multichannel television markets," *Econometrica* 86(3), 891-954.

Crawford, Gregory S., and Ali Yurukoglu (2012). "The Welfare Effects of Bundling in Multi-channel Television Markets," *American Economic Review*, 102(2), 643-85.

De Clippel, Geoffroy, and Roberto Serrano (2008). "Marginal contributions and externalities in the value," *Econometrica*, *76*(6), 1413-1436.

De Fontenay, Catherine C., and Joshua S. Gans (2014). "Bilateral Bargaining with Externalities," *Journal of Industrial Economics, 62(4), 756-788.*

Farrell, Joseph (2017). Written Rebuttal Testimony of Joseph Farrell, in re "Determination of Royalty Rates and Terms for Transmission of Sound Recordings by Satellite Radio and 'Preexisting' Subscription Services (SDARS III)," Docket No. 16-CRB-0001 SR/PSSR (2018-2022), available at https://www.crb.gov/rate/16-CRB-0001-SR-PSSR-SDARSIII/rebuttals/2-21-17-sirius-xm.pdf

Federal Communications Commision (2011). "Applications of Comcast Corporation, General Electric Company and NBC Universal, Inc. For Consent to Assign Licenses and Transfer Control of Licensees," Memorandum Opinion and Order, FCC 11-4.

Gowrisankaran, Gautam, Aviv Nevo, and Robert Town (2015). "Mergers When Prices Are Negotiated: Evidence from the Hospital Industry," *American Economic Review*, 105(1), 172-203.

Ho, Kate, and Robin S. Lee (2017). "Insurer Competition in Health Care Markets," *Econometrica*, 85(2), 379-417.

Horn, Henrick, and Asher Wolinsky (1988). "Bilateral Monopolies and Incentives for Merger," *The RAND Journal of Economics*, 19(3), 408-419.

Jackson, Matthew O. (2010). "Social and economic networks," Princeton University Press.

Jackson, Mattew O, and Asher Wolinsky (1996). "A Strategic Model of Social and Economic Networks," *Journal of Economic Theory* 71, 44-74.

Myerson, Roger B. (1977a). "Graphs and Cooperation in Games," *Mathematics of Operations Research*, 6(1), 23-31.

Myerson, Roger B. (1977b). "Values of Games in Partition Function Form," *Int. Journal of Game Theory*, 6(1), 23-31.

Nash, John (1950). "The Bargaining Problem," *Econometrica*, 18(2), 155-162.

Navarro, Noemí (2007). "Fair allocation in networks with externalities," *Games and Economic Behavior*, 58, 354-364.

Navarro, Noemí, and Andres Perea (2013). "A simple bargaining procedure for the Myerson value," The BE Journal of Theoretical Economics, 13(1), 131-150.

Osborne, Martin J., and Ariel Rubinstein (1990). "*Bargaining and markets,"* Academic press.

Raskovich, Alexander (2003). "Pivotal buyers and bargaining position," *The Journal of Industrial Economics*, 51(4), 405-426.

Stole, Lars A., and Jeffrey Zwiebel (1996). "Intra-firm bargaining under non-binding contracts," *The Review of Economic Studies*, 63(3), 375-410.

Willig, Robert (2016). Written Direct Testimony of Robert Willig, in the matter of "Determination of Royalty Rates and Terms for Transmission of Sound Recordings by Satellite Radio and 'Preexisting' Subscription Services (SDARS III)," Docket No. 16-CRB-0001 SR/PSSR (2018-2022), available at https://www.crb.gov/rate/16-CRB-0001-SR-PSSR-SDARSIII/statements/soundexchange/testimony-stefan-boedeker.pdf

Appendix A. Proofs

To prove Proposition 1, we will first prove a few lemmas.

Lemma 1. *Let* $S = \{1, 2, \dots, s\}$ *be a set of agents and* g *be a connected network of* S. *Let* $U \in R^{|S|}$ *be the solution to the following set of linear equations:*

$$U_i - U_j = \Delta_{ij}, \qquad \forall ij \in g, i < j \tag{A1}$$

$$\sum_{i \in S} U_i = \alpha, \tag{A2}$$

where $\alpha, \Delta_{ij} \in R$. *If* U *exists then it is unique.*

Proof. For any $k, l \in S$, there is a path from k to l since g is connected. Let the path be $k_1, \dots, k_m$ such that $k_1 = k$ and $k_m = l$. Using Eq. (A1) repeatedly along the path, we have

$$U_k - U_l = (U_{k_1} - U_{k_2}) + (U_{k_2} - U_{k_3}) + \dots + (U_{k_{m-1}} - U_{k_m})$$

$$= \Delta_{k_1k_2} + \Delta_{k_2k_3} + \cdots + \Delta_{k_{m-1}k_m} \equiv \delta_{kl}, \qquad (A3)$$

where $\Delta_{ij} \equiv -\Delta_{ji}$ for any $ij \in g$ with $i > j$.

There may be other paths from k to l, but if U exists then $U_k - U_l$ are the same regardless the path used for the above calculation.

If U exists, then it is the solution to the following set of linear equations:

$$\begin{cases} U_m - U_{m+1} = \delta_{m,m+1}, \qquad m = 1, \dots, s-1, & (A4) \\ \sum_{m \in S} U_m = \alpha, & (A2) \end{cases}$$

This is because Eq. (A1) can be derived from Eq. (A4) for any ij in g and Eq. (A4) can be derived from Eq. (A1) for any $m = 1, \dots, s-1$.

Eqs. (A2) and (A4) can be written in matrix form as

$$B\ U\ = \delta, \qquad (A5)$$

where δ is a vector with s components such that $\delta_m = \delta_{m,m+1}$ for $m = 1, \dots, s-1$ and $\delta_m = \alpha$, for $m = s$; $B = \{b_{i,j}\}_{i,j \in S}$ is an s-by-s matrix such that all components are zero except that $b_{m,m} = 1, b_{m,m+1} = -1$ for $m = 1, \dots, s-1$ and $b_{s,m} = 1$ for $m = 1, \dots, s$.

Let $C = \{c_{i,j}\}_{i,j \in S}$ be an s-by-s matrix such that $c_{i,j} = (s-j)/s$ for $j \le s-1$ and $i \le j$, $c_{i,j} = -j/s$ for $j \le s-1$ and $i > j$, and $c_{i,j} = 1/s$ for $j = s$ and $i = 1, \dots, s$. Note that BC equals the identity matrix. Therefore, B is invertible. Hence, Eq. (A5) has a unique solution and thus if U exists then it is unique. ∎

Lemma 2. $Sh_i(v^{g'}) - Sh_i(v^{g'\setminus ij}) = Sh_j(v^{g'}) - Sh_j(v^{g'\setminus ij}), \forall g' \subseteq g, \forall ij \in g'$.

Proof. $\forall g' \subseteq g, ij \in g'$, define a cooperative game in characteristic function form $(N, \hat{v}^{ij})$ as follows: $\hat{v}^{ij}(S) = v^{g'}(S) - v^{g'\setminus ij}(S), \forall S \subseteq N$. By the linearity of Shapley values, $Sh_i(\hat{v}^{ij}) = Sh_i(v^{g'}) - Sh_i(v^{g'\setminus ij}), \forall i \in N$.

$\forall S \subseteq N$ such that only one of i and j is in S, $g'|_S = (g'\setminus ij)|_S$ and thus $\hat{v}^{ij}(S) = v^{g'}(S) - v^{g'\setminus ij}(S) = \sum_{k \in S} \pi_k(g'|_S) - \sum_{k \in S} \pi_k((g'\setminus ij)|_S) = 0$. By the symmetry property of the Shapley value, $Sh_i(\hat{v}^{ij}) = Sh_j(\hat{v}^{ij})$. Therefore, $Sh_i(v^{g'}) - Sh_i(v^{g'\setminus ij}) = Sh_j(v^{g'}) - Sh_j(v^{g'\setminus ij})$.∎

Lemma 3. *Suppose Assumption 1 (No externality across components) holds. Then* $\sum_{i \in C} Sh_i(v^{g'}) = \sum_{i \in C} \pi_i(g'), \forall g' \subseteq g, \forall C \in Q_{g'}$.

Proof. Take an arbitrary $g' \subseteq g$, $C \in Q_{g'}$, $i \in C$. Firstly, we will show that $Sh_i(v^{g'}) = Sh_i\left(v^{g'|_C}\right)$. Consider i's marginal contribution in game $(N, v^{g'})$. When adding i to a coalition, the gross profits of the agents in other bargaining groups do not change and the gross profits of the agents in i's bargaining group change in the same way as if the game is $\left(N, v^{g'|_C}\right)$ by Assumption 1. Similarly, in game $\left(N, v^{g'|_C}\right)$, when adding i to a coalition, the gross profits of the agents in other bargaining groups do not change by Assumption 1. Therefore, i's marginal contribution to any coalition is the same in $(N, v^{g'})$ and $\left(N, v^{g'|_C}\right)$. Hence, i's Shapely value in these two games are the same.

Secondly, we will show that $\sum_{i\in C} Sh_i\left(v^{g'|_C}\right) = \sum_{i\in C} \pi_i(g'|_C)$. In game $\left(N, v^{g'|_C}\right)$, the marginal contribution of an agent j who is not in C to any coalition is simply his gross profit $\pi_j(g'|_C)$ by the definition of $v^{g'|_C}$. Therefore, the Shapley value of these agents are simply their gross profits given the set of agreements $g'|_C$. Since the sum of all agents' Shapley values are their total gross profits given agreements $g'|_C$, we have $\sum_{i\in C} Sh_i\left(v^{g'|_C}\right) = \sum_{i\in C} \pi_i(g'|_C)$.

Lastly, $\sum_{i\in C} \pi_i(g'|_C) = \sum_{i\in C} \pi_i(g')$ by Assumption 1 and we have shown that $\sum_{i\in C} Sh_i(v^g) = \sum_{i\in C} Sh_i\left(v^{g|_C}\right)$. Therefore, $\sum_{i\in C} Sh_i(v^{g'}) = \sum_{i\in C} \pi_i(g')$.■

Lemma 4. *Suppose Assumptions 1 (No externality across components) and 2 (Monotonicity) hold. Then* $Sh_i(v^{g'}) \geq Sh_i(v^{g' \setminus ij})$ *and* $Sh_j(v^{g'}) \geq Sh_j(v^{g' \setminus ij})$, $\forall g' \subseteq g, \forall ij \in g'$.

Proof. Take an arbitrary $g' \subseteq g$ and $ij \in g'$. Let C be the bargaining group that ij is in given g'. Firstly, note that a coalition's total gross profit is the same in games $(N, v^{g'})$ and $(N, v^{g' \setminus ij})$ if the coalition does not include both i and j. This is because the only difference between the two games is whether agreement ij is possible and a coalition without both ij does not allow ij to reach an agreement.

Secondly, the total gross profit of a coalition S with both i and j in $(N, v^{g'})$ is weakly larger than in $(N, v^{g' \setminus ij})$. This is because the total gross profit of all agents in N is weakly larger given agreements $g'|_S$ than given agreements $g' \setminus ij|_S$ by Assumption 2 and the total gross profit for agents not in coalition S is the same given agreements $g'|_S$ and $(g' \setminus ij)|_S$ by Assumption 1.

Given these two observations, i's marginal contribution to any coalition is weakly larger in $(N, v^{g'})$ than in $(N, v^{g' \setminus ij})$. Therefore, its Shapely value is also weakly larger in $(N, v^{g'})$ than in $(N, v^{g' \setminus ij})$. The same arguments can be used to show $Sh_j(v^{g'}) \geq Sh_j(v^{g' \setminus ij})$. ■

Proof of Proposition 1. Proof by induction.

First, we will show that $\forall g_1 \subseteq g$ with $|g_1| = 1$, the recursive Nash-in-Nash bargaining solution given N, g_1 and π is the same as the Shapley values for the cooperative game (N, v^{g_1}).

Let the negotiating pair in g_1 be i and j. Solving the two equations defined by the component balance and fairness conditions, we get $U_i^{g_1} = \pi_i(\emptyset) + \left(\pi_i(g_1) + \pi_j(g_1) - \pi_i(\emptyset) - \pi_j(\emptyset)\right)/2$, $U_j^{g_1} = \pi_j(\emptyset) + \left(\pi_i(g_1) + \pi_j(g_1) - \pi_i(\emptyset) - \pi_j(\emptyset)\right)/2$, and $U_k^{g_1} = \pi_k(g_1), \forall k \neq i, j$. Moreover,

$$\begin{aligned} &\left(U_i^{g_1} + U_j^{g_1}\right) - \left(\pi_i(\emptyset) + \pi_j(\emptyset)\right) \\ &= \pi_i(g_1) + \pi_j(g_1) - \pi_i(\emptyset) - \pi_j(\emptyset) \\ &= \left(\pi_i(g_1) + \pi_j(g_1) + \sum_{k \neq i,j} \pi_k(g_1)\right) - \left(\pi_i(\emptyset) + \pi_j(\emptyset) + \sum_{k \neq i,j} \pi_k(g_1 \backslash ij)\right), \end{aligned}$$

where the last equality uses Assumption 1. Therefore, $\left(U_i^{g_1} + U_j^{g_1}\right) - \left(\pi_i(\emptyset) + \pi_j(\emptyset)\right) \geq 0$ by Assumption 2. Hence, U^{g_1} is the recursive Nash-in-Nash bargaining solution given N, g_1 and π. Solving the Shapley values for the cooperative game (N, v^{g_1}) yields that the Shapley values are the same as U^{g_1}.

Now suppose we have shown that $\forall g_m \subseteq g$ with $|g_m| = m$, the recursive Nash-in-Nash bargaining solution given N, g_m and π is the same as the Shapley values for the cooperative game (N, v^{g_m}). By Lemmas 2-4, the Shapely values for the cooperative game $(N, v^{g_{m+1}})$, $\forall g_{m+1} \subseteq g$ with $|g_{m+1}| = m + 1$ satisfies component balance, fairness and individual rationality. By Lemma 1, the recursive Nash-in-Nash bargaining solution is unique if it exists. Therefore, the Shapley values for game (N, v^g) are the same as the recursive Nash-in-Nash bargaining solution given N, g, and π.■

To prove Proposition 2, we will first prove Lemmas 5 and 6.

Lemma 5. $\Phi_i(w^{g'}) - \Phi_i\left(w^{g' \backslash ij}\right) = \Phi_j(w^{g'}) - \Phi_j\left(w^{g' \backslash ij}\right), \forall g' \subseteq g, \forall ij \in g'$.

Proof. Similar to the proof of Lemma 2, $\forall g' \subseteq g, \forall ij \in g'$, define a cooperative game in partition function form w^{ij} as follows: $\widehat{w}_{S,Q}^{ij} = w_{S,Q}^{g'} - w_{S,Q}^{g' \backslash ij}, \forall (S, Q) \in ECL$. By the linearity of Myerson values, $\Phi_k\left(\widehat{w}^{ij}\right) = \Phi_k(w^{g'}) - \Phi_k\left(w^{g' \backslash ij}\right)$, $\forall k \in N$.

$\forall (S, Q) \in ECL$ such that i and j are in different elements of Q, $g'|Q = (g' \backslash ij)|Q$ and thus $\widehat{w}_{S,Q}^{ij} = w_{S,Q}^{g'} - w_{S,Q}^{g' \backslash ij} = \sum_{k \in S} \pi_i(g'|Q) - \sum_{k \in S} \pi_i\left((g' \backslash ij)\middle|Q\right) = 0$. By the symmetry property of the Myerson value (implied by Value Axiom 1 of Myerson (1977b)), $\Phi_i\left(\widehat{w}^{ij}\right) = \Phi_j(\widehat{w}^{ij})$. Therefore, $\Phi_i(w^{g'}) - \Phi_i\left(w^{g' \backslash ij}\right) = \Phi_j(w^{g'}) - \Phi_j\left(w^{g' \backslash ij}\right)$. ■

Lemma 6. For any $g' \subseteq g$ and for any $C \in Q_{g'}$, $\sum_{i \in C} \Phi_i(w^{g'}) = \sum_{i \in C} \pi_i(g')$.

Proof. Recall the definition of decomposability in Myerson (1977b): Given $Q \in PT$ and $w \in R^{|ECL|}$, we say that w is *Q-decomposable* if and only if:

$$\forall(\tilde{S},\tilde{Q}) \in ECL, w_{\tilde{S},\tilde{Q}} = \sum_{S\in Q} w_{\tilde{S}\cap S,\tilde{Q}\cap Q},$$

where for any $Q \in PT$ and $\tilde{Q} \in PT$, $\tilde{Q} \cap Q \in PT$ is defined as $\tilde{Q} \cap Q = \{\tilde{S} \cap S | \tilde{S} \in \tilde{Q}, S \in Q, \tilde{S} \cap S \neq \emptyset\}$. Also recall Corollary 1 of Myerson (1977b): If $w \in R^{ECL}$ is Q-decomposable, then, for any $S \in Q$, $\sum_{n\in S} \Phi_n(w) = w_{S,Q}$.

Note that $Q_{g'}$ is a partition of N and $g'|Q = g'|Q \cap Q_{g'}$, $\forall Q \in PT$. $w^{g'}$ is $Q_{g'}$-decomposable because for any $(S,Q) \in ECL$,

$$w^{g'}_{S,Q} = \sum_{i\in S} \pi_i(g'|Q) = \sum_{i\in S} \pi_i(g'|Q \cap Q_{g'}) = \sum_{C\in Q_{g'}} \sum_{i\in S\cap C} \pi_i(g'|Q \cap Q_{g'}) = \sum_{C\in Q_{g'}} w^{g'}_{S\cap C,Q\cap Q_{g'}}.$$

By Corollary 1 of Myerson (1977b), for any $C \in Q_{g'}$, $\sum_{i\in C} \Phi_i(w^{g'}) = w^{g'}_{C,Q_{g'}}$. Since $w^{g'}_{C,Q_{g'}} = \sum_{i=c} \pi_i(g'|Q_{g'}) = \sum_{i=c} \pi_i(g')$, we have $\sum_{i\in C} \Phi_i(w^{g'}) = \sum_{i\in C} \pi_i(g')$.■

Proof of Proposition 2. This proof is similar to the proof of Proposition 1. By Lemma 1, the recursive Nash-in-Nash bargaining solution is unique if it exists and it is given recursively by the component balance and fairness conditions. By Lemmas 5 and 6, the Myerson values of the corresponding cooperative games satisfy the component balance and fairness conditions. Therefore, the recursive Nash-in-Nash bargaining solution is the same as the Myerson value if it exists.■