Divide and conquer in two-sided markets: a potential-game approach*

Lester T. Chan

Department of Economics

Boston University

lctlabc@bu.edu

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Abstract

Positive cross-side externalities in two-sided markets typically lead to multiple equilibria. To overcome the methodological challenge in selecting a suitable equilibrium, this paper shows that many two-sided market models are weighted potential games, and thus potential-maximizer selection (Monderer and Shapley 1996) can always select a unique equilibrium in these models. As proved in the game-theory literature, the selected equilibrium coincides with the unique equilibrium under global-game selection, p-dominance selection, perfect foresight dynamics, and log-linear dynamics; it is also robust against incomplete information and widely supported by experimental results. Under potential-maximizer selection, platforms often subsidize one side and charge the other side, i.e., divide and conquer. The fundamental determinant of which side to subsidize or monetize is the cross-side externalities only. This divide-and-conquer strategy implies that platforms are often designed to favor the money side much more than the subsidy side.

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1 Introduction

Two groups of agents often interact via platforms: men and women meet in a nightclub, buyers and sellers trade on a marketplace, consumers and merchants transact through a payment card, etc. These markets are known as two-sided markets. Typically, positive cross-side externalities are present in these markets, which lead to strategic complementarities among the agents. For example, a man (woman) will join a heterosexual nightclub only if there are some women (men) joining the nightclub. Therefore, to attract men, the nightclub needs a lot of women, but to attract women, the nightclub needs a lot of men. This issue is known as the classic "chicken and egg" problem in two-sided markets (Caillaud and Jullien 2003) — one of the most difficult challenges for many two-sided platforms, and a methodological challenge for researchers on two-sided markets.

Formally speaking, in a typical two-sided market model where platforms set prices in stage 1 and all agents simultaneously make their joining decisions in stage 2, agents often engage in a coordination game with multiple equilibria in stage 2. For example, if there is a monopoly platform and the agents from the same side are identical, there can be two equilibria in stage 2: (1) all agents joining the platform, and (2) no one joining the platform. If there are competing platforms, all agents will coordinate on one of the platforms in the equilibrium when the cross-side externalities are strong enough, but which platform will they coordinate on?

In the two-sided market literature, researchers impose various equilibrium selection criteria to deal with multiple equilibria in stage 2. A popular selection criterion is *Pareto-dominance selection*, i.e., to select the Pareto-dominant equilibrium whenever there are multiple equilibria. However, this criterion often fails to select a unique equilibrium under platform competition because coordinating on one of the platforms does not necessarily Pareto-dominate the others. For duopoly markets with platforms A and B, a popular selection criterion is to assume that all agents always coordinate on platform A whenever there are multiple equilibria (Hagiu 2006; Jullien 2011). For convenience, I call this selection criterion *focal-platform selection*.[1] The drawback of this criterion is that we have to impose this asymmetry between the competing platforms. Another natural criterion is to assume that the number of agents from each side joining a platform decreases with the two prices charged by the platform (Caillaud and Jullien 2003; Armstrong and Wright 2007).[2] Nevertheless, this criterion is often too weak to select a unique equilibrium, even for the monopoly-platform markets.[3] More recently proposed selection criteria include the concept of coalitional rationalizability (Ambrus and Argenziano 2009) and insulating tariffs (Weyl 2010; White and Weyl 2016). Nevertheless, both criteria sometimes fail to select a unique equilibrium under platform competition.[4]

In response to the limitations of the existing equilibrium selection criteria in the two-sided market literature, this paper proposes using another approach — the potential-game approach — to resolve the multiple equilibria issue in two-sided markets. This approach has many microfoundations in the game theory literature, and it can select a unique equilibrium for many two-sided market models.

The concept of potential games was formally defined by Monderer and Shapley (1996), which I will explain in detail in Section 3.[5] In short, a game is called a (weighted) potential game if it is strategically equivalent to an identical interest game, in which all players share the same payoff function P. Note that the maximizer of P (called the *potential maximizer*) always exists and it is generically unique. In the game theory literature, many selection criteria select the potential maximizer if a game is a potential game. For example, the unique

[1] In the literature, this selection criterion is also called favorable/unfavorable expectations (Hagiu 2006; Jullien 2011), optimistic/pessimistic beliefs (Halaburda and Yehezkel 2013), focality (Halaburda and Yehezkel 2019), or incumbency advantage (Biglaiser *et al.* 2019).

[2] This criterion imposes no restriction when the price of one side increases but that of the other side decreases.

[3] For instance, both selecting the Pareto-dominant equilibrium whenever there are multiple equilibria and selecting the Pareto-dominated equilibrium whenever there are multiple equilibria satisfy this criterion.

[4] Coalitional rationalizability fails when the agents are sufficiently heterogeneous (Ambrus and Argenziano 2009, Abstract). Insulating tariffs fail when the agents from the same side are homogeneous and the cross-side externalities are sufficiently strong (White and Weyl 2016, Proposition 4c).

[5] The use of potential games appeared in several earlier papers such as Rosenthal (1973) and Blume (1993).

equilibrium under global-game selection (Frankel *et al.* 2003, Theorem 4) and p-dominance selection (Morris and Ui 2005, Lemma 7) is the potential maximizer.[6] The potential maximizer is also the unique state that is absorbing and globally accessible under perfect foresight dynamics (Hofbauer and Sorger 1999; Oyama *et al.* 2008) and the unique stochastically stable state under the log-linear dynamics (Blume 1993, Theorem 6.3, 6.5; Alos-Ferrer and Netzer 2010; Okada and Tercieux 2012).[7] Even without relying on other selection criteria, the potential maximizer itself is also robust against incomplete information (Ui 2001; Morris and Ui 2005). Given all these microfoundations in the game theory literature, *potential-maximizer selection*, which was also first proposed by Monderer and Shapley (1996, Section 5), is to select the potential maximizer whenever there are multiple equilibria.[8] Moreover, potential-maximizer selection is widely supported by experimental results (Van Huyck *et al.* 1990; Goeree and Holt 2005; Chen and Chen 2011).[9]

Although potential-maximizer selection is applicable only to potential games, many existing two-sided market models are indeed potential games.[10] In particular, for the top four

[6] For games with strategic complements, this result holds for a more general class of potential games, called monotone potential games (see Morris and Ui 2005, Section 6). For the concept of global games, see Carlsson and van Damme (1993) and Morris and Shin (2003). Jullien and Pavan (2018) study two-sided markets under the global-game setting, but their paper is not about equilibrium selection (see p. 5 of their paper). By contrast, Sakovics and Steiner (2012) study one-sided markets with network externalities under global-game selection. For the concept of p-dominance, see Kajii and Morris (1997).

[7] For games with strategic complements, the results under perfect foresight dynamics and log-linear dynamics hold for monotone potential games and local potential games (which is a special case of monotone potential games; see Morris and Ui (2005, Section 6) for details) respectively. For the concept of perfect foresight dynamics, see Matsui and Matsuyama (1995).

[8] We can apply other selection criteria (e.g. global-game selection) to a potential game and obtain the potential maximizer as the unique equilibrium. Nevertheless, potential-maximizer selection involves minimal computation because we can directly work on a complete information game as demonstrated in this paper.

[9] Anderson *et al.* (2001) introduce the notion of stochastic potential by adding some noise to the standard potential (the former converges to the latter as the noise goes to zero; see p. 194 of their paper for details). Both Goeree and Holt (2005) and Chen and Chen (2011) find that experimenters often end up at the maximizer of the stochastic potential.

[10] It means that every subgame in stage 2 of these two-sided market models is a potential game.

most-cited papers on two-sided markets (namely, Rochet and Tirole 2003, 2006; Armstrong 2006; Caillaud and Jullien 2003), all of their main models with strategic complements are (weighted) potential games. Therefore, potential-maximizer selection is applicable to these two-sided market models.[11]

The purpose of this paper is to demonstrate how potential-maximizer selection can resolve the multiple equilibria issue and derive novel insights into two-sided markets. To achieve this goal, I study a few variants of Armstrong's (2006) models. I now outline the results.

Sections 2 and 3 analyze the baseline model, which is a special case of Armstrong's (2006, Section 3) monopoly-platform model where agents from the same side are identical. Under potential-maximizer selection, the platform has to leave enough surplus to the agents by setting sufficiently low prices in stage 1, so that all agents will join the platform in stage 2. It turns out that the platform's optimal pricing strategy is to fully subsidize one side and set the highest possible price on the other side, i.e., to *divide and conquer* (Caillaud and Jullien 2003). The only determinant of which side to monetize or subsidize is the relative size of the cross-side externalities, i.e., the platform monetizes the side that enjoys a larger *per-interaction benefit*. In other words, the *money/subsidy side* is independent of the total number of agents on each side and the costs for serving the agents. This divide-and-conquer strategy implies that the optimal design of the monopoly platform is to favor the money side only, which is socially suboptimal.

Two-sided platforms do often divide and conquer in reality: women enjoy free admission on ladies' nights, and men pay an admission fee; shoppers pay nothing to shopping malls, and retailers pay the rent; consumers are paid to use credit cards, and merchants pay for the service, etc. The existing literature shows that a monopoly platform would divide and conquer if the elasticities of demand or the cross-side externalities of the two sides are significantly different (Armstrong 2006; Rochet and Tirole 2003, 2006).[12] By contrast, the

[11] Another contribution of this paper is to unveil the significant applications of potential games on two-sided markets.

[12] Alternatively, Caillaud and Jullien (2003) and Jullien (2011) derive the divide-and-conquer strategy under platform competition together with focal-platform selection.

monopoly platform's divide-and-conquer strategy is ubiquitous in my baseline model under potential-maximizer selection.

Section 4 extends the baseline model to allow for heterogeneous agents, in which agents incur idiosyncratic personal costs from joining the platform. When agents from one side are homogeneous and those from the other side are heterogeneous, the platform has an additional incentive to lower the price on the heterogeneous side to attract more agents from this side. Nonetheless, under potential-maximizer selection, the platform monetizes the homogeneous side and subsidizes the heterogeneous side if and only if the per-interaction benefit of the former is larger than that of the latter — the same feature as in the baseline model — and this is irrespective of the details of the heterogeneities among the agents as long as some regularity conditions are satisfied. Moreover, the optimal design of the platform tends to favor one side (not necessary the heterogeneous side) much more than the other side. Hence, all the key implications of the baseline model are naturally extended to this richer framework.

When agents are sufficiently heterogeneous on both sides as in Armstrong's (2006, Section 3) original model, the platform has high incentives to lower the prices on both sides to attract more agents from both sides, which in turn leaves a lot of surplus to the agents. Hence, unlike the previous models, it is possible that the equilibrium outcome under potential-maximizer selection coincides with that under Pareto-dominance selection.

Section 5 analyzes a variant of Armstrong's (2006, Section 4) duopoly-platform model, in which the platforms are vertically (but not horizontally) differentiated. Under potential-maximizer selection, the platforms' price-setting stage is similar to the standard Bertrand competition with vertical differentiation. The market tips to a dominant platform in the equilibrium. The dominant platform always divides and conquers as in the baseline model. By contrast, the money/subsidy side depends on the relative size of the average per-interaction benefits across the competing platforms, rather than its own per-interaction benefits of the two sides. The optimal design of the platforms tends to favor both sides (one side) when the two platforms are very (not) competitive. I further extend the analysis by allowing the platforms to use alternative pricing instruments and derive more novel results.

2 Monopoly platform: homogeneous agents

Section 2.1 presents the baseline model, which is a special case of Armstrong's (2006, Section 3) model where agents from the same side are identical. Section 2.2 analyzes the model under Pareto-dominance selection as a benchmark.[13] The next section analyzes the same model under potential-maximizer selection and shows that the equilibrium outcome is very different from that under Pareto-dominance selection. This section studies the more realistic finite-agent model, and Section 4 studies the more popular continuum-agent model.

2.1 The baseline model

There is a monopoly platform that serves two groups of agents, namely, group 1 and group 2, and there are $N_1 \in \mathbb{N}$ group-1 agents and $N_2 \in \mathbb{N}$ group-2 agents. The game has two stages. In stage 1, the platform sets subscription fees $(p_1, p_2) \in \mathbb{R}^2$ to the two groups. In stage 2, all group-1 and group-2 agents observe the platform's prices and simultaneously decide whether to join the platform. Let $a_i^k \in \{0 \equiv \text{Not join}, 1 \equiv \text{Join}\}$ denote the action of agent $k \in \{1, \dots, N_i\}$ from group $i = 1, 2$.

If the platform attracts $n_1 \equiv \sum_{k=1}^{N_1} a_1^k$ group-1 agents and $n_2 \equiv \sum_{k=1}^{N_2} a_2^k$ group-2 agents, the payoff of a group-i agent from joining the platform is

$$u_i(n_j, p_i) = v_i n_j - p_i, \quad (j = 1, 2;\ j \neq i) \tag{1}$$

where $v_i \in \mathbb{R}_{++}$ is the *per-interaction benefit* of a group-i participant from interacting with each group-j participant. If an agent does not join the platform, his payoff is normalized to zero.

The platform's payoff is equal to its profit:

$$\pi(n_1, n_2, p_1, p_2) = (p_1 - c_1)\, n_1 + (p_2 - c_2)\, n_2, \tag{2}$$

[13] Armstrong (2006, p. 672) allows the platform to choose the agents' utility levels directly rather than setting prices, which is equivalent to imposing Pareto-dominance selection in his model. This equivalence applies to most two-sided market models, including all the models in this paper.

where $c_i \in \mathbb{R}_+$ is the (sufficiently low) marginal cost for serving each group-i participant.

In what follows, I am interested in the pure strategy subgame-perfect equilibria of this two-stage game. Similar to most two-sided market models, there are multiple equilibria in this model. The rest of this section analyzes the game under Pareto-dominance selection. This benchmark serves the purpose of illustrating the novel findings under potential-maximizer selection in the next section.

2.2 Pareto-dominance selection

I solve this game backwards, starting from stage 2. Multiple equilibria often arise in stage 2 due to the cross-side externalities, and thus the strategic complementarities among the agents. Because agents from the same side are identical, if a group-i agent joins the platform in an equilibrium, all other group-i agents will also join the platform in that equilibrium.[14] Hence, by (1), there are two equilibria in stage 2 if and only if $(p_1, p_2) \in [0, v_1 N_2] \times [0, v_2 N_1]$,[15] and they are:

1. all agents joining the platform;

2. no one joining the platform.

In the first equilibrium, the payoff of a group-i agent is $v_i N_j - p_i (\geq 0)$; in the second equilibrium, every agent has zero payoff. Clearly, the former Pareto-dominates the latter for all agents (as well as the platform).

When there are multiple equilibria in stage 2, we cannot make a sharp prediction on the platform's pricing strategy in stage 1. As a benchmark, I apply *Pareto-dominance selection* to the current model, i.e., to assume that all agents always join the platform whenever there

[14] There can be equilibria with some (but not all) group-i participants when they are indifferent between joining the platform or not. Nevertheless, the platform can always eliminate these equilibria by perturbing the prices p_1 and/or p_2 a bit. These equilibria play an inconsequential role in the subsequent analysis; therefore, I will not discuss them in detail throughout this paper.

[15] If $p_i > v_i N_j$, the price for group i is too high that there is a unique equilibrium with no group-i participants. Similarly, there is a unique equilibrium with N_i group-i participants if $p_i < 0$.

are multiple equilibria in stage 2. Under this selection criterion, the platform charges both groups the highest possible prices in stage 1, such that all agents will join the platform with zero surplus in stage 2, i.e.,

$$p_1^* = v_1 N_2; \quad p_2^* = v_2 N_1. \tag{3}$$

Therefore, from (2), the platform's equilibrium profit is

$$\pi^* = (v_1 + v_2) N_1 N_2 - c_1 N_1 - c_2 N_2. \tag{4}$$

Under Pareto-dominance selection, the platform charges the maximum prices and makes the maximum equilibrium profit. The next section applies potential-maximizer selection instead of Pareto-dominance selection in stage 2. As we will see, the platform's optimal pricing strategy is very different from (3); its equilibrium profit is also significantly lower than (4).

3 Potential-maximizer selection

This section analyzes the baseline model under potential-maximizer selection. Section 3.1 illustrates the potential-game approach with the simplest case where there are only one group-1 agent and one group-2 agent. Section 3.2 analyzes the general case with N_1 group-1 agents and N_2 group-2 agents. Section 3.3 compares and discusses the results under potential-maximizer selection with the benchmark results in Section 2.2.

3.1 Simplest case: $N_1 = N_2 = 1$

When there are only two agents, the subgame in stage 2 can be represented by the following payoff matrix:

	Join	Not join
Join	$v_1 - p_1, v_2 - p_2$	$-p_1, 0$
Not join	$0, -p_2$	$0, 0$

(5)

Consider a function P defined on the strategy space of the same game as below:

	Join	Not join
Join	$1 - \frac{p_1}{v_1} - \frac{p_2}{v_2}$	$-\frac{p_1}{v_1}$
Not join	$-\frac{p_2}{v_2}$	0

(6)

P is constructed in a way that the change in the group-i agent's payoff from unilaterally switching actions in (5) is proportional (with proportion v_i) to the change in P. To see this, the group-1 agent's payoff difference between (Join, Join) and (Not join, Join) is $v_1 - p_1$, and the corresponding difference in P is $1 - \frac{p_1}{v_1} = \frac{1}{v_1}(v_1 - p_1)$. Similarly, his payoff difference between (Join, Not join) and (Not join, Not join) is $-p_1$, and the corresponding difference in P is $-\frac{p_1}{v_1} = \frac{1}{v_1}(-p_1)$. The same logic applies to the group-2 agent.

Hence, if we view (6) as an identical interest game in which the two agents share the same payoff function P, then this game is strategically equivalent to (5).[16] In particular, the best-response correspondence and the set of equilibria for these two games are identical. Therefore, we can view P as the sufficient statistic for the equilibrium analysis of (5): P is also called the *potential* of (5).[17] A game is a *weighted potential game* if there exists such a P function, and thus (5) is clearly a weighted potential game. The formal definition of weighted potential games is given as follows. The mathematical definition is given in Appendix B.

Definition 1 *A game is a weighted potential game if there exists a function P defined on the strategy space of the game, such that the change in any player's payoff from unilaterally switching actions is (positively) proportional to the corresponding change in P. P is called the game's potential.*

For a weighted potential game, the maximizer of the game's potential (called the *po-*

[16] In fact, these two games are von Neumann-Morgenstern equivalent as defined by Morris and Ui (2004).

[17] The potential P of a game is unique up to positive affine transformations, i.e., $P' \equiv C_1 P + C_2$ ($C_1 \in \mathbb{R}_{++}$; $C_2 \in \mathbb{R}$) is also a potential of the game.

tential maximizer) always exists, and it is generically unique.[18] For example, the potential maximizer in (6) when $p_1, p_2 \geq 0$ is

$$\begin{array}{ll} \text{(Join, Join)} & \text{if } \dfrac{p_1}{v_1} + \dfrac{p_2}{v_2} \leq 1; \\ \text{(Not join, Not join)} & \text{if } \dfrac{p_1}{v_1} + \dfrac{p_2}{v_2} \geq 1. \end{array} \tag{7}$$

Note that the potential maximizer must be a pure strategy Nash equilibrium of the game: if an agent deviates from the potential maximizer, the potential will decrease, and, by definition, the deviator will have a lower payoff.

Clearly, there can be multiple equilibria in a weighted potential game: both (Join, Join) and (Not join, Not join) in (5) are equilibria when $(p_1, p_2) \in [0, v_1] \times [0, v_2]$. Nevertheless, (generically) only one of the equilibrium is the potential maximizer as shown in (7). As stated in the Introduction, the equilibrium selection criterion based on potential games is to select the potential maximizer (see p. 3–4 of the Introduction for the microfoundations and justifications of this selection criterion). The formal definition is given as follows.

Definition 2 *Potential-maximizer selection is to select the potential-maximizing equilibrium of a weighted potential game.*

Under potential-maximizer selection, the (generically) unique equilibrium of the subgame for this simplest case is given by (7) when $p_1, p_2 \geq 0$. It implies that the two agents will join the platform only when the prices (p_1, p_2) set by the platform in stage 1 are sufficiently low. Otherwise, they will not join the platform.

3.2 General case

Section 3.1 shows that every subgame in stage 2 is a weighted potential game when there are one group-1 agent and one group-2 agent. Now, I prove the same result for the general

[18] As explained in footnote 17, the potential of a game is unique up to positive affine transformations. Thus, the potential maximizer is invariant to the choice of the potential. Ultimately, we are only interested in the potential maximizer. Therefore, it suffices to identify one potential of a game, and then identify its potential maximizer.

case with N_1 group-1 agents and N_2 group-2 agents.

Lemma 1 *Every subgame in stage 2 is a weighted potential game with the potential*

$$P(n_1, n_2|p_1, p_2) = n_1 n_2 - \frac{p_1}{v_1} n_1 - \frac{p_2}{v_2} n_2. \tag{8}$$

As shown in Lemma 1, the potential P of the subgame in stage 2 depends on the prices $(p_1, p_2) \in \mathbb{R}^2$ set by the platform in stage 1. Because agents from the same side are identical, P is symmetric in the sense that it depends only on the number of participants n_1 and n_2.

Proof. The proof is simply to verify the definition of a weighted potential game. For a group-1 agent, suppose there are n_1 (excluding himself) and n_2 participants. If he joins the platform, his payoff is $u_1(n_2, p_1)$, and the potential is $P(n_1 + 1, n_2|p_1, p_2)$. If he does not join the platform, his payoff is 0, and the potential is $P(n_1, n_2|p_1, p_2)$. Hence, the payoff difference between joining the platform or not for the group-1 agent is

$$u_1(n_2, p_1) - 0 = v_1 n_2 - p_1. \quad \text{(by (1))}$$

The corresponding difference in the potential is

$$\begin{aligned} & P(n_1 + 1, n_2|p_1, p_2) - P(n_1, n_2|p_1, p_2) \\ = & \left((n_1 + 1)n_2 - \frac{p_1}{v_1}(n_1 + 1) - \frac{p_2}{v_2} n_2\right) - \left(n_1 n_2 - \frac{p_1}{v_1} n_1 - \frac{p_2}{v_2} n_2\right) \quad \text{(by (8))} \\ = & n_2 - \frac{p_1}{v_1} \\ = & \frac{1}{v_1}(u_1(n_2, p_1) - 0). \end{aligned}$$

Thus, the change in the group-1 agent's payoff from unilaterally switching actions is proportional (with proportion v_1) to the change in the potential. The same logic applies to a group-2 agent (with proportion v_2 for him). ■

After identifying the game's potential, the next step is to identify the potential maximizer, i.e., the potential-maximizing equilibrium. Clearly, if there is a unique equilibrium in the subgame (i.e., $(p_1, p_2) \notin [0, v_1 N_2] \times [0, v_2 N_1]$), the potential maximizer is the unique equilibrium. If there are two equilibria in the subgame (i.e., $(p_1, p_2) \in [0, v_1 N_2] \times [0, v_2 N_1]$), the potential maximizer is the equilibrium with a higher potential. By Lemma 1, the potentials of the two equilibria are

$$\begin{aligned} P(N_1, N_2|p_1, p_2) &= N_1 N_2 - \frac{p_1}{v_1} N_1 - \frac{p_2}{v_2} N_2; \\ P(0, 0|p_1, p_2) &= 0. \end{aligned}$$

The former corresponds to all agents joining the platform, and the latter corresponds to no one joining the platform. Given the above analysis, the potential maximizer of the subgame, which is the selected equilibrium in stage 2 under potential-maximizer selection, is summarized by the following lemma.[19]

Lemma 2 *When $p_1, p_2 \geq 0$, the unique equilibrium of the subgame in stage 2 under potential-maximizer selection is*

$$\begin{array}{ll} \textit{all agents joining the platform} & \textit{if } \dfrac{p_1}{v_1 N_2} + \dfrac{p_2}{v_2 N_1} \leq 1; \\ \textit{no one joining the platform} & \textit{otherwise.} \end{array}$$ [20]

As shown in Lemma 2, under potential-maximizer selection, the platform has to leave enough surplus to the participants by setting sufficiently low prices (p_1, p_2) in stage 1, so that all agents will join the platform in stage 2.[21] Otherwise, no agent will join the platform in stage 2, and the platform makes zero profit. Therefore, from (2) and Lemma 2, the

[19] I omit the cases when p_1 and/or p_2 are strictly negative in Lemma 2 because the platform will not set such prices in the equilibrium: the (weakly) dominant strategy for the agents is to join the platform whenever it is free to do so.

[20] Both (Join, Join) and (Not join, Not join) are potential maximizers when $\frac{p_1}{v_1 N_2} + \frac{p_2}{v_2 N_1} = 1$. Nevertheless, the platform can always lower the prices p_1 or p_2 a bit in stage 1, so that (Join, Join) is the unique potential maximizer in stage 2. Therefore, we can assume for simplicity that the equilibrium in stage 2 is (Join, Join) when $\frac{p_1}{v_1 N_2} + \frac{p_2}{v_2 N_1} = 1$.

[21] The terms $\frac{p_1}{v_1 N_2}$ and $\frac{p_2}{v_2 N_1}$ in Lemma 2 represent the proportion of surplus extracted from group 1 and group 2 by the platform. As shown in Lemma 2, the sum of the proportion of surplus extracted from both sides can at most be one. By contrast, the platform extracts all surplus from both sides under Pareto-dominance selection, in which $\frac{p_1^*}{v_1 N_2} + \frac{p_2^*}{v_2 N_1} = 1 + 1 > 1$.

platform's profit-maximization problem in stage 1 becomes

$$\max_{p_1,p_2\geq 0}(p_1-c_1)N_1+(p_2-c_2)N_2 \quad \text{s.t.} \quad \frac{p_1}{v_1N_2}+\frac{p_2}{v_2N_1}\leq 1.$$

Generically and w.l.o.g., assume that group-1 agents enjoy less per-interaction benefits than group-2 agents do, i.e., $v_1 < v_2$. Solving the above optimization problem shows that the platform's optimal pricing strategy is to set zero price for group 1 and the highest possible price for group 2, i.e.,

$$p_1^*=0; \quad p_2^*=v_2N_1.$$

Therefore, the platform's equilibrium profit is

$$\pi^*=v_2N_1N_2-c_1N_1-c_2N_2.$$

3.3 Comparisons and discussions

Now, let us compare the results with the benchmark results.

Pareto-dominance selection	Potential-maximizer selection
$p_1^*=v_1N_2; \quad p_2^*=v_2N_1$	$p_1^*=0; \quad p_2^*=v_2N_1$
$\pi^*=(v_1+v_2)N_1N_2-c_1N_1-c_2N_2$	$\pi^*=v_2N_1N_2-c_1N_1-c_2N_2$
Division of total surplus:	
Group 1: 0; Group 2: 0	Group 1: $v_1N_1N_2$; Group 2: 0
Platform: $(v_1+v_2)N_1N_2-c_1N_1-c_2N_2$	Platform: $v_2N_1N_2-c_1N_1-c_2N_2$

Table 1: Comparisons between the two selection criteria (with $v_1 < v_2$)

Under both selection criteria, the platform charges group-2 agents the same maximum price and fully extracts their surplus. By contrast, the platform provides free access for group-1 agents and leaves them a lot of surplus under potential-maximizer selection. Hence, the platform's equilibrium profit is significantly lower than that of the benchmark. In this case, I call group 1 the *subsidy side* and group 2 the *money side*.

There are three key implications in this model.

Divide-and-conquer strategy The platform's *divide-and-conquer* strategy that subsidizes one side and monetizes the other side is ubiquitous because the per-interaction benefits of the two sides are (generically) different.[22] As mentioned in the Introduction (p. 5 and footnote 12), two-sided platforms often divide and conquer in reality, and the existing two-sided market literature typically derives this pricing strategy under platform competition together with focal-platform selection. To derive the divide-and-conquer strategy under the current monopoly-platform framework (and without using potential-maximizer selection), one would need to assume that all agents always coordinate on the Pareto-dominated equilibrium (i.e., no one joining the platform) whenever there are multiple equilibria. However, this selection criterion (for convenience, I call it *Pareto-dominated selection*) is often regarded as even "less plausible" than Pareto-dominance selection. Surprisingly, the equilibrium outcome under Pareto-dominated selection actually coincides with that under potential-maximizer selection in the current model.[23] Nevertheless, this equivalence no longer holds when I extend the model in the next two sections.

Money/subsidy side The *money/subsidy side* of the platform depends only on whether the per-interaction benefits v_1 or v_2 is larger. In other words, the money/subsidy side is independent of the total number of agents N_1 and N_2 on each side, i.e., the platform does not necessarily monetize the group with more agents (e.g. shopping malls have more shoppers

[22] For example, the per-interaction benefit of a man is typically higher than that of a woman in a nightclub. Besides, two-sided platforms are often designed to favor one side much more than the other side, as I will argue soon. Moreover, the platform's divide-and-conquer strategy remains optimal even when $v_1 = v_2$.

[23] Under Pareto-dominated selection, the platform needs to guarantee participation from one side by providing free access for that side. Then, the platform can charge the highest possible price on the other side. Clearly, the money side and the subsidy side under Pareto-dominated selection is the same as those under potential-maximizer selection.

than retailers, but only the latter are charged). When there are more, say, group-1 agents, the platform can extract more surplus from group 1 by increasing p_1. Nevertheless, having more group-1 agents also increases the group-2 agents' benefits from joining the platform. Hence, the platform can also extract more surplus from group 2 by increasing p_2. These two effects cancel out perfectly in this model. Thus, the money/subsidy side is independent of the number of agents.

Similarly, the money/subsidy side is independent of the marginal costs c_1 and c_2 for serving the agents (e.g. for open-access academic journals, the marginal cost of an additional reader is zero and reviewing a paper is costly, but these journals only charge authors for submission fees). Hence, the platform does not necessarily monetize the "more profitable" side. The reason is that all agents either join or not join the platform in the equilibrium. Therefore, the total cost $c_1N_1 + c_2N_2$ incurred by the platform can be interpreted as a fixed cost, which does not affect the decision on the money/subsidy side.

Optimal design Oftentimes, the agents' per-interaction benefits v_1 and v_2 are not exogenous, but rather the platform's endogenous choice. For example, shopping malls are often designed to maximize shoppers' travel distances (e.g. locating anchor stores far from each other, placing escalators at opposite ends); this benefits retailers but harms shoppers. Hence, the following discussion investigates the comparative statics by varying v_1 and v_2.

As shown in Table 1, under Pareto-dominance selection, the optimal design of the platform that maximizes its equilibrium profit is to favor both sides, i.e., to increase both v_1 and v_2. This is not true under potential-maximizer selection. The platform's equilibrium profit is independent of v_1 as long as $v_1 < v_2$. Therefore, the platform only has the incentive to increase the per-interaction benefit of the money side; the optimal design of the platform is to favor one side only.

Besides, the social surplus is equal to the platform's equilibrium profit under Pareto-dominance selection as shown in Table 1. Therefore, the optimal design of the platform maximizes social welfare. By contrast, the optimal design of the platform is likely to be socially suboptimal under potential-maximizer selection: the platform has no incentive to increase group-1 agents' surplus by increasing v_1.

I conclude the discussion with an example interpreted under the current framework. Suppose there is a monopoly marketplace with a million potential buyers and a thousand potential sellers. When a buyer and a seller meet on the marketplace, there are some potential trades between them, which create some surplus. By the first key implication, the platform will monetize one side and provide free access for the other side. By the second key implication, the money/subsidy side depends only on whether the (expected) consumer surplus or producer surplus is larger within each potential trade; this may be easy to identify empirically. Oftentimes, the marketplace can be designed to encourage or discourage competition among sellers (e.g. facilitating or inhibiting the buyers to compare the products from different sellers). Competition benefits buyers but harms sellers. Suppose that sellers are the money side and buyers are the subsidy side. Then, by the third key implication, the optimal design of the platform is to increase producer surplus by discouraging competition among sellers. After that, the platform will capture all producer surplus by monetizing sellers and provide free access for buyers.

4 Monopoly platform: heterogeneous agents

This section extends the baseline model to allow for heterogeneous agents and derives further insights into two-sided markets. Sections 4.1–4.2 analyze the model with heterogeneous agents on one side as in Armstrong and Wright (2007, Section 4), where agents from one side incur idiosyncratic personal costs from joining the platform (e.g. for physical platforms such as shopping malls or trade fairs, transport cost is likely to be a major consideration for buyers but not for sellers).[24] An alternative interpretation is that agents from one side have different reservation values; see p. 354 and 361 of Armstrong and Wright (2007) for more

[24] Hagiu and Spulber (2013) study a similar model to mine, and they analyze it under both Pareto-dominance selection and Pareto-dominated selection.

examples and interpretations. Section 4.3 extends the analysis to Armstrong's (2006) original model, in which agents are heterogeneous on both sides. Under these richer frameworks, every subgame in stage 2 remains a weighted potential game; potential-maximizer selection remains applicable. For convenience, this section presents the continuum-agent model, and Appendix C analyzes the corresponding limiting case of the finite-agent model.

Sometimes same-side externalities are present on one or both sides of a platform; they can also be positive (e.g. peer effect) or negative (e.g. competition/congestion effect). Appendix D shows that the models in this section are equivalent to the homogeneous-agent models with separable[25] negative same-side externalities from the platform's point of view. In particular, the platform's profit-maximization problem in stage 1 is identical under these two frameworks. Thus, all the results and implications in this section carry over to the homogeneous-agent models with negative same-side externalities. Appendix D also analyzes the homogeneous-agent models with positive same-side externalities and shows that all the three key implications of the baseline model carry over to this richer framework. In particular, the money/subsidy side of the platform is independent of the positive same-side externalities.

4.1 Model

There are a continuum $[0, N_1]$ ($N_1 \in \mathbb{R}_{++}$) of identical group-1 agents and a continuum $[0, \overline{N}_2]$ ($\overline{N}_2 \in \mathbb{R}_{++}$) of heterogeneous group-2 agents. If the platform attracts $n_1 \equiv \int_0^{N_1} a_1^k dk$ group-1 agents and $n_2 \equiv \int_0^{\overline{N}_2} a_2^k dk$ group-2 agents, the payoffs of a group-1 agent and agent k (agent k always refers to a group-2 agent in this section) from joining the platform are

$$u_1(n_2, p_1) = v_1 n_2 - p_1; \quad u_2^k(n_1, p_2) = v_2 n_1 - p_2 - t(k), \tag{9}$$

where the function $t : [0, \overline{N}_2] \to \mathbb{R}_+$ specifies each group-2 agent's transport cost from joining the platform. I permute the group-2 agents such that t is increasing. For technical convenience, I assume that t is twice-differentiable, strictly increasing, convex, log-concave,

[25] It means that the same-side externalities depend only on the mass of participants from the same side, but not the mass of participants on the other side.

$t(0) = 0$ and $t(\overline{N}_2) \to \infty$.[26]

For simplicity, I assume away the marginal costs for serving the participants. Hence, the platform's profit is

$$\pi(n_1, n_2, p_1, p_2) = p_1 n_1 + p_2 n_2. \tag{10}$$

The rest of the model setup is the same as that of the baseline model.

4.2 Analysis

Compared to the baseline model, there is an additional demand effect on group 2 in the current model. More precisely, for any group-2 price $p_2 \leq v_2 N_1$ set by the platform in stage 1, only some group-2 agents whose transport costs are sufficiently low will consider joining the platform or not in stage 2.[27] In other words, not joining the platform is the (strictly) dominant strategy for agent $k \in (N_2, \overline{N}_2]$, where (by (9))

$$0 \leq N_2 \equiv t^{-1}(v_2 N_1 - p_2) < \overline{N}_2. \tag{11}$$

I call these group-2 agents *irrelevant agents* and the remaining group-1 and group-2 agents *relevant agents*. In addition, if p_2 is negative (i.e., the platform subsidizes group 2),[28] some group-2 agents with really low transport costs will join the platform for sure. More precisely, joining the platform is the (strictly) dominant strategy for agent $k \in [0, \underline{N}_2)$, where (by (9))

$$0 \leq \underline{N}_2 \equiv t^{-1}(-p_2) < N_2. \tag{12}$$

[26] These assumptions are not fundamental and can be relaxed. Even if all these assumptions are violated, every subgame in stage 2 remains a weighted potential game, and thus potential-maximizer selection remains applicable. Besides, the analysis can be easily extended to allow negative values of t, i.e., group-2 agents can derive some stand-alone benefits from joining the platform.

[27] If $p_2 > v_2 N_1$, not joining the platform is the (strictly) dominant strategy for all group-2 agents. Because the platform never sets such a high group-2 price in the equilibrium, I omit the discussion on these prices throughout this section.

[28] Armstrong (2006, footnote 5) points out that it is often unrealistic to suppose negative subscription fees are feasible. For completeness, this section analyzes both cases, i.e., with or without the non-negative price constraint (footnote 35 states the equilibrium outcome under the non-negative price constraint). When I study platform competition in Section 5, I only analyze the model under the non-negative price constraint.

From the above discussions, it is clear that multiple equilibria often arise in stage 2. To be exact, there are two stable[29] equilibria and an unstable equilibrium in stage 2 if and only if

Case 1: $(p_1, p_2) \in [0, v_1 N_2] \times [0, v_2 N_1]$, or

Case 2: $(p_1, p_2) \in [v_1 \underline{N}_2, v_1 N_2] \times (-\infty, 0]$.

Denote $\mathbf{a}_i$ as group-i agents' action profile, $\mathbf{a}_i = \mathbf{0}$ and $\mathbf{a}_i = \mathbf{1}$ as no agent and all agents from group i joining the platform respectively, and $\mathbf{1}_S$ as the indicator function where

$$\mathbf{1}_S(k) \equiv \begin{cases} 1 & \text{if } k \in S; \\ 0 & \text{if } k \notin S. \end{cases}$$

As shown in Figure 1, the three equilibria in stage 2 for Case 1 are:

1. Pareto-dominant equilibrium: $(\mathbf{a}_1^*, \mathbf{a}_2^*) = (\mathbf{1}, \mathbf{1}_{[0,N_2]})$;
2. unstable equilibrium: $(\mathbf{a}_1^*, \mathbf{a}_2^*) = (\mathbf{1}_{[0,\widehat{N}_1]}, \mathbf{1}_{[0,\widehat{N}_2]})$;[30]
3. Pareto-dominated equilibrium: $(\mathbf{a}_1^*, \mathbf{a}_2^*) = (\mathbf{0}, \mathbf{0})$.

As shown in Figure 2, the three equilibria in stage 2 for Case 2 are:

1. Pareto-dominant equilibrium: $(\mathbf{a}_1^*, \mathbf{a}_2^*) = (\mathbf{1}, \mathbf{1}_{[0,N_2]})$;
2. unstable equilibrium: $(\mathbf{a}_1^*, \mathbf{a}_2^*) = (\mathbf{1}_{[0,\widehat{N}_1]}, \mathbf{1}_{[0,\widehat{N}_2]})$;
3. Pareto-dominated equilibrium: $(\mathbf{a}_1^*, \mathbf{a}_2^*) = (\mathbf{0}, \mathbf{1}_{[0,\underline{N}_2]})$.

Clearly, both Pareto-dominance selection and Pareto-dominated selection are applicable under the current framework. Appendix E analyzes these two benchmarks and shows that the equilibrium outcomes under these two benchmarks always differ from that under potential-maximizer selection.

[29] An equilibrium is stable (unstable) if it can (cannot) be reached by some dynamic process.

[30] Because group-1 agents are identical, $\mathbf{a}_1^* = \mathbf{1}_{[0,\widehat{N}_1]}$ simply means that the mass of group-1 participants is $\widehat{N}_1$. Thus, strictly speaking, there is a continuum of unstable equilibria.

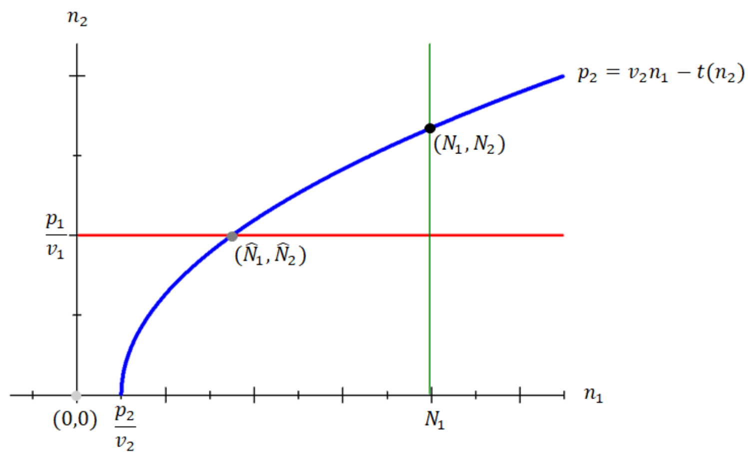

Figure 1: The equilibria of the subgame in stage 2 for Case 1

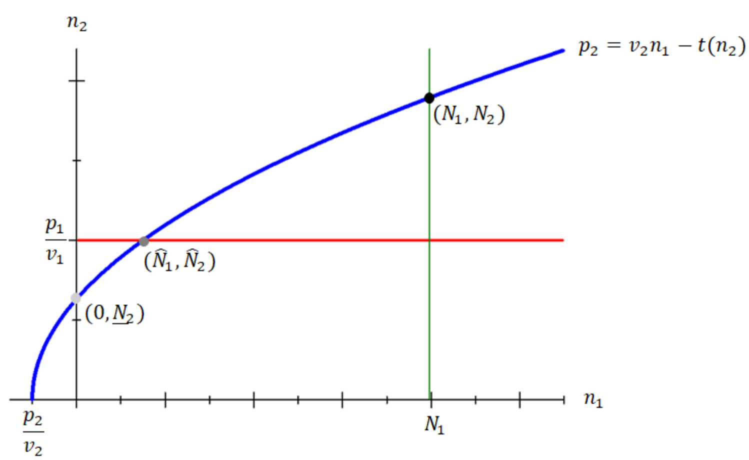

Figure 2: The equilibria of the subgame in stage 2 for Case 2

Now, I analyze the model under potential-maximizer selection. As I will show, the unstable equilibrium is never the potential maximizer, and therefore it is never selected under potential-maximizer selection. First, I show that every subgame in stage 2 is a weighted potential game.

Lemma 3 *Every subgame in stage 2 is a weighted potential game with the potential*

$$P(n_1, \mathbf{a}_2|p_1, p_2) = n_1 n_2 - \frac{p_1}{v_1} n_1 - \frac{p_2}{v_2} n_2 - \frac{1}{v_2}\int_0^{\overline{N}_2} t(k) a_2^k dk.^{31} \tag{13}$$

Proof. See Appendix C. ■

Compared to Lemma 1 in Section 3.2, the extra term $\frac{1}{v_2}\int_0^{\overline{N}_2} t(k)a_2^k dk$ captures the aggregate transport cost incurred by group-2 participants. Note that Lemma 3 remains true even if the transport cost function t violates all the imposed assumptions in Section 4.1.

After identifying the game's potential, the next step is to identify the potential maximizer. Recall from Sections 3.1–3.2 that if there is a unique equilibrium in the subgame (i.e., neither Case 1 nor Case 2), the potential maximizer is the unique equilibrium. If there are multiple equilibria (i.e., either Case 1 or Case 2), we can follow the same approach as in Section 3.2 to identify the potential maximizer. Thus, the unique equilibrium in stage 2 under potential-maximizer selection is summarized by the following lemma.

Lemma 4 *Under potential-maximizer selection, the unique equilibrium of the subgame in stage 2 is*

1. *when* $0 \leq p_2 \leq v_2 N_1$*:*

$$\begin{aligned} (\mathbf{a}_1^*, \mathbf{a}_2^*) &= (\mathbf{1}, \mathbf{1}_{[0,N_2]}) \quad \text{if } p_1 \leq \frac{v_1}{v_2 N_1}\int_0^{N_2} (t(N_2) - t(k))dk; \\ (\mathbf{a}_1^*, \mathbf{a}_2^*) &= (\mathbf{0}, \mathbf{0}) \qquad\qquad \text{otherwise.} \end{aligned}$$

[31] For Lemma 1 in Section 3.2, agents from the same side are identical, and thus the potential depends only on the number of participants n_1 and n_2. By contrast, group-2 agents are now heterogeneous, and thus the potential of the subgame depends on group-2 agents' action profile $\mathbf{a}_2$.

2. when $p_2 \leq 0$*:*

$$\begin{aligned}(\mathbf{a}_1^*, \mathbf{a}_2^*) &= (\mathbf{1}, \mathbf{1}_{[0,N_2]}) \quad \text{if } p_1 \leq v_1 \underline{N}_2 + \frac{v_1}{v_2 N_1} \int_{\underline{N}_2}^{N_2} (t(N_2) - t(k))\, dk; \\ (\mathbf{a}_1^*, \mathbf{a}_2^*) &= (\mathbf{0}, \mathbf{1}_{[0,\underline{N}_2]}) \quad \text{otherwise.}\end{aligned}$$

Proof. See Appendix A1. ■

Now, I derive the platform's optimal pricing strategy in stage 1. Similar to Lemma 2 in Section 3.2, potential-maximizer selection implies that the platform has to leave the participants enough surplus by setting sufficiently low prices (p_1, p_2) in stage 1, so that all relevant agents will join the platform in stage 2.[32] Therefore, for any group-2 price $p_2 \leq v_2 N_1$ (which determines the values of N_2 and $\underline{N}_2$ by (11) and (12) respectively) set by the platform, the platform optimally sets the highest possible group-1 price until the constraint in Lemma 4 binds, i.e.,

$$p_1^* = \begin{cases} \frac{v_1}{v_2 N_1} \int_0^{N_2} (t(N_2) - t(k)) dk & \text{if } 0 \leq p_2 \leq v_2 N_1; \\ v_1 \underline{N}_2 + \frac{v_1}{v_2 N_1} \int_{\underline{N}_2}^{N_2} (t(N_2) - t(k))\, dk & \text{if } p_2 \leq 0. \end{cases} \tag{14}$$

Now, it remains to derive the platform's optimal group-2 price p_2^*. From (10) and (14), the platform's profit-maximization problem in stage 1 becomes

$$\max_{p_2 \leq v_2 N_1} \pi = \begin{cases} \frac{v_1}{v_2} \int_0^{N_2} (t(N_2) - t(k))\, dk + p_2 N_2 & \text{if } 0 \leq p_2 \leq v_2 N_1; \\ v_1 N_1 \underline{N}_2 + \frac{v_1}{v_2} \int_{\underline{N}_2}^{N_2} (t(N_2) - t(k))\, dk + p_2 N_2 & \text{if } p_2 \leq 0. \end{cases} \tag{15}$$

Solving the above optimization problem gives us the platform's optimal group-2 price p_2^* and the equilibrium mass of group-2 participants N_2^*. After that, we can derive the platform's optimal group-1 price p_1^* and its equilibrium profit π^* from (14) and (15) respectively. Thus, the equilibrium outcome of this game is characterized by the following proposition.

[32] We can easily verify that the terms $\frac{v_1}{v_2 N_1} \int_0^{N_2} (t(N_2) - t(k)) dk$ and $v_1 \underline{N}_2 + \frac{v_1}{v_2 N_1} \int_{\underline{N}_2}^{N_2} (t(N_2) - t(k))\, dk$ in Lemma 4 are decreasing in p_2. Thus, lowering the group-2 price makes it easier for all relevant agents to coordinate on joining the platform.

Proposition 1 *Under potential-maximizer selection, there is a unique equilibrium in this model. When $v_1 \leq v_2$, the platform's optimal group-2 price p_2^* and the equilibrium mass of group-2 participants N_2^* are implicitly given by*

$$p_2^* = v_2 N_1 - t(N_2^*) = \left(1 - \frac{v_1}{v_2}\right) N_2^* t'(N_2^*) \geq 0,^{33} \tag{16}$$

and the platform's optimal group-1 price p_1^* and its equilibrium profit π^* are given by

$$\begin{aligned} p_1^* &= \frac{v_1}{v_2 N_1} \int_0^{N_2^*} (t(N_2^*) - t(k))\, dk; \\ \pi^* &= \frac{v_1}{v_2} \int_0^{N_2^*} (t(N_2^*) - t(k))\, dk + p_2^* N_2^*. \end{aligned}$$

When $v_1 \geq v_2$, p_2^* and N_2^* are implicitly given by

$$p_2^* = -t(\underline{N}_2^*) = v_2 N_1 - t(N_2^*) = \left(N_2^* - \frac{v_1}{v_2}\left(N_2^* - \underline{N}_2^*\right)\right) t'(N_2^*) \leq 0,^{34} \tag{17}$$

and p_1^* and π^* are given by

$$\begin{aligned} p_1^* &= v_1 \underline{N}_2^* + \frac{v_1}{v_2 N_1} \int_{\underline{N}_2^*}^{N_2^*} (t(N_2^*) - t(k))\, dk; \\ \pi^* &= v_1 N_1 \underline{N}_2^* + \frac{v_1}{v_2} \int_{\underline{N}_2^*}^{N_2^*} (t(N_2^*) - t(k))\, dk + p_2^* N_2^*. \end{aligned}$$

In contrast to the baseline model, the platform may strictly subsidize group-2 agents in the current model because of the additional demand effect on group 2. Owing to this additional incentive to lower group-2 price, the platform always charges group 1 (i.e., $p_1^* > 0$).

As shown in Proposition 1, the platform's optimal pricing strategy depends crucially on the per-interaction benefits v_1 and v_2. When $v_1 \leq v_2$, the optimal group-2 price is equal to the standard monopoly markup $N_2^* t'(N_2^*)$, adjusted downward by the fraction $1 - \frac{v_1}{v_2}$ due to the cross-side externalities group-1 participants enjoy. When $v_1 \geq v_2$, the platform strictly

[33] We can easily show that p_2^* and N_2^* exist. They are also unique because t is strictly increasing and convex. p_2^* is positive because $v_1 \leq v_2$ and t is increasing.

[34] We can easily show that p_2^* and N_2^* exist. They are also unique because t is strictly increasing, convex, and log-concave. We can also verify that p_2^* is negative. See Appendix A2 for the formal proof.

subsidizes group-2 agents.[35] The higher the v_1, the more the platform's incentive to subsidize group 2. In short, the platform subsidizes group 2 if and only if $v_1 \geq v_2$ — the same feature as in the baseline model — and this is irrespective of the number of group-1 agents N_1 and the group-2 agents' transport cost function t (as long as it satisfies the assumptions in Section 4.1).

To derive more properties on the equilibrium outcome, the following two tables summarize how the equilibrium values of the variables change with the parameter values. Again, the results do not depend on the exact form of the transport cost function t.

	N_2^*	p_1^*	p_2^*	$p_1^* N_1$	$p_2^* N_2^*$	π^*
N_1	+	+/−	+	+	+	+
v_1	+	+	−	+	−	+
v_2	+/−	+/−	+	+/−	+	+

Table 2a: Comparative statics when $v_1 \leq v_2$ (with $p_2^* \geq 0$)

	N_2^*	p_1^*	p_2^*	$p_1^* N_1$	$p_2^* N_2^*$	π^*
N_1	+	+	−	+	−	+
v_1	+	+	−	+	−	+
v_2	+	+/−	+	+/−	+	+

Table 2b: Comparative statics when $v_1 \geq v_2$ (with $p_2^* \leq 0$)

Proof. See Appendix A3. ■

As shown in Table 2, different parameters have different effects on the variables. When the mass of group-1 agents N_1 increases, the platform can generate more revenue from group

[35] If the platform is not allowed to strictly subsidize the agents (as mentioned in footnote 28), the platform will optimally set $p_2^* = 0$. The equilibrium mass of group-2 participants will be $N_2^* = t^{-1}(v_2 N_1)$ by (11), and p_1^* and π^* will be given by the same expressions in Proposition 1.

1. Meanwhile, group-2 agents enjoy more benefits from joining the platform, and thus the platform can charge them more. When $v_1 \leq v_2$ and N_1 increases, the platform does both: it generates more revenue from both sides (i.e., both $p_1^* N_1$ and $p_2^* N_2^*$ increase). On the other hand, the platform subsidizes group 2 when $v_1 \geq v_2$. When N_1 increases, it subsidizes each group-2 agent more (i.e., p_2^* decreases) to attract more group-2 participants, and then recover the loss by charging group-1 participants more.

When the per-interaction benefit v_1 (v_2) increases, the platform can generate more revenue by charging group-1 (group-2) participants more, and it does so as shown in Table 2. When v_1 increases, the platform also lowers the group-2 price to attract more group-2 participants, so it can charge group-1 participants even more. This is the well-known "seesaw principle" in the two-sided market literature (Rochet and Tirole 2006).[36] On the other hand, more group-2 agents will consider joining the platform or not (i.e., more relevant group-2 agents) when v_2 increases, and thus oftentimes there are more group-2 participants even though the platform sets a higher group-2 price. Owing to these complex tradeoffs in group 2, depending on the parameter values, the platform may increase or decrease the group-1 price when v_2 increases.

Now, I discuss the optimal design of the platform, i.e., the comparative statics of the platform's equilibrium profit π^* by varying v_1 and v_2. In contrast to the previous section, π^* always (strictly) increases with both v_1 and v_2 as shown in Table 2. Nevertheless, when v_1 (v_2) is relatively large, π^* increases more with respect to an increase in v_1 (v_2). The following corollary formally states the result.

[36] The seesaw principle is defined by Rochet and Tirole (2006, p. 659) as follows: a factor that is conducive to a high price on one side, to the extent that it raises the platform's margin on that side, tends also to call for a low price on the other side as attracting members on that other side becomes more profitable.

Corollary 1 *For any fixed equilibrium mass of group-2 participants N_2^*,*[37]

$$\frac{\partial \pi^*}{\partial v_1} \leq \frac{\partial \pi^*}{\partial v_2} \quad \text{if and only if} \quad \frac{v_1}{v_2} \leq \frac{N_2^* t'(N_2^*) + \frac{1}{N_2^*}\int_0^{N_2^*} t(k)dk}{N_2^* t'(N_2^*) + t(N_2^*) - \frac{1}{N_2^*}\int_0^{N_2^*} t(k)dk}.$$

Proof. See Appendix A4. ■

As shown in Corollary 1, there is a cutoff for $\frac{v_1}{v_2}$ that determines whether the platform can make a higher profit when v_1 or v_2 increases. When the ratio $\frac{v_1}{v_2}$ is high (low), the platform has more incentive to be designed in a way that further increase v_1 (v_2) rather than increasing v_2 (v_1). Therefore, the optimal design of the two-sided platforms tends to favor one side much more than the other side.

This section demonstrates how potential-maximizer selection can be applied to Armstrong's model with heterogeneous agents on one side, and shows that all the three key implications of the baseline model are naturally extended to the current model. Despite the additional demand effect on group 2 in the current model, under both models, the platform subsidizes group 2 and monetizes group 1 (i.e., divides and conquers) when the per-interaction benefits $v_1 \geq v_2$.

As shown in Proposition 1, the current model does not have a closed-form solution in general. Appendix F provides two examples with closed-form solutions: the model with linear transport cost and that with quadratic transport cost. These examples verify all the above results (e.g. Table 2 and Corollary 1) and derive further implications.

4.3 Armstrong's original model (2006, Section 3)

After analyzing Armstrong's model with heterogeneous agents on one side, we can easily extend the analysis to Armstrong's original model, in which agents are heterogeneous on both sides. The rest of this section discusses the key insights of applying potential-maximizer selection to Armstrong's original model. The formal analysis is given in Appendix G.

[37] In general, N_2^* varies as v_1 and v_2 vary. To compare $\frac{\partial \pi^*}{\partial v_1}$ and $\frac{\partial \pi^*}{\partial v_2}$ in a meaningful way, we need to keep the value of N_2^* fixed as v_1 and v_2 vary. This can be done by adjusting the mass of group-1 agents N_1 accordingly as shown in (16) and (17) of Proposition 1.

In Armstrong's original model, both group-1 and group-2 agents incur idiosyncratic personal costs from joining the platform. Therefore, demand effects are present on both sides, and thus the equilibrium masses of group-1 and group-2 participants in stage 2 are simultaneously determined by the prices (p_1, p_2) set by the platform in stage 1.

Similar to the previous models, multiple equilibria often arise in stage 2. For simplicity, the discussion below assumes that there are only two (stable) equilibria: the Pareto-dominant equilibrium with high participation, and the Pareto-dominated equilibrium with low/no participation.

As explained in footnote 13, Armstrong (2006, Section 3) implicitly imposes Pareto-dominance selection and derives the platform's optimal prices (p_1^*, p_2^*) in stage 1. If we analyze Armstrong's original model under potential-maximizer selection, the platform faces an additional constraint in stage 1: it has to set the prices (p_1, p_2) such that the Pareto-dominant equilibrium in stage 2 is also the potential maximizer. As shown in Sections 3.2 and 4.2, this additional constraint always binds in the equilibrium if agents are identical on at least one side. By contrast, when agents are heterogeneous on both sides, sometimes this additional constraint does not bind in the equilibrium. The reason is that there are now demand effects on both sides, and therefore the platform has the incentive to lower both p_1 and p_2 in order to attract more agents from both sides.[38] Thus, the platform's optimal prices (p_1^*, p_2^*) under Pareto-dominance selection may be sufficiently low so that the Pareto-dominant equilibrium in stage 2 is also the potential maximizer. If this is the case, the equilibrium outcome under potential-maximizer selection coincides with that under Pareto-dominance selection; otherwise, the equilibrium outcomes under these two selection criteria differ.

Appendix G shows that when the transport-cost functions on both sides are monomials[39] with the same degree $\alpha > 1$, the equilibrium outcome under potential-maximizer selection coincides with that under Pareto-dominance selection if and only if $\frac{v_1}{v_2} \leq \frac{1}{\alpha}$ or $\frac{v_1}{v_2} \geq \alpha$. In other

[38] By contrast, if agents are identical on one side, the platform always fully extracts these agents' surplus under Pareto-dominance selection.

[39] Monomial stands for a polynomial which has only one term.

words, the equilibrium outcomes under these two selection criteria are different whenever the per-interaction benefits of the two sides do not differ too much, i.e., $\frac{1}{\alpha} < \frac{v_1}{v_2} < \alpha$.

5 Platform competition

This section demonstrates how potential-maximizer selection can resolve the multiple equilibria issue under platform competition and derives further insights into two-sided markets. The baseline model is naturally extended to a duopoly-platform model, which is a special case of Armstrong's (2006, Section 4) duopoly-platform model and almost equivalent to Caillaud and Jullien's (2003, Section 5) model.[40] Multiple equilibria naturally arise in both of their models, but they do not attempt to select an equilibrium.[41] By contrast, I apply potential-maximizer selection to the model and derive the unique equilibrium. Section 5.3 extends the analysis by allowing the platforms to use alternative pricing instruments and derive further implications.

5.1 Model

There are two competing platforms, indexed by A and B. The payoff of a group-i agent from joining platform $m \in \{A, B\}$ depends on the number of group-j agents who join the same platform, i.e.,

$$u_i^m(n_j^m, p_i^m) = v_i^m n_j^m - p_i^m. \tag{18}$$

[40] The only difference to Caillaud and Jullien's model is that their agents can choose not to join any platforms, while mine have to join one of the platforms as in Armstrong's (2006, Section 4) model. My model is a special case of Armstrong's model with no transport cost (i.e., $t_1 = t_2 = 0$ in his model). Note that the platforms in my model can be vertically differentiated. In this sense, my model is more general than theirs.

[41] Armstrong (2006) actually does not analyze this special case; he only studies the cases where transport costs (i.e., t_1 and t_2 in his model) are sufficiently high so that there is a unique market-sharing equilibrium for the competing platforms. In fact, White and Weyl (2016, p. 3) call this special case the "previously intractable parameter values", and it remains intractable in their paper.

As shown in the above payoff function, the model allows group-i participants to enjoy different per-interaction benefits $v_i^m \in \mathbb{R}_{++}$ at different platforms.[42]

For simplicity, I assume away the marginal costs for serving the participants. Hence, platform m's profit is

$$\pi^m(n_1^m, n_2^m, p_1^m, p_2^m) = p_1^m n_1^m + p_2^m n_2^m. \tag{19}$$

Following Armstrong and Wright (2007), I assume that the subscription fees p_1^m and p_2^m set by the platforms are non-negative.[43] As they have argued (on p. 356), this is a reasonable restriction for pure-subscription models because strictly subsidizing the participants will create obvious adverse selection and moral hazard problems. Armstrong (2006, footnote 5) also makes a similar argument.

The timing of the game is the same as before. In stage 1, the platforms simultaneously set prices $(p_1^m, p_2^m) \in \mathbb{R}_+^2$ to the two groups. In stage 2, after observing the platforms' prices, all agents simultaneously decide which platform to join (they have to join one, and only one), and the game ends. The rest of the model setup is the same as that of the baseline model.

5.2 Analysis

Similar to the previous sections, multiple equilibria often arise owing to the strategic complementarities among the agents. Because agents from the same side are identical, if a group-i agent joins the platform in an equilibrium, all other group-i agents will also join the platform in that equilibrium. In other words, all agents will coordinate on one of the platforms in the equilibrium. Pareto-dominance selection is not applicable to the current model: coordinating on one platform does not necessarily Pareto-dominate the other. By the same token, Pareto-dominated selection is not applicable either. By contrast, potential-maximizer

[42] As explained in Section 3.3, different designs of the platforms can lead to different per-interaction benefits for the two sides. In reality, no two platforms share exactly the same design. Therefore, I allow the competing platforms to deliver different per-interaction benefits for each side.

[43] In Sections 2 and 3, the platform will not set negative price(s) in the equilibrium as explained in footnote 19. Section 4 analyzes both cases (i.e., with or without the non-negative price constraint) for completeness as explained in footnote 28.

selection remains valid. First, I show that every subgame in stage 2 is a weighted potential game. Denote the price differences between the two platforms as

$$\Delta p_1 \equiv p_1^A - p_1^B; \quad \Delta p_2 \equiv p_2^A - p_2^B.$$

Lemma 5 *Every subgame in stage 2 is a weighted potential game with the potential*

$$P(n_1^A, n_2^A | \Delta p_1, \Delta p_2) = n_1^A n_2^A - \frac{v_1^B N_2 + \Delta p_1}{v_1^A + v_1^B} n_1^A - \frac{v_2^B N_1 + \Delta p_2}{v_2^A + v_2^B} n_2^A. \tag{20}$$

Proof. See Appendix A5. ■

After identifying the game's potential, the next step is to identify the potential maximizer. The only non-trivial (and interesting) cases are the subgames with $\Delta p_i \in [-v_i^B N_j, v_i^A N_j]$ $(i, j = 1, 2;\ j \neq i)$, in which all agents coordinating on either platform is an equilibrium.[44] By Lemma 5, the potentials of the two equilibria are

$$\begin{aligned} P(N_1, N_2 | \Delta p_1, \Delta p_2) &= \frac{v_1^A v_2^A - v_1^B v_2^B}{(v_1^A + v_1^B)(v_2^A + v_2^B)} N_1 N_2 - \frac{\Delta p_1}{v_1^A + v_1^B} N_1 - \frac{\Delta p_2}{v_2^A + v_2^B} N_2; \\ P(0, 0 | \Delta p_1, \Delta p_2) &= 0.^{45} \end{aligned}$$

Given that the potential maximizer is the equilibrium with the higher potential, the unique equilibrium in stage 2 under potential-maximizer selection is summarized by the following lemma.

Lemma 6 *When $\Delta p_i \in [-v_i^B N_j, v_i^A N_j]$ for all $i, j = 1, 2;\ j \neq i$, the unique equilibrium of the subgame in stage 2 under potential-maximizer selection is*

$$\begin{array}{ll} \textit{all agents joining platform } A & \textit{if } \dfrac{v_2^A + v_2^B}{N_2} \Delta p_1 + \dfrac{v_1^A + v_1^B}{N_1} \Delta p_2 \leq v_1^A v_2^A - v_1^B v_2^B; \\ \textit{all agents joining platform } B & \textit{otherwise.} \end{array} \tag{21}$$

[44] Clearly, if a platform charges too much, the potential maximizer will be the unique equilibrium with all agents joining the other platform.

[45] The former corresponds to all agents joining platform A; the latter corresponds to all agents joining platform B.

Under potential-maximizer selection, we can view all group-1 and group-2 agents as a "single agent" who either joins A or B in stage 2. To see this, the inequality in (21) can be re-expressed as

$$v_1^A v_2^A - \left(\frac{v_2^A + v_2^B}{N_2} p_1^A + \frac{v_1^A + v_1^B}{N_1} p_2^A \right) \geq v_1^B v_2^B - \left(\frac{v_2^A + v_2^B}{N_2} p_1^B + \frac{v_1^A + v_1^B}{N_1} p_2^B \right). \tag{22}$$

As shown in the above expression, the "value" of platform $m \in \{A, B\}$ is $v_1^m v_2^m$, its "price" is $\frac{v_2^A + v_2^B}{N_2} p_1^m + \frac{v_1^A + v_1^B}{N_1} p_2^m$, and the single agent joins the platform that offers the higher "net value".

Given that all agents (or the "single agent") will join the "better" platform in stage 2, stage 1 is equivalent to Bertrand competition with vertical differentiation. Generically and w.l.o.g., assume that the "value" of platform A is higher, i.e., $v_1^A v_2^A > v_1^B v_2^B$. Standard analysis for Bertrand competition implies that B charges the minimum prices (i.e., $p_1^{B*} = p_2^{B*} = 0$ under the non-negative price constraint), and A slightly undercuts B to capture the entire market. From (22), this implies that

$$\frac{v_2^A + v_2^B}{N_2} p_1^{A*} + \frac{v_1^A + v_1^B}{N_1} p_2^{A*} = v_1^A v_2^A - v_1^B v_2^B. \tag{23}$$

Subject to the above constraint, A maximizes its profit by optimally allocating the prices to the two sides, i.e.,

$$\max_{p_1^A, p_2^A \geq 0} p_1^A N_1 + p_2^A N_2 \quad \text{s.t.} \quad \frac{v_2^A + v_2^B}{N_2} p_1^A + \frac{v_1^A + v_1^B}{N_1} p_2^A = v_1^A v_2^A - v_1^B v_2^B. \tag{24}$$

Generically and w.l.o.g., assume that the average per-interaction benefit across the competing platforms for a group-1 agent is smaller than that of a group-2 agent, i.e., $v_1^A + v_1^B < v_2^A + v_2^B$. Solving (24) shows that A's optimal pricing strategy is to set zero group-1 price and a positive group-2 price such that (23) holds, i.e.,

$$p_1^{A*} = 0; \quad p_2^{A*} = \frac{v_1^A v_2^A - v_1^B v_2^B}{v_1^A + v_1^B} N_1.$$

Hence, A's equilibrium profit is

$$\pi^{A*} = \frac{v_1^A v_2^A - v_1^B v_2^B}{v_1^A + v_1^B} N_1 N_2.$$

Given the above analysis, the unique equilibrium of this game under potential-maximizer selection is summarized by the following proposition.

Proposition 2 *(Generically and w.l.o.g.,) suppose* $v_1^A v_2^A > v_1^B v_2^B$ *and* $v_1^A + v_1^B < v_2^A + v_2^B$*. Under potential-maximizer selection, there is a unique equilibrium in this model. Stage 1 is a Bertrand equilibrium with*

$$p_1^{A*} = 0; \quad p_2^{A*} = \frac{v_1^A v_2^A - v_1^B v_2^B}{v_1^A + v_1^B} N_1; \quad p_1^{B*} = 0; \quad p_2^{B*} = 0.$$

All agents join platform A in stage 2, and platform A's equilibrium profit is

$$\pi^{A*} = \frac{v_1^A v_2^A - v_1^B v_2^B}{v_1^A + v_1^B} N_1 N_2.$$

As shown in Proposition 2, the market tips to a dominant platform with the higher value of $v_1^m v_2^m$, irrespective of the total number of agents N_1 and N_2 on each side.[46] An immediate implication is that the optimal design of competing platforms is likely to favor both sides instead of only one side because the platform with the lower value of $v_1^m v_2^m$ has zero market share in the equilibrium. Before I further elaborate on the platforms' optimal design, first I discuss the other two key implications of the baseline model under the current framework.

Divide-and-conquer strategy Similar to the baseline model, the dominant platform (A) always extracts surplus from one side and provides free access for the other side as shown in Proposition 2. The weaker (in terms of v_1^B and v_2^B) the competitor, the more surplus the dominant platform can extract.

Money/subsidy side In contrast to the monopoly-platform models, the money/subsidy side of the dominant platform depends on the average per-interaction benefits $v_1^A + v_1^B$ and $v_2^A + v_2^B$ across the competing platforms, rather than its own per-interaction benefits v_1^A and v_2^A. It implies that the decision on the money/subsidy side for the dominant platform is

[46] The rationale is the same as that of the money/subsidy side being independent of the total number of agents as discussed in Section 3.3.

significantly affected by the per-interaction benefits delivered by other competing platforms, even if the competitors' market shares are negligible. Therefore, there can be a reversal of the money/subsidy side for the dominant platform under competition. To see this, consider the following example:

$$v_1^A = 3; \quad v_2^A = 2; \quad v_1^B = 1; \quad v_2^B = 5. \tag{25}$$

In this example, platform A favors group 1 more than group 2, but platform B favors group 2 much more than group 1. If initially, A is a monopolist, it monetizes group 1 and subsidizes group 2 according to the analysis of the baseline model. By Table 1 (with group 1 and group 2 interchanged in the table), A's optimal pricing strategy and its equilibrium profit are

$$p_1^{A*} = 3N_2; \quad p_2^{A*} = 0; \quad \pi^{A*} = 3N_1N_2. \tag{26}$$

Suppose now B enters the market. Under competition, A still dominates the market by Proposition 2, and its optimal pricing strategy and equilibrium profit are

$$p_1^{A*} = 0; \quad p_2^{A*} = \frac{1}{4}N_1; \quad \pi^{A*} = \frac{1}{4}N_1N_2.$$

Now, A subsidizes group 1 and monetizes group 2: the money/subsidy side of the dominant platform is reversed. Besides, if B is the monopolist, its optimal pricing strategy and equilibrium profit (by Table 1) are

$$p_1^{B*} = 0; \quad p_2^{B*} = 5N_1; \quad \pi^{B*} = 5N_1N_2.$$

Compared to (26), B actually makes a higher profit if A and B are separate monopolists because B can extract more surplus from one side. In other words, the optimal design for a monopoly platform might not work well under platform competition; this leads us back to the discussion on the optimal design of competing platforms.

Optimal design As shown in Proposition 2, the equilibrium profit of the dominant platform is non-linear in its per-interaction benefits v_1^A and v_2^A; it also depends on the competitor's per-interaction benefits v_1^B and v_2^B. When the platforms are highly competitive (say,

$v_1^A v_2^A \approx v_1^B v_2^B$), the optimal design of the competing platforms tends to favor both sides (i.e., to maximize the product of its own per-interaction benefits $v_1^m v_2^m$) in order to capture the entire market. On the other hand, when one of the platforms is inferior (say, $v_1^B \approx v_2^B \approx 0$), the optimal design of the superior platform tends to favor only one side in order to capture the more surplus from the money side. In fact, when $v_1^B = v_2^B = 0$, platform A's optimal pricing strategy and equilibrium profit in Proposition 2 are equal to those of the monopoly platform in Table 1.

When platform A dominates the market (i.e., $v_1^A v_2^A > v_1^B v_2^B$), the social surplus is $(v_1^A + v_2^A)N_1 N_2$; it can be less than the social surplus $(v_1^B + v_2^B)N_1 N_2$ if all agents coordinate on joining platform B instead (see (25) as an example). Moreover, the optimal design of the dominant platform (A) is likely to be socially suboptimal: when the platforms are highly competitive (say, $v_1^A v_2^A \approx v_1^B v_2^B$), the optimal design of A maximizes $v_1^A v_2^A$ instead of $v_1^A + v_2^A$; when B is inferior (say, $v_1^B \approx v_2^B \approx 0$), the optimal design of A favors only the money side.

When the competing platforms are identical (i.e., $v_1^A = v_1^B$ and $v_2^A = v_2^B$), the model is almost equivalent to Caillaud and Jullien's (2003, Section 5) pure-subscription model as mentioned before. In their paper, they report (on p. 322) that there are multiple dominant-platform equilibria with strictly positive profit for the dominant platform. By contrast, if we apply potential-maximizer selection to the current model, there is a unique Bertrand equilibrium in stage 1 with zero profit for both platforms.[47] Caillaud and Jullien (2003, Section 2) actually derive a similar result (Proposition 1) in their two-part-tariff model. Nevertheless, the platforms in their model charge the agents the maximum transaction fees in the equilibrium, while those in my model cannot charge any transaction fees.

5.3 Alternative tariffs

In what follows, I study a model in which platforms charge transaction fees instead of subscription fees. After that, I discuss how the analysis can be extended to two-part tariffs.

[47] Proposition 2 only characterizes the unique equilibrium for generic cases, but we can easily deduce that if the competing platforms are identical, both of them will set zero price for both sides in the equilibrium.

The previous models assume that agents are charged a fixed subscription fee to join a platform. There are two reasons for that. First, many two-sided platforms do charge subscription fees only (e.g. nightclubs, shopping malls, exhibitions), because monitoring each transaction is too costly or impossible, and the agents might conduct their subsequent transactions outside the platforms (known as disintermediation). Second, if the monopoly platform in the baseline model charges transaction fees instead of subscription fees, it is easy to see that there is a unique equilibrium in the model.[48] Nevertheless, the primary purpose of the baseline model is to illustrate how potential-maximizer selection can resolve the multiple equilibria issue in two-sided markets. Therefore, the previous models restrict the monopoly platform to use subscription fees only. By contrast, under platform competition, there can be multiple equilibria even if the competing platforms use transaction fees. Therefore, potential-maximizer selection can be applied meaningfully to these cases.

Model

The platforms set transaction fees instead of subscription fees to the two groups. If a group-i agent joins platform m, he pays a transaction fee $p_i^m \in \mathbb{R}$ per each interaction with group-j agents who join the same platform. Thus, his payoff from joining platform m is

$$u_i^m(n_j^m, p_i^m) = (v_i^m - p_i^m)n_j^m. \tag{27}$$

If platform m attracts n_1^m group-1 participants and n_2^m group-2 participants, there is a total of $n_1^m n_2^m$ interactions within the platform. Platform m's profit is the total transaction fees collected from both sides, i.e.,

$$\pi^m(n_1^m, n_2^m, p_1^m, p_2^m) = (p_1^m + p_2^m)n_1^m n_2^m. \tag{28}$$

I assume that both platforms do not use any weakly dominated strategies, i.e., the sum of transaction fees on both sides $p_1^m + p_2^m$ is non-negative. The rest of the model setup is the same as that in Section 5.1.

[48] In this unique equilibrium, the monopoly platform charges the maximum transaction fees on both sides and capture all agents' surplus.

Analysis

Pareto-dominance selection remains not applicable to this alternative duopoly-platform model for the same reason as before, while potential-maximizer selection remains valid. Note that the agent's payoff function (27) in this model can be obtained by replacing v_i^m and p_i^m in (18) with $v_i^m - p_i^m$ and 0 respectively. Hence, it is immediate that the current model is a weighted potential game, and its potential and the potential maximizer can be obtained directly by applying the above replacements to Lemmas 5 and 6. Thus, we can immediately characterize the unique equilibrium in stage 2 under potential-maximizer selection.

Lemma 7 *When $p_i^m \leq v_i^m$ for all $m \in \{A, B\}$ and $i = 1, 2$,*[49] *the unique equilibrium of the subgame in stage 2 under potential-maximizer selection is*

all agents joining platform A	*if* $(v_1^A - p_1^A)(v_2^A - p_2^A) \geq (v_1^B - p_1^B)(v_2^B - p_2^B)$;
all agents joining platform B	*otherwise.*

Under potential-maximizer selection, all agents will coordinate on the platform that delivers a higher product of net per-interaction benefits $(v_1^m - p_1^m)(v_2^m - p_2^m)$ for the two sides in stage 2. Thus, stage 1 is very similar to Bertrand competition as in the duopoly-platform pure-subscription model. Generically and w.l.o.g., assume that A is the dominant platform (and thus B is dominated) in the equilibrium. Standard analysis for Bertrand competition and Lemma 7 imply that B charges the minimum prices to maximize $(v_1^B - p_1^B)(v_2^B - p_2^B)$ subject to the non-negative profit constraint $p_1^B + p_2^B \geq 0$. Thus, B's equilibrium pricing strategy is

$$p_1^{B*} = \frac{v_1^B - v_2^B}{2}; \quad p_2^{B*} = \frac{v_2^B - v_1^B}{2}. \tag{29}$$

As shown in the above expression, B rebalances the net per-interaction benefits of the two sides by monetizing the side with higher per-interaction benefit and subsidizing the other side. After the rebalance, the net per-interaction benefits are $\frac{v_1^B + v_2^B}{2}$ for both sides. Standard

[49] The only interesting cases are the subgames with $p_i^m \leq v_i^m$ because both platforms will not set a transaction fee higher than the respective per-interaction benefit in the equilibrium.

analysis for Bertrand competition also implies that A slightly undercuts B to capture the entire market. By Lemma 7 and (29), this implies that

$$(v_1^A - p_1^{A*})(v_2^A - p_2^{A*}) = \left(\frac{v_1^B + v_2^B}{2}\right)^2. \tag{30}$$

Under the non-negative profit constraint $p_1^A + p_2^A \geq 0$, it is easy to see that A can successfully undercut B if and only if $v_1^A + v_2^A > v_1^B + v_2^B$. Hence, assuming $v_1^A + v_2^A > v_1^B + v_2^B$, A maximizes its profit by optimally allocating the prices to the two sides subject to the constraint in (30), i.e.,

$$\max_{\left\{(p_1^A, p_2^A) \in (-\infty, v_1^A] \times (-\infty, v_2^A] : p_1^A + p_2^A \geq 0\right\}} (p_1^A + p_2^A) N_1 N_2 \quad \text{s.t.} \quad (v_1^A - p_1^A)(v_2^A - p_2^A) = \left(\frac{v_1^B + v_2^B}{2}\right)^2.$$

Solving the above optimization problem shows that A adjusts the net per-interaction benefits of the two sides with transaction fees, so that the net per-interaction benefits for both sides are also equal to $\frac{v_1^B + v_2^B}{2}$, i.e.,

$$p_1^{A*} = v_1^A - \frac{v_1^B + v_2^B}{2}; \quad p_2^{A*} = v_2^A - \frac{v_1^B + v_2^B}{2}.$$

Hence, A's equilibrium profit is

$$\pi^{A*} = \left(v_1^A + v_2^A - v_1^B - v_2^B\right) N_1 N_2.$$

Given the above analysis, the unique equilibrium of this alternative duopoly-platform model under potential-maximizer selection is summarized by the following proposition.

Proposition 3 *(Generically and w.l.o.g.,) suppose $v_1^A + v_2^A > v_1^B + v_2^B$. Under potential-maximizer selection, there is a unique equilibrium in this model. Stage 1 is a Bertrand equilibrium with*

$$p_1^{A*} = v_1^A - \frac{v_1^B + v_2^B}{2}; \quad p_2^{A*} = v_2^A - \frac{v_1^B + v_2^B}{2}; \quad p_1^{B*} = \frac{v_1^B - v_2^B}{2}; \quad p_2^{B*} = \frac{v_2^B - v_1^B}{2}.$$

All agents join platform A in stage 2, and platform A's equilibrium profit is

$$\pi^{A*} = \left(v_1^A + v_2^A - v_1^B - v_2^B\right) N_1 N_2.$$

Similar to the duopoly-platform pure-subscription model, the market tips to a dominant platform. Nevertheless, the dominant platform is the one with a higher sum of per-interaction benefits $v_1^m + v_2^m$ in the current model, but not the product of them $v_1^m v_2^m$ as in the pure-subscription model. Similar to the previous sections, I discuss the three key implications one-by-one under the current framework.

Divide-and-conquer strategy As shown in Proposition 3, the dominated platform (B) monetizes the side with higher per-interaction benefit and subsidizes the side with lower per-interaction benefit. However, the rationale of this pricing strategy differs from that of the previous divide-and-conquer strategy. As explained, B's pricing strategy is to rebalance the net per-interaction benefits for the two sides. Hence, if the two sides enjoy similar per-interaction benefits (i.e., $v_1^B \approx v_2^B$), B will approximately charge zero price for both sides as shown in Proposition 3. By contrast, a platform adopting the previous divide-and-conquer strategy (e.g. the monopolist in the baseline model) sets very different prices for the two sides even if $v_1 \approx v_2$.

The dominant platform's pricing strategy departs further from the divide-and-conquer strategy. Depending on how competitive the two platforms are, the dominant firm may or may not monetize one side and subsidize the other side. As shown in Proposition 3, if the platforms are highly competitive (say, $v_1^A + v_2^A \approx v_1^B + v_2^B$), the dominant platform (A) will divide and conquer. On the other hand, if one of the platforms is inferior (say, $v_1^B \approx v_2^B \approx 0$), the superior platform will monetize both sides.[50]

Money/subsidy side As shown in Proposition 3, the money/subsidy side of the dominated platform (B) depends only on whether its own per-interaction benefits v_1^B or v_2^B is larger. As explained, the dominant platform (A) might monetize both sides if its competitor is inferior. Nevertheless, when the dominant platform monetizes one side and subsidizes the

[50] In fact, when $v_1^B = v_2^B = 0$, platform A's optimal pricing strategy and equilibrium profit in Proposition 3 are equal to those of the monopoly platform in Section 2 that uses transaction fees instead of subscription fees.

other side, the money/subsidy side depends only on whether its own per-interaction benefits v_1^A or v_2^A is larger.[51] This differs from that of the duopoly-platform pure-subscription model, in which the money/subsidy side of the dominant platform depends on the competitor's per-interaction benefits v_1^B and v_2^B for the two sides.

Optimal design Because the platforms can always rebalance the net per-interaction benefits with transaction fees, the optimal design of the platforms is to maximize the sum of per-interaction benefits $v_1^m + v_2^m$ that they can deliver to the two sides as shown in Proposition 3. As we can see, the platforms' optimal design differs from that of the pure-subscription model. In other words, given the same set of parameter values, the dominant platform may differ under these two duopoly-platform models. Take (25) as an example; A is the dominant platform in the pure-subscription model, but B is the dominant platform in the current model.

In contrast to the pure-subscription model, all agents always coordinate on joining the platform that delivers the higher social surplus $(v_1^m + v_2^m)N_1N_2$ in the current model as shown in Proposition 3. As explained, the optimal design of the competing platforms is to maximize the sum of per-interaction benefits $v_1^m + v_2^m$. Therefore, the optimal design of the dominant platform also maximizes the social surplus.

Section 5.3 demonstrates how potential-maximizer selection can be applied to two-sided market models with alternative tariffs, and derives some novel implications that are absent in pure-subscription models. The most distinctive feature of the current model is that both platforms adjust the net per-interaction benefits with transaction fees, such that the two sides have the same net per-interaction benefit from either platform. To achieve this, the platforms may divide and conquer, but the rationale differs from that in the pure-subscription models.

[51] As shown in Proposition 3, if A subsidizes group 1 ($p_1^{A*} < 0$) and monetizes group 2 ($p_2^{A*} > 0$), it implies that $v_1^A < v_2^A$, irrespective of the competitor's per-interaction benefits v_1^B and v_2^B.

Two-part tariffs

Although I restrict the platforms to use only transaction fees in Section 5.3, the analysis can be easily extended to two-part tariffs. Suppose the agent's payoff from joining the platform takes the form

$$u_i^m(n_j^m, p_i^m, r_i^m) = (v_i^m - r_i^m)n_j^m - p_i^m,$$

where $p_i^m \in \mathbb{R}$ is the subscription fee and $r_i^m \in \mathbb{R}$ is the transaction fee. Similar to the analysis of the transaction-fee model, this game is a weighted potential game, and its potential and the potential maximizer can be obtained directly by replacing v_i^m in Lemmas 5 and 6 with $v_i^m - r_i^m$. Thus, we have already characterized the unique equilibrium in stage 2, even without discussing the reasonable constraints on the prices p_i^m and r_i^m. In fact, the non-negative price constraint in Section 5.1 and the non-negative profit constraint in Section 5.3 are imposed to guarantee the existence of a unique equilibrium in stage 1. Even without these constraints, every subgame in stage 2 remains a weighted potential game, and thus potential-maximizer selection can always resolve the multiple equilibria issue in stage 2.

6 Conclusion

Through analyzing a few variants of Armstrong's (2006) models, this paper demonstrates how potential-maximizer selection can resolve the multiple equilibria issue and derives novel insights into two-sided markets. In particular, two-sided platforms often divide and conquer, and the fundamental determinant of the money/subsidy side is the cross-side externalities only. This divide-and-conquer strategy also has significant implications on the optimal design of the platforms, e.g. the platforms are often designed to favor the money side much more than the subsidy side. A natural direction for future research is to apply potential-maximizer selection to other two-sided market models and derive further novel results.

In contrast to some equilibrium selection criteria in the two-sided market literature, potential-maximizer selection has many microfoundations in the game theory literature, and it is surprisingly easy to compute as demonstrated in this paper. Although potential-maximizer selection is applicable only to potential games, this paper shows that many two-sided market models are weighted potential games, and thus potential-maximizer selection can resolve the multiple equilibria issue for these two-sided market models. In fact, another contribution of this paper is to unveil the significant applications of potential games on two-sided markets.

Note that not all two-sided market models are weighted potential games. For example, if agents can multihome under platform competition, the model is not a weighted potential game in general. Nevertheless, potential-maximizer selection is not restricted only to weighted potential games, but also applicable to monotone potential games, which is a superset of weighted potential games (as mentioned in footnotes 6 and 7). Yet, checking whether a game is a monotone potential game is computationally more challenging. Hence, extending potential-maximizer selection to two-sided market models which are monotone potential games is another fruitful direction for future research.

References

[1] Alos-Ferrer, C., & Netzer, N. (2010). The logit-response dynamics. *Games and Economic Behavior*, *68*(2), 413–427.

[2] Ambrus, A., & Argenziano, R. (2009). Asymmetric networks in two-sided markets. *American Economic Journal: Microeconomics*, *1*(1), 17–52.

[3] Anderson, S. P., Goeree, J. K., & Holt, C. A. (2001). Minimum-effort coordination games: Stochastic potential and logit equilibrium. *Games and Economic Behavior*, *34*(2), 177–199.

[4] Armstrong, M. (2006). Competition in two-sided markets. *The RAND Journal of Economics*, *37*(3), 668–691.

[5] Armstrong, M., & Wright, J. (2007). Two-sided markets, competitive bottlenecks and exclusive contracts. *Economic Theory*, *32*(2), 353–380.

[6] Biglaiser, G., Calvano, E., & Cremer, J. (2019). Incumbency advantage and its value. *Journal of Economics & Management Strategy*, *28*, 41–48.

[7] Blume, L. E. (1993). The statistical mechanics of strategic interaction. *Games and Economic Behavior*, *5*(3), 387–424.

[8] Caillaud, B., & Jullien, B. (2003). Chicken & egg: Competition among intermediation service providers. *The RAND Journal of Economics*, 309–328.

[9] Chen, R., & Chen, Y. (2011). The potential of social identity for equilibrium selection. *American Economic Review*, *101*(6), 2562–89.

[10] Carlsson, H., & Van Damme, E. (1993). Global games and equilibrium selection. *Econometrica*, *61*(5), 989–1018.

[11] Frankel, D. M., Morris, S., & Pauzner, A. (2003). Equilibrium selection in global games with strategic complementarities. *Journal of Economic Theory*, *108*(1), 1–44.

[12] Goeree, J. K., & Holt, C. A. (2005). An experimental study of costly coordination. *Games and Economic Behavior*, *51*(2), 349–364.

[13] Hagiu, A. (2006). Pricing and commitment by two-sided platforms. *The RAND Journal of Economics*, *37*(3), 720–737.

[14] Hagiu, A., & Spulber, D. (2013). First-party content and coordination in two-sided markets. *Management Science*, *59*(4), 933–949.

[15] Halaburda, H., & Yehezkel, Y. (2013). Platform competition under asymmetric information. *American Economic Journal: Microeconomics*, *5*(3), 22–68.

[16] Halaburda, H., & Yehezkel, Y. (2019). Focality advantage in platform competition. *Journal of Economics & Management Strategy*, *28*, 49–59.

[17] Hofbauer, J., & Sorger, G. (1999). Perfect foresight and equilibrium selection in symmetric potential games. *Journal of Economic Theory*, *85*(1), 1–23.

[18] Jullien, B. (2011). Competition in multi-sided markets: Divide and conquer. *American Economic Journal: Microeconomics*, *3*(4), 186–220.

[19] Jullien, B., & Pavan, A. (2018). Information Management and Pricing in Platform Markets. *Review of Economic Studies*, forthcoming.

[20] Kajii, A., & Morris, S. (1997). The robustness of equilibria to incomplete information. *Econometrica*, 65, 1283–1309.

[21] Matsui, A., & Matsuyama, K. (1995). An approach to equilibrium selection. *Journal of Economic Theory*, *65*(2), 415–434.

[22] Monderer, D., & Shapley, L. S. (1996). Potential games. *Games and Economic Behavior*, *14*(1), 124–143.

[23] Morris, S., & Shin, H. S. (2003). Global Games: Theory and Applications. *Advances in Economics and Econometrics*, Eighth World Congress (M. Dewatripont, L. Hansen, and S. Turnovsky, eds.). Cambridge University Press.

[24] Morris, S., & Ui, T. (2004). Best response equivalence. *Games and Economic Behavior*, *49*(2), 260–287.

[25] Morris, S., & Ui, T. (2005). Generalized potentials and robust sets of equilibria. *Journal of Economic Theory*, *124*(1), 45–78.

[26] Okada, D., & Tercieux, O. (2012). Log-linear dynamics and local potential. *Journal of Economic Theory*, *147*(3), 1140–1164.

[27] Oyama, D., Takahashi, S., & Hofbauer, J. (2008). Monotone methods for equilibrium selection under perfect foresight dynamics. *Theoretical Economics*, 3, 155–192.

[28] Rochet, J. C., & Tirole, J. (2003). Platform competition in two-sided markets. *Journal of the European Economic Association*, *1*(4), 990–1029.

[29] Rochet, J. C., & Tirole, J. (2006). Two-sided markets: a progress report. *The RAND Journal of Economics*, *37*(3), 645–667.

[30] Rosenthal, R. W. (1973). A class of games possessing pure-strategy Nash equilibria. *International Journal of Game Theory*, *2*(1), 65–67.

[31] Sakovics, J., & Steiner, J. (2012). Who matters in coordination problems?. *American Economic Review*, *102*(7), 3439–61.

[32] Sandholm, W. H. (2001). Potential games with continuous player sets. *Journal of Economic Theory*, *97*(1), 81–108.

[33] Ui, T. (2001). Robust equilibria of potential games. *Econometrica*, *69*(5), 1373–1380.

[34] Van Huyck, J. B., Battalio, R. C., & Beil, R. O. (1990). Tacit coordination games, strategic uncertainty, and coordination failure. *American Economic Review*, *80*(1), 234–248.

[35] Weyl, E. G. (2010). A price theory of multi-sided platforms. *American Economic Review*, *100*(4), 1642–72.

[36] White, A., & Weyl, E. G. (2016). Insulated platform competition. *Available at SSRN 1694317*.

Appendix

A. Proofs

A1. Lemma 4

First, I identify the potential maximizer for Case 1, and then for Case 2. For Case 1, by Lemma 3, the respective potentials of the three equilibria are

$$\begin{aligned} P(N_1, \mathbf{1}_{[0,N_2]}|p_1,p_2) &= N_1N_2 - \frac{p_1}{v_1}N_1 - \frac{p_2}{v_2}N_2 - \frac{1}{v_2}\int_0^{N_2} t(k)dk \\ &= -\frac{p_1}{v_1}N_1 + \frac{1}{v_2}\int_0^{N_2}(t(N_2)-t(k))dk; \quad \text{(by (11))} \qquad (31) \\ P(\widehat{N}_1, \mathbf{1}_{[0,\widehat{N}_2]}|p_1,p_2) &= \widehat{N}_1\widehat{N}_2 - \frac{p_1}{v_1}\widehat{N}_1 - \frac{p_2}{v_2}\widehat{N}_2 - \frac{1}{v_2}\int_0^{\widehat{N}_2} t(k)dk \\ &= -\frac{p_2}{v_2}\widehat{N}_2 - \frac{1}{v_2}\int_0^{\widehat{N}_2} t(k)dk \ (\frac{p_1}{v_1} = \widehat{N}_2 \text{ as shown in Figure 1}) \qquad (32) \\ &\leq 0; \\ P(0, \mathbf{0}|p_1,p_2) &= 0. \end{aligned}$$

Given that the potential maximizer is the equilibrium with the higher potential, the unique equilibrium in stage 2 under potential-maximizer selection for Case 1 is

$$\begin{aligned} (\mathbf{a}_1^*, \mathbf{a}_2^*) &= (\mathbf{1}, \mathbf{1}_{[0,N_2]}) \quad \text{if } p_1 \leq \frac{v_1}{v_2N_1}\int_0^{N_2}(t(N_2)-t(k))dk; \\ (\mathbf{a}_1^*, \mathbf{a}_2^*) &= (\mathbf{0}, \mathbf{0}) \qquad \text{otherwise.}^{52} \end{aligned}$$

Now, I study Case 2. The potentials of the Pareto-dominant equilibrium and the unstable equilibrium are the same as those in Case 1, i.e., given by (31) and (32) respectively. By Lemma 3, the potential of the Pareto-dominated equilibrium is

$$P(0, \mathbf{1}_{[0,\underline{N}_2]}|p_1,p_2) = -\frac{p_2}{v_2}\underline{N}_2 - \frac{1}{v_2}\int_0^{\underline{N}_2} t(k)dk.$$

[52] Note that the constraint $p_1 \leq v_1N_2$ never binds in the Pareto-dominant equilibrium because $p_1 \leq \frac{v_1}{v_2N_1}\int_0^{N_2}(t(N_2)-t(k))dk \leq v_1N_2\frac{t(N_2)}{v_2N_1} \leq v_1N_2$. The last inequality comes from the fact that $p_2 = v_2N_1 - t(N_2) \geq 0$ in Case 1.

The potential of the unstable equilibrium is less than that of the Pareto-dominated equilibrium because $-p_2 < t(k)$ for all $k \in (\underline{N}_2, \widehat{N}_2]$ by (12). Similar to Case 1, the potential maximizer is the equilibrium with the higher potential, i.e.,

$$(\mathbf{a}_1^*, \mathbf{a}_2^*) = (\mathbf{1}, \mathbf{1}_{[0,N_2]}) \quad \text{if } p_1 \le v_1 \underline{N}_2 + \frac{v_1}{v_2 N_1} \int_{\underline{N}_2}^{N_2} (t(N_2) - t(k))\, dk;$$
$$(\mathbf{a}_1^*, \mathbf{a}_2^*) = (\mathbf{0}, \mathbf{1}_{[0,\underline{N}_2]}) \quad \text{otherwise.}^{53}$$

A2. Proposition 1

First, I show that p_2^* and N_2^* are unique when $v_1 \ge v_2$. After that, it is obvious that p_2^* is negative. To prove the first part, it suffices to show that the right-hand side of (17) decreases with p_2^*:

$$\begin{aligned}
& \frac{d\left(\left(N_2^* - \frac{v_1}{v_2}\left(N_2^* - \underline{N}_2^*\right)\right) t'(N_2^*)\right)}{dp_2^*} \\
= & \left(\left(\frac{v_1}{v_2} - 1\right)\frac{1}{t'(N_2^*)} - \frac{v_1}{v_2}\frac{1}{t'(\underline{N}_2^*)}\right) t'(N_2^*) - \left(N_2^* - \frac{v_1}{v_2}\left(N_2^* - \underline{N}_2^*\right)\right)\frac{t''(N_2^*)}{t'(N_2^*)} \\
= & -1 - \frac{v_1}{v_2}\left(\frac{t'(N_2^*)}{t'(\underline{N}_2^*)} - 1\right) + \frac{t(\underline{N}_2^*) t''(N_2^*)}{(t'(N_2^*))^2} \quad \text{(by (17))} \\
\le & -1 + \frac{t(\underline{N}_2^*) t''(N_2^*)}{(t'(N_2^*))^2} \quad (t \text{ is increasing and convex}) \\
\le & -1 + \frac{t(\underline{N}_2^*)}{t(N_2^*)} \quad (t \text{ is log-concave, i.e., } t(N_2^*) t''(N_2^*) \le (t'(N_2^*))^2) \\
\le & \ 0. \quad (t \text{ is increasing})
\end{aligned}$$

Now, I show that $p_2^* \le 0$. When $p_2^* = 0$, the left-hand side of (17) (i.e., p_2^*) is zero, while the right-hand side of (17) is negative. Given that the left-hand side increases with p_2^* and the right-hand side decreases with p_2^*, the unique optimal group-2 price p_2^* must be negative.

A3. Table 2

This appendix computes all the comparative statics one-by-one: first for Table 2a, and then for Table 2b.

[53] Note that the constraint $p_1 \le v_1 N_2$ never binds in the Pareto-dominant equilibrium because $p_1 \le v_1 \underline{N}_2 + \frac{v_1}{v_2 N_1} \int_{\underline{N}_2}^{N_2} (t(N_2) - t(k))\, dk \le v_1 \underline{N}_2 + v_1 (N_2 - \underline{N}_2) \frac{t(N_2) - t(\underline{N}_2)}{v_2 N_1} = v_1 \underline{N}_2 + v_1 (N_2 - \underline{N}_2) = v_1 N_2$.

Table 2a Figures 3 and 5 in Appendix F demonstrate that the signs of $\frac{\partial N_2^*}{\partial v_2}$, $\frac{\partial p_1^*}{\partial v_2}$, and $\frac{\partial p_1^* N_1}{\partial v_2}$ are ambiguous. Now, I compute the comparative statics in the following order:

	N_2^*	p_1^*	p_2^*	$p_1^* N_1$	$p_2^* N_2^*$	π^*
N_1	[1]$+$	[9] $+/-$	[3]$+$	[11]$+$	[6]$+$	[13]$+$
v_1	[2]$+$	[10]$+$	[4]$-$	[12]$+$	[7]$-$	[14]$+$
v_2	$+/-$	$+/-$	[5]$+$	$+/-$	[8]$+$	[15]$+$

N_2^* is characterized by (16) in Proposition 1, and thus its comparative statics is also based on (16):

[1] $\frac{\partial N_2^*}{\partial N_1} = \frac{v_2^2}{(2v_2-v_1)t'(N_2^*)+(v_2-v_1)N_2^* t''(N_2^*)} \geq 0.$

[2] $\frac{\partial N_2^*}{\partial v_1} = \frac{N_2^* t'(N_2^*)}{(2v_2-v_1)t'(N_2^*)+(v_2-v_1)N_2^* t''(N_2^*)} \geq 0.$

As shown in (11), p_2^* can be expressed as a function of N_2^*. Hence, its comparative statics is based on that of N_2^*:

[3] $\frac{\partial p_2^*}{\partial N_1} = v_2 - t'(N_2^*)\frac{\partial N_2^*}{\partial N_1} = \frac{v_2(v_2-v_1)(t'(N_2^*)+N_2^* t''(N_2^*))}{(2v_2-v_1)t'(N_2^*)+(v_2-v_1)N_2^* t''(N_2^*)} \geq 0.$

[4] $\frac{\partial p_2^*}{\partial v_1} = -t'(N_2^*)\frac{\partial N_2^*}{\partial v_1} \leq 0.$

[5] $\frac{\partial p_2^*}{\partial v_2} = N_1 - t'(N_2^*)\frac{\partial N_2^*}{\partial v_2} = \frac{(v_2-v_1)N_1(t'(N_2^*)+N_2^* t''(N_2^*))+\frac{v_1}{v_2}N_2^*(t'(N_2^*))^2}{(2v_2-v_1)t'(N_2^*)+(v_2-v_1)N_2^* t''(N_2^*)} \geq 0.$

The comparative statics of $p_2^* N_2^*$ is based on those of p_2^* and N_2^*:

[6] $\frac{\partial p_2^* N_2^*}{\partial N_1} = N_2^* \frac{\partial p_2^*}{\partial N_1} + p_2^* \frac{\partial N_2^*}{\partial N_1} \geq 0.$

[7] $\frac{\partial p_2^* N_2^*}{\partial v_1} = -\frac{v_1}{v_2} N_2^* t'(N_2^*)\frac{\partial N_2^*}{\partial v_1} \leq 0.$

[8] $\frac{\partial p_2^* N_2^*}{\partial v_2} = \frac{(v_2-v_1)N_1 N_2^*(2t'(N_2^*)+N_2^* t''(N_2^*))+\left(\frac{v_1}{v_2}N_2^* t'(N_2^*)\right)^2}{(2v_2-v_1)t'(N_2^*)+(v_2-v_1)N_2^* t''(N_2^*)} \geq 0.$

As shown in Proposition 1, p_1^* is a function of N_2^*. Hence, its comparative statics is based on that of N_2^*:

[9] $\frac{\partial p_1^*}{\partial N_1} = \frac{v_1}{v_2 N_1^2}\left(N_1 N_2^* t'(N_2^*)\frac{\partial N_2^*}{\partial N_1} - \int_0^{N_2^*}(t(N_2^*) - t(k))\,dk\right)$

$= \frac{v_1}{v_2 N_1^2}\left(N_2^* t'(N_2^*)\frac{t(N_2^*)+\left(1-\frac{v_1}{v_2}\right)N_2^* t'(N_2^*)}{\left(2-\frac{v_1}{v_2}\right)t'(N_2^*)+\left(1-\frac{v_1}{v_2}\right)N_2^* t''(N_2^*)} - N_2^* t(N_2^*) + \int_0^{N_2^*} t(k)dk\right).$

$\frac{\partial p_1^*}{\partial N_1}$ can be positive as shown in (43) of Appendix F. Now, I show with an example that $\frac{\partial p_1^*}{\partial N_1}$ can also be negative. Suppose $N_1 = 2.6111$, $v_1 = 1$, $v_2 = 10$, and define

$$\widehat{t}(k) = \begin{cases} e^k & \text{if } 3 \leq k \leq \overline{N}_2; \\ e^3(x-2) & \text{if } 2 \leq k \leq 3; \\ 0 & \text{if } 0 \leq k \leq 2. \end{cases}$$

Note that the transport-cost function $\widehat{t}$ does not satisfy all the imposed assumptions in Section 4.1 because it is not strictly increasing nor twice-differentiable. Nevertheless, we can always "perturb" $\widehat{t}$ a little bit to another function t, such that t satisfies all the imposed assumptions in Section 4.1, $t(3) = t'(3) = t''(3) = e^3$, and $\int_0^3 t(k)dk$ is arbitrarily close to $\int_0^3 \widehat{t}(k)dk = \frac{e^3}{2}$. Given this transport-cost function t, by Proposition 1, the equilibrium mass of group-2 participants is $N_2^* = 3$. Hence, by (A9), $\frac{\partial p_1^*}{\partial N_1} = -2.5618 \times 10^{-2} < 0$.

[10] $\frac{\partial p_1^*}{\partial v_1} = \frac{1}{v_2 N_1}\left(\int_0^{N_2^*}(t(N_2^*) - t(k))\,dk + v_1 N_2^* t'(N_2^*)\frac{\partial N_2^*}{\partial v_1}\right) \geq 0.$

The comparative statics of $p_1^* N_1$ is based on that of p_1^*:

[11] $\frac{\partial p_1^* N_1}{\partial N_1} = \frac{v_1}{v_2} N_2^* t'(N_2^*)\frac{\partial N_2^*}{\partial N_1} \geq 0.$

[12] $\frac{\partial p_1^* N_1}{\partial v_1} = N_1 \frac{\partial p_1^*}{\partial v_1} \geq 0.$

π^* is given by Proposition 1, and its comparative statics can be easily obtained by applying the envelope theorem:

[13] $\frac{\partial \pi^*}{\partial N_1} = v_2 N_2^* \geq 0.$

[14] $\frac{\partial \pi^*}{\partial v_1} = \frac{1}{v_2}\int_0^{N_2^*}(t(N_2^*) - t(k))\,dk \geq 0.$

[15] $\frac{\partial \pi^*}{\partial v_2} = \frac{N_2^*}{v_2}\left(1 - \frac{v_1}{v_2}\right)(t(N_2^*) + N_2^* t'(N_2^*)) + \frac{v_1}{v_2^2}\int_0^{N_2^*} t(k)dk \geq 0.$

Table 2b When computing the comparative statics for $v_1 \geq v_2$, two expressions based on re-organizing (17) in Proposition 1 are frequently used:

$$
\begin{aligned}
v_2 N_1 &= t(N_2^*) - t(\underline{N}_2^*); && (33)\\
\frac{v_1}{v_2} &= \frac{t(\underline{N}_2^*) + N_2^* t'(N_2^*)}{(N_2^* - \underline{N}_2^*) t'(N_2^*)}. && (34)
\end{aligned}
$$

Now, I compute the comparative statics in the following order (starting from [16]):

	N_2^*	p_1^*	p_2^*	$p_1^* N_1$	$p_2^* N_2^*$	π^*
N_1	[19]+	[25]+	[16]−	[28]+	[22]−	[31]+
v_1	[20]+	[26]+	[17]−	[29]+	[23]−	[32]+
v_2	[21]+	[27] +/−	[18]+	[30] +/−	[24]+	[33]+

p_2^* is characterized by (17) in Proposition 1, and thus its comparative statics is also based on (17):

[16] $$\frac{\partial p_2^*}{\partial N_1} = \frac{-(v_1-v_2)+(v_1\underline{N}_2^*-(v_1-v_2)N_2^*)\frac{t''(N_2^*)}{t'(N_2^*)}}{2+\frac{v_1}{v_2}\left(\frac{t'(N_2^*)}{t'(\underline{N}_2^*)}-1\right)-\frac{t(\underline{N}_2^*)t''(N_2^*)}{(t'(N_2^*))^2}} \leq 0.^{54}$$

[17] $$\frac{\partial p_2^*}{\partial v_1} = -\frac{\frac{N_2^*-\underline{N}_2^*}{v_2}t'(N_2^*)}{2+\frac{v_1}{v_2}\left(\frac{t'(N_2^*)}{t'(\underline{N}_2^*)}-1\right)-\frac{t(\underline{N}_2^*)t''(N_2^*)}{(t'(N_2^*))^2}} \leq 0.$$

[18] $$\begin{aligned}
\frac{\partial p_2^*}{\partial v_2} &= \frac{\left(\frac{v_1}{v_2^2}(N_2^*-\underline{N}_2^*)-\left(\frac{v_1}{v_2}-1\right)\frac{N_1}{t'(N_2^*)}\right)t'(N_2^*)+\left(\frac{v_1}{v_2^2}\underline{N}_2^*-\left(\frac{v_1}{v_2}-1\right)N_2^*\right)\frac{N_1 t''(N_2^*)}{t'(N_2^*)}}{2+\frac{v_1}{v_2}\left(\frac{t'(N_2^*)}{t'(\underline{N}_2^*)}-1\right)-\frac{t(\underline{N}_2^*)t''(N_2^*)}{(t'(N_2^*))^2}}\\
&= \frac{\frac{v_1}{v_2}\left((N_2^*-\underline{N}_2^*)t'(N_2^*)-v_2N_1\right)+v_2N_1\left(1-\left(\frac{v_1}{v_2}(N_2^*-\underline{N}_2^*)-N_2^*\right)\frac{t''(N_2^*)}{t'(N_2^*)}\right)}{v_2\left(2+\frac{v_1}{v_2}\left(\frac{t'(N_2^*)}{t'(\underline{N}_2^*)}-1\right)-\frac{t(\underline{N}_2^*)t''(N_2^*)}{(t'(N_2^*))^2}\right)}\\
&= \frac{\frac{v_1}{v_2}\left(N_2^*-\underline{N}_2^*\right)\left(t'(N_2^*)-\frac{t(N_2^*)-t(\underline{N}_2^*)}{N_2^*-\underline{N}_2^*}\right)+v_2N_1\left(1-\frac{t(\underline{N}_2^*)t''(N_2^*)}{\left(t'(N_2^*)\right)^2}\right)}{v_2\left(2+\frac{v_1}{v_2}\left(\frac{t'(N_2^*)}{t'(\underline{N}_2^*)}-1\right)-\frac{t(\underline{N}_2^*)t''(N_2^*)}{(t'(N_2^*))^2}\right)} \quad \text{(by (33) and (34))}
\end{aligned}$$

[54] As shown in Appendix A2, the denominator $2 + \frac{v_1}{v_2}\left(\frac{t'(N_2^*)}{t'(\underline{N}_2^*)} - 1\right) - \frac{t(\underline{N}_2^*)t''(N_2^*)}{(t'(N_2^*))^2}$ is positive.

$$= N_1 \frac{1+\frac{v_1}{v_2}\left(\frac{t'(N_2^*)}{\frac{t(N_2^*)-t(\underline{N}_2^*)}{N_2^*-\underline{N}_2^*}}-1\right)-\frac{t(\underline{N}_2^*)t''(N_2^*)}{\left(t'(N_2^*)\right)^2}}{2+\frac{v_1}{v_2}\left(\frac{t'(N_2^*)}{t'(\underline{N}_2^*)}-1\right)-\frac{t(\underline{N}_2^*)t''(N_2^*)}{(t'(N_2^*))^2}} \text{ (by (33))}$$

$$\geq \frac{N_1\left(1-\frac{t(\underline{N}_2^*)}{t(N_2^*)}\right)}{2+\frac{v_1}{v_2}\left(\frac{t'(N_2^*)}{t'(\underline{N}_2^*)}-1\right)-\frac{t(\underline{N}_2^*)t''(N_2^*)}{(t'(N_2^*))^2}} \geq 0. \text{ (t is convex and log-concave)}$$

As shown in (11), N_2^* can be expressed as a function of p_2^*. Hence, its comparative statics is based on that of p_2^*:

[19] $\frac{\partial N_2^*}{\partial N_1} = \frac{1}{t'(N_2^*)}\left(v_2 - \frac{\partial p_2^*}{\partial N_1}\right) \geq 0.$

[20] $\frac{\partial N_2^*}{\partial v_1} = -\frac{1}{t'(N_2^*)}\frac{\partial p_2^*}{\partial v_1} \geq 0.$

[21] $\frac{\partial N_2^*}{\partial v_2} = \frac{1}{t'(N_2^*)}\left(N_1 - \frac{\partial p_2^*}{\partial v_2}\right)$

$$= \frac{N_1}{t'(N_2^*)} \frac{2+\frac{v_1}{v_2}\left(\frac{t'(N_2^*)}{t'(\underline{N}_2^*)}-1\right)-\frac{t(\underline{N}_2^*)t''(N_2^*)}{(t'(N_2^*))^2}-\left(1+\frac{v_1}{v_2}\left(\frac{t'(N_2^*)}{\frac{t(N_2^*)-t(\underline{N}_2^*)}{N_2^*-\underline{N}_2^*}}-1\right)-\frac{t(\underline{N}_2^*)t''(N_2^*)}{\left(t'(N_2^*)\right)^2}\right)}{2+\frac{v_1}{v_2}\left(\frac{t'(N_2^*)}{t'(\underline{N}_2^*)}-1\right)-\frac{t(\underline{N}_2^*)t''(N_2^*)}{(t'(N_2^*))^2}}$$

$$= \frac{N_1}{t'(N_2^*)} \frac{1+\frac{v_1}{v_2}t'(N_2^*)\left(\frac{1}{t'(\underline{N}_2^*)}-\frac{1}{\frac{t(N_2^*)-t(\underline{N}_2^*)}{N_2^*-\underline{N}_2^*}}\right)}{2+\frac{v_1}{v_2}\left(\frac{t'(N_2^*)}{t'(\underline{N}_2^*)}-1\right)-\frac{t(\underline{N}_2^*)t''(N_2^*)}{(t'(N_2^*))^2}} \geq 0.$$

The comparative statics of $p_2^* N_2^*$ is based on those of p_2^* and N_2^*:

[22] $\frac{\partial p_2^* N_2^*}{\partial N_1} = \frac{\partial p_2^*}{\partial N_1}N_2^* + p_2^*\frac{\partial N_2^*}{\partial N_1} \leq 0.$

[23] $\frac{\partial p_2^* N_2^*}{\partial v_1} = \frac{\partial p_2^*}{\partial v_1}N_2^* + p_2^*\frac{\partial N_2^*}{\partial v_1} \leq 0.$

[24] $\frac{\partial p_2^* N_2^*}{\partial v_2} = \left(N_2^* - \frac{p_2^*}{t'(N_2^*)}\right)\frac{\partial p_2^*}{\partial v_2} + \frac{p_2^* N_1}{t'(N_2^*)}$

$$= \left(\frac{t(\underline{N}_2^*)+N_2^* t'(N_2^*)}{t'(N_2^*)}\right) N_1 \frac{1+\frac{v_1}{v_2}\left(\frac{t'(N_2^*)}{\frac{t(N_2^*)-t(\underline{N}_2^*)}{N_2^*-\underline{N}_2^*}}-1\right)-\frac{t(\underline{N}_2^*)t''(N_2^*)}{\left(t'(N_2^*)\right)^2}}{2+\frac{v_1}{v_2}\left(\frac{t'(N_2^*)}{t'(\underline{N}_2^*)}-1\right)-\frac{t(\underline{N}_2^*)t''(N_2^*)}{(t'(N_2^*))^2}} - \frac{t(\underline{N}_2^*)N_1}{t'(N_2^*)} \text{ (by (12))}$$

$$= \frac{N_1 t(\underline{N}_2^*)}{t'(N_2^*)} \frac{-1+\frac{v_1}{v_2}t'(N_2^*)\left(\left(1+\frac{N_2^* t'(N_2^*)}{t(\underline{N}_2^*)}\right)\frac{1}{\frac{t(N_2^*)-t(\underline{N}_2^*)}{N_2^*-\underline{N}_2^*}}-\frac{1}{t'(\underline{N}_2^*)}\right)-\frac{N_2^* t''(N_2^*)}{t'(N_2^*)}}{2+\frac{v_1}{v_2}\left(\frac{t'(N_2^*)}{t'(\underline{N}_2^*)}-1\right)-\frac{t(\underline{N}_2^*)t''(N_2^*)}{(t'(N_2^*))^2}}$$

$$\geq \frac{N_1 t(\underline{N}_2^*)}{t'(N_2^*)} \frac{-1+\frac{v_1}{v_2} t'(N_2^*)\left(\frac{1}{t'(N_2^*)}+\frac{N_2^*}{t(\underline{N}_2^*)}-\frac{\underline{N}_2^*}{t(\underline{N}_2^*)}\right)-\frac{N_2^* t'(N_2^*)}{t(N_2^*)}}{2+\frac{v_1}{v_2}\left(\frac{t'(N_2^*)}{t'(\underline{N}_2^*)}-1\right)-\frac{t(\underline{N}_2^*)t''(N_2^*)}{(t'(N_2^*))^2}} \text{ (t is convex)}$$

$$\geq \frac{N_1 t(\underline{N}_2^*)}{t'(N_2^*)} \frac{\frac{t(\underline{N}_2^*)+N_2^* t'(N_2^*)}{(N_2^*-\underline{N}_2^*)t'(N_2^*)} \frac{t'(N_2^*)}{t(\underline{N}_2^*)}(N_2^*-\underline{N}_2^*)-\frac{N_2^* t'(N_2^*)}{t(N_2^*)}}{2+\frac{v_1}{v_2}\left(\frac{t'(N_2^*)}{t'(\underline{N}_2^*)}-1\right)-\frac{t(\underline{N}_2^*)t''(N_2^*)}{(t'(N_2^*))^2}} \text{ (by (34) and } v_1 \geq v_2)$$

$$= \frac{N_1 t(\underline{N}_2^*)}{t'(N_2^*)} \frac{1+\frac{N_2^* t'(N_2^*)}{t(\underline{N}_2^*)}-\frac{N_2^* t'(N_2^*)}{t(N_2^*)}}{2+\frac{v_1}{v_2}\left(\frac{t'(N_2^*)}{t'(\underline{N}_2^*)}-1\right)-\frac{t(\underline{N}_2^*)t''(N_2^*)}{(t'(N_2^*))^2}} \geq 0. \text{ (t is increasing)}$$

As shown in Proposition 1, p_1^* is a function of N_2^*. Hence, its comparative statics is based on that of N_2^*:

[25] $\frac{\partial p_1^*}{\partial N_1} = v_1 \frac{\partial \underline{N}_2^*}{\partial N_1} - \frac{v_1}{v_2 N_1}(t(N_2^*) - t(\underline{N}_2^*))\frac{\partial \underline{N}_2^*}{\partial N_1}$
$+\frac{v_1}{v_2 N_1}(N_2^* - \underline{N}_2^*)t'(N_2^*)\frac{\partial N_2^*}{\partial N_1} - \frac{v_1}{v_2 N_1^2}\int_{\underline{N}_2^*}^{N_2^*}(t(N_2^*) - t(k))dk$
$= \frac{v_1}{v_2 N_1}(N_2^* - \underline{N}_2^*)\left(v_2 - \frac{\partial p_2^*}{\partial N_1}\right) - \frac{v_1}{v_2 N_1^2}\int_{\underline{N}_2^*}^{N_2^*}(t(N_2^*) - t(k))dk$ (by (33))
$\geq \frac{v_1}{v_2 N_1^2}\left((t(N_2^*) - t(\underline{N}_2^*))(N_2^* - \underline{N}_2^*) - \int_{\underline{N}_2^*}^{N_2^*}(t(N_2^*) - t(k))dk\right)$ (by (33) and $\frac{\partial p_2^*}{\partial N_1} \leq 0$)
$= \frac{v_1}{v_2 N_1}\int_{\underline{N}_2^*}^{N_2^*}(t(k) - t(\underline{N}_2^*))dk \geq 0.$

[26] $\frac{\partial p_1^*}{\partial v_1} = \underline{N}_2^* + v_1 \frac{\partial \underline{N}_2^*}{\partial v_1} + \frac{1}{v_2 N_1}\int_{\underline{N}_2^*}^{N_2^*}(t(N_2^*) - t(k))dk$
$-\frac{v_1}{v_2 N_1}(t(N_2^*) - t(\underline{N}_2^*))\frac{\partial \underline{N}_2^*}{\partial v_1} + \frac{v_1}{v_2 N_1}(N_2^* - \underline{N}_2^*)t'(N_2^*)\frac{\partial N_2^*}{\partial v_1}$
$= \underline{N}_2^* + \frac{1}{v_2 N_1}\int_{\underline{N}_2^*}^{N_2^*}(t(N_2^*) - t(k))dk + \frac{v_1}{v_2 N_1}(N_2^* - \underline{N}_2^*)t'(N_2^*)\frac{\partial N_2^*}{\partial v_1} \geq 0.$

[27] $\frac{\partial p_1^*}{\partial v_2}$ can be positive as shown in Figures 3 and 5 of Appendix F. Now, I show with an example that $\frac{\partial p_1^*}{\partial v_2}$ can also be negative. Suppose $N_1 = v_1 = 1$, and define

$$\widehat{t}(k) = \begin{cases} 2k - \frac{1}{2} & \text{if } \frac{1}{2} \leq k \leq \overline{N}_2; \\ k & \text{if } 0 \leq k \leq \frac{1}{2}. \end{cases}$$

Note that the transport-cost function $\widehat{t}$ does not satisfy all the imposed assumptions in Section 4.1 because it is not twice-differentiable. Nevertheless, we can always "perturb" $\widehat{t}$ a little bit to a smooth function t, such that t satisfies all the imposed assumptions in Section 4.1, and $t(k) = \widehat{t}(k)$ for all $k \in [0, \overline{N}_2]$ except for those which are arbitrarily close to $\frac{1}{2}$. Given this transport-cost function t, by (17), we have

$$p_2^* = \frac{v_2}{2} - \frac{1}{2}; \quad N_2^* = \frac{1}{2} + \frac{v_2}{4}; \quad \underline{N}_2^* = \frac{1}{2} - \frac{v_2}{2}.$$

Hence, by Proposition 1, the platform's optimal group-1 price is $p_1^* = \frac{1}{2} - \frac{v_2}{16}$, which is decreasing in v_2.

The comparative statics of $p_1^* N_1$ is based on that of p_1^*:

[28] $\frac{\partial p_1^* N_1}{\partial N_1} = \frac{\partial p_1^*}{\partial N_1} + p_1^* \geq 0.$

[29] $\frac{\partial p_1^* N_1}{\partial v_1} = N_1 \frac{\partial p_1^*}{\partial v_1} \geq 0.$

[30] $\frac{\partial p_1^* N_1}{\partial v_2} = N_1 \frac{\partial p_1^*}{\partial v_2}$, the sign is ambiguous.

π^* is given by Proposition 1, and its comparative statics can be easily obtained by applying the envelope theorem:

[31] $\frac{\partial \pi^*}{\partial N_1} = v_1 N_2^* + \frac{v_2 p_2^*}{t'(N_2^*)} = v_1 \underline{N}_2^* + v_2 N_2^* \geq 0.$

[32] $\frac{\partial \pi^*}{\partial v_1} = N_1 \underline{N}_2^* + \frac{1}{v_2} \int_{\underline{N}_2^*}^{N_2^*} (t(N_2^*) - t(k))\, dk \geq 0.$

To show $\frac{\partial \pi^*}{\partial v_2} \geq 0$, first I prove that $p_2^* \geq -\frac{v_1 N_1}{2}$ when $v_2 \to 0$. As shown in (17),

$$v_2 p_2^* = \left(v_1 t^{-1}(-p_2^*) - (v_1 - v_2)\, t^{-1}(v_2 N_1 - p_2^*)\right) t'(t^{-1}(v_2 N_1 - p_2^*)).$$

Because both sides converge to zero when $v_2 \to 0$, by L'Hospital's rule, we can differentiate both sides of the above equation with respect to v_2 and obtain

$$p_2^* = \left(N_2^* - \frac{(v_1 - v_2)}{t'(N_2^*)} N_1\right) t'(N_2^*) + (v_1 \underline{N}_2^* - (v_1 - v_2)\, N_2)\, \frac{t''(N_2^*)}{t'(N_2^*)} N_1).$$

As $v_2 \to 0$, both $\underline{N}_2^*$ and N_2^* converge to $t^{-1}(-p_2^*)$. The above equation becomes

$$v_1 N_1 = t(N_2^*) + N_2^* t'(N_2^*) \geq 2t(N_2^*) = -2p_2^*. \quad (t \text{ is convex})$$

Therefore, we have $p_2^* \geq -\frac{v_1 N_1}{2}$ as $v_2 \to 0$. By [18] (i.e., $\frac{\partial p_2^*}{\partial v_2} \geq 0$ when $v_1 \geq v_2$), we have $p_2^* \geq -\frac{v_1 N_1}{2}$ for all $v_1 \geq v_2$. Now, I show that $\frac{\partial \pi^*}{\partial v_2} \geq 0$.

[33] $\frac{\partial \pi^*}{\partial v_2} = \frac{v_1}{v_2^2} \int_{\underline{N}_2^*}^{N_2^*} (t(k) - t(\underline{N}_2^*))\, dk + \frac{p_2^* N_1}{t'(N_2^*)}$
$\geq \frac{v_1}{v_2^2} \left(\int_{\underline{N}_2^*}^{N_2^*} (t(k) - t(\underline{N}_2^*))\, dk + \frac{(t(N_2^*) - t(\underline{N}_2^*))^2}{2t'(N_2^*)} \right)$ (by (33))
$\geq \frac{v_1}{v_2^2} \left(\frac{(t(N_2^*) - t(\underline{N}_2^*))^2}{2t'(N_2^*)} - \frac{(t(N_2^*) - t(\underline{N}_2^*))^2}{2t'(N_2^*)} \right) = 0.$ (t is convex)

A4. Corollary 1

Note that the cutoff for $\frac{v_1}{v_2}$ in Corollary 1 is less than one because t is convex (which implies $\frac{t(N_2^*)}{2} \geq \frac{1}{N_2^*}\int_0^{N_2^*} t(k)dk$). Therefore, first I show that Corollary 1 is true when $v_1 \leq v_2$, and then I show that $\frac{\partial \pi^*}{\partial v_1} \geq \frac{\partial \pi^*}{\partial v_2}$ when $v_1 \geq v_2$. When $v_1 \leq v_2$, $\frac{\partial \pi^*}{\partial v_1}$ and $\frac{\partial \pi^*}{\partial v_2}$ are given by [14] and [15] in Appendix A3 respectively, i.e.,

$$\begin{aligned}
\frac{\partial \pi^*}{\partial v_2} - \frac{\partial \pi^*}{\partial v_1} &= \frac{N_2^*}{v_2}\left(1-\frac{v_1}{v_2}\right)\left(t(N_2^*)+N_2^* t'(N_2^*)\right) + \frac{v_1}{v_2^2}\int_0^{N_2^*} t(k)dk - \frac{1}{v_2}\int_0^{N_2^*}\left(t(N_2^*)-t(k)\right)dk \\
&= \frac{1}{v_2}\left(N_2^* t'(N_2^*) + \frac{1}{N_2^*}\int_0^{N_2^*} t(k)dk - \frac{v_1}{v_2}\left(t(N_2^*)+N_2^* t'(N_2^*) - \frac{1}{N_2^*}\int_0^{N_2^*} t(k)dk\right)\right).
\end{aligned}$$

Hence, Corollary 1 is true when $v_1 \leq v_2$ as

$$\frac{\partial \pi^*}{\partial v_1} \leq \frac{\partial \pi^*}{\partial v_2} \quad \text{if and only if} \quad \frac{v_1}{v_2} \leq \frac{N_2^* t'(N_2^*) + \frac{1}{N_2^*}\int_0^{N_2^*} t(k)dk}{N_2^* t'(N_2^*) + t(N_2^*) - \frac{1}{N_2^*}\int_0^{N_2^*} t(k)dk}.$$

On the other hand, when $v_1 \geq v_2$, $\frac{\partial \pi^*}{\partial v_1}$ and $\frac{\partial \pi^*}{\partial v_2}$ are given by [32] and [33] in Appendix A3 respectively, i.e.,

$$\begin{aligned}
\frac{\partial \pi^*}{\partial v_1} - \frac{\partial \pi^*}{\partial v_2} &= N_1 \underline{N}_2^* + \frac{1}{v_2}\int_{\underline{N}_2^*}^{N_2^*}\left(t(N_2^*)-t(k)\right)dk - \frac{v_1}{v_2^2}\int_{\underline{N}_2^*}^{N_2^*}\left(t(k)-t(\underline{N}_2^*)\right)dk + \frac{p_2^* N_1}{t'(N_2^*)} \\
&= \frac{1}{v_2}\begin{pmatrix} v_2 N_1 \left(\frac{t(\underline{N}_2^*)+\underline{N}_2^* t'(N_2^*)}{t'(N_2^*)}\right) \\ + \int_{\underline{N}_2^*}^{N_2^*}\left(t(N_2^*)-t(k)\right)dk - \frac{v_1}{v_2}\int_{\underline{N}_2^*}^{N_2^*}\left(t(k)-t(\underline{N}_2^*)\right)dk \end{pmatrix} \quad \text{(by (16))} \\
&= \frac{1}{v_2}\begin{pmatrix} \left(t(N_2^*)-t(\underline{N}_2^*)\right)\left(\frac{t(\underline{N}_2^*)+N_2^* t'(N_2^*)}{t'(N_2^*)} - (N_2^* - \underline{N}_2^*)\right) \\ + \int_{\underline{N}_2^*}^{N_2^*}\left(t(N_2^*)-t(k)\right)dk - \frac{v_1}{v_2}\int_{\underline{N}_2^*}^{N_2^*}\left(t(k)-t(\underline{N}_2^*)\right)dk \end{pmatrix} \quad \text{(by (33))} \\
&= \frac{1}{v_2}\begin{pmatrix} \left(\frac{v_1}{v_2}-1\right)(N_2^* - \underline{N}_2^*)(t(N_2^*)-t(\underline{N}_2^*)) \\ + \int_{\underline{N}_2^*}^{N_2^*}\left(t(N_2^*)-t(k)\right)dk - \frac{v_1}{v_2}\int_{\underline{N}_2^*}^{N_2^*}\left(t(k)-t(\underline{N}_2^*)\right)dk \end{pmatrix} \quad \text{(by (34))} \\
&= \frac{1}{v_2}\left(\frac{v_1}{v_2}\int_{\underline{N}_2^*}^{N_2^*}\left(t(N_2^*)-t(k)\right)dk - \int_{\underline{N}_2^*}^{N_2^*}\left(t(k)-t(\underline{N}_2^*)\right)dk\right) \\
&\geq \frac{1}{v_2}\left(\int_{\underline{N}_2^*}^{N_2^*}\left(t(N_2^*)-t(k)\right)dk - \int_{\underline{N}_2^*}^{N_2^*}\left(t(k)-t(\underline{N}_2^*)\right)dk\right) \geq 0. \quad (t \text{ is convex})
\end{aligned}$$

A5. Lemma 5

For a group-1 agent, suppose there are n_1^A group-1 agents (excluding himself) and n_2^A group-2 agents joining platform A, the payoff difference between joining platform A or B for the group-1 agent is

$$\begin{aligned}
& u_1^A(n_2^A, p_1^A) - u_1^B(n_2^B, p_1^B) \\
= \; & v_1^A n_2^A - p_1^A - v_1^B n_2^B + p_1^B \quad \text{(by (18))} \\
= \; & (v_1^A + v_1^B) n_2^A - v_1^B N_2 - \Delta p_1.
\end{aligned}$$

The corresponding difference in the potentials is

$$\begin{aligned}
& P(n_1^A + 1, n_2^A | \Delta p_1, \Delta p_2) - P(n_1^A, n_2^A | \Delta p_1, \Delta p_2) \\
= \; & n_2^A - \frac{v_1^B N_2 + \Delta p_1}{v_1^A + v_1^B} \quad \text{(by (20))} \\
= \; & \frac{1}{v_1^A + v_1^B}(u_1^A(n_2^A, p_1^A) - u_1^B(n_2^B, p_1^B)).
\end{aligned}$$

Thus, the change in the group-1 agent's payoff from unilaterally switching actions is proportional (with proportion $v_1^A + v_1^B$) to the change in the potential. The same logic applies to a group-2 agent (with proportion $v_2^A + v_2^B$ for him).

B. The mathematical definition of weighted potential games

The definition can be found in Monderer and Shapley's (1996, p. 127–128) paper. Let $N = \{1, \ldots, n\}$ be the set of players, A_i the set of actions for player i, and $u_i : A \to \mathbb{R}$ the payoff function for player i, where $A \equiv A_1 \times \cdots \times A_n$. A game $G = (N, A, (u_i)_{i \in N})$ is a weighted potential game if there is a function $P : A \to \mathbb{R}$ and a vector $(w_i)_{i \in N} \in \mathbb{R}^n_{++}$ such that, for every $i \in N$ and $a_{-i} \in A_{-i}$,

$$u_i(a_i, a_{-i}) - u_i(a_i', a_{-i}) = w_i(P(a_i, a_{-i}) - P(a_i', a_{-i})), \quad \forall a_i, a_i' \in A_i.$$

C. The finite-agent model of Section 4.1 and the proof of Lemma 3

In this appendix, first I present a finite-agent model that converges to the continuum-agent model in Section 4.1 in the limit. Then, I show that this finite-agent model is a weighted potential game with the potential converging to (13) in Lemma 3. The subsequent analysis for the limiting case of the finite-agent model is identical to that of the continuum-agent model in Section 4.2 and thus omitted.

In the finite-agent model, there are $\gamma N_1 \in \mathbb{N}$ ($\gamma, N_1 \in \mathbb{R}_{++}$) identical group-1 agents and $\gamma \overline{N}_2 \in \mathbb{N}$ ($N_2 \in \mathbb{R}_{++}$) heterogeneous group-2 agents. If the platform attracts $n_1 \equiv \sum_{k=1}^{\gamma N_1} a_1^k$ group-1 agents and $n_2 \equiv \sum_{k=1}^{\gamma \overline{N}_2} a_2^k$ group-2 agents, the payoffs of a group-1 agent and agent k (agent k always refers to a group-2 agent in this appendix) from joining the platform are

$$u_1(n_2, p_1; \gamma) = \frac{v_1}{\gamma} n_2 - p_1; \quad u_2^k(n_1, p_2; \gamma) = \frac{v_2}{\gamma} n_1 - p_2 - t\left(\frac{k}{\gamma}\right), \tag{35}$$

where the function $t : \{\frac{1}{\gamma}, \frac{2}{\gamma}, \ldots, \overline{N}_2\} \to \mathbb{R}_+$ specifies each group-2 agent's transport cost from joining the platform.[55]

In this finite-agent model, the platform's profit is

$$\pi(n_1, n_2, p_1, p_2; \gamma) = \frac{1}{\gamma}(p_1 n_1 + p_2 n_2).$$

The rest of the model setup is the same as that of the baseline model.

Clearly, this finite-agent model converges to the continuum-agent model in Section 4.1 as $\gamma \to \infty$. Hence, it remains to show that this finite-agent model is a weighted potential game with the potential converging to (13) in Lemma 3 as $\gamma \to \infty$. Now, I prove that every subgame in stage 2 is a weighted potential game with the potential

$$P(n_1, \mathbf{a}_2 | p_1, p_2; \gamma) = \frac{1}{\gamma^2}\left(n_1 n_2 - \frac{\gamma p_1}{v_1} n_1 - \frac{\gamma p_2}{v_2} n_2 - \frac{\gamma}{v_2} \sum_{k=1}^{\gamma \overline{N}_2} t\left(\frac{k}{\gamma}\right) a_2^k\right). \tag{36}$$

Proof. For a group-1 agent, suppose there are n_1 group-1 participants (excluding himself) and the group-2 agents' action profile is $\mathbf{a}_2$, the payoff difference between joining the platform or not for the agent is

$$u_1(n_2, p_1; \gamma) - 0 = \frac{v_1}{\gamma} n_2 - p_1. \quad \text{(by (35))}$$

[55] Section 4.1 imposes some assumptions on the function t. Nevertheless, these assumptions play no role in this appendix and thus omitted.

The corresponding difference in the potentials is

$$
\begin{aligned}
& P(n_1+1, \mathbf{a}_2|p_1,p_2;\gamma) - P(n_1, \mathbf{a}_2|p_1,p_2;\gamma) \\
= \quad & \frac{1}{\gamma^2}\left(n_2 - \frac{\gamma p_1}{v_1}\right) \quad \text{(by (36))} \\
= \quad & \frac{1}{\gamma v_1}(u_1(n_2,p_1;\gamma) - 0).
\end{aligned}
$$

Thus, the change in the group-1 agent's payoff from unilaterally switching actions is proportional (with proportion γv_1) to the change in the potential.

Similarly, for agent k, suppose there are n_1 group-1 participants and the group-2 agents' action profile (except himself) is $\mathbf{a}_2^{-k}$, the payoff difference between joining the platform or not for agent k is

$$
u_2^k(n_1,p_2;\gamma) - 0 = \frac{v_2}{\gamma}n_1 - p_2 - t\left(\frac{k}{\gamma}\right). \quad \text{(by (35))}
$$

The corresponding difference in the potentials is

$$
\begin{aligned}
& P(n_1, \mathbf{a}_2^{-k}, a_2^k = 1|p_1,p_2;\gamma) - P(n_1, \mathbf{a}_2^{-k}, a_2^k = 0|p_1,p_2;\gamma) \\
= \quad & \frac{1}{\gamma^2}\left(n_1 - \frac{\gamma p_2}{v_2} - \frac{\gamma}{v_2}t\left(\frac{k}{\gamma}\right)\right) \quad \text{(by (36))} \\
= \quad & \frac{1}{\gamma v_2}(u_2^k(n_1,p_2;\gamma) - 0).
\end{aligned}
$$

Thus, the change in agent k's payoff from unilaterally switching actions is proportional (with proportion γv_2) to the change in the potential. ■

Clearly, (36) converges to (13) in Lemma 3 as $\gamma \to \infty$. This completes the proof of Lemma 3. The subsequent analysis of the finite-agent model as $\gamma \to \infty$ is identical to that of the continuum-agent model in Section 4.2 and thus omitted.

D. Monopoly platform: homogeneous agents with same-side externalities

This appendix first shows that the homogeneous-agent model with negative same-side externalities on one side is equivalent to the model in Section 4.1. By the same token, the model with negative same-side externalities on both sides is equivalent to Armstrong's (2006) original model. After that, I analyze the homogeneous-agent model with positive same-side externalities on one side. Extending the analysis to positive same-side externalities on both sides is straightforward, and thus omitted. Likewise, the analysis of the model with positive same-side externalities on one side and negative same-side externalities on the other side is very similar to that with negative same-side externalities on one side, and thus omitted.

Negative same-side externalities

Consider a model which is identical to the model in Section 4.1, except the payoff of a group-2 agent from joining the platform is now

$$u_2(n_1, n_2, p_2) = v_2 n_1 - p_2 - t(n_2), \tag{37}$$

where $t : [0, \overline{N}_2] \to \mathbb{R}_+$ is an increasing function that measures the *congestion cost* each group-2 participant suffers.

Similar to the model in Section 4.1, there is a demand effect on group 2 owing to the congestion cost. More precisely, for any group-2 price $p_2 \leq v_2 N_1$ set by the platform in stage 1, there is at most a mass of N_2 group-2 participants in the equilibrium, where N_2 is given by (11). Similarly, if p_2 is negative, there is at least a mass of $\underline{N}_2$ group-2 participants in the equilibrium, where $\underline{N}_2$ is given by (12). Hence, it is clear that the set of equilibria in this model is closely related to that of the model in Section 4.1. More precisely, there are two sets of stable equilibria and a set of unstable equilibria in stage 2 under *Case 1* or *Case 2* (as defined in p. 20 of Section 4.2), and the masses of participants (n_1^*, n_2^*) for each set of equilibria are characterized by Figures 1 and 2 in Section 4.2 respectively. For Case 1, the masses of participants of the three sets of equilibria are:

1. Pareto-dominant equilibrium: $(n_1^*, n_2^*) = (N_1, N_2)$;
2. unstable equilibrium: $(n_1^*, n_2^*) = (\widehat{N}_1, \widehat{N}_2)$;
3. Pareto-dominated equilibrium: $(n_1^*, n_2^*) = (0, 0)$.

For Case 2, the masses of participants of the three sets of equilibria are:

1. Pareto-dominant equilibrium: $(n_1^*, n_2^*) = (N_1, N_2)$;

2. unstable equilibrium: $(n_1^*, n_2^*) = (\widehat{N}_1, \widehat{N}_2)$;

3. Pareto-dominated equilibrium: $(n_1^*, n_2^*) = (0, \underline{N}_2)$.

From the platform's point of view, the set of equilibria in the current model is equivalent to that of the model in Section 4.1: ultimately it only cares about the masses of participants, but not the identity nor the utility of each participant. Now, I show that the potential maximizers of these two models are also equivalent. First, I prove that every subgame in stage 2 of the current model is a weighted potential game.

Lemma 8 *Every subgame in stage 2 is a weighted potential game with the potential*

$$P(n_1, n_2|p_1, p_2) = n_1 n_2 - \frac{p_1}{v_1} n_1 - \frac{p_2}{v_2} n_2 - \frac{1}{v_2} \int_0^{n_2} t(k) dk. \tag{38}$$

Proof. We can follow the same approach in proving Lemma 3 in Section 4.2 (i.e., Appendix C) to prove this lemma. Alternatively, we can directly prove that every subgame in stage 2 is a continuum-agent weighted potential game as defined by Sandholm (2001, p. 85).[56]

Similar to the definition of a finite-agent weighted potential game (i.e., Definition 1), a continuum-agent game is a continuum-agent weighted potential game if there exists a function P defined on the strategy space of the game, such that the change in any player's payoff from unilaterally switching actions is (positively) proportional to the corresponding differential change in P.

Now, I prove that the current model is a continuum-agent weighted potential game. Suppose there are a mass of n_1 group-1 participants and a mass of n_2 group-2 participants.

[56] Sandholm (2001) only defines continuum-agent exact potential games, but the definition and the results can be naturally extended to continuum-agent weighted potential games. As proved by Sandholm (2001, Theorem 6.1), continuum-agent potential games are the limits of convergent sequences of the finite-agent potential games.

The payoff difference between joining the platform or not for a group-1 agent is

$$u_1(n_2, p_1) - 0 = v_1 n_2 - p_1. \quad \text{(by (1))}$$

The corresponding differential change in the potential is

$$\begin{aligned} \frac{\partial P(n_1, n_2 | p_1, p_2)}{\partial n_1} &= n_2 - \frac{p_1}{v_1} \quad \text{(by (38))} \\ &= \frac{1}{v_1} \left(u_1(n_2, p_1) - 0 \right). \end{aligned}$$

Similarly, the payoff difference between joining the platform or not for a group-2 agent is

$$u_2(n_1, n_2, p_2) - 0 = v_2 n_1 - p_2 - t(n_2). \quad \text{(by (37))}$$

The corresponding differential change in the potential is

$$\begin{aligned} \frac{\partial P(n_1, n_2 | p_1, p_2)}{\partial n_2} &= n_2 - \frac{p_2}{v_2} - \frac{t(n_2)}{v_2} \quad \text{(by (38))} \\ &= \frac{1}{v_2} \left(u_2(n_1, n_2, p_2) - 0 \right). \end{aligned}$$

Thus, the change in a group-i agent's payoff from unilaterally switching actions is proportional (with proportion v_i) to the differential change in the potential. ■

Compared to Lemma 3 in Section 4.2, the term $\frac{1}{v_2} \int_0^{\overline{N}_2} t(k) a_2^k dk$ is now replaced by $\frac{1}{v_2} \int_0^{n_2} t(k) dk$ in Lemma 8. Nevertheless, it is apparent that the potentials of the three sets of equilibria are equal to the respective potentials of the three equilibria in Section 4.2. In other words, the potential maximizers of these two models are equivalent from the platform's point of view, and thus the subsequent analysis and the results are also identical. This implies a surprising result: there is no distinction between the agents' transport costs and congestion costs from the platform's point of view. By contrast, group-2 participants have more surplus in the equilibrium for the former because their aggregate transport cost is $\int_0^{N_2} t(k) dk$, while each of them suffers a congestion cost of $t(N_2)$ (and thus their aggregate congestion cost is $N_2 t(N_2)$) for the latter.

Positive same-side externalities

Consider a model which is identical to the previous model with negative same-side externalities, except the payoff of a group-2 agent from joining the platform is now

$$u_2(n_1, n_2, p_2) = v_2 n_1 - p_2 + t(n_2),$$

where $t : [0, N_2] \to \mathbb{R}_+$ is an increasing function that measures the positive same-side externalities each group-2 participant enjoys.[57]

Unlike the previous model with negative same-side externalities, there is no demand effect on group 2 when the same-side externalities are positive. Thus, the set of multiple equilibria of the current model is similar to that of the baseline model: there are two (stable) equilibria in stage 2 when $(p_1, p_2) \in [0, v_1 N_2] \times [0, v_2 N_1 + t(N_2)]$,[58] and they are:

1. all agents joining the platform;

2. no one joining the platform.

By Lemma 8 (with the sign of t reversed) of the previous model with negative same-side externalities, every subgame in stage 2 of the current model is a weighted potential game with the potential

$$P(n_1, n_2 | p_1, p_2) = n_1 n_2 - \frac{p_1}{v_1} n_1 - \frac{p_2}{v_2} n_2 + \frac{1}{v_2} \int_0^{n_2} t(k) dk.$$

Following the same approach in proving Lemma 2 in Section 3.2, we can easily identify the potential maximizer when $(p_1, p_2) \in [0, v_1 N_2] \times [0, v_2 N_1 + t(N_2)]$, and it is

$$\begin{array}{ll} \text{all agents joining the platform} & \text{if } \dfrac{p_1}{v_1 N_2} + \dfrac{p_2}{v_2 N_1} \leq 1 + \dfrac{1}{v_2 N_1 N_2} \displaystyle\int_0^{N_2} t(k) dk; \\ \text{no one joining the platform} & \text{otherwise.} \end{array} \quad (39)$$

[57] In this model, the mass of group-2 agents is N_2 (but not $\overline{N}_2$), and I do not impose any additional assumptions on the function t.

[58] Multiple equilibria also exist when $(p_1, p_2) \in (-\infty, 0) \times [v_2 N_1, v_2 N_1 + t(N_2)] \cup (v_1 N_2, \infty) \times [0, t(N_2)]$, but clearly the platform will not set these prices in the equilibrium. There are also unstable equilibria when $(p_1, p_2) \in [0, v_1 N_2] \times [0, v_2 N_1 + t(N_2)]$, but these equilibria are never the potential maximizer by the same reason as before, and thus ignored.

Compared to Lemma 2, the platform can charge the participants more under potential-maximizer selection because of the additional positive same-side externalities among group-2 participants. From (10) and (39), the platform's profit-maximization problem in stage 1 becomes

$$\max_{(p_1,p_2)\in[0,v_1N_2]\times[0,v_2N_1+t(N_2)]} p_1N_1+p_2N_2 \quad \text{s.t.} \quad \frac{p_1}{v_1N_2}+\frac{p_2}{v_2N_1}\leq 1+\frac{1}{v_2N_1N_2}\int_0^{N_2} t(k)dk.$$

Solving the above optimization problem gives us the platform's optimal pricing strategy: when $v_1 < v_2$, it is

$$p_1^* = 0; \quad p_2^* = v_2N_1 + \frac{1}{N_2}\int_0^{N_2} t(k)dk;$$

when $v_1 > v_2$, it is

$$p_1^* = v_1N_2; \quad p_2^* = \frac{1}{N_2}\int_0^{N_2} t(k)dk.$$

As we can see, the additional positive same-side externalities do not affect the platform's money/subsidy side. Compared to the baseline model, the platform marks up the group-2 price by $\frac{1}{N_2}\int_0^{N_2} t(k)dk$ because of this additional positive same-side externalities they enjoy. Still, group-2 agents have more surplus in the equilibrium under the current framework because the additional positive same-side externalities $t(N_2)$ each group-2 agent enjoys is higher than the price markup $\frac{1}{N_2}\int_0^{N_2} t(k)dk$.

E. Two benchmarks for Armstrong's model with heterogeneous agents on one side

This appendix first analyzes the model in Section 4.1 under Pareto-dominance selection, then analyzes it under Pareto-dominated selection. As we will see, the equilibrium outcomes under both benchmarks differ from that under potential-maximizer selection.

Benchmark 1: Pareto-dominance selection

Under this selection criterion, all relevant agents always coordinate on the Pareto-dominant equilibrium whenever there are multiple equilibria. Given that group-1 agents are identical, the platform charges the maximum group-1 price in stage 1, such that all group-1 agents will join the platform with zero surplus in stage 2, i.e., $p_1^* = v_1 N_2$.

Now, it remains to derive the platform's optimal group-2 price p_2^*. From (10), the platform's profit-maximization problem in stage 1 becomes

$$\max_{p_2 \leq v_2 N_1} v_1 N_2 N_1 + p_2 N_2. \tag{40}$$

Solving the above optimization problem gives us p_2^* and the equilibrium mass of group-2 participants N_2^*:

$$p_2^* = v_2 N_1 - t(N_2^*) = N_2^* t'(N_2^*) - v_1 N_1.$$

The above expression actually appears in Armstrong's (2006, expression 3) paper:[59] the platform's optimal group-2 price p_2^* under Pareto-dominance selection is equal to the standard monopoly markup $N_2^* t'(N_2^*)$, adjusted downward by the cross-side externalities $v_1 N_1$ to group-1 agents.

After identifying p_2^* and N_2^*, we can derive the platform's optimal group-1 price and its equilibrium profit from (40):

$$p_1^* = v_1 N_2^*; \quad \pi^* = (v_1 + v_2) N_1 N_2^* - N_2^* t(N_2^*).$$

As compared to Proposition 1, the equilibrium outcome under Pareto-dominance selection is very different from that under potential-maximizer selection.

Benchmark 2: Pareto-dominated selection

The second benchmark applies Pareto-dominated selection to the model. Under this selection criterion, all relevant agents always coordinate on the Pareto-dominated equilibrium whenever there are multiple equilibria. In order to make a positive profit, the platform has to guarantee participation from one side by subsidizing that side and then monetizes the other side. In other words, the platform either subsidizes group 1 and monetizes group 2, or subsidizes group 2 and monetizes group 1. First, I discuss the former strategy, and then the latter strategy.

[59] The terms N_2^* and $t'(N_2^*)$ correspond to $\phi_2(u_2)$ and $\frac{1}{\phi_2'(u_2)}$ in his paper.

Group-1 subsidy strategy If the platform subsidizes group 1 and monetizes group 2, it provides free access for group 1 (i.e., $p_1^* = 0$) such that all group-1 agents will join the platform for sure. Thus, the coordination problem no longer exists among the agents: for any group-2 price $p_2 \leq v_2 N_1$ set by the platform in stage 1, the continuum $[0, N_2]$ of group-2 agents will join the platform. As shown in (11), there is a one-to-one correspondence between $p_2 \leq v_2 N_1$ and $N_2 \in [0, \overline{N}_2)$. Hence, we can assume that the platform chooses $N_2 \in [0, \overline{N}_2)$ rather than $p_2 \leq v_2 N_1$ to maximize its profit in (10):

$$\max_{N_2 \in [0, \overline{N}_2)} (v_2 N_1 - t(N_2)) N_2. \tag{41}$$

Solving the above optimization problem gives us the platform's optimal group-2 price p_2^* and the equilibrium mass of group-2 participants N_2^*:

$$p_2^* = v_2 N_1 - t(N_2^*) = N_2^* t'(N_2^*).$$

After identifying p_2^* and N_2^*, we can derive the platform's equilibrium profit from (41):

$$\pi^* = (v_2 N_1 - t(N_2^*)) N_2^*.$$

Group-2 subsidy strategy If the platform subsidizes group 2 and monetizes group 1, it strictly subsidizes group-2 participants such that the continuum $[0, \underline{N}_2]$ of group-2 agents, whose transport costs are really low, will join the platform for sure. All other group-2 agents will not join the platform under Pareto-dominated selection. Therefore, the platform optimally sets $p_1^* = v_1 \underline{N}_2$ such that all group-1 agents will join the platform with zero surplus in stage 2. As shown in (12), there is a one-to-one correspondence between $p_2 \leq 0$ and $\underline{N}_2 \in [0, \overline{N}_2)$. Hence, we can assume that the platform chooses $\underline{N}_2 \in [0, \overline{N}_2)$ rather than $p_2 \leq 0$ to maximize its profit in (10):

$$\max_{\underline{N}_2 \in [0, \overline{N}_2)} (v_1 N_1 - t(\underline{N}_2)) \underline{N}_2. \tag{42}$$

Solving the above optimization problem gives us the platform's optimal group-2 price p_2^* and the equilibrium mass of group-2 participants $\underline{N}_2^*$:

$$p_2^* = -t(\underline{N}_2^*) = \underline{N}_2^* t'(\underline{N}_2^*) - v_1 N_1.$$

After identifying p_2^* and $\underline{N}_2^*$, we can derive the platform's optimal group-1 price and its equilibrium profit from (42):

$$p_1^* = v_1 \underline{N}_2^*; \quad \pi^* = (v_1 N_1 - t(\underline{N}_2^*))\underline{N}_2^*.$$

By comparing the two optimization problems (41) and (42), it is clear that the group-1 subsidy strategy yields higher profit than group-2 subsidy strategy if and only if $v_1 \leq v_2$, irrespective of the number of group-1 agents N_1 and the group-2 agents' transport cost function t (as long as it satisfies the assumptions in Section 4.1). Thus, the platform's optimal group-2 price and the equilibrium mass of group-2 participants are characterized by the following proposition.

Proposition 4 *Under Pareto-dominated selection, the platform adopts group-1 subsidy strategy when $v_1 \leq v_2$, and group-2 subsidy strategy when $v_1 \geq v_2$. Under group-1 subsidy strategy, the platform's group-1 price is $p_1^* = 0$, and the optimal group-2 price p_2^* and the equilibrium mass of group-2 participants N_2^* are implicitly given by*

$$p_2^* = v_2 N_1 - t(N_2^*) = N_2^* t'(N_2^*) > 0.$$

Under group-2 subsidy strategy, the platform's optimal group-1 price is $p_1^* = v_1 \underline{N}_2^*$, where the optimal group-2 price p_2^* and the equilibrium mass of group-2 participants $\underline{N}_2^*$ are implicitly given by

$$p_2^* = -t(\underline{N}_2^*) = \underline{N}_2^* t'(\underline{N}_2^*) - v_1 N_1 < 0.$$

As compared to Proposition 1, the equilibrium outcome under Pareto-dominated selection differs from that under potential-maximizer selection. Although the equilibrium outcomes share some common features (e.g. under both selection criteria, the platform monetizes group 1 and subsidizes group 2 if and only if $v_1 \geq v_2$), the platform's optimal pricing strategy under Pareto-dominated selection is discontinuous and takes a "more extreme" form compared to that under potential-maximizer selection.

F. Two examples of Armstrong's model with heterogeneous agents on one side

This appendix first analyzes the model with linear transport cost $t(k) = tk$, then analyzes the model with quadratic transport cost $t(k) = tk^2$. As we will see, these examples have closed-form solutions, and they can verify all the results derived in Section 4.2.

Linear transport cost: $t(k) = tk$

By Proposition 1, the equilibrium outcome under linear transport cost is:

1. when $v_1 \le v_2$:

$$p_1^* = \frac{v_1 v_2^3 N_1}{2t\left(2v_2 - v_1\right)^2};\ p_2^* = \frac{(v_2 - v_1)v_2 N_1}{2v_2 - v_1};\ N_2^* = \frac{v_2^2 N_1}{t(2v_2 - v_1)};\ \pi^* = \frac{v_2^3 N_1^2}{2t\left(2v_2 - v_1\right)}; \quad (43)$$

2. when $v_1 \ge v_2$:

$$p_1^* = \frac{v_1^2 N_1}{2t};\ p_2^* = -\frac{(v_1 - v_2)N_1}{2};\ N_2^* = \frac{\left(v_1 + v_2\right) N_1}{2t};\ \pi^* = \frac{\left(v_1^2 + v_2^2\right) N_1^2}{4t}. \quad (44)$$

Figure 3 sketches the equilibrium outcome when $v_1 = N_1 = t = 1$.[60] As shown in Figure 3, the platform's optimal group-1 price p_1^* is non-monotonic in v_2. Nevertheless, the proportion of surplus $\frac{p_1^*}{v_1 N_2^*}$ extracted from group 1 by the platform always decreases with v_2 (and it is true for any values of v_1, N_1, and t). To see this, by (43) and (44),

$$\frac{p_1^*}{v_1 N_2^*} = \begin{cases} \dfrac{\frac{v_2}{v_1}}{2\left(2\frac{v_2}{v_1} - 1\right)} & \text{if } v_1 \le v_2; \\[2ex] \dfrac{1}{1 + \frac{v_2}{v_1}} & \text{if } v_1 \ge v_2. \end{cases}$$

As shown in the above expression, $\frac{p_1^*}{v_1 N_2^*}$ is independent of N_1, t, and the exact values of v_1 and v_2 (only their ratio matters). In other words, $\frac{p_1^*}{v_1 N_2^*}$ is a function of $\frac{v_2}{v_1}$ as shown in Figure 4. The dashed line in Figure 4 represents the proportion of surplus extracted from group 1

[60] Because the mass of group-1 agents N_1 is normalized to one, p_1^* represents both the group-1 price and the platform's revenue from group 1. Thus, $\pi^* - p_1^*$ in Figure 3 represents the platform's revenue (or loss if this term is negative) from group 2.

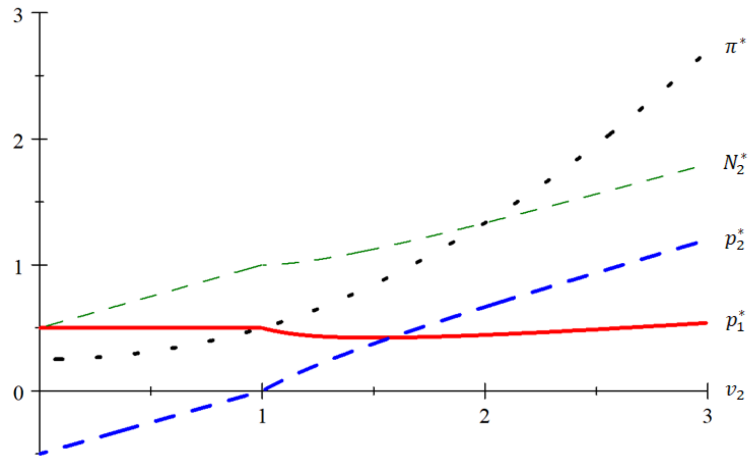

Figure 3: The equilibrium under linear transport cost (with $v_1 = N_1 = t = 1$)

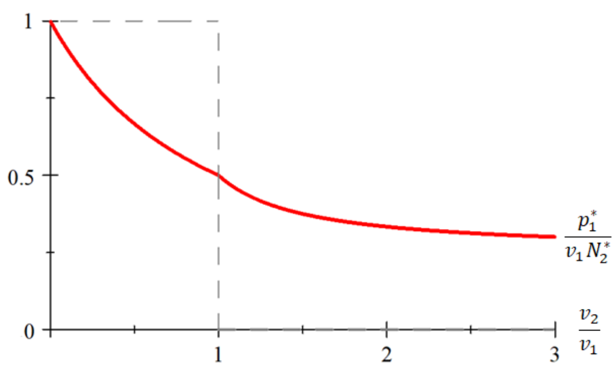

Figure 4: Proportion of surplus extracted from group 1 under linear transport cost

by the monopoly platform in the baseline model: it extracts all surplus from group 1 when $v_1 \geq v_2$, and leaves all surplus to group 1 when $v_1 \leq v_2$. By contrast, there is a smooth demand effect on group 2 in Section 4.1, and therefore $\frac{p_1^*}{v_1 N_2^*}$ decreases gradually with $\frac{v_2}{v_1}$.[61]

Quadratic transport cost: $t(k) = tk^2$

Similar to the previous example, the equilibrium outcome under quadratic transport cost is given by Proposition 1:

1. when $v_1 \leq v_2$:

$$p_1^* = \frac{2v_1 v_2^2}{3(3v_2 - 2v_1)^{\frac{3}{2}}}\sqrt{\frac{N_1}{t}}; \quad p_2^* = \frac{2(v_2 - v_1)v_2 N_1}{3v_2 - 2v_1};$$
$$N_2^* = v_2\sqrt{\frac{N_1}{t(3v_2 - 2v_1)}}; \quad \pi^* = \frac{2v_2^2 N_1^{\frac{3}{2}}}{3\sqrt{t(3v_2 - 2v_1)}};$$

2. when $v_1 \geq v_2$:

$$p_2^* = -\frac{2N_1}{3}\frac{v_1\sqrt{v_1^2 - 2v_2^2 + 2v_1 v_2} + v_1^2 + 3v_2^2 - 5v_1 v_2}{4v_1 - 3v_2},$$

   and other variables are functions of p_2^* as described in Proposition 1.

Figure 5 sketches the equilibrium outcome when $v_1 = N_1 = t = 1$. Figure 5 is similar to Figure 3 in general. Note that the equilibrium mass of group-2 participants N_2^* is non-monotonic in v_2. Under quadratic transport cost, the proportion of surplus extracted from group 1 $\frac{p_1^*}{v_1 N_2^*}$ remains a decreasing function of $\frac{v_2}{v_1}$ (and it is true for any values of v_1, N_1, and t) as shown in Figure 6. Figure 6 is similar to Figure 4 in general. Similar to the previous example, there is a smooth demand effect on group 2, and therefore $\frac{p_1^*}{v_1 N_2^*}$ decreases gradually with $\frac{v_2}{v_1}$.

[61] Compared to the model in Section 3, group-2 agents incur additional transport costs in the current model. Therefore, the proportion of surplus extracted from group 2 cannot be compared meaningfully in a similar fashion.

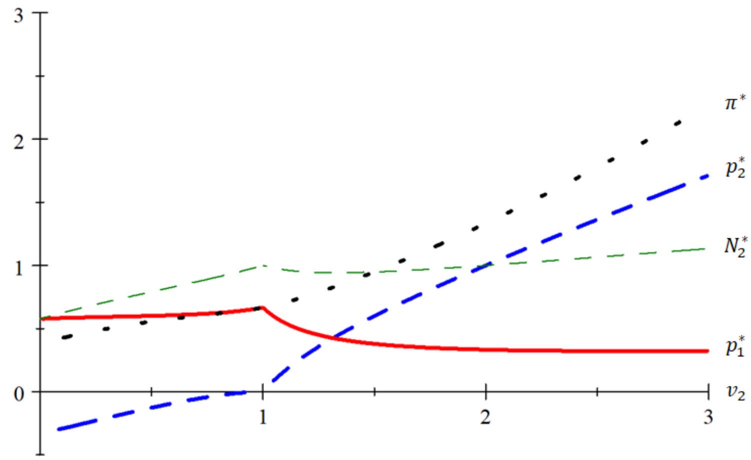

Figure 5: The equilibrium under quadratic transport cost (with $v_1 = N_1 = t = 1$)

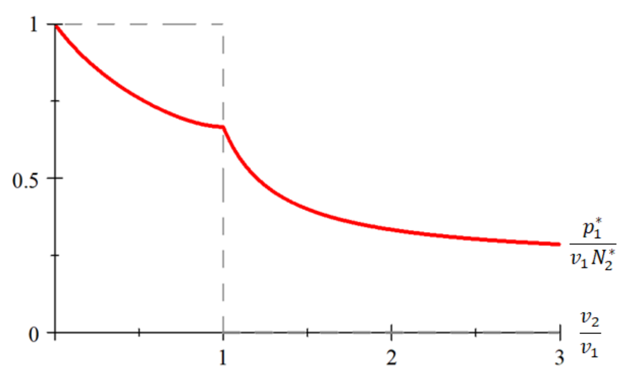

Figure 6: Proportion of surplus extracted from group 1 under quadratic transport cost

G. Formal analysis of Armstrong's original model

Model

In Armstrong's original model, there are a continuum $[0, \overline{N}_1]$ ($\overline{N}_1 \in \mathbb{R}_{++}$) of heterogeneous group-1 agents and a continuum $[0, \overline{N}_2]$ ($\overline{N}_2 \in \mathbb{R}_{++}$) of heterogeneous group-2 agents. If the platform attracts $n_1 \equiv \int_0^{\overline{N}_1} a_1^k dk$ group-1 agents and $n_2 \equiv \int_0^{\overline{N}_2} a_2^k dk$ group-2 agents, the payoff from joining the platform for agent $k \in [0, \overline{N}_i]$ from group $i = 1, 2$ is

$$u_i^k(n_j, p_i) = v_i n_j - p_i - t_i(k), \tag{45}$$

where the function $t_i : [0, \overline{N}_i] \to \mathbb{R}_+$ specifies each group-i agent's transport cost from joining the platform. I permute the agents such that t_1 and t_2 are increasing. For technical convenience, I assume that t_1 and t_2 are twice-differentiable, strictly convex, $t_1(0) = t_2(0) = 0$ and $t_1(\overline{N}_1), t_2(\overline{N}_2) \to \infty$.[62] The rest of the model setup is the same as that in Section 4.1.

Analysis

Compared to the model in Section 4.1, demand effects are present on both sides in the current model. Therefore, the equilibrium masses of group-1 and group-2 participants in stage 2 are simultaneously determined by the prices (p_1, p_2) set by the platform in stage 1.

Similar to Section 4.2, there are two stable equilibria and an unstable equilibrium in stage 2 when both prices p_1 and p_2 are positive and sufficiently low. As shown in Figure 7, the three equilibria in stage 2 are:

1. Pareto-dominant equilibrium: $(\mathbf{a}_1^*, \mathbf{a}_2^*) = (\mathbf{1}_{[0,N_1]}, \mathbf{1}_{[0,N_2]})$;
2. unstable equilibrium: $(\mathbf{a}_1^*, \mathbf{a}_2^*) = (\mathbf{1}_{[0,\widehat{N}_1]}, \mathbf{1}_{[0,\widehat{N}_2]})$;
3. Pareto-dominated equilibrium: $(\mathbf{a}_1^*, \mathbf{a}_2^*) = (\mathbf{0}, \mathbf{0})$,

where (from (45)) (N_1, N_2) and $(\widehat{N}_1, \widehat{N}_2)$ are the solutions to the system of equations

$$p_1 = v_1 n_2 - t_1(n_1); \quad p_2 = v_2 n_1 - t_2(n_2). \tag{46}$$

[62] These assumptions are not fundamental and can be relaxed as explained in footnote 26.

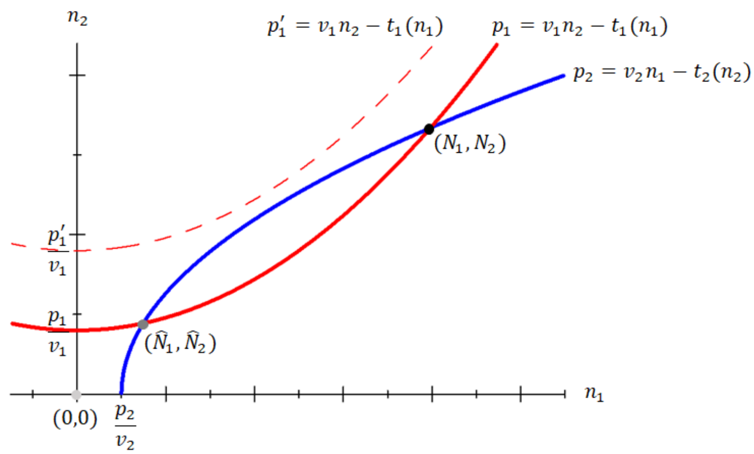

Figure 7: The equilibria of the subgame in stage 2 when $p_1, p_2 \geq 0$

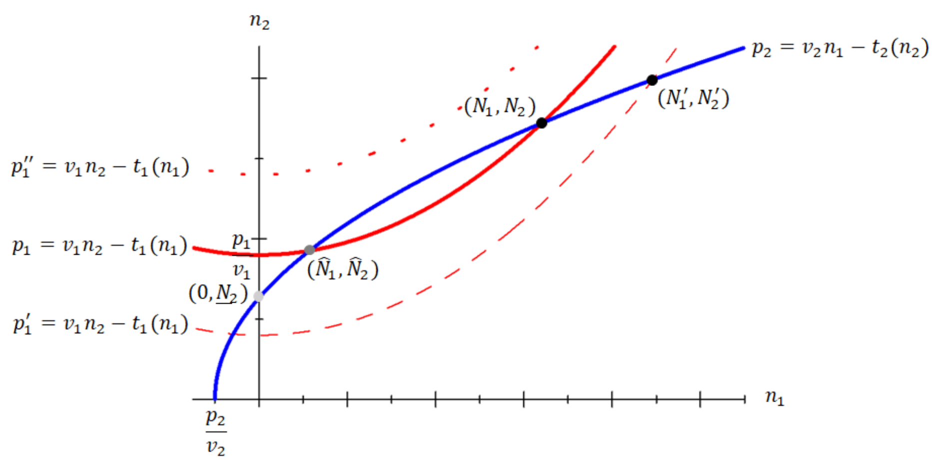

Figure 8: The equilibria of the subgame in stage 2 when $p_2 \leq 0 \leq p_1$

The unique equilibrium is the Pareto-dominated equilibrium if the platform's prices are too high (as shown in the dashed line of Figure 7).

Similar to Section 4.2, when one of the platform's price, say, p_2, is negative (i.e., the platform subsidizes group 2), some group-2 agents with really low transport costs will join the platform for sure.[63] More precisely, joining the platform is the (strictly) dominant strategy for agent $k \in [0, \underline{N}_2)$ from group 2, where $\underline{N}_2 \equiv t_2^{-1}(-p_2)$.

As shown in Figure 8, there are at most three equilibria in stage 2 when $p_2 \leq 0 \leq p_1$, and they are:

1. Pareto-dominant equilibrium: $(\mathbf{a}_1^*, \mathbf{a}_2^*) = (\mathbf{1}_{[0,N_1]}, \mathbf{1}_{[0,N_2]})$;
2. unstable equilibrium: $(\mathbf{a}_1^*, \mathbf{a}_2^*) = (\mathbf{1}_{[0,\widehat{N}_1]}, \mathbf{1}_{[0,\widehat{N}_2]})$;
3. Pareto-dominated equilibrium: $(\mathbf{a}_1^*, \mathbf{a}_2^*) = (\mathbf{0}, \mathbf{1}_{[0,\underline{N}_2]})$.

All of the three equilibria are present when p_1 is not too high or too low (for a fixed value of p_2) as shown in (the solid line of) Figure 8. The unstable equilibrium disappears if p_1 is too low (as shown the dashed line of Figure 8).[64] The unique equilibrium is the Pareto-dominated equilibrium if p_1 is too high (as shown in the dotted line of Figure 8).

Clearly, Pareto-dominance selection is applicable to this model. As explained in footnote 13, Armstrong's (2006, Section 3) analysis implicitly imposes Pareto-dominance selection, i.e., he assumes that all agents always coordinate on the Pareto-dominant equilibrium whenever there are multiple equilibria. Under Pareto-dominance selection, we can assume that the platform chooses the masses of participants $(N_1, N_2) \in [0, \overline{N}_1) \times [0, \overline{N}_2)$ directly (rather than setting prices $(p_1, p_2) \in \mathbb{R}^2$) to maximize its profit. From (10) and (46), the platform's

[63] The analysis for $p_1 \leq 0$ is analogous to that for $p_2 \leq 0$, and thus omitted in this appendix. Because setting negative prices on both sides is a (weakly) dominant strategy for the platform, I also omit the discussion on these prices.

[64] The Pareto-dominant equilibrium becomes (N_1', N_2') in Figure 8, and the Pareto-dominated equilibrium is still $(0, \underline{N}_2)$.

profit-maximization problem in stage 1 becomes

$$\max_{(N_1,N_2)\in[0,\overline{N}_1)\times[0,\overline{N}_2)} (v_1N_2 - t_1(N_1))N_1 + (v_2N_1 - t_2(N_2))N_2. \quad (47)$$

Solving the above optimization problem gives us the platform's optimal prices (p_1^*, p_2^*) and the equilibrium masses of participants (N_1^*, N_2^*):

$$\begin{aligned} p_1^* &= v_1N_2^* - t_1(N_1^*) = N_1^*t_1'(N_1^*) - v_2N_2^*; \\ p_2^* &= v_2N_1^* - t_2(N_2^*) = N_2^*t_2'(N_2^*) - v_1N_1^*. \end{aligned} \quad (48)$$

The above expressions actually appear in Armstrong's (2006, expression 3) paper:[65] the platform's optimal prices p_1^* and p_2^* under Pareto-dominance selection are equal to the standard monopoly markups $N_1^*t_1'(N_1^*)$ and $N_2^*t_2'(N_2^*)$, adjusted downward by the cross-side externalities $v_2N_2^*$ and $v_1N_1^*$ to the other side.

Now, I analyze the model under potential-maximizer selection. Similar to Section 4.2, the unstable equilibrium is never the potential maximizer, and therefore it is never selected under potential-maximizer selection. As the first step of the analysis, I show that every subgame in stage 2 is a weighted potential game.

Lemma 9 *Every subgame in stage 2 is a weighted potential game with the potential*

$$P(\mathbf{a}_1, \mathbf{a}_2|p_1, p_2) = n_1n_2 - \frac{p_1}{v_1}n_1 - \frac{p_2}{v_2}n_2 - \frac{1}{v_1}\int_0^{\overline{N}_1} t_1(k)a_1^k dk - \frac{1}{v_2}\int_0^{\overline{N}_2} t_2(k)a_2^k dk.$$

Proof. The proof is very similar to that of Lemma 3 in Section 4.2 (i.e., Appendix C) and thus omitted. ■

Compared to Lemma 3, the extra term $\frac{1}{v_1}\int_0^{\overline{N}_1} t_1(k)a_1^k dk$ captures the aggregate transport cost incurred by group-1 participants.

After identifying the game's potential, the next step is to identify the potential maximizer. Following the same approach in proving Lemma 4 in Section 4.2 (i.e., Appendix A1), we can identify the potential maximizer, which is summarized by the following lemma.

[65] The terms N_i^* and $t_i'(N_i^*)$ correspond to $\phi_i(u_i)$ and $\frac{1}{\phi_i'(u_i)}$ in his paper.

Lemma 10 *Under potential-maximizer selection, the unique equilibrium of the subgame in stage 2 if there are multiple equilibria is*

1. *when* $p_1, p_2 \geq 0$*:*

$$\begin{aligned}(\mathbf{a}_1^*, \mathbf{a}_2^*) &= (\mathbf{1}_{[0,N_1]}, \mathbf{1}_{[0,N_2]}) \text{ if } \frac{1}{v_1}\int_0^{N_1}(p_1 + t_1(k))dk + \frac{1}{v_2}\int_0^{N_2}(p_2 + t_2(k))dk \leq N_1 N_2; \\ (\mathbf{a}_1^*, \mathbf{a}_2^*) &= (\mathbf{0}, \mathbf{0}) \qquad \text{otherwise.}\end{aligned} \tag{49}$$

2. *when* $p_2 \leq 0 \leq p_1$*:*

$$\begin{aligned}(\mathbf{a}_1^*, \mathbf{a}_2^*) &= (\mathbf{1}_{[0,N_1]}, \mathbf{1}_{[0,N_2]}) \text{ if } \frac{1}{v_1}\int_0^{N_1}(p_1 + t_1(k))dk + \frac{1}{v_2}\int_{\underline{N}_2}^{N_2}(p_2 + t_2(k))dk \leq N_1 N_2; \\ (\mathbf{a}_1^*, \mathbf{a}_2^*) &= (\mathbf{0}, \mathbf{1}_{[0,\underline{N}_2]}) \qquad \text{otherwise.}\end{aligned} \tag{50}$$

Proof. The proof is very similar to Appendix A1. Here, I only prove the non-trivial part: to show that the unstable equilibrium is never the potential maximizer. I prove for the case $p_1, p_2 \geq 0$; the proof for the case $p_2 \leq 0 \leq p_1$ is analogous. By Lemma 9, the potential of the unstable equilibrium is

$$P(\mathbf{1}_{[0,\widehat{N}_1]}, \mathbf{1}_{[0,\widehat{N}_2]}|p_1, p_2) = \widehat{N}_1\widehat{N}_2 - \frac{1}{v_1}\int_0^{\widehat{N}_1}(p_1 + t_1(k))dk - \frac{1}{v_2}\int_0^{\widehat{N}_2}(p_2 + t_2(k))dk.$$

To prove that the unstable equilibrium is never the potential maximizer, it suffices to show that its potential is always less than that of the Pareto-dominated equilibrium, i.e., $\frac{1}{v_1}\int_0^{\widehat{N}_1}(p_1 + t_1(k))dk + \frac{1}{v_2}\int_0^{\widehat{N}_2}(p_2 + t_2(k))dk \geq \widehat{N}_1\widehat{N}_2$. This can be clearly seen in Figure 9 (which is edited from Figure 7): the area $A + B$ is equal to $\frac{1}{v_1}\int_0^{\widehat{N}_1}(p_1 + t_1(k))dk$; the area $B + C$ is equal to $\frac{1}{v_2}\int_0^{\widehat{N}_2}(p_2 + t_2(k))dk$; the area $A + B + C$ is equal to $\widehat{N}_1\widehat{N}_2$.[66] ■

Now, I discuss the platform's optimal pricing strategy in stage 1. Similar to Lemma 4 in Section 4.2, potential-maximizer selection implies that the platform has to leave the participants enough surplus by setting sufficiently low prices (p_1, p_2) in stage 1 so that all agents

[66] By the same token, we can identify the potential maximizer directly from Figure 9. The area $A+B+F$ is equal to $\frac{1}{v_1}\int_0^{N_1}(p_1+t_1(k))dk$; the area $B+C+D$ is equal to $\frac{1}{v_2}\int_0^{N_2}(p_2+t_2(k))dk$; the area $A+B+C+D+E+F$ is equal to N_1N_2. Thus, when $p_1, p_2 \geq 0$, the potential maximizer is the Pareto-dominant (Pareto-dominated) equilibrium when $E \geq B$ ($E \leq B$).

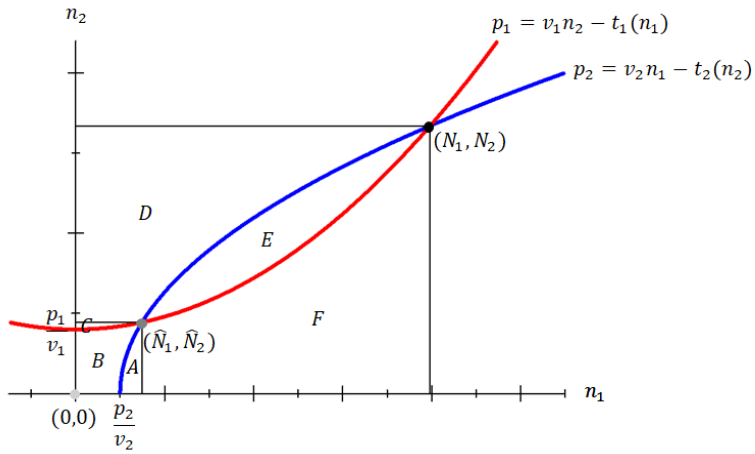

Figure 9: The equilibria of the subgame in stage 2 when $p_1, p_2 \geq 0$

will coordinate on the Pareto-dominant equilibrium in stage 2. In other words, potential-maximizer selection essentially imposes an additional constraint (i.e., the inequalities in (49) and (50) of Lemma 10) to the platform's profit-maximization problem under Pareto-dominance selection in (47).

As explained in Section 4.3, sometimes this additional constraint does not bind in the equilibrium. As shown in Lemma 10, this happens when the platform's prices (p_1, p_2) and the participants' aggregate transport cost $\int_0^{N_i} t_i(k)dk$ are sufficiently low.[67] If this is the case, the equilibrium outcome under potential-maximizer selection coincides with that under Pareto-dominance selection; otherwise, the equilibrium outcomes under these two selection criteria differ.

Example

In what follows, I analyze an example in which the transport cost functions take the form

$$t_1(k) = t_1 k^\alpha; \quad t_2(k) = t_2 k^\alpha. \quad (\alpha > 1)$$

[67] If $p_2 \leq 0 \leq p_1$, only the aggregate transport cost of group-2 participants who do not have a dominant strategy to join the platform matters, i.e., only $\int_{\underline{N}_2}^{N_2} t_2(k)dk$ (but not $\int_0^{N_2} t_2(k)dk$) matters.

As I will show, the equilibrium outcome under Pareto-dominance selection coincides with that under potential-maximizer selection if and only if $\frac{v_1}{v_2} \leq \frac{1}{\alpha}$ or $\frac{v_1}{v_2} \geq \alpha$. In other words, the equilibrium outcomes under these two selection criteria are different if and only if the per-interaction benefits of the two sides do not differ too much, i.e., $\frac{1}{\alpha} < \frac{v_1}{v_2} < \alpha$.

First of all, the equilibrium masses of participants under Pareto-dominance selection are given by (48):

$$
\begin{aligned}
(v_1 + v_2) N_2^* &= t_1(\alpha + 1) \left(N_1^*\right)^{\alpha}; \\
(v_1 + v_2) N_1^* &= t_2(\alpha + 1) \left(N_2^*\right)^{\alpha}.
\end{aligned}
$$

Solving the above system of equations gives us

$$
\begin{aligned}
N_1^* &= \left(\frac{1}{t_1^{\alpha} t_2}\right)^{\frac{1}{\alpha^2 - 1}} \left(\frac{v_1 + v_2}{\alpha + 1}\right)^{\frac{1}{\alpha - 1}}; \\
N_2^* &= \left(\frac{1}{t_1 t_2^{\alpha}}\right)^{\frac{1}{\alpha^2 - 1}} \left(\frac{v_1 + v_2}{\alpha + 1}\right)^{\frac{1}{\alpha - 1}}.
\end{aligned}
\tag{51}
$$

Hence, by (48), the platform's optimal prices are

$$
\begin{aligned}
p_1^* &= \frac{\alpha v_1 - v_2}{\alpha + 1} \left(\frac{1}{t_1 t_2^{\alpha}}\right)^{\frac{1}{\alpha^2 - 1}} \left(\frac{v_1 + v_2}{\alpha + 1}\right)^{\frac{1}{\alpha - 1}}; \\
p_2^* &= \frac{\alpha v_2 - v_1}{\alpha + 1} \left(\frac{1}{t_1^{\alpha} t_2}\right)^{\frac{1}{\alpha^2 - 1}} \left(\frac{v_1 + v_2}{\alpha + 1}\right)^{\frac{1}{\alpha - 1}}.
\end{aligned}
\tag{52}
$$

Now, I show that the equilibrium outcomes under Pareto-dominant selection and potential-maximizer selection are different if and only if $\frac{1}{\alpha} < \frac{v_1}{v_2} < \alpha$.

Proof. When $\frac{1}{\alpha} < \frac{v_1}{v_2} < \alpha$, the platform's optimal prices p_1^* and p_2^* are positive under Pareto-dominance selection as shown in (52). By substituting (51) and (52) into the inequality in (49) and with some simplifications, we have

$$
(\alpha v_1 - v_2)(v_1 - \alpha v_2) \leq 0. \tag{53}
$$

The above inequality is violated if and only if $\frac{1}{\alpha} < \frac{v_1}{v_2} < \alpha$, i.e., the Pareto-dominant equilibrium is not the potential maximizer when $\frac{1}{\alpha} < \frac{v_1}{v_2} < \alpha$. Therefore, the equilibrium outcomes under Pareto-dominance selection and potential-maximizer selection must differ.

By contrast, when $\frac{v_1}{v_2} \geq \alpha$ (i.e., $p_2^* \leq 0$), (53) is satisfied. Moreover, the term $\int_{\underline{N}_2}^{N_2}(p_2 + t_2(k))dk$ in (50) is less than the term $\int_0^{N_2}(p_2 + t_2(k))dk$ in (49). Hence, the Pareto-dominant equilibrium is the potential maximizer when $\frac{v_1}{v_2} \geq \alpha$. The same logic applies to the case $\frac{v_1}{v_2} \leq \frac{1}{\alpha}$. ■

In what follows, I characterize the equilibrium under potential-maximizer selection when $\frac{1}{\alpha} < \frac{v_1}{v_2} < \alpha$. It turns out that the inequality in (49) binds under potential-maximizer selection. Hence, from (47), the platform's profit-maximization problem in stage 1 becomes

$$\begin{aligned} \max_{(N_1,N_2)\in[0,\overline{N}_1)\times[0,\overline{N}_2)} \quad & (v_1 N_2 - t_1 N_1^{\alpha})N_1 + (v_2 N_1 - t_2 N_2^{\alpha})N_2 \\ \text{s.t.} \quad & \frac{t_1}{v_1}N_1^{\alpha+1} + \frac{t_2}{v_2}N_2^{\alpha+1} = \frac{\alpha+1}{\alpha}N_1 N_2. \end{aligned}$$

Solving the above optimization problem gives us the equilibrium masses of participants (N_1^*, N_2^*) (λ is the Lagrange multiplier):

$$\begin{aligned} (v_1+v_2)N_2^* - t_1(\alpha+1)\left(N_1^*\right)^{\alpha} &= \lambda\left(\frac{\alpha+1}{\alpha}N_2^* - \frac{t_1}{v_1}(\alpha+1)\left(N_1^*\right)^{\alpha}\right); \\ (v_1+v_2)N_1^* - t_2(\alpha+1)\left(N_2^*\right)^{\alpha} &= \lambda\left(\frac{\alpha+1}{\alpha}N_1^* - \frac{t_2}{v_2}(\alpha+1)\left(N_2^*\right)^{\alpha}\right); \\ \frac{t_1}{v_1}\left(N_1^*\right)^{\alpha+1} + \frac{t_2}{v_2}\left(N_2^*\right)^{\alpha+1} &= \frac{\alpha+1}{\alpha}N_1^* N_2^*. \end{aligned}$$

There are closed-form solutions to the above system of equations. Here, I present the closed-form solutions when $v_1 = t_1 = t_2 = 1$ and $\alpha = 2$ (i.e., with quadratic transport cost):

$$\begin{aligned} N_1^* &= \left(\frac{v_2\left(3 - 3v_2 + \sqrt{7v_2^2 - 13v_2 + 7}\right)}{2 - v_2}\right)^{\frac{1}{3}} \frac{4v_2 + \sqrt{7v_2^2 - 13v_2 + 7} - 5}{6(v_2 - 1)}; \\ N_2^* &= \left(\frac{v_2\left(3 - 3v_2 + \sqrt{7v_2^2 - 13v_2 + 7}\right)}{2 - v_2}\right)^{\frac{2}{3}} \frac{4v_2 + \sqrt{7v_2^2 - 13v_2 + 7} - 5}{6(v_2 - 1)}, \end{aligned}$$

and the platform's optimal prices (p_1^*, p_2^*) can be computed from (46). Figure 10 sketches the equilibrium outcome for this case. As shown in Figure 10, the equilibrium outcome when $\frac{1}{2} \leq v_2 \leq 2$ (where the additional constraint under potential-maximizer selection binds) differs from that when $v_2 \leq \frac{1}{2}$ or $v_2 \geq 2$ (where the additional constraint does not bind).

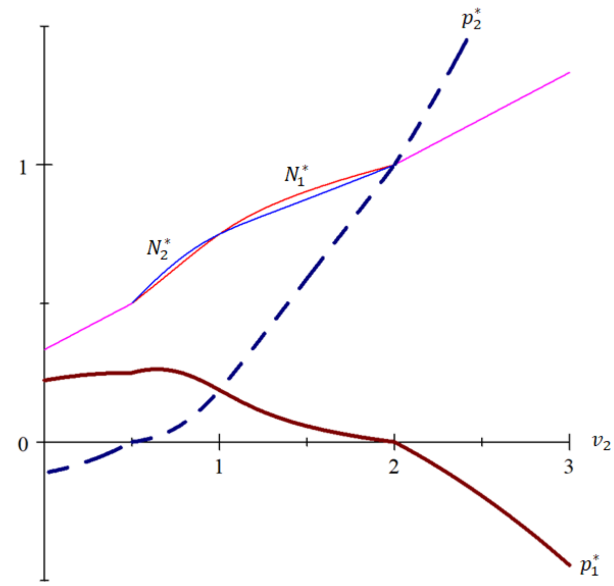

Figure 10: The equilibrium under quadratic transport cost (with $v_1 = t_1 = t_2 = 1$; $\alpha = 2$)

In particular, there are more (less) group-2 participants than group-1 participants when $\frac{1}{2} < v_2 < 1$ $(1 < v_2 < 2)$, but the equilibrium masses of participants on both sides are the same when $v_2 \leq \frac{1}{2}$ or $v_2 \geq 2$. Nevertheless, the general pattern of the equilibrium outcome is consistent: the platform's optimal group-1 price and the equilibrium masses of participants increase with v_2, while the optimal group-2 price tends to decrease with v_2.

As illustrated in this example, the equilibrium outcomes under Pareto-dominance selection and potential-maximizer selection are different when the per-interaction benefits of the two sides do not differ too much.