Market Structure and the Speed of R&D*

(Preliminary and Incomplete)

Dawei Fang
University of Gothenburg

Thomas Noe
University of Oxford

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Abstract

We study an innovation race in which firms compete in the speed of discovering an innovation by choosing risky research strategies. We show that market competition biases firms against risk-taking. Increasing the number of competing firms increases risk-taking and increases innovation speed via both a statistical and a strategic channel. However, for difficult innovations that society needs urgently, social welfare is higher under monopoly (no competition) than under oligopoly. This last result holds even if there are more innovation attempts under oligopoly and there is no cost of conducting research.

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1 Introduction

Innovation races in which firms compete in the speed of discovering an innovation by choosing risky research strategies are ubiquitous. Such a race has a *winner-take-all* reward structure: The firm that discovers the innovation first wins all of the reward. Society only benefits from the first discovery. Since society discounts the future, society wants the innovation to be discovered as quick as possible. In fact, under continuous time discounting, social welfare is convex in every firm's discovery date of the innovation. To see this, let t_i be the discovery date for firm i and t^*_{-i} be the earliest discovery date attained by firms excluding i. Social welfare is in proportion to

$$e^{-r_s \min[t_i, t^*_{-i}]}, \tag{1}$$

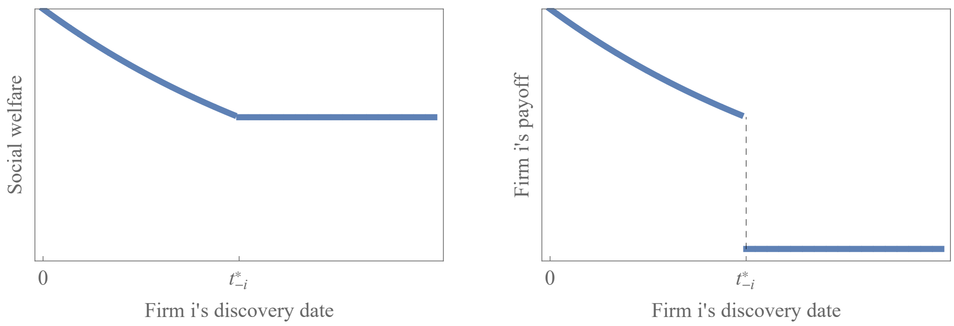
where $r_s \geq 0$ is the social discount rate. The function given by (1) is clearly convex in t_i (cf. Figure 1.A). Given such convexity, society prefers every firm to adopt high-risk research strategies, strategies that can produce high dispersion of the discovery date of the innovation.

A natural question to ask is whether market competition biases firms against risk taking. This question captured many researchers' attention in the 1980s (cf. Dasgupta and Stiglitz, 1980; Klette and de Meza, 1986; Bhattacharya and Mookherjee, 1986; Dasgupta and Maskin, 1987). A common conclusion they reach is that, in winner-take-all R&D competitions, the market is generally not biased against risky projects.[1] Some later papers show that, if the R&D competition is not winner-take-all, the market is often biased against risky projects (cf. Cabral, 1994; Tse, 2001; Kwon, 2010; Färnstrand Damsgaard et al., 2017). Combining these existing results seems to suggest that it is the winner-take-all reward structure of an innovation race that leads to efficient risk taking.

However, we show that, in contrast to the existing literature, even if an innovation race is winner-take-all, risk taking is insufficient. Insufficient risk taking stems from the fact that, in winner-take-all innovation races, reward allocation is based on the ranking of the discovery dates of the innovation, i.e., the firm that discovers the innovation first wins. Thus, for any given firm, holding its rivals' discovery dates of the innovation fixed, the firm's payoff is not convex in its discovery date of the innovation but has a discontinuity point—the marginal gain from incrementally besting the best-performing rival relative to being incrementally bested by the best-performing rival is infinitely large. More specifically, firm i's payoff is in proportion to $e^{-r_f t_i}$ if $t_i < t^*_{-i}$ and equals

[1] Dasgupta and Stiglitz (1980) is an exception, but as pointed out by Klette and de Meza (1986), Dasgupta and Stiglitz (1980) made an error in their analysis, so their result is misleading.

0 if $t_i > t^*_{-i}$, where $r_f \geq 0$ is firm i's discount rate (cf. Figure 1.B).[2] Consequently, to exploit this discontinuity, firms focus too much on winning by a small margin and are thus biased against high-risk strategies. The reason why Klette and de Meza (1986), Bhattacharya and Mookherjee (1986), and Dasgupta and Maskin (1987) do not obtain a similar result is because they commonly restrict a firm's risk choice to mean-preserving spreads of performance with a symmetric density function that has no mass point. The symmetry assumption constrains the firms' ability to exploit the discontinuity and the no-mass point assumption restricts the discontinuity in such a way that a discontinuity point can only exist ex post (when the earliest discovery date attained by a firm's rivals is realized) but not ex ante (when research strategies have been chosen but discovery dates have not been realized). Consequently, these two assumptions substantially weaken the role of the rank-dependency nature of rewards in determining firms' risk-taking strategies.[3] By following the setting used by Klette and de Meza (1986), which is close to the settings used by Bhattacharya and Mookherjee (1986) and Dasgupta and Maskin (1987), while relaxing the symmetry and the continuity restrictions, we uncover the rank dependency nature of rewards as the driver of insufficient risk-taking in R&D races.



A. Social welfare function B. Firm welfare function

Figure 1: Social welfare and firm i's welfare as functions of t_i, firm i's discovery date of the innovation, holding t^*_{-i}, the earliest discovery date attained by firms other than i, fixed.

Our model has at least two policy implications. First, increasing the number of competing firms increases risk taking and, thus, benefits society both statistically (as having

[2]Firm i's payoff when $t_i = t^*_{-i}$ (i.e., when firm i ties with its best-performing rival) depends on the tie-breaking rule and the number of firms associated with this tie. However, regardless of what tie-breaking rule we choose and regardless of the number of firms associated with the tie, $t_i = t^*_{-i}$ is always a discontinuity point of firm i's payoff function.

[3]Gaba et al. (2004) show that, in symmetric risk-taking contests where prizes are identical and each contestant is free to choose any distribution of random performance that is symmetric about the same mean, contestants choose the riskiest strategy if the fraction of winners is less than one half and choose the safest strategy if the fraction of winners is more than one half. As shown by Fang and Noe (2016), such a "bang-bang" strategy no longer sustains an equilibrium if one allows for the play of skewed distributions of performance.

more firms conduct R&D leads to more innovation attempts) and strategically. Second, more importantly, because risk taking is inefficient under oligopoly while efficient under monopoly, social welfare can be higher under monopoly than under oligopoly even though there are more innovation attempts under oligopoly. Particularly, we find that, for difficult innovations that society needs urgently, social welfare is higher under monopoly than under oligopoly. Since we assume zero cost of conducting research in our model, the fact that monopoly can be more efficient than oligopoly is not due to waste of resources caused by duplications of research efforts under oligopoly but purely due to insufficient risk taking under oligopoly.

2 The Model

The structure of the model follows Dasgupta and Stiglitz (1980) and Klette and de Meza (1986). There are n identical laboratories (firms) working independently of each other. They compete in the speed of an innovation. The case where $n = 1$ corresponds to the case of a monopolist firm (no competition). Following Klette and de Meza (1986), we assume that every firm will discovery the innovation by date $T > 0$ for sure but can speed up the discovery.[4] If a firm discovers the innovation at date t, its *performance*, defined as the time saved from a quicker discovery, is $x = T - t$. We interpret x as *innovation speed*, with $x = 0$ corresponding to the lowest possible speed and $x = T$ to the highest possible speed. Translating discovery date, t, to performance, $x = T - t$, is just for expositional convenience. None of our results depends on this translation.

Each firm's performance is a *nonnegative* random variable. Each firm simultaneously chooses a research strategy that gives a distribution of performance (equivalently, a distribution of the discovery date of the innovation) following the strategy. We assume that each firm can choose *any* distribution of performance supported by the closed interval, $[0, T]$, subject to a *capacity constraint*, i.e., the expected performance equals a fixed value, $\mu \in (0, T)$. Each firm's realized performance is independently drawn according to its distributional choice. The best-performing firm wins and receives a prize of $V_f > 0$ at discovery. The present discounted prize value if the winning performance is x equals $e^{-r_f(T-x)}V_f$, where $r_f \geq 0$ is each firm's discount rate. All of the losers receive 0. Ties are broken randomly. Society only values the quickest innovation. The social value of the quickest innovation equals $V_s > 0$ at discovery and the present dis-

[4]For equilibrium existence, we require the support of each firm's feasible distributional choice of the random discovery date of the innovation to be bounded. This boundedness restriction is also imposed, albeit implicitly, by Klette and de Meza (1986). Specifically, in Klette and de Meza (1986), because the discovery date is bounded below by 0 and because firms can only choose among symmetric density functions subject to the same finite mean, the upper bound is implied by the lower bound and symmetry.

counted social value equals $e^{-r_s(T-x)}V_s$ if the winning performance is x, where $r_s \geq 0$ is the social discount rate.

Our model is less restrictive than the models in Dasgupta and Stiglitz (1980) and Klette and de Meza (1986) in the sense that Dasgupta and Stiglitz (1980) and Klette and de Meza (1986), in addition to imposing the capacity constraint and the nonnegativity constraint on feasible distributional choice, assume that every feasible research strategy produces a continuously distributed random performance and all of the feasible research strategies can be ordered based on their "riskiness" under the notion of mean-preserving spreads. Moreover, Klette and de Meza (1986) further assume that every feasible research strategy produces a distribution of performance that is symmetric about its mean. In contrast, we allow for discontinuous and skewed distributions of performance and we do not require the feasible distributions to be ordered by mean-preserving spreads.

3 Socially optimal strategy

We start our analysis by investigating the socially optimal strategy. For each firm $i = 1, \ldots, n$, let $\tilde{x}_i$ be its random performance (i.e., random innovation speed). The present discounted social value of innovation thus equals $e^{-r_s(T-\max\{\tilde{x}_1,\ldots,\tilde{x}_n\})}V_s$, where $\max\{\tilde{x}_1, \ldots, \tilde{x}_n\}$ represents the best firm performance. Note that, for any firm $j \in \{1, \ldots, n\}$, fixing x_i for all $i \neq j$, the map, $x_j \hookrightarrow e^{-r_s(T-\max\{x_1,\ldots,x_n\})}V_s$, is convex, implying that social welfare is convex in every firm's performance. It is well-known that, for two random variables, X and Y, if Y is riskier than X in the sense that Y differs from X by a mean-preserving spread, $\mathbb{E}[\phi(Y)] \geq \mathbb{E}[\phi(X)]$ for all convex functions $\phi : \mathbb{R} \to \mathbb{R}$, provided the expectations exist (cf. Shaked and Shanthikumar, 2007, p.109).[5] Thus, a mean-preserving spread of $\tilde{x}_j$ increases social welfare, implying that social welfare is increasing in the riskiness of every firm's strategy. Since the riskiest research strategy in our setting is the one that leads to either immediate discovery or the slowest discovery, fixing the number of firms, social welfare is maximized if every firm plays the riskiest strategy, i.e., the two-point support distribution of performance that places all mass on 0 (lowest possible speed) and T (highest possible speed).

If $n = 1$, we have a monopolist firm. The present discounted value of the innovation for the monopolist firm if the monopolist firm's performance equals x is given by $e^{-r_f(T-x)}V_f$, which is convex in x. Thus, by the same argument as the one used above, the socially optimal strategy, i.e., the riskiest strategy, is also optimal for the monopolist

[5] In probability theory, Y differing from X by a mean-preserving spread is often called Y being larger than X in the convex order.

firm.

If $n \geq 2$, we have an innovation race among firms. Maximizing social welfare requires each firm to play the riskiest strategy. However, firms all playing the riskiest strategy cannot sustain an equilibrium. This is because, if all of the firms played the riskiest strategy, that is, the performance distribution that places all mass on 0 (lowest possible speed) and T (highest possible speed), then with a positive probability, all of them discovered the innovation at the lowest speed. A firm would, however, be better off transferring mass from 0 to ε, for $\varepsilon > 0$ sufficiently small, so that when all its rivals ended up with the lowest innovation speed, the firm would win with certainty by marginally besting all of its rivals. Such a transfer's effect on the firm's capacity constraint can be made arbitrarily small by shrinking ε to 0 while, for all $\varepsilon > 0$, no matter how small, the transfer generates a gain to the firm that is bounded below by a strictly positive number. Therefore, when $n \geq 2$, there is no equilibrium in which all of the firms play the socially optimal strategy. The next result is thus straightforward.

Proposition 1. *It is socially optimal to have every firm play the riskiest strategy, the performance distribution that places all mass on 0 and T. Fixing n, this socially optimal strategy is played in equilibrium if and only if $n = 1$.*

Proposition 1 implies that market competition biases firms against risk taking, which is the drawback of competition. However, competition also provides a natural advantage via a statistical channel: Everything else being equal, the innovation will be discovered quicker if more firms conduct research. What is the overall effect of competition? This is the question we aim to tackle in the next two sections.

4 Market Competition

In this section, we study the effect of market competition on social welfare with $n \geq 2$ firms. We focus on symmetric Nash equilibria in which all of the firms play the same strategy. Let $F : [0, T] \to [0, 1]$ be the equilibrium performance distribution for each firm. Our next lemma shows that, in any symmetric equilibrium, F has no mass point over $[0, T)$.

Lemma 1. *In any symmetric equilibrium, the equilibrium performance distribution, F, has no mass point over the interval $[0, T)$.*

Proof. The reason for Lemma 1 is similar to the reason for why the socially optimal strategy cannot be sustained in equilibrium. If F had a mass point at $x' \in [0, T)$, then in a symmetric equilibrium, every firm has a positive probability of having performance equal to x'. In this case, with a positive probability, there would be a tie at x'. A firm would then be better off transferring mass from x' to $x'+\varepsilon$, for $\varepsilon > 0$ sufficiently small. Clearly, such an argument does not apply to performance level T, since T corresponds to the highest possible speed, which cannot be topped. Thus, it is possible that F has a mass point at T. $\square$

Let $P : [0, T] \to [0, 1]$ be a firm's *probability of winning function*, with $P(x)$ representing a firm's probability of winning the prize when its realized performance equals x. Note that, in any symmetric equilibrium, P and F satisfy that

$$P(x) = \begin{cases} F(x)^{n-1} & \text{if } x < T \\ \sum_{i=0}^{n-1} \left(\frac{1}{i+1}\right) \binom{n-1}{i} (1-F(T-))^i F(T-)^{n-1-i} & \text{if } x = T \end{cases}. \tag{2}$$

The complexity in the relation between $P(T)$ and $F(T)$ is driven by the fact that F may have a mass point at $x = T$ and, thus, a firm may tie with other firms at $x = T$ (highest possible speed). With a probability $\binom{n-1}{i}(1-F(T-))^i F(T-)^{n-1-i}$, exactly i out of a firm's $n-1$ rivals reach the highest possible speed, where $1-F(T-)$ represents the probability that a firm reaches the highest possible speed.[6] In this case, by also reaching the highest possible speed, the firm's probability of winning equals $1/(i+1)$ under the random tie-breaking rule. Summing over i from $i = 0$ to $i = n-1$ gives the relation between $P(T)$ and $F(T)$.

Let $\pi : [0, T] \to [0, V_f]$ be a firm's *contest payoff function*, with $\pi(x)$ representing a firm's expected payoff when its realized performance equals x. Since $x = T - t$, if a firm wins at performance level x, the present discounted value of the prize equals $e^{-r_f t} V_f = e^{-r_f (T-x)} V_f$. Thus, given that losers always receive zero payoff, the contest payoff function, π, and the probability of winning function, P, satisfy that

$$\pi(x) = P(x) e^{-r_f (T-x)} V_f, \quad x \in [0, T]. \tag{3}$$

A firm's problem is to choose a performance distribution, $\hat{F}$, for its nonnegative random performance $\tilde{x}$ to maximize $\mathbb{E}[\pi(\tilde{x})]$ subject to the capacity constraint, $\mathbb{E}[\tilde{x}] = \mu$. In fact, it is more convenient to express each firm's problem as one of choosing a measure over the nonnegative real line rather than choosing a random variable on a measure space. Thus, if $d\hat{F}$ denotes the probability measure associated with $\hat{F}$, we can formulate each firm's problem as one of choosing a performance measure, $d\hat{F}$, to use against the contest payoff function, π. The performance measure has to satisfy

[6] Throughout, we use $F(a-)$ as a shorthand for $\lim_{x \uparrow a} F(x)$.

two constraints: (a) it has to be a probability measure supported by $[0, T]$ and (b) its expectation equals μ. The solution to this problem coincides with the solution to the following relaxed problem:

$$\max_{d\hat{F}\geq 0} \int_{0-}^{T} \pi(x)\, d\hat{F}(x) \qquad \text{s.t. (i)} \int_{0-}^{T} d\hat{F}(x) \leq 1 \text{ \& (ii)} \int_{0-}^{T} x\, d\hat{F}(x) \leq \mu. \tag{P$_\mathrm{F}$}$$

The Lagrangian associated with this problem is given by

$$\mathscr{L}(d\hat{F}, \alpha, \beta) = \int_{0-}^{T} \pi(x) d\hat{F}(x) - \alpha \left(\int_{0-}^{T} d\hat{F}(x) - 1 \right) - \beta \left(\int_{0-}^{T} x d\hat{F}(x) - \mu \right), \tag{4}$$

where α and β are nonnegative dual variables.

Problem ($\mathrm{P_F}$) must have a solution. This is because (a) the set of measures satisfying ($\mathrm{P_F}$–i,ii) and supported by the compact interval, $[0, T]$, is nonempty and compact, and (b) the contest payoff function, $\pi : [0, T] \to [0, V_f]$, is upper semicontinuous, implying that the map, $d\hat{F} \hookrightarrow \int_{0-}^{T} \pi\, d\hat{F}$, is upper semicontinuous.[7] Upper semicontinuity of π follows simply because F, being a probability distribution function, is upper semicontinuous, implying, by equation (2), upper semicontinuity of the probability of winning function, $P : [0, T] \to [0, 1]$, and, hence, by equation (3), upper semicontinuity of π.

The next result, which is given by Lemma A-1 in Fang and Noe (2018), provides a general characterization of the solution to problem ($\mathrm{P_F}$).

Lemma 2. *A probability distribution function, $\hat{F} : [0, T] \to [0, 1]$, solving problem* ($\mathrm{P_F}$) *exists. For any such solution, there exist dual variables $\alpha \geq 0$ and $\beta > 0$ such that α and β satisfy that*

$$\pi(x) \leq \alpha + \beta x \quad \forall\, x \in [0, T] \quad \& \quad d\hat{F}\{x \in [0, T] : \pi(x) < \alpha + \beta x\} = 0, \tag{5}$$

and, if $\mathbf{v}(\pi, \mu)$ *represents the optimal value of problem* ($\mathrm{P_F}$),

$$v(\pi, \mu) = \alpha + \beta \mu. \tag{6}$$

Conversely, if a probability distribution, $\hat{F}$, supported by $[0, T]$*, satisfies* (5) *and makes the capacity constraint, (P_F–ii), bind, it is a solution to* ($\mathrm{P_F}$).

Proof. See the proof of Lemma A-1 in Fang and Noe (2018). □

Lemma 2 implies that $\alpha + \beta x$ is an *upper support line* for the contest payoff func-

[7]See Ash (1972, Theorems 4.5.1 and 8.2.4).

tion, π. If we rewrite equation (4) as

$$\mathscr{L}(d\hat{F},\alpha,\beta) = \int_{0-}^{T} [\pi(x) - (\alpha + \beta x)]\, d\hat{F}(x) + \alpha + \beta\mu, \tag{7}$$

we see that placing any probability weight on points at which $\pi(x) - (\alpha + \beta x) < 0$ lowers the Lagrangian. Thus, as shown by Lemma 2, the optimal performance distribution must place no weight on such points. Hence, the optimal performance measure is always concentrated on points at which the upper support line, $\alpha + \beta x$, meets the contest payoff function, π.

Note that, in problem ($\mathrm{P_F}$), the contest payoff function, π, is determined by the equilibrium performance distribution, F, through equations (2) and (3). Since, in a symmetric equilibrium, F must be a best reply when used against π, F itself must be a solution to problem ($\mathrm{P_F}$) and, hence, must satisfy the conditions for a solution given by Lemma 2. This equilibrium property allows us to further characterize the equilibrium strategy, F. Our next lemma shows that, in any symmetric equilibrium, with the highest performance level, T, excluded, the support of F is an interval with 0 as the lower bound, and the optimal dual variable associated with the unit mass constraint, α, equals 0.

Lemma 3. *Let* $\mathrm{supp}(F)$ *be the support of the performance distribution, F, in any symmetric equilibrium. Define* $\hat{x}$ *as*

$$\hat{x} = \sup \mathrm{supp}(F) \backslash \{T\}. \tag{8}$$

The following results hold:

i. $\min \mathrm{supp}(F) = 0$*;*

ii. the optimal dual variable associated with the unit mass constraint, (P_F–i), equals 0, i.e., $\alpha = 0$*;*

iii. either $\mathrm{supp}(F) = [0,\hat{x}]$ *or* $\mathrm{supp}(F) = [0,\hat{x}] \cup \{T\}$.

Proof. See the Appendix. □

The intuition of parts i and ii of Lemma 3 are as follows. Each firm mixes between different performance levels subject to a capacity constraint. Although performance choices are costless, they use up capacity. Thus, the capacity constraint imposes a shadow price on performance, which can be seen from Lemma 2 that the shadow price, represented by the optimal dual variable associated with the capacity constraint, ($\mathrm{P_F}$–ii), is strictly positive, i.e., $\beta > 0$. If the lower bound of the support of the equilibrium performance distribution, F, equaled an intermediate value, say $x' \in (0,T)$, given that firms place no mass point over $[0, T)$ in any symmetric equilibrium (cf. Lemma 1), the performance level x' would produce zero probability of winning. Because of the shadow price, each firm would be better off mixing with zero performance instead of with $x' \in (0, T)$. Thus, the lower bound of the support of F cannot be an intermediate value. Clearly, it cannot be T either, because the expected performance, μ, is assumed to be strictly less than T. Thus, as shown by part i, the lower bound of the support of F must be zero. Because zero is part of a firm's mixed performances, by Lemma 2, the upper support line, $\alpha + \beta x$, must meet the contest payoff function, π, at the origin, implying that $\alpha = \pi(0)$. Given that no firm places mass point at 0, it must be that $\pi(0) = 0$. Thus, as shown by part ii, $\alpha = 0$.

Part iii of Lemma 3 implies that, excluding T, the support of F must be connected. To understand this result, note that, for an individual firm, it benefits from higher performance through two channels: a *prize-winning channel*, i.e., higher performance (weakly) increases the firm's probability of winning the prize, and a *value-discounting channel*, i.e., as the winner's prize is less discounted when the winner's performance is higher, higher performance increases the firm's prize conditional on winning. Differentiating the contest payoff function, π, using equation (3), at any point at which the probability of winning function, P, is differentiable, we can decompose the marginal benefit of performance according to these two channels as:

$$\pi'(x) = \underbrace{k e^{r_f x} P'(x)}_{\text{the prize-winning effect}} + \underbrace{k r_f e^{r_f x} P(x)}_{\text{the value-discounting effect}}, \tag{9}$$

where $k = e^{-r_f T} V_f$ is a constant. Clearly, the value-discounting effect is always strictly increasing in performance, whereas the prize-winning effect is zero outside the support of F (because outside the support of F, $F'(x) = 0$ and thus $P'(x) = 0$).

If, excluding T, the support of F were not connected, there would have to exist two disconnected intervals, $[a, b]$ and $[c, d)$, with $0 \leq a < b < c < d < T$, such that both intervals are part of the support of F while the open interval between them, i.e., the interval (b, c), is not part of the support of F. By Lemma 2, the support of F must be included in the set of points where the contest payoff function, π, meets the upper support line, $\alpha + \beta x$. Thus, over both (a, b) and (c, d), $\pi'(x) = \beta$. This would imply that the value-discounting effect is smaller than β over the higher performance range (c, d), which would further imply that the value-discounting effect is strictly smaller than β over the lower performance range (b, c). Because, by hypothesis, (b, c) is not part of the support of F, the prize-winning effect is zero over (b, c). Thus, the marginal benefit of performance, π', would be strictly smaller than β over (b, c). Hence, over

(b, c), π would have to grow at a strictly lower rate than the upper support line. Given that contestants place no mass point over $[0, T)$, implying the continuity of F, P, and π over (a, d), and given the hypothesis that π meets the upper support line everywhere over (a, b), if π grows at a strictly lower rate than the upper support line over (b, c), π cannot meet the upper support line again everywhere over (c, d), a contradiction. Thus, excluding T, the support of F must be connected. Given that 0 is part of the support, the support of F, excluding T, must be a connected interval with 0 as the lower bound.

Lemma 3.ii implies that the upper support line, $\alpha + \beta x$, simply equals βx. Because the contest payoff function, π, must meet the upper support line, βx, over the support of the equilibrium performance distribution, F, Lemma 3.iii implies that, in any symmetric equilibrium, π can only take on one of the two configurations illustrated in Figure 2. Figure 2.A illustrates the configuration of π in which π has no discontinuity point. In this configuration, π meets the upper support line, βx, over a closed interval, $[0, \hat{x}]$, and above this interval, π, albeit being still increasing due to the value-discounting effect, falls below the upper support line, βx. Under this configuration, it is optimal for a firm to place weight only over $[0, \hat{x}]$. Figure 2.B illustrates the contrasting configuration of π in which π has a discontinuity point at T. In this alternative configuration, π meets the upper support line, βx, both over a closed interval, $[0, \hat{x}]$, and at T, and over $(\hat{x}, T)$, π falls below the upper support line, βx. Under this alternative configuration, it is optimal for a firm to place weight only over $[0, \hat{x}] \cup \{T\}$.

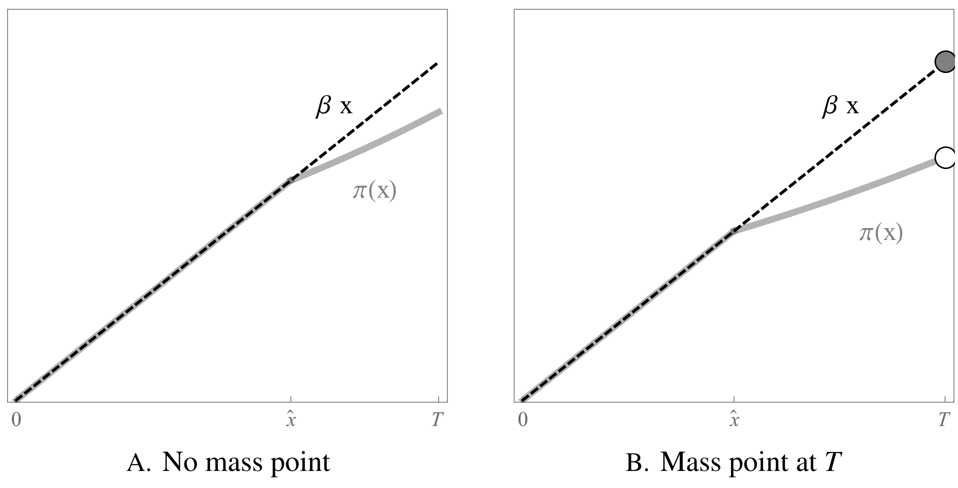


Figure 2: Two possible equilibrium configurations of the contest payoff function, π. The upper support line is given by βx.

The fact that the contest payoff function, π, must meet the upper support line, βx, over the support of F implies a relation between π and β. Thus, the fact that π is determined by F via equations (2) and (3) further implies a relation between F and β.

Exploiting these relations and noting that the mean of F must equal μ and the support of F must satisfy the condition presented in Lemma 3.iii, we can fully characterize F. The next proposition provides this characterization.

Proposition 2. *There exists a unique symmetric equilibrium. The performance distribution in this equilibrium is determined as follows: For any fixed* $n \geq 2$, $r_f \geq 0$, *and* $T > 0$, *there exists a capacity threshold,* $\bar{\mu}(n, r_f, T) > 0$, *such that*

i. if $\mu \in (0, \bar{\mu}]$, *then* $\text{supp}(F) = [0, \hat{x}]$ *and, over this support,*

$$F(x) = \left(\frac{e^{r_f \hat{x}}}{\hat{x}}\right)^{\frac{1}{n-1}} \left(\frac{x}{e^{r_f x}}\right)^{\frac{1}{n-1}}, \tag{10}$$

where $\hat{x}$ *is determined in a way such that the mean of* F *given by* (10) *equals* μ.

ii. If $\mu \in (\bar{\mu}, T)$, *then* $\text{supp}(F) = [0, \hat{x}] \cup \{T\}$. $F(T) = 1$, *while over* $[0, \hat{x}]$, F *satisfies*

$$F(x) = \left(\frac{e^{r_f T}}{T} \sum_{i=0}^{n-1} \frac{1}{i+1} \binom{n-1}{i} F(\hat{x})^{n-1-i} (1 - F(\hat{x}))^i \right)^{\frac{1}{n-1}} \left(\frac{x}{e^{r_f x}}\right)^{\frac{1}{n-1}}, \tag{11}$$

where $\hat{x}$ *and* $F(\hat{x})$ *are jointly determined through two equations, one from the fact that the mean of* F *given by* (11) *equals* μ *and the other from the fact that equation* (11) *holds for* $x = \hat{x}$.

Proof. See the Appendix. □

Proposition 2 implies that, in contrast to the monopoly case, where a monopolist firm always gambles on the highest possible speed of innovation by placing point mass on the highest performance level, T, firms under market competition are much more conservative; they gamble on the highest possible speed only if they have sufficiently high capacity (i.e., only if $\mu > \bar{\mu}$). The intuition is easy to understand. For a monopolist firm, the marginal benefit of performance comes purely from the value-discounting channel, i.e., the value of the innovation is discounted less if the innovation is discovered quicker. In contrast, for a firm in a race, as shown by equation (9), the marginal benefit of performance not only comes from the value-discounting channel but also the prize-winning channel. Although gambling on the highest possible speed gives a firm a chance of winning the prize at its highest value, it also makes the firm difficult to win whenever the firm fails to obtain an immediate discovery. When firms lack capacity, it is hard for a firm to obtain a successful gamble on immediate discovery. In this case, to win the prize at its highest value, a firm has to sacrifice its winning probability significantly. Consequently, all of the competing firms refrain from reaching for the highest prize value and, instead, use a conservative strategy of randomizing performance over a low performance range to boost the probability of winning the prize, despite at a lower value.

Consistent with the observation in Proposition 1, Proposition 2 also implies that, under market competition, firms never take the socially optimal risk (i.e., the maximal risk), even if they have high capacity. As has been discussed in Section 3, taking the socially optimal risk requires a firm to place mass point at 0 (the lowest possible speed of innovation), which can be topped by an infinitesimal performance of $\varepsilon > 0$. Consequently, over a low performance range, firms randomize their performances continuously to prevent themselves from being exploited by such an ε-topping strategy. At this point, it is worth pointing out that Klette and de Meza (1986) admit that, to show that competition does not bias firms against risk-taking, the symmetric distribution assumption plays a crucial role in their analysis. However, they also argue that, even if firms can play skewed distributions, whenever the number of competing firms exceeds some threshold, firms always play the riskiest strategy in equilibrium. In contrast, Proposition 1 implies that competition always induces insufficient risk-taking whenever the number of competing firms is finite. The difference between our result and theirs is mainly driven by the fact that Klette and de Meza (1986) impose continuity on performance distributions whereas we allow for discontinuous performance distributions. Given the continuity restriction, the socially optimal strategy is forced to be a continuous performance distribution with no mass point. In this case, for a given firm, if all of its competitors choose the socially optimal strategy, the rank-dependency nature of rewards only creates a discontinuity point in the firm's payoff function in an ex post but not an ex ante sense. Also, the continuity restriction forbids the firm from placing mass point on any performance level the firm aims to top. Consequently, given the continuity restriction, it is hard for a firm to exploit the rank-dependency nature of rewards by using any strategy similar to the ε-topping strategy. Thus, playing the socially optimal strategy can possibly be sustained in equilibrium in their setting.

While market competition biases firms against risk-taking, our next proposition shows that increasing competitiveness of R&D races by increasing the number of competing firms reduces such a bias and increases social welfare.

Proposition 3. *Suppose $n \geq 2$. An increase in n induces the equilibrium performance distribution to undergo a mean-preserving spread and strictly increases social welfare.*

Proof. See the Appendix. □

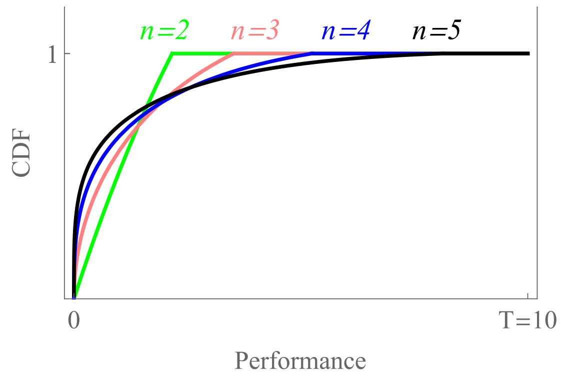


Figure 3: The change in the equilibrium performance distribution as n increases from 2 to 5, with $r_f = 0.1$, $T = 10$, and $\mu = 1$.

Figure 3 illustrates the effect of increasing the number of competing firms on each firm's performance distribution. As the figure shows, increasing competition makes each firm's performance distribution more spread out.

Proposition 3 implies that increasing the number of competing firms induces firms to take higher risks. Thus, increasing the number of competing firms benefits society from two respects. First, having more firms compete produces a positive *statistical effect* on social welfare: everything else being equal, when more firms conduct research, the innovation will be discovered quicker. Second, more firms joining the race also produces a positive *strategic effect* on social welfare. Recall that social welfare is highly convex, i.e., a convex function of the highest firm performance. Thus, everything else being equal, society benefits from riskier research strategies.

5 Monopoly vs oligopoly

Proposition 3 implies that, when competition is already in place, an increase in the number of competing firms benefits society via both a statistical and a strategic channel. In contrast, as implied by our discussion in Section 3, the directional alignment between the statistical and the strategic effect on social welfare of an increase in the number of firms breaks when the number of firms increases from one (monopoly) to more than one (oligopoly), making the overall effect ambiguous when the number of firms is increased from one. Thus, how efficient monopoly is, relative to oligopoly, is unclear and requires a further analysis.

To simplify the algebra, in this section, we follow Dasgupta and Stiglitz (1980) and Klette and de Meza (1986) by assuming that the social discount rate equals each firm's discount rate, i.e., $r_s = r_f = r$. Note that, under the assumption that firms and society discount the future at the same rate, the present value of social benefits always equals the present value of the prize multiplied by V_s/V_f, a positive constant. Thus, social welfare equals the expected winner payoff multiplied by the positive constant, V_s/V_f. Since the expected winner payoff simply equals the sum of all firms' expected payoffs, comparing social welfare under monopoly with social welfare under oligopoly with $n \geq 2$ firms is equivalent to comparing the monopoly payoff with the sum of the $n \geq 2$ firms' payoffs.

Let v_n be a firm's expected payoff when there are n firms, where $n \geq 1$. Social welfare equals the sum of firms' expected payoffs, $n v_n$, multiplied by the constant scalar, V_s/V_f. To compute the monopolist firm's expected payoff, v_1, note that the monopolist firm only places probability weight on the two extreme points, 0 and T. Given the capacity constraint, the weight on T is μ/T and the weight on 0 is $1-(\mu/T)$. Thus,

$$v_1 = \frac{\mu}{T} e^{-r(T-T)} V_f + \left[1 - \left(\frac{\mu}{T}\right)\right] e^{-r(T-0)} V_f = \left[\frac{\mu}{T} + e^{-rT}\left(1 - \frac{\mu}{T}\right)\right] V_f. \tag{12}$$

To find out the expression for v_n when there are $n \geq 2$ competing firms, note that, by Lemma 2, each competing firm's payoff equals $\alpha + \beta\mu$, where α and β are the optimal dual variables. By Lemma 3, in equilibrium, $\alpha = 0$. Thus,

$$v_n = \beta\mu, \quad \text{for } n \geq 2, \tag{13}$$

where β is the optimal dual variable associated with the capacity constraint in equilibrium. Society is better off under monopoly than under oligopoly with $n \geq 2$ firms if $v_1 > n v_n$. Expressing this condition using only exogenous variables is impossible, because there is no general closed-form expression for the optimal dual variable, β, under market competition. However, we are still able to derive a pretty general sufficient condition under which monopoly generates higher social benefits than oligopoly with $n \geq 2$ firms.

Proposition 4. *For any $n \geq 2$, $r > 0$, and $T > 0$, with r and T satisfying that*

$$rT > k(n), \quad \text{where } k(n) > 0 \text{ is uniquely determined by } e^k - 1 - nk = 0, \tag{14}$$

there exists $\mu^*(n, r, T) > 0$ (with expression given by equation (67) in the Appendix) such that, for all $\mu < \mu^*$, social welfare is strictly higher under monopoly than under oligopoly with $n \geq 2$ firms.

Proof. See the Appendix. □

Proposition 4 implies that, fixing T, the longest time period that the innovation could possibly take, monopoly is more efficient than oligopoly with $n \geq 2$ firms when the discount rate, r, is sufficiently high and when each firm's capacity, μ, is sufficiently low. The intuition of this result is as follows: Compared to monopoly, market competition has a positive statistical effect but a negative strategic effect on social welfare. As we discussed in Section 4, when firms compete but lack capacity, they will only randomize performance over a low performance range but will never gamble on immediate discovery. In this case, when society sufficiently discounts the future, the negative strategic effect will dominate the positive statistical effect.

We illustrate the above result using a simple example. Suppose that $r = 0.4$, $T = 5$, and $\mu = 0.4$ and consider the case of monopoly ($n = 1$) and the case of duopoly ($n = 2$). Figure 4 depicts the performance distribution under monopoly (in black) and the distribution of the highest performance under duopoly (in purple), where the gray dotted line is the social payoff function with $V_s = 1$. As the figure shows, while the highest performance under duopoly is very likely to be better than a monopolist firm's performance, given that the monopolist firm gambles on immediate discovery whereas neither of the two firms under duopoly gambles on such high innovation speed, the monopolist firm's performance can sometimes far exceed the highest performance under duopoly. As society discounts the future, high innovation speed is disproportionately valued by society compared to low innovation speed. Consequently, in this example, while the expected performance (i.e., the expected innovation speed) under monopoly, which equals capacity $\mu = 0.4$, is less than the expected highest performance under duopoly, which, by numerical computation, equals approximately 0.57, social welfare is higher under monopoly (≈ 0.21) than under duopoly (≈ 0.17) and the difference is economically significant ($\sim 10\%$).[8]

It is worth pointing out that, in our model, there is no cost of R&D for a firm. Thus, the result that monopoly can be welfare improving compared to oligopoly with any finite number of competing firms is not driven by the duplication of research efforts under oligopoly but purely by the effect of competition on firms' risk-taking strategies. As has been mentioned before, we relaxed several restrictions imposed by Dasgupta and Stiglitz (1980) and Klette and de Meza (1986) on feasible risk-taking strategies in R&D races. This relaxation has turned out to be productive, since the result that risk-taking can make monopoly more efficient than oligopoly cannot be derived under the models of Dasgupta and Stiglitz (1980) or Klette and de Meza (1986). In Klette and de Meza (1986), under the restriction that firms can only choose among performance distributions with symmetric density functions, firms always choose the riskiest strategy and,

[8]It is clear that the expected highest performance under competition is higher than each individual firm's expected performance. Thus, in our model, the expected highest performance under competition is always higher than the expected performance under monopoly. Hence, for monopoly to be more efficient than oligopoly, we require a strictly positive discount rate, $r > 0$.

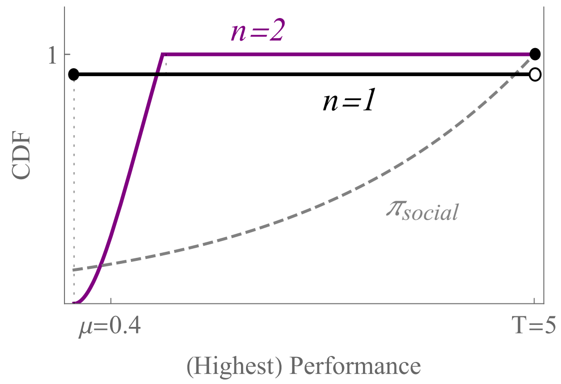


Figure 4: The CDF of (highest) performance under monopoly ($n = 1$) and under duopoly ($n = 2$), with $r = 0.4$, $T = 5$, and $\mu = 0.4$. The gray dotted line is the social payoff function. If highest performance equals x, social payoff equals $e^{-r(T-x)} V_s$. In the graph, $V_s = 1$.

hence, do not vary riskiness according to a change in market structure. Although Dasgupta and Stiglitz (1980) argue that, compared to market competition, monopoly produces a positive strategic effect on social welfare, as pointed out by Klette and de Meza (1986), there is an error in the analysis of Dasgupta and Stiglitz (1980), which makes it questionable whether monopoly really produces a positive strategic effect. Moreover, the setting with the mean-preserving spread restriction used by Dasgupta and Stiglitz (1980) is less tractable than our setting with only the mean constraint. Thus, unlike our setting, it is very hard, if not fully impossible, to evaluate the magnitude of the strategic effect relative to the statistical effect under the setting of Dasgupta and Stiglitz (1980).

6 Conclusion

In this paper, we studied firms' risk-taking strategies in an R&D race in which firms compete in the speed of an innovation. By using an unrestrictive risk-taking R&D race model in which each firm is free to choose any random, mean-preserving innovation speed with a bounded support, we showed that, even though R&D races often feature a winner-take-all reward structure that has been commonly considered as inducing aggressive strategies in the literature (cf. Klette and de Meza, 1986; Bhattacharya and Mookherjee, 1986; Dasgupta and Maskin, 1987), the rank-dependency nature of rewards biases firms against risk-taking. Increasing the number of competing firms induces more socially efficient risk taking. However, social welfare can be higher under monopoly than under oligopoly, especially when the innovation is urgently needed by society (higher discount rate r) but difficult to discover quickly (low capacity μ).

Throughout our analysis, we abstracted from any cost of R&D for a firm. Thus, the result that monopoly can sometimes be more efficient than oligopoly was not driven by waste of resources caused by duplications of research efforts under oligopoly but purely driven by insufficient risk-taking under oligopoly. Introducing a fixed cost, $c > 0$, of conducting research would introduce duplications of research efforts as another disadvantage of oligopoly, which would further favor monopoly as being more efficient than oligopoly. Also, throughout our analysis, we treated the number of firms, n, fixed. A simple way to endogenize n is to assume free entry and a fixed cost, $c > 0$, of conducting research. Then it is obvious that, for every $c > 0$, the equilibrium number of competing firms is finite and weakly decreasing in c. As implied by Proposition 4, for breakthrough innovations that society needs urgently but firms face great challenge to discover them quickly, monopoly can be socially optimal even absent of any cost of R&D. In this case, unless c is sufficiently high such that monopoly naturally arises, the equilibrium number of firms can be greater than social optimum. Whenever the latter happens, policies that *fully eliminate* competition can increase social welfare. In the cases in which monopoly is not socially optimal, policies that encourage competition (such as subsidizing firms' costs of R&D) can be welfare improving. As Proposition 3 implies, such policies can benefit society via both a statistical and a strategic channel. Missing the strategic channel would lead to an underestimation of the benefits of such policies against their costs (such as the increased cost of R&D).

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Appendix

Proof of Lemma 3. We first establish the following result.

Result 1. *For any $a \in (0,T)$, if $\pi(a) = \alpha + \beta a$, we must have $\pi(x) = \alpha + \beta x$ for all $x \in [0,a]$.*

Proof. We prove the result by way of contradiction. Suppose, contrary to the result, that there exists $x_o \in [0,a)$ such that $\pi(x_o) \neq \alpha + \beta x_o$. Let $x_1 = \min\{x \in (x_o, a] : \pi(x) = \alpha + \beta x\}$. Because, by hypothesis, $\pi(a) = \alpha + \beta a$ and because, by Lemma 1, π is continuous over $[0,T)$, x_1 is well defined. By the definition of x_1 and the hypothesis that $\pi(x_o) \neq \alpha + \beta x_o$, the result, established in Lemma 2, that $\pi(x) \leq \alpha + \beta x$ for all $x \geq 0$ implies that

$$\pi(x) < \alpha + \beta x, \quad x \in [x_o, x_1). \tag{15}$$

Because all the firms face the same contest payoff function, π, in a symmetric equilibrium, by (15) and Lemma 2, no firm places any weight over $[x_o, x_1)$. Thus, the probability of winning function, P, is constant over $[x_o, x_1)$, i.e., $P(x) = P(x_o)$ for all $x \in [x_o, x_1)$. Also, because $x_1 \leq a < T$, by Lemma 1, no firm places point mass on x_1. Thus, $P(x) = P(x_o)$ for all $x \in [x_o, x_1]$. Thus, given that $x_1 \leq a < T$, by equation (3),

$$\pi(x) = P(x_o) e^{-r_f (T-x)} V_f, \quad x \in [x_o, x_1]. \tag{16}$$

Inspection of equation (16) shows that, over $[x_o, x_1]$, $x \hookrightarrow \pi(x)$ is strictly increasing and convex. Note that $\alpha + \beta x$ is affine and is, by (15) and the definition of x_1, strictly greater than π at x_o while equal to π at x_1. Thus, $\partial_- \pi(x_1) > \beta$. Moreover, by equations (3) and (16) and the fact that π is continuous over $[0,T)$, it must be that $\partial_+ \pi(x_1) \geq \partial_- \pi(x_1)$. Thus, given that $\partial_- \pi(x_1) > \beta$, it must be that $\partial_+ \pi(x_1) > \beta$. However, because π meets $\alpha + \beta x$ at x_1 and is bounded above by $\alpha + \beta x$, it cannot be that $\partial_+ \pi(x_1) > \beta$, a contradiction. This contradiction establishes the result. □

Given Result 1, we now prove the lemma. Note that there must exist $x' \in (0,T)$ such that $\pi(x') = \alpha + \beta x'$, because otherwise, by Lemma 2, no firm would place weight over $(0,T)$, which would imply that the firms place point mass on 0, contradicting Lemma 1. Given that $\pi(x') = \alpha + \beta x'$, $x' \in (0,T)$, by Result 1, we must have

$$\pi(x) = \alpha + \beta x, \quad x \leq x'. \tag{17}$$

Equations (3) and (17) imply that the probability of winning function, P, has no flat region over $[0, x']$. Thus, given that P is nondecreasing in any symmetric equilibrium, it must be that P is everywhere increasing over $[0,x']$. Thus, $[0,x'] \subset \text{supp}(F)$, which, given nonnegativity of performance, implies that $\min \text{supp}(F) = 0$. This establishes part i. By a similar argument, it must be that $\text{supp}(F) \backslash \{T\}$ is an interval with zero as the lower bound. Part iii thus follows.

To establish part ii, simply note that, by equation (17), $\pi(0) = \alpha$. By Lemma 1, $P(0) = 0$. Thus, by equation (3), $\pi(0) = 0$. Hence, given that $\pi(0) = \alpha$, it must be that $\alpha = 0$. $\square$

Proof of Proposition 2. We first establish part (i). To show this part, it suffices to show the following result.

Result 2. *For fixed $r_f > 0$ and $T > 0$, there uniquely exists $\bar{x} \in (0, 1/r_f]$ such that*

$$\frac{e^{r_f \bar{x}}}{\bar{x}} = \frac{e^{r_f T}}{T}. \tag{18}$$

Define $\bar{F}$ as

$$\bar{F}(x) = \left(\frac{x}{\bar{x}} e^{r_f(\bar{x}-x)}\right)^{\frac{1}{n-1}}, \quad x \in [0, \bar{x}]. \tag{19}$$

Define $\bar{\mu}$ as

$$\bar{\mu} = \int_0^{\bar{x}} x d\bar{F}(x). \tag{20}$$

If $\mu \in (0, \bar{\mu}]$, then in a symmetric equilibrium, $\text{supp}(F) = [0, \hat{x}]$ *and, over this support, F takes the expression in equation* (10).

Proof. We prove the result in three steps.

Step 1: We show that, in any symmetric equilibrium,

$$\text{supp}(F) = [0, \hat{x}] \quad \Longrightarrow \quad F \text{ takes the expression in equation (10).} \tag{21}$$

Suppose, in a symmetric equilibrium, $\text{supp}(F) = [0, \hat{x}]$. By Lemmas 2 and 3, the hypothesis that $\text{supp}(F) = [0, \hat{x}]$ implies that

$$\pi(x) = \beta x, \quad x \in [0, \hat{x}]. \tag{22}$$

Equation (22) implies that π is continuous over $\text{supp}(F)$ and thus no firm places any point mass. Hence, the probability of winning function, P, satisfies that

$$P(x) = F(x)^{n-1}, \quad x \in [0, \hat{x}]. \tag{23}$$

Thus, by equation (3), we have

$$\pi(x) = P(x)e^{-r_f(T-x)}V_f, \quad x \in [0, \hat{x}]. \tag{24}$$

The hypothesis that $\text{supp}(F) = [0, \hat{x}]$ implies that $F(\hat{x}) = 1$. Thus, by equation (24), we have

$$\pi(\hat{x}) = e^{-r_f(T-\hat{x})}V_f. \tag{25}$$

Equations (22) and (25) imply that

$$\beta = \frac{e^{-r_f(T-\hat{x})}}{\hat{x}}V_f, \quad x \in [0, \hat{x}] \tag{26}$$

$$\pi(x) = \frac{e^{-r_f(T-\hat{x})}}{\hat{x}}V_f x, \quad x \in [0, \hat{x}]. \tag{27}$$

Equations (24) and (27) imply that

$$P(x) = \frac{x}{\hat{x}}e^{r_f(\hat{x}-x)}, \quad x \in [0, \hat{x}]. \tag{28}$$

Equations (23) and (28) imply (21).

Step 2: By analyzing the equation,

$$\frac{e^{r_f x}}{x} = \frac{e^{r_f T}}{T}, \tag{29}$$

we show that $\bar{x}$, defined in (18), and $\bar{\mu}$, defined in (20), both uniquely exist and we present a result in (31) below, that assists our later proof. Note that, for $r_f > 0$, over the positive real line, the graph of $x \hookrightarrow e^{r_f x}/x$ is continuous and U-shaped with its minimum located at $x = 1/r_f$ and tends to infinity as $x \to 0$ or as $x \to \infty$. This implies that, for fixed $r_f > 0$ and $T > 0$, equation (29) has two positive solutions (identical only when $T = 1/r_f$), with one solution no greater than $1/r_f$ and the other no less than $1/r_f$. Thus, $\bar{x}$, defined in (18), simply represents the solution that is no greater than $1/r_f$. Thus, $\bar{x}$ uniquely exists. This implies that $\bar{F}$, defined in (19), uniquely exists. Thus, by (20), $\bar{\mu}$ also uniquely exists. Let x_o represent the solution to equation (29) that is no less than $1/r_f$, i.e., x_o is implicitly determined by the following:

$$\frac{e^{r_f x_o}}{x_o} = \frac{e^{r_f T}}{T} \quad \& \quad x_o \geq \frac{1}{r_f}. \tag{30}$$

Given the definitions of $\bar{x}$ and x_o and given the fact that the graph of $x \hookrightarrow e^{r_f x}/x$ is U-shaped with its minimum located at $x = 1/r_f$, we must have

$$\frac{e^{r_f T}}{T} \leq \frac{e^{r_f x}}{x} \iff x \in (0, \bar{x}] \cup [x_o, \infty), \tag{31}$$

where $\bar{x}$ and x_o are defined by (18) and (30), respectively.

Step 3: We show that

$$\mu \leq \bar{\mu} \iff \text{supp}(F) = [0, \hat{x}] \text{ sustains a symmetric equilibrium,} \tag{32}$$

where $\bar{\mu}$ is defined by (20). Note that, by Lemmas 2 and 3, π is bounded above by the upper support line, βx, and supp(F) is bounded above by T. Thus, by (21), supp$(F) = [0, \hat{x}]$ sustains a symmetric equilibrium if and only if the following conditions are satisfied:

a. F, expressed in equation (10), is a CDF and is thus nondecreasing in x for all $x \in [0, \hat{x}]$;

b. $\hat{x}$, which makes the mean of F, expressed in equation (10), equal to μ, satisfies that $\hat{x} \leq T$;

c. $\pi(x) \leq \beta x$ for all $x \in [0, T]$, where π is defined in (3) and β is given by (26).

Note that F, expressed in (10), is nondecreasing in x for all $x \in [0, \hat{x}]$ if and only if

$$\hat{x} \leq 1/r_f. \tag{33}$$

Thus, condition (a) holds if and only if (33) holds. Given equation (22), condition (c) holds if and only if $\pi(x) \leq \beta x$ for all $x \in (\hat{x}, T]$. Note that, given supp$(F) = [0, \hat{x}]$, $P(x) = 1$ for all $x \in (\hat{x}, T]$. Thus, by equation (3), $\pi(x) = e^{-r_f(T-x)} V_f$ for all $x \in (\hat{x}, T]$. Because the mapping, $x \hookrightarrow e^{-r_f(T-x)} V_f$, is convex and increasing and because π meets the upper support line at $\hat{x}$, π is bounded above by the upper support line, βx, over $(\hat{x}, T]$ if and only if π is bounded above by βx at T. Hence, condition (c) holds if and only if

$$e^{-r_f(T-T)} V_f = V_f \leq \frac{e^{-r_f(T-\hat{x})}}{\hat{x}} V_f T = \beta T, \tag{34}$$

where the last equality follows from equation (26). By rearranging, we see that condition (c) holds if and only if

$$\frac{e^{r_f T}}{T} \leq \frac{e^{r_f \hat{x}}}{\hat{x}}. \tag{35}$$

By (31), the condition expressed in (35) holds if and only if

$$\hat{x} \in (0, \bar{x}] \cup [x_o, \infty), \tag{36}$$

where $\bar{x}$ is defined in (18) and x_o in (30). By the definitions of $\bar{x}$ and x_o, $\bar{x} \le 1/r_f \le x_o$. Thus, (33) and (36) imply that

$$\text{conditions (a) and (c) hold simultaneously} \iff \hat{x} \in (0, \bar{x}], \tag{37}$$

where $\bar{x}$ is defined in (18). In fact, when $\hat{x} \in (0, \bar{x}]$, condition (b) is also satisfied. To see this, note that, as discussed in Step 2, equation (29) has two solutions (possibly identical) and $\bar{x}$ represents the smaller solution. Since it is clear that $x = T$ solves equation (29), it must be that

$$\bar{x} \le T. \tag{38}$$

Hence, $\hat{x} \in (0, \bar{x}]$ implies $\hat{x} \le T$ and thus implies the satisfaction of condition (b). Therefore,

$$\text{conditions (a), (b), and (c) hold simultaneously} \iff \hat{x} \in (0, \bar{x}], \tag{39}$$

where $\bar{x}$ is defined in (18).

Note that, given $\hat{x} \le \bar{x}$ and thus $\hat{x} \le 1/r_f$ and given the expression for F in (10), we have $\partial F / \partial \hat{x} \le 0$. Thus, it is clear that, when F takes the expression in (10) with $\text{supp}(F) = [0, \hat{x}]$, increasing $\hat{x}$ leads to a new distribution that first-order stochastically dominates the original distribution. Thus, given $\hat{x} \le \bar{x}$, $\hat{x}$ and the mean of F, μ, have a one-to-one correspondence, and a higher $\hat{x}$ corresponds to a higher μ. Therefore,

$$\hat{x} \in (0, \bar{x}] \iff \mu \le \bar{\mu}, \tag{40}$$

where $\bar{x}$ is defined by equation (18) and $\bar{\mu}$ by equation (20). Hence, given (39), (40), and the fact that $\text{supp}(F) = [0, \hat{x}]$ sustains a symmetric equilibrium if and only if conditions (a), (b), and (c) hold simultaneously, the result presented in (32) must hold.

Result 2 then follows immediately from (21) and (32). □

Next, we show part (ii) of the proposition by showing the following result.

Result 3. *If* $\mu \in (\bar{\mu}, T)$*, where* $\bar{\mu}$ *is defined by equation* (20)*, then, in a symmetric equilibrium,* $\text{supp}(F) = [0, \hat{x}] \cup \{T\}$ *and, over* $[0, \hat{x}]$*,* F *takes the expression in equation* (11).

Proof. Lemma 3 and equation (32) imply that, when $\mu \in (\bar{\mu}, T)$, in a symmetric equilibrium, we must have $\text{supp}(F) = [0, \hat{x}] \cup \{T\}$. Then Lemmas 2 and 3 imply that

$$\pi(x) = \beta x, \quad x \in [0, \hat{x}] \cup \{T\}. \tag{41}$$

By equations (2) and (3) and the fact that, when $\text{supp}(F) = [0, \hat{x}] \cup \{T\}$, it must be that $F(T-) = F(\hat{x})$, we have that

$$\pi(T) = \sum_{i=0}^{n-1} \left(\frac{1}{i+1}\right) \binom{n-1}{i} (1 - F(\hat{x}))^i F(\hat{x})^{n-1-i} V_f. \tag{42}$$

Equation (41) implies that $\pi(T) = \beta T$. Thus, by (42), we must have

$$\beta = \frac{1}{T} \sum_{i=0}^{n-1} \left(\frac{1}{i+1}\right) \binom{n-1}{i} (1 - F(\hat{x}))^i F(\hat{x})^{n-1-i} V_f. \tag{43}$$

Equations (2), (3), (41), and (43) imply equation (11). □

The proposition follows immediately from Results 2 and 3. □

Proof of Proposition 3. Let F and F^o be the equilibrium performance distributions given n and n^o competing firms, respectively, where $2 \leq n < n^o$. Define $\hat{x}$ and $\hat{x}^o$ for F and F^o, respectively, by equation (8). Let β and β^o be the optimal dual variables associated with the capacity constraint for F and F^o, respectively. By Lemmas 2 and 3 and equations (2) and (3),

$$\left(\frac{F(x)}{F^o(x)}\right)^{n-1} = \frac{\beta}{\beta^o} F^o(x)^{n^o - n} \quad x \in (0, \min[\hat{x}, \hat{x}^o]]. \tag{44}$$

Because, by Proposition 2, F^o is continuous and strictly increasing on $[0, \min[\hat{x}, \hat{x}^o]]$ and $F^o(0) = 0$ and because, by Lemma 2, β and β^o are both strictly positive, given that $n^o - n > 0$, there must exist $\varepsilon > 0$ such that $(\beta/\beta^o) F^o(x)^{n^o - n} \in (0, 1)$, $x \in (0, \varepsilon)$. Thus, equation (44) implies that

$$F^o(x) > F(x), \quad x \in (0, \varepsilon). \tag{45}$$

Based on equation (44), we establish the following result.

Result 4. *If $F^o(x) - F(x)$ changes its sign on $(0, \min[\hat{x}, \hat{x}^o]]$, where $\hat{x}$ and $\hat{x}^o$ are defined for F and F^o, respectively, by equation* (8)*, then the following results hold:*

i. there exists $x' \in (0, \min[\hat{x}, \hat{x}^o])$ such that $F^o(x) - F(x) > 0$ if $x \in (0, x')$, $F^o(x) - F(x) = 0$ if $x = x'$, and $F^o(x) - F(x) < 0$ if $x \in (x', \min[\hat{x}, \hat{x}^o])$;

ii. if $F(T-) < 1$*, then* $F^o(T-) < F(T-)$.

Proof. We first establish part (i). By Lemma 2, β and β^o are both strictly positive. By Proposition 2, F^o is continuous and strictly increasing on $[0, \min[\hat{x}, \hat{x}^o]]$. Thus, by equation (44), $x \hookrightarrow F(x)/F^o(x)$ is continuous and strictly increasing on $(0, \min[\hat{x}, \hat{x}^o]]$. Then the single-crossing result in part (i) follows immediately from the hypothesis that $F^o(x) - F(x)$ changes its sign on $(0, \min[\hat{x}, \hat{x}^o]]$.

Now we establish part (ii). By part (i), which has been established above, if $F^o(x) - F(x)$ changes its sign on $(0, \min[\hat{x}, \hat{x}^o]]$, there must exist $x' \in (0, \min[\hat{x}, \hat{x}^o])$ such that $F(x') = F^o(x')$. Then equation (44) implies that

$$\frac{\beta}{\beta^o} = \frac{1}{F^o(x')^{n^o - n}}. \tag{46}$$

Let π and π^o be the contest payoff functions given n and n^o competing firms, respectively. The hypothesis in part (ii) implies that F has point mass on T. Thus, by Lemma 2, it must be that $\pi(T) = \beta T$ and $\pi^o(T) \leq \beta^o T$. Hence, by equation (46),

$$\frac{\pi(T)}{\pi^o(T)} \geq \frac{\beta}{\beta^o} = \frac{1}{F^o(x')^{n^o - n}}. \tag{47}$$

Now consider a given firm in a conjectured R&D race in which the firm has $n^o - 1$ rivals, $n - 1$ of them playing F and the remaining $n^o - n$ playing F^o. With some abuse of notation, let $\hat{\pi}(T)$ and $\hat{P}(T)$ be the expected payoff and the probability of winning for this given firm in this conjectured race, respectively, when its realized performance equals T. Let P be the probability of winning function given n competing firms. Note that

$$\hat{\pi}(T) = \hat{P}(T)V_f \geq P(T)F^o(T-)^{n^o - n}V_f = \pi(T)F^o(T-)^{n^o - n}, \tag{48}$$

where the two equalities follow from equation (3) and the inequality from the fact that $\hat{P}(T)$ represents the probability that the firm with realized performance T wins the conjectured contest while $P(T)F^o(T-)^{n^o - n}$ represents the probability that the firm with realized performance T wins the conjectured contest and, at the same time, strictly outperforms the $n^o - n$ rivals who play F^o. Clearly, the former probability is no less than the latter probability.

Below we prove part (ii) by way of contradiction. Suppose, contrary to part (ii), that $F(T-) \leq F^o(T-)$. Then, clearly, the firm with performance T is weakly better off when all its rivals play F^o than when only $n^o - n$ of them play F^o while the rest play F. Thus,

$$\hat{\pi}(T) \leq \pi^o(T). \tag{49}$$

Equations (48) and (49) imply that

$$\frac{\pi(T)}{\pi^o(T)} \leq \frac{1}{F^o(T-)^{n^o-n}}. \tag{50}$$

Equations (47) and (50) then imply that

$$F^o(T-) \leq F^o(x'). \tag{51}$$

However, Lemma 3 implies that

$$F^o(T-) = F^o(\hat{x}^o). \tag{52}$$

Since $x' \in (0, \min[\hat{x}, \hat{x}^o])$, we must have $x' < \hat{x}^o$. Since, by Proposition 2, F^o is strictly increasing on $(0, \hat{x}^o)$, we must have

$$F^o(\hat{x}^o) > F^o(x'). \tag{53}$$

Equations (52) and (53) imply that $F^o(T-) > F^o(x')$, which contradicts (51). This contradiction establishes part (ii). $\square$

Equation (45) and Result 4 imply that, over $(0, \min[\hat{x}, \hat{x}^o]]$, either a. $F^o(x) - F(x) \geq 0$ (with strict inequality holding for some $x \in (0, \min[\hat{x}, \hat{x}^o]]$), or b. $F^o(x) - F(x)$ changes its sign from positive to negative. In what follows, we show that, regardless of which of these two cases happens in a symmetric equilibrium, the mean-preserving spread result holds.

First, suppose case (a) occurs in a symmetric equilibrium, in which F^o never falls below F over $(0, \min[\hat{x}, \hat{x}^o]]$. In this case, there must exist $x_1 \in (\min[\hat{x}, \hat{x}^o], T)$ such that $F^o(x_1) - F(x_1) < 0$, because otherwise, given the fact that $F^o(0) = F(0) = 0$ and $F^o(T) = F(T) = 1$, F^o would never fall below F and, hence, would be first-order stochastically dominated by F, contradicting the fact that F^o and F have the same mean. Then given the hypothesis in case (a) that F^o never falls below F over $(0, \min[\hat{x}, \hat{x}^o]]$, the result that $F^o(x_1) - F(x_1) < 0$ and the fact that F^o is nondecreasing imply that $F(x_1) > F(\min[\hat{x}, \hat{x}^o])$, which, by Lemma 3, further implies that $\hat{x} > \hat{x}^o$. Thus, $\min[\hat{x}, \hat{x}^o] = \hat{x}^o$, which, by Lemma 3, implies that F^o is constant over $(\min[\hat{x}, \hat{x}^o], T)$. These results imply that, as x increases from 0 to T, $F^o(x) - F(x)$ changes sign only once, from positive to negative, and the sign change occurs somewhere over $(\min[\hat{x}, \hat{x}^o], T)$. This single-crossing result, combined with the fact that F and F^o have the same mean, implies that F^o is a mean-preserving spread of F.

Second, suppose case (b) occurs in a symmetric equilibrium, in which $F^o(x) - F(x)$ changes its sign from positive to negative over $(0, \min[\hat{x}, \hat{x}^o]]$. Given the facts that $F^o(0) = F(0) = 0$ and $F^o(T) = F(T) = 1$, to show the mean-preserving spread relation, it suffices to show that

$$F^o(x) \leq F(x), \quad x \in (\min[\hat{x}, \hat{x}^o], T). \tag{54}$$

We show (54) by way of contradiction. Suppose

$$\exists x' \in (\min[\hat{x}, \hat{x}^o], T) \quad \text{such that} \quad F^o(x') > F(x'). \tag{55}$$

Then, given the hypothesis in case (b), F^o must increase somewhere over $(\min[\hat{x}, \hat{x}^o], x']$, which implies, by Lemma 3, that $\min[\hat{x}, \hat{x}^o] = \hat{x}$. Thus, by Lemma 3, F must be constant over $[\min[\hat{x}, \hat{x}^o], T)$. Note that $x' \in [\min[\hat{x}, \hat{x}^o], T)$. Thus, by (55), $F(\hat{x}) = F(\min[\hat{x}, \hat{x}^o]) = F(x') < F^o(x')$. Hence, given $F^o(x') \leq 1$, we must have $F(\hat{x}) < 1$, which, by Lemma 3, implies that $F(T-) < 1$. By Result 4, $F(T-) < 1$ implies that $F^o(T-) < F(T-)$. Hence, given that F is constant over $[\min[\hat{x}, \hat{x}^o], T)$, we must have

$$F(\min[\hat{x}, \hat{x}^o]) = F(T-) > F^o(T-). \tag{56}$$

Then, since $x' \in (\min[\hat{x}, \hat{x}^o], T)$ and since F and F^o are nondecreasing, we must have

$$F(x') \geq F(\min[\hat{x}, \hat{x}^o]) \quad \& \quad F^o(T-) \geq F^o(x'). \tag{57}$$

However, equations (56) and (57) imply that $F(x') > F^o(x')$, which contradicts (55). This contradiction establishes (54) and completes the proof. □

Proof of Proposition 4. To prove the proposition, we only need to focus on the situation in which μ is sufficiently small. Proposition 2 shows that, when $\mu \leq \bar{\mu}$, where μ is defined in Result 2, the equilibrium performance distribution, F, is a continuous distribution given by equation (10). In what follows, we assume that $\mu \leq \bar{\mu}$. Under this assumption, F is given by equation (10), in which case, by equation (26), $\beta = \frac{e^{-r_f(T-\hat{x})}}{\hat{x}} V_f$. Thus, by equation (13),

$$v_n = \frac{e^{-r_f(T-\hat{x})}}{\hat{x}} V_f \mu. \tag{58}$$

Because the above expression for v_n includes $\hat{x}$, which depends on parameterization of the model, and because there is, in general, no closed-form expression for $\hat{x}$, to proceed, we first identify a lower bound on $\hat{x}$.

Result 5. *Define $\bar{\mu}$ as in Result 2. Suppose $n \geq 2$ and $\mu \leq \bar{\mu}$, in which case* $\mathrm{supp}(F) = [0, \hat{x}]$. *It must be that*

$$\hat{x} > n\mu. \tag{59}$$

Proof. By Proposition 2, given $\mu \leq \bar{\mu}$, the equilibrium performance distribution, F, is given by equation (10). Now consider another distribution, $\hat{F}$, such that $\mathrm{supp}(\hat{F}) = \mathrm{supp}(F) = [0, \hat{x}]$ and

$$\hat{F}(x) = \left(\frac{x}{\hat{x}}\right)^{\frac{1}{n-1}}, \quad x \in [0, \hat{x}]. \tag{60}$$

The mean of $\hat{F}$ is given by

$$\int_0^{\hat{x}} x d\hat{F}(x) = \frac{\hat{x}}{n}. \tag{61}$$

Note that, given $r_f > 0$, equations (10) and (60) imply that

$$\hat{F}(x) < F(x), \quad x \in (0, \hat{x}). \tag{62}$$

Thus, given that $\mathrm{supp}(\hat{F}) = \mathrm{supp}(F) = [0, \hat{x}]$ and both F and $\hat{F}$ are continuous distributions, the mean of $\hat{F}$, $\hat{x}/n$, must be strictly greater than the mean of F, μ. The result thus follows. □

By equations (12) and (58), we can express the difference between the monopoly payoff and the aggregate firm payoff under competition with $n \geq 2$ firms as

$$v_1 - n v_n = \left[\frac{\mu}{T} + e^{-rT}\left(1 - \frac{\mu}{T}\right)\right] V_f - \frac{n e^{-r(T-\hat{x})}}{\hat{x}} V_f \mu$$
$$= e^{-rT} V_f \left[1 + \left(\frac{e^{rT} - 1}{T} - \frac{n e^{r\hat{x}}}{\hat{x}}\right)\mu\right]. \tag{63}$$

Because we have assumed that $\mu \leq \bar{\mu}$, by (40), we have $\hat{x} \leq 1/r$, in which case $\hat{x} \hookrightarrow e^{r\hat{x}}/\hat{x}$ is decreasing and thus, by (63), $\partial(v_1 - n v_n)/\partial\hat{x} \geq 0$. Hence, given that $\hat{x} > n\mu$, which has been established in Result 5, we must have

$$v_1 - n v_n \geq e^{-rT} V_f \left[1 + \left(\frac{e^{rT} - 1}{T}\right)\mu - e^{rn\mu}\right]. \tag{64}$$

Note that the right hand side of (64) tends to 0 as $\mu \to 0$. Also, the right hand side of (64) is strictly increasing in μ if

$$\mu < \frac{1}{rn} \ln\left(\frac{e^{rT} - 1}{rTn}\right).$$

Thus, given the assumption that $\mu \leq \bar{\mu}$, we have

$$\mu \in \left(0, \frac{1}{rn} \ln \left(\frac{e^{rT} - 1}{rTn}\right)\right) \quad \Longrightarrow \quad v_1 - n v_n > 0. \tag{65}$$

Note that the interval, $\left(0, \frac{1}{rn} \ln \left(\frac{e^{rT}-1}{rTn}\right)\right)$, is nonempty if and only if

$$e^{rT} - 1 - rTn > 0. \tag{66}$$

On the nonnegative real line, the map, $x \hookrightarrow e^x - 1 - xn$, is strictly convex and U-shaped, equals 0 when $x = 0$, and tends to infinity when $x \to \infty$. Thus, for any fixed $n \geq 2$, there uniquely exists $k > 0$ such that $e^k - 1 - kn = 0$, and, on the nonnegative real line, $e^x - 1 - xn > 0$ if and only if $x > k$. Thus, condition (66) is equivalent to condition (14). Hence, conditions (14) and (65) and the assumption that $\mu \leq \bar{\mu}$ imply that $v_1 - n v_n > 0$ if (14) holds and if

$$0 < \mu < \mu^* = \min \left[\bar{\mu}, \frac{1}{rn} \ln \left(\frac{e^{rT} - 1}{rTn}\right)\right]. \tag{67}$$

This completes the proof. □