Collusion, Mergers and Antitrust Policy

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January 2019

Abstract

This paper develops a model that formalizes several connections between mergers, collusion and antitrust policy. In equilibrium, firms may merge to make collusion sustainable when it cannot be sustained with the original set of firms. A rise in the probability of detecting and prosecuting collusion could induce a wave of mergers, so firms can sustain collusion again. Indeed, mergers could fully neutralize the pro-competitive effect of an improvement in collusion detection and prosecution. From a normative perspective, we show that merger policy is crucial when cost synergies are small (or nonexistent) and the competition authority can only deter collusion by restricting mergers. Finally, we highlight that mergers could be more harmful (less beneficial) than expected if the impact that mergers have on the competition regime is properly considered, which suggests a formal definition for the notions of unilateral and coordinated effects associated with mergers.

JEL classification codes: D43, L12, L13, L41

Keywords: collusion, mergers, antitrust policy, unilateral and coordinated effects

1 Introduction

There are two typical channels through which firms can gain market power: mergers and collusion.[1] Two extensive bodies of theoretical literature in industrial organization have studied mergers and collusion, separately. Surprisingly, only very few models have considered both simultaneously, even when historical cases and the available empirical evidence suggest important connections between mergers and collusion. For example, after the U.S. Congress passed the Sherman Act of 1890, significantly increasing the cost of collusion, there was a wave of mergers, which eventually, led to the Clayton Act of 1914, dealing with mergers and other anticompetitive practices (see, for example, Bittlingmayer (1985)).[2] In this paper, we develop an infinitely dynamic model of oligopoly to study the connections between mergers and collusion.

In the model, there is a set of firms that supply a market with linear demand. In each period firms are randomly matched in pairs and have the chance to merge. After merger decisions have been made, firms simultaneously and independently select the quantity produced. If firms decide to collude, with

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[1] For simplicity, we use the term mergers to refer to mergers and acquisitions.

[2] For more recent evidence on mergers and collusion see Davies et al. (2015).

some probability the competition authority detects collusion, in which case it bans mergers and forces firms to compete in all subsequent periods. Moreover, we assume that the competition authority does not allow firms to merge if only two firms are left, i.e., if the merger would create a monopoly. We study two possible settings. First, we consider the case of n homogeneous firms when mergers have no effect on the cost function of the firms. Then, we explore the case of four initially identical firms when mergers produce cost reductions (for example, due to synergies), which may generate an industry with heterogeneous firms. Both settings help formalizing several connections between mergers and collusion.

In the model with homogeneous firms and no synergies, firms do not have a priori any incentive to merge. Either when they are forced to compete or when they are colluding, there are no mergers. However, in equilibrium, they may merge to move the industry from competition to collusion. That is, firms may choose to merge to reduce the number of firms in the industry and, hence, be able to sustain collusion when collusion could not have been sustained with the original number of firms. An important implication is that a rise in the probability of detecting and prosecuting collusion could induce a wave of mergers. If the improvement in detecting and prosecuting collusion destabilize the collusive agreement, firms react merging until the number of firms remaining in the industry is low enough to sustain collusion again. Thus, mergers could fully neutralize the pro-competitive effect of an improvement in the probability of detecting and prosecuting collusion. An immediate policy implication is that improving antitrust enforcement without simultaneously strengthening merger policy could be ineffective to promote competition. Indeed, in our model with homogeneous firms, as no cost-synergy arises from mergers, the optimal policy is to ban mergers. The model has also an important normative implication. We show that the welfare loss caused by mergers is higher when firms can collude than when they are forced to compete. That is, under competition, mergers produce a reduction in the total surplus generated in the industry, but if mergers lead to collusion, the reduction in total surplus is even larger. Thus, if the effect that mergers may have on the competition regime is not properly considered, the negative welfare effect of mergers will be underestimated.

In the setting with four firms and positive synergies, we also find that firms could merge in order to make collusion sustainable. Positive synergies also bring novel results. Contrary to the case of homogeneous firms and no synergies, if firms are forced to compete and the cost reductions generated by a merger are high enough, there could be mergers in equilibrium. Thus, while in the case of homogeneous firms and no synergies the only reason firms choose to merge is to sustain collusion, once we introduce positive synergies there are two motives: exploiting cost synergies and colluding. Moreover, if firms are forced to compete, it is perfectly possible that the total surplus generated in the industry increases with mergers. In other words, under competition, mergers could be welfare improving. However, we show that this is less likely to hold when mergers lead to collusion. Indeed, even when mergers under competition are welfare improving, the total effect of mergers on total surplus could be negative. This has significant implications for merger policy. Indeed, we show that merger policy plays a key role when the probability of detecting and prosecuting collusion adopts intermediate values. The reason is that in such circumstances the probability of detection is high enough to dissuade collusion when firms do not merge, but not sufficiently high to deter collusion after they merge. Thus, merger policy ultimately determines whether the industry operates under collusion or competition. Specifically, we also show that when cost-synergies are below some threshold, it is optimal to restrict mergers and, hence, induce firms to compete, while when cost-synergies are above such threshold, the total surplus is higher if firms merge and then collude.

Finally, the model suggests a formal definition for the notions of unilateral and coordinated effects associated with mergers, two terms commonly employed in antitrust policy circles, which have not been rigorously defined. The unilateral welfare effect associated with a set of mergers is the effect on total surplus that those mergers will produce under competition, that is, assuming that firms will not collude. The coordinated welfare effect associated with a set of mergers is the effect on total surplus exclusively attributed to a change in the competition regime, that is, a move from competition to collusion. Our decomposition of unilateral and coordinated effects is closely related to the normative implications of our model. In the setting with homogeneous firms and no synergies, both effects are negative. In the setting with four firms and positive synergies, the sign of the unilateral effect is ambiguous, and the coordinated effect is always negative.

1.1 Related literature

As we have already mentioned, few papers have studied the connections between mergers and collusion. Compte et al. (2002) considers the effects of horizontal mergers in a repeated Bertrand model with firms having different capacity constraints. Vasconcelos (2005) explores the same scenario but employing a Cournot model with capital asymmetries. Both studies conclude that the more symmetric firms are, the easier is to sustain collusion.[3] Ivaldi and Lagos (2017) develops a model in which mergers strengthen the incentives to collude of the merged firm but weaken the incentives to collude of the non-merging firms (see also Kovacic et al. (2009)). We build on these works, endogenizing merger decisions. That is, we do not only consider the effects that mergers would have on the sustainability of collusion, but we also study the merger decisions of the firms. In our model, if a merger facilitates collusion, firms take this into account in their merger decisions. Closer to this paper is Ganslandt et al. (2012). They propose a model where collusion is more sustainable for moderately asymmetric industries and study the welfare effects of adopting a merger policy that bans mergers that lead to symmetric industries.[4] Our model, however, differs in several ways. First, we allow mergers in every period, which can induce mergers waves. Second, we consider an unrestricted optimal merger policy, which highlights the negative welfare impact arising from coordinated effects. Third, we study the connections between merger and competition policies.

Our paper is also related to the literature on horizontal mergers under oligopoly. Salant et al. (1983) show that in a Cournot model with identical firms, each with constant marginal cost, mergers are only profitable if they involve at least 80% of firms (see also Perry and Porter (1985) and McAfee and Williams (1992)).[5] In our setting with n homogeneous firms and no synergies we use a sequential version of this result. We assume that in each period firms are matched in pairs and each pair of matched firms have the chance to merge. Thus, in each period at most 50% of the firms can merge. Since we also assume that the competition authority does not allow a monopoly, the 80% threshold is never reached. Thus,

[3]One difference is that in Compte et al. (2002) larger firms have more incentives to deviate from collusion while in Vasconcelos (2005) smaller firms are more prone to deviate.

[4]In particular, they show that such a policy reduces welfare because it induces firms to merge to asymmetric markets.

[5]Perry and Porter (1985) extends Salant et al. (1983), allowing that the cost function of each firm also depends on its capital stock, which generates increasing marginal cost curves. They use this extension to study how the distribution of the industry capital stock among the firms affects the incentives to merge. McAfee and Williams (1992) employ Perry and Porter (1985)'s model and find that horizontal mergers are more likely to be welfare enhancing the more concentrated is the ownership of the non-merging firms. For example, they show that if mergers create the new largest firm, or increase the size of the largest firm, they will reduce welfare.

under competition, firms do not find mergers profitable. Nevertheless, in equilibrium, firms may merge to make collusion sustainable.

Farrell and Shapiro (1990) study horizontal mergers using a Cournot model with general cost functions. They find that if mergers do not generate synergies, they will increase the equilibrium price. However, mergers can still improve total welfare if the merging firms have small shares and the non-merging firms have larger shares. Since we employ a simple model with constant marginal cost, in our setting with homogeneous firms and no synergies, mergers always induce a rise in the equilibrium price and a reduction in total welfare. On the other hand, in our setting with four firms and cost synergies, mergers could be welfare enhancing, but it is important to note that this is less likely to hold when mergers lead to collusion.

Our work also relates to Amir et al. (2009) who study the feasibility of mergers once there is uncertainty over the ability of merging firms to achieve efficiency gains. They show that "a bilateral merger is profitable provided the non-merged firms sufficiently believe that the merger will generate large enough efficiency gains, even if ex post none actually materialize". In our case, mergers are profitable also even in the absence of efficiency gains, if merging moves the market from competition to collusion. Qiu and Zhou (2007) model sequential mergers using a Cournot model with different marginal costs. They show that mergers are strategic complements in the sense that firms' incentives to merge increase when other firms also merge. This is because mergers reduce the number of free riders for a given merger, making the merger more profitable. Thus, forward-looking firms may engage in mergers strategically. That is, firms may carry out an otherwise unprofitable merger to facilitate some further mergers. In our setting with homogeneous firms and no synergies, if firms are forced to compete, they never find mergers profitable, regardless what other firms are doing. However, strategic mergers can occur in equilibrium because firms have incentives to make collusion sustainable in the future. Thus, our model provides an alternative mechanism to produce a wave of mergers. In our setting with four firms and cost synergies, if firms are forced to compete and synergies adopt intermediate values, mergers are more profitable if other firms also merge, while if synergies are high, mergers are more profitable when other firms do not merge.[6] In any case, the possibility of facilitating collusion is always a strategic motive for firms to merge.

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 characterizes the equilibrium and perform welfare analysis for the case of homogeneous firms and no synergies. Section 4 deals with the case with four firms and positive synergies. Section 5 concludes.

2 A model of mergers and collusion

Consider a market with inverse demand given by $P_t = a - Q_t$, where P_t and Q_t are the price and quantity in period t, respectively. Time is infinite, discrete and indexed by $t \in \{0, 1, ...\}$. In each period, the market is supplied by a finite number of firms, all of which have a common discount factor $\delta \in (0, 1)$. Specifically, let N_{t-1} denote the set of firms at the beginning of period t and assume that N_{t-1} is a partition of $\{1, ..., n\}$ with $n > 2$. Moreover, assume that, initially, the industry is integrated by n independent firms, i.e., $N_{-1} = \{\{1\}, ..., \{n\}\}$.

[6] If synergies are low, under competition, it is still the case that firms do not find mergers profitable regardless of the decisions of other firms.

2.1 Matching and mergers

At the beginning of period t, if the number of firms in N_{t-1} is greater than 2, firms are randomly matched in pairs.[7] Matched firms have the chance of merging. Let $\mathcal{M}_t$ be the set of all possible ways in which firms in N_{t-1} can be matched and denote by M_t a particular set of matches. Furthermore, assume that every $M_t \in \mathcal{M}_t$ occurs with the same probability.[8] For example, if $N_{t-1} = \{\{1,2\},\{3\},\{4\}\}$, then the possible set of matches are $M_t = \{\{1,2,3\},\{4\}\}$, $M_t = \{\{1,2,4\},\{3\}\}$, and $M_t = \{\{1,2\},\{3,4\}\}$, each of which occurs with probability 1/3.

Each pair of firms in M_t must decide whether they merge or not. Let $m_t^x(y) \in \{0,1\}$ denote the merging decision of firm $x \in N_{t-1}$ when it is matched with firm $y \in N_{t-1}$, where $m_t^x(y) = 1$ indicates that firm x accepts the merger and $m_t^x(y) = 0$ that it does not. Only if both firms agree a merge occurs. If firms x and y decide to merge, they form a new firm denoted by $z = x \cup y$. Given the set of matched firms (M_t) and the merging decisions of the firms ($m_t^x(y)$ for all $x, y \in N_{t-1}$), we can easily compute N_t, the set of firms after merger decisions have been made in period t. For example, suppose that $N_{t-1} = \{\{1,2\},\{3\},\{4\}\}$, $M_t = \{\{1,2\},\{3,4\}\}$, $m_t^{\{3\}}(\{4\}) = m_t^{\{4\}}(\{3\}) = 1$. Then, $N_t = \{\{1,2\},\{3,4\}\}$. That is, at the beginning of period t there are 3 firms, firm $\{1,2\}$ remains unmatched (equivalently, it is matched with itself), while firms $\{3\}$ and $\{4\}$ are matched and decide to merge. Then, after merger decisions in period t have been made, there are two firms, namely, $\{1,2\}$, and $\{3,4\}$. It will also be useful to define $\mathcal{N}_t(N_{t-1})$, the set all feasible N_t that can be reached from N_{t-1}. For example, if $N_{t-1} = \{\{1,2\},\{3\},\{4\}\}$, then $\mathcal{N}_t(N_{t-1}) = \{\{1,2,3\},\{4\}\},\{\{1,2,4\},\{3\}\},\{\{1,2\},\{3,4\}\},\{\{1,2\},\{3\},\{4\}\}$.

The cost function of the n original firms is $C(q) = cq$ where $c \in (0,a)$ and $q \geq 0$ is the quantity produced. For any other firm, the cost function is $C(q) = (1-s)\,cq$, where $s \in [0,1)$ captures a synergy effect. Thus, the first time two of the original firms decide to merge, they enjoy a cost reduction, for example, due to technological synergies. Afterwards, if they merge with another firm, the new firm will inherit the low unit cost, but there will be no additional reduction in costs.[9]

2.2 Competition, collusion and antitrust policy

Let q_t^z indicate the quantity produced by firm $z \in N_t$ in period t and π_t^z the profits that firm $z \in N_t$ obtains in period t. Then:

$$\pi_t^z = \pi^z\left((q_t^x)_{x\in N_t}\right) = \left(a - \sum\nolimits_{x\in N_t} q_t^x - c^z\right) q_t^z,$$

where $c^z = c$ if z is one of the original firms and, otherwise, $c^z = (1-s)\,c$. Also assume that, if firm z is the outcome of the merger of firms $x \in N_t$ and $y \in N_t$ in period t, then the original firms split the profits of the new firm equally. Thus, $\pi_t^x = \pi_t^y = \pi_t^z/2$.

The quantity produced by each firm in period t is determined as follows. Firms in N_t simultaneously decide whether to compete, collude or deviate from collusion. If firm z chooses to compete

[7] If the number of firms is odd, then one firm remains unmatched or, which is equivalent, it is matched with itself. If only two firms remain, then each firm is matched with itself.

[8] Let n_{t-1} be the number of firms in N_{t-1}. Then, the number of possible ways of matching the firms in N_{t-1} is given by $M = \prod_{j=0}^{j=\frac{n_{t-1}-2}{2}} (n_{t-1} - 1 - 2j)$ if n_{t-1} is even and $M = \prod_{j=0}^{j=\frac{n_{t-1}-3}{2}} (n_{t-1} - 1 - 2j) + \prod_{j=1}^{j=\frac{n-1}{2}} (n_{t-1} - 2j)$ if n_{t-1} is odd. Thus, the probability that a particular $M_t \in \mathcal{M}_t$ occurs is $(M)^{-1}$.

[9] New firms have the same discount factor as the original ones.

in period t, then it selects $q_t^z = q^{z,COM}(N_t)$, where $q^{z,COM}(N_t)$ is the quantity associated with a Cournot oligopoly. If firm z chooses to collude in period t, then it selects $q_t^z = q^{z,COL}(N_t)$, where $q^{z,COL}(N_t) = \left[q^{z,COM}(N_t) / \sum_{x \in N_t} q^{x,COM}(N_t)\right] Q_t^M$ and Q_t^M is the quantity produced by a monopolist with unit cost c ($c(1-s)$) if all firms have unit cost c (if at least one firm has unit cost $c(1-s)$). Thus, under collusion all firms reduce production to induce the monopoly quantity, but keeping the market shares that firms would obtain under competition. Finally, if firm z decides to deviate from collusion, it selects $q_t^z = q^{z,DEV}(N_t)$, where $q^{z,DEV}(N_t)$ is the best reaction of firm z when all the other firms in N_t are producing the collusion quantity, i.e., $q_t^x = q^{x,COL}(N_t)$ for all $x \neq z$ and $x \in N_t$.

Let $\pi^{z,COM}(N_t)$, $\pi^{z,COL}(N_t)$, and $\pi^{z,DEV}(N_t)$ denote the profits of firm z in period t under competition, collusion and deviation, respectively. Then:

$$\begin{aligned}
\pi^{z,COM}(N_t) &= \pi^z\left(\left(q^{x,COM}(N_t)\right)_{x \in N_t}\right) = \left(a - \sum\nolimits_{x \in N_t} q^{x,COM}(N_t) - c^z\right) q^{z,COM}(N_t) \\
\pi^{z,COL}(N_t) &= \pi^z\left(\left(q^{x,COL}(N_t)\right)_{x \in N_t}\right) = \left(a - \sum\nolimits_{x \in N_t} q^{x,COL}(N_t) - c^z\right) q^{z,COL}(N_t) \\
\pi^{z,DEV}(N_t) &= \max_{q^z} \pi^z\left(\left(q^{x,COL}(N_t)\right)_{x \in N_t, x \neq z}, q^z\right) \\
&= \left(a - \sum\nolimits_{x \in N_t, x \neq z} q^{x,COL}(N_t) - q^{z,DEV}(N_t) - c^z\right) q^{z,DEV}(N_t)
\end{aligned}$$

Finally, suppose that there is a competition authority in charge of detecting and prosecuting collusion. Specifically, if firms are colluding in period t, with probability $\alpha \in [0,1)$, the competition authority detects it; in which case, it bans any future merger and forces firms to compete in all subsequent periods.

2.3 Period-t game

Summing up, the timing of events in period t is as follows:

1. Suppose that collusion has not been detected in previous periods. Then:
   (a) Firms in N_{t-1} are randomly matched in pairs. Each pair of matched firms must decide whether they merge or not. Let N_t denote the set of firms after merger decisions have been made in period t.
   (b) The firms in N_t simultaneously and independently decide if they compete, collude or deviate from collusion and the corresponding quantities produced in period t.
   (c) With probability $\alpha \in (0,1)$ collusion is detected.
2. Suppose that collusion has been detected in previous periods. Then, $N_t = N_{t-1}$ and the competition authority forces firms in N_t to compete.

2.4 Strategies, equilibrium and selection

Let $m_t^z(y) \in \{0,1\}$ denote the merger decision of firm $z \in N_{t-1}$ when matched with firm $y \in N_{t-1}$ in period t and $q_t^z \in \left\{q^{z,COM}(N_t), q^{z,COL}(N_t), q^{z,DEV}(N_t)\right\}$ the quantity decision of firm $z \in N_t$ in period t. Let $e_t = 1$ ($e_t = 0$) indicate that collusion has (not) been detected in period t. Then, a t-history is a sequence $h_t = \left\{M_\tau, \left(m_\tau^x(y)\right)_{x \in N_{\tau-1}, x \cup y \in M_\tau}, N_\tau, \left(q_\tau^x\right)_{x \in N_\tau}, e_\tau\right\}_{\tau=0}^{\tau=t}$, where $N_{-1} = \{\{1\}, ..., \{n\}\}$. Denote by $\mathcal{H}_t$ the set of all t-histories.

2.4.1 Strategies

A period-t merger strategy for firm $z \in N_{t-1}$ is a function $m_t^z : \mathcal{H}_{t-1} \times N_{t-1} \to \{0,1\}$. That is, m_t^z indicates for each history $h_{t-1} \in \mathcal{H}_{t-1}$ and possible match $y \in N_{t-1}$, the merger decision of firm $z \in N_{t-1}$. A profile of merger strategies is a sequence of functions $\left\{ (m_t^z)_{z \in N_{t-1}} \right\}_{t=0}^{\infty}$. Antitrust policy imposes the following restriction on $\left\{ (m_t^z)_{z \in N_{t-1}} \right\}_{t=0}^{\infty}$. If $e_\tau = 1$ for $\tau < t$, then $m_t^z(h_{t-1}, y) = 0$ for all $y, z \in N_{t-1}$ and $h_{t-1} \in \mathcal{H}_{t-1}$.

A period-t quantity strategy for firm $z \in N_t \in \mathcal{N}_t(N_{t-1})$ is a function $q_t^z(N_t) : \mathcal{H}_{t-1} \to \left\{ q^{z,COM}(N_t), q^{z,COL}(N_t), q^{z,DEV}(N_t) \right\}$, where recall that $\mathcal{N}_t(N_{t-1})$ is the set all the feasible sets of firms N_t that can be reached from N_{t-1}. That is, $q_t^z(N_t)$ indicates for each history $h_{t-1} \in \mathcal{H}_{t-1}$, the quantity decision of firm $z \in N_t \in \mathcal{N}_t(N_{t-1})$. A profile of quantity strategies is a sequence of functions $\left\{ (q_t^z(N_t))_{z \in N_t \in \mathcal{N}_t(N_{t-1})} \right\}_{t=0}^{\infty}$. Antitrust policy imposes the following restriction on $\left\{ (q_t^z(N_t))_{z \in N_t \in \mathcal{N}_t(N_{t-1})} \right\}_{t=0}^{\infty}$. If $e_\tau = 1$ for $\tau < t$, then $q_t^z(N_t)(h_{t-1}) = q^{z,COM}(N_t)$ for all $z \in N_t \in \mathcal{N}_t(N_{t-1})$ and $h_{t-1} \in \mathcal{H}_{t-1}$.

A profile of strategies $\left\{ (m_t^z)_{z \in N_{t-1}}, (q_t^z(N_t))_{z \in N_t \in \mathcal{N}_t(N_{t-1})} \right\}_{t=0}^{\infty}$ induces an outcome path $O = \left\{ N_t, (q_t^z)_{z \in N_t} \right\}_{t=0}^{\infty}$, that is, a set of firms N_t and a quantity supplied for each firm in N_t for each period $t \geq 0$ until collusion is detected by the competition authority. Thereafter, mergers are not allowed and firms are forced to compete. That is, if collusion is detected in period T, then the outcome path O is stopped and $\left(N_\tau, (q_\tau^z)_{z \in N_\tau} \right) = \left(N_T, \left(q^{COM}(N_T) \right)_{z \in N_T} \right)$ for all $\tau > T$. Associated with the outcome path O, let $v_t^z(O)$ indicate the expected discounted continuation profits of firm z from period t onward. Then, $v_t^z(O) = \mathbf{E}_t \left[\sum_{\tau=t}^{\infty} \delta^{\tau-t} \pi_\tau^z \left((q_\tau^x)_{x \in N_\tau} \right) \right]$, where the expectation operator $\mathbf{E}_t$ is required because the time at which collusion is detected is a random variable.[10]

We assume that firms employ Nash reversion strategies. Formally, a Nash reversion strategy profile calls for playing outcome path $O = \left\{ N_t, (q_t^z)_{z \in N_t} \right\}_{t=0}^{\infty}$ prior to any deviation or detection of collusion and no mergers and competition thereafter. Note that no mergers, i.e., $m_t^z(y) = 0$ for all $y, z \in N_{t-1}$, and competition for any set of firms, i.e., $q_t^z(N_t) = q^{z,COM}(N_t)$ for all $z \in N_t \in \mathcal{N}_t(N_{t-1})$, is always a subgame perfect Nash equilibrium of period-t game. Thus, if in period t a firm deviates from the merging decisions implicitly prescribed by outcome path O, then all firms move to competition immediately and stay there forever. That is, suppose that after merger decisions have been made in period t, the set of firms is $\hat{N} \neq N_t$. Then, firms will choose $q_\tau^z = q^{z,COM}\left(\hat{N}\right)$ for all $z \in \hat{N}$ and $\tau \geq t$ and $m_\tau^z(y) = 0$ for all $y, z \in \hat{N}$ and $\tau > t$. If in period t, a firm deviates from the quantity decisions prescribed by outcome path O, then all firms move to no mergers and competition from period $t+1$ onward.

2.4.2 Notion of equilibrium and selection criteria

For the notion of equilibrium we employ subgame perfect Nash equilibrium (SPNE) assuming that firms use a Nash reversion strategy profile that punishes any deviation with no mergers and competition forever. In other words, we look for outcome paths that can be supported as an SPNE using a Nash reversion strategy profile that calls for punishing any deviation with no mergers and competition thereafter.

[10]Formally, the time at which collusion is detected T is a stopping time.

Finally, when more than one outcome path can be supported as an SPNE, we focus on the outcome path the induces the highest expected discounted profits for the firms. In other words, we assume that firms can coordinate their behavior to avoid playing Pareto dominated equilibria.

2.5 Welfare

Let Q_t^L and Q_t^H denote the aggregate quantity produced in period t by firms with unit cost $c(1-s)$ and c, respectively. Given the linear demand $P_t = a - Q_t$, the aggregate consumer surplus in period t is $CS_t = \left(a - Q_t^L - Q_t^H\right)\left(Q_t^L + Q_t^H\right)/2$, while aggregate profits in period t are $\Pi_t = \left(a - Q_t^L - Q_t^H - c\right)\left(Q_t^L + Q_t^H\right) + scQ_t^L$. Then, the total surplus generated in period t is

$$TS_t = CS_t + \Pi_t = \frac{\left(3a - 2c - 3Q_t^L - 3Q_t^H\right)\left(Q_t^L + Q_t^H\right)}{2} + scQ_t^L$$

The expected discounted total surplus from period t onward is $W_t = \mathbf{E}_t\left[\sum_{\tau=t}^{\infty} \delta^{\tau-t} TS_\tau\right]$.

3 Equilibrium and welfare with no synergies ($s = 0$)

This section studies the case in which there are no synergies and, therefore, all firms are homogeneous. This is an important benchmark, because in a setting with no synergies firms have no apparent incentive to merge. Indeed, under competition or collusion, if firms are matched in pairs, mergers are not part of a SPNE outcome path. However, we find that in some situations firms merge in equilibrium in order to make collusion sustainable. The section is organized as follows. First, we study mergers under competition and collusion and show that, in equilibrium, there are no mergers. Then, we focus on how the number of firms operating in the market affects the sustainability of collusion. Finally, we study merger decisions when a change in the number of firms affects the incentive to collude.

3.1 No mergers under competition

In order to gain intuition, it is useful to study mergers under competition in a given period. For this purpose, consider the game in period t and suppose that whatever set of firms emerge from the merger process, they will be forced to compete. Then, the quantity produced by firm $z \in N_t$ will be $q_t^z = q^{COM}(n_t) = (a-c)/(n_t+1)$, where N_t (n_t) is the set (number) of firms after merger decisions have been made.[11] The profits obtained by each firm will be $\pi_t^z = \pi^{COM}(n_t) = (a-c)^2/(n_t+1)^2$ and the industry profits will be $n_t\pi^{COM}(n_t) = n_t(a-c)^2/(n_t+1)^2$. Note that $n_t\pi^{COM}(n_t)$ is decreasing in n_t. Thus, as a group, firms maximize their joint profits if each pair of matched firms in period t decide to merge.[12] However, mergers are not a Nash equilibrium. In order to see this, assume that firms x and $y \in N_{t-1}$ are matched in period t. If x and y decide to merge, then the profits of x and y will be given by $\pi_t^x = \pi_t^y = \pi^{COM}(n_t)/2 = (a-c)^2/2(n_t+1)^2$, where n_t is the number of firms in period t after merger decisions have been made when x and y merge. On the contrary, if firms x and y do not merge, then the

[11]Since mergers do not generate any cost reduction, all firms have the same unit cost c and, hence, all we need to know in each period to compute equilibrium quantities and profits is the number of firms remaining, which is denoted by $n_t \geq 2$.

[12]$\pi^{COM}(n_t)$ is also decreasing in n_t. Thus, each of the remaining firms maximizes its profits when each pair of matched firms in period t decide to merge.

profits of x and y will be $\pi_t^x = \pi_t^y = \pi^{COM}(n_t+1) = (a-c)^2/(n_t+2)^2$, where n_t+1 is the number of firms in period t after merger decisions have been made when x and y do not merge. It is easy to verify that

$$n_t \geq 2 \Rightarrow \pi^{COM}(n_t+1) > \frac{\pi^{COM}(n_t)}{2} \tag{1}$$

That is, regardless of the merging decisions of other firms, under competition, a firm always prefers to do not merge. In other words, under competition, pairwise mergers have the structure of a prisoner's dilemma. If all pairs of matched firms decide to merge, profits are higher for each firm, but each firm can earn higher profits if it does not merge and allow other firms to merge. Thus, if we only consider the game in a given period and firm must compete, there are no mergers in equilibrium. Formally, the unique SPNE outcome path of the game in period t is $\left(n_t, (q_t^z)_{z\in N_t}\right) = \left(n_{t-1}, \left(q^{COM}(n_{t-1})\right)_{z\in N_{t-1}}\right)$.

Lemma 1 extends this result to a situation in which firms will be forced to compete in all future periods.

Lemma 1 ***No mergers under competition.*** *Suppose that for all $\tau \geq t$ firms are forced to compete. Then, the unique SPNE outcome path from period t onward is $\left(n_\tau, (q_\tau^z)_{z\in N_\tau}\right) = \left(n_{t-1}, \left(q^{COM}(n_{t-1})\right)_{z\in N_{t-1}}\right)$ for all $\tau \geq t$.* ***Proof:*** *See Appendix A.1.* ■

The intuition behind Lemma 1 is simple. Naturally, repeating the SPNE outcome of the game in every period $\tau \geq t$ is an SPNE outcome path from period t onward. The surprising result is that this is the unique SPNE outcome path. The reason is as follows. If after a pair of matched firms merge, only 3 firms left, each firm in the pair of matched firms will prefer to do not merge because $\pi^{COM}(3) > \pi^{COM}(2)/2$. All firms anticipate this and, hence, if after a pair of matched firms merge, only 4 firms left, $\pi^{COM}(4) > \pi^{COM}(3)/2$ implies that each firm in the pair of matched firms will prefer to do not merge. In general, if after a pair of matched firms merge, only n_t firms left and, once n_t firms left there will be no more mergers, then $\pi^{COM}(n_t+1) > \pi^{COM}(n_t)/2$ implies that each firm in the pair of matched firms will prefer to do not merge.

Since every firm would earn higher profits if all firms merge, it is also interesting to explore why the folk theorem does not work for mergers. In order to see this, suppose that initially there are 4 firms. If each pair of matched firms merge, then there will be two firms. Since $\pi^{COM}(2)/2 > \pi^{COM}(4)$, every firm will earn higher profits. However, this is not an SPNE outcome path because $\pi^{COM}(3) > \pi^{COM}(2)/2$ implies that each firm in a match prefers to deviate and do not merge. Moreover, note that threatening the defector with no future mergers has no effect on its incentives. Indeed, the defector is already anticipating that there will be no more mergers.

From a normative point of view, the lack of incentives to merge under competition is good news. If firms merge, consumers will be worse off, firms as a group will be better off and the aggregate surplus generated in the market will be lower. Formally, under competition, the aggregate consumer surplus is $CS^{COM}(n_t) = (a-c)^2(n_t)^2/2(n_t+1)^2$, which is increasing in n_t; aggregate profits are $\Pi^{COM}(n_t) = (a-c)^2 n_t/(n_t+1)^2$, which is decreasing in n_t; and the total surplus is $TS^{COM}(n_t) = CS^{COM}(n_t) + \Pi^{COM}(n_t) = (a-c)^2 n_t(n_t+2)/2(n_t+1)^2$, which is increasing in n_t.[13]

[13] Note that $\frac{\partial \ln CS^{COM}(n_t)}{\partial n_t} = \frac{2}{n_t(n_t+1)} > 0$, $\frac{\partial \ln \Pi^{COM}(n_t)}{\partial n_t} = \frac{-(n_t-1)}{n_t(n_t+1)} < 0$, and $\frac{\partial \ln TS^{COM}(n_t)}{\partial n_t} = \frac{2}{n_t(n_t+2)(n_t+1)} > 0$.

3.2 No mergers under collusion

Once again, in order to gain intuition, consider the game in period t and suppose that firms will be forced to collude. Then, the quantity produced by firm $z \in N_t$ will be $q_t^z = q^{COL}(n_t) = (a-c)/2n_t$ and the profits obtained by each firm will be $\pi_t^z = \pi^{COL}(n_t) = (a-c)^2/4n_t$. Following the same logic we used to study mergers under competition, it is easy to verify that

$$n_t \geq 2 \Rightarrow \pi^{COL}(n_t+1) > \frac{\pi^{COL}(n_t)}{2} \tag{2}$$

That is, if collusion is expected in period t, no firm has an incentive to merge. Intuitively, each firm earns higher profits as a single member of a collusive agreement with n_t+1 participants than as one of the two equal partners of a combined firm in a collusive agreement with n_t participants. Formally, the unique SPNE outcome of the game in period t is $\left(n_t, (q_t^z)_{z\in N_t}\right) = \left(n_{t-1}, \left(q^{COL}(n_{\tau-1})\right)_{z\in N_{t-1}}\right)$.

Lemma 2 extends this result to a situation in which firms will be forced to collude in all future periods.

Lemma 2 ***No mergers under collusion.*** *Suppose that for all $\tau \geq t$ firms are forced to collude until they are detected, whereupon they will not be allowed to merge anymore and will be forced to compete. Then, the unique SPNE outcome path from period t onward is $\left(n_\tau, (q_\tau^z)_{z\in N_\tau}\right) = \left(n_{t-1}, \left(q^{COL}(n_{t-1})\right)_{z\in N_{t-1}}\right)$ until collusion is detected and $\left(n_\tau, (q_\tau^z)_{z\in N_\tau}\right) = \left(n_{t-1}, \left(q^{COM}(n_{t-1})\right)_{z\in N_{t-1}}\right)$, thereafter.* ***Proof:*** *See Appendix A.1.* ■

The intuition behind Lemma 2 is simple. When firms are forced to collude, no firm has an incentive to merge because a merger implies lower profits while collusion lasts and also lower profits when collusion is detected and firms are forced to compete.

From a normative point of view, under collusion, the number of firms is irrelevant because they always reduce production to induce the monopoly quantity. Formally, under collusion, the aggregate consumer surplus is $CS^{COL} = (a-c)^2/8$, aggregate profits are $\Pi^{COL} = (a-c)^2/4$, and the total surplus is $TS^{COL} = 3(a-c)^2/8$, which do not depend on n_t.

To sum up, in our setting without synergies, there are no intrinsic incentives to merge when mergers do not affect the competition regime. Whether firms expect competition (Lemma 1) or collusion (Lemma 2) for all future periods, in equilibrium, firms have no incentive to merger.

3.3 Sustainability of collusion

Consider the situation in period t after merger decisions have been made. The following lemma characterizes the conditions under which n_t firms can sustain collusion.

Lemma 3 ***Sustainability of collusion.*** *Suppose that before period t firms did not collude. Assume that after merger decisions have been made in period t, there are $n_t \geq 2$ firms. Then, for $\tau \geq t$, $\left(n_\tau, (q_\tau^z)_{z\in N_\tau}\right) = \left(n_t, \left(q^{COL}(n_t)\right)_{z\in N_t}\right)$ until collusion is detected and $\left(n_\tau, (q_\tau^z)_{z\in N_\tau}\right) = \left(n_t, \left(q^{COM}(n_t)\right)_{z\in N_t}\right)$ thereafter is an SPNE outcome path from t onward if and only if $\alpha \leq \bar{\alpha}(n_t)$, where:*

$$\bar{\alpha}(n_t) = 1 - \frac{(n_t-1)^2(n_t+1)^2}{\delta\left[(n_t+1)^4 - 16(n_t)^2\right]}. \tag{3}$$

Moreover, $\bar{\alpha}(n_t)$ is decreasing in n_t. ***Proof**: See Appendix A.1.* ■

The intuition behind Lemma 3 is simple. As the number of firms increases, it is more difficult to sustain collusion because each firm has more incentives to deviate from collusion. Indeed, $\pi^{DEV}(n_t) - \pi^{COL}(n_t) = (a-c)^2(n_t-1)^2/16(n_t)^2$ is increasing in n_t. As a consequence, if given the detection probability α, n_t firms are able to sustain collusion, then less than n_t firms will also be able to sustain collusion.[14] Figure 1 illustrates Lemma 3. For $n_t = 10$, collusion cannot be sustained even when it cannot be detected by the competition authority. For $n_t < 10$, firms can collude but only if the detection probability is lower than $\bar{\alpha}(n_t)$. Finally, note that as the number of firms decreases, the competition authority requires a higher probability of detection in order to dissuade firms to collude.

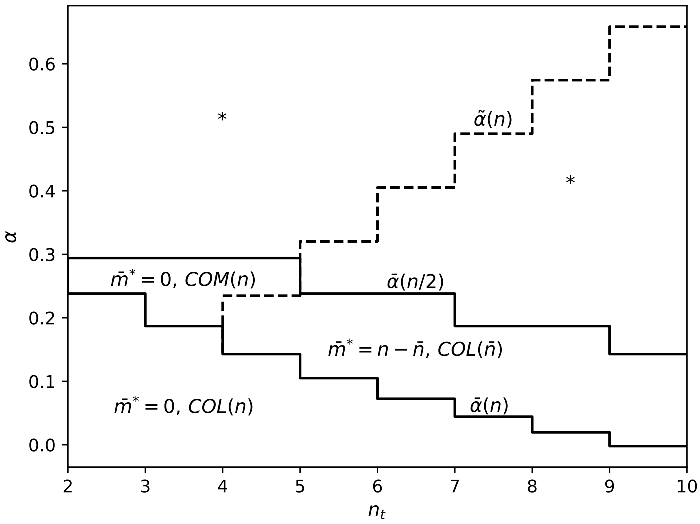

Figure 1: Sustainability of collusion. Note: $\delta = 3/4$.

3.4 Mergers and collusion

Although firms do not have incentives to merge under competition or while they are colluding, Lemma 3 suggests that mergers could be profitable if collusion cannot be sustained with the existing number of firms. The following proposition formally explores this possibility.

Proposition 1 *Assume that $\delta > \frac{(n-1)^2(n+1)^2}{(n+1)^4 - 16n^2}$. Let $\bar{n} = \lceil \bar{\alpha}^{-1}(\alpha) \rceil$ and $\tilde{\alpha}(n_t) = \left(\frac{1-\delta}{\delta}\right)\frac{(n_t+1)^2(n_t-2)^2}{4n_t\left[(n_t)^2-2\right]}$, where $\lceil x \rceil$ indicates the integer part of x.*

1. *Suppose that $\alpha \leq \bar{\alpha}(n)$. Then, the n original firms collude from $t = 0$ until they are detected. Thereafter, there is competition with n firms operating in the market.*

2. *Suppose that $\bar{\alpha}(n) < \alpha \leq \bar{\alpha}(n/2)$.*

[14] Formally, since $\bar{\alpha}(n_t)$ is a decreasing function, it has an inverse and, hence, the condition to sustain collusion can be rewritten as $n_t \leq \bar{\alpha}^{-1}(\alpha)$.

(a) If $\alpha \leq \tilde{\alpha}(\bar{n})$, in period $t = 0$ there are $m = n - \bar{n}$ mergers and the $\bar{n}$ new firms collude until they are detected. Thereafter, there is competition with $\bar{n}$ firms operating in the market.

(b) If $\alpha > \tilde{\alpha}(\bar{n})$, in every period there is competition with n firms. ***Proof****: See Appendix A.1.* ■

Proposition 1 Part 1 states that if the original n firms can sustain collusion (formally, $\alpha \leq \bar{\alpha}(n)$), then they do not merge and begin colluding immediately. Thus, firms have no incentives to merge if mergers are not necessary to sustain collusion. Intuitively, when collusion can be sustained with n firms, it can also be sustained with less than n firms and, therefore, mergers have no impact on the sustainability of collusion. Moreover, from Lemma 2, in such a situation, a firm will get lower expected profits if it merges than if it does not merge. Intuitively, a merger implies lower profits while collusion lasts and also lower profits when collusion is detected and firms are forced to compete.

Proposition 1 Part 2 considers a case in which mergers are required to sustain collusion. Two possible equilibria emerge. First, firms do not merge and, therefore, there is competition with n firms. Second, there are just enough mergers to reduce the number of remaining firms to $\bar{n} = \lceil \bar{\alpha}^{-1}(\alpha) \rceil$, i.e., the maximum number required to sustain collusion. The first equilibrium always exists because if all firms but one refuse to merge, nothing can be done to induce a merger. The second equilibrium only exists when $\alpha \leq \tilde{\alpha}(\bar{n})$. The intuition is as follows. On the one hand, when the number of mergers is $m = n - \bar{n}$, if one merger fails, collusion will not be sustainable and, hence, there will be competition with $\bar{n}+1$ firms. On the other hand, if the number of mergers is $m = n - \bar{n}$, firms will collude but, eventually, they will be detected and forced to compete. Moreover, after collusion is detected there will be competition with $\bar{n}$ firms and a firm that merged to make collusion sustainable will get lower profits than it would have otherwise obtained if it had refused to merge. More formally, suppose that if firms x and y merge there will be $\bar{n}$ firms. Then, if x and y merge, firms can sustain collusion and, hence, the expected discounted profits of x and y will be

$$\frac{v^{COL}(\bar{n})}{2} = \frac{\frac{\pi^{COL}(\bar{n})}{2} + \frac{\delta\alpha\pi^{COM}(\bar{n})}{(1-\delta)2}}{1 - \delta(1-\alpha)}.$$

On the contrary, if x and y do not merge, then there will be competition among $\bar{n}+1$ firms and, hence, the expected discounted profits of x and y will be

$$v^{COM}(\bar{n}+1) = \frac{\pi^{COM}(\bar{n}+1)}{1-\delta}.$$

Note that $\pi^{COL}(\bar{n})/2 \geq \pi^{COM}(\bar{n}+1)$. That is, while collusion lasts firms earn higher profits if they merge to make collusion sustainable. But, also note that $\pi^{COM}(n_t)/2 < \pi^{COM}(n_t+1)$. That is, when collusion is detected and firms are forced to compete, firms would have earned higher profits if they had not merged. Therefore, the second equilibrium exists only when α is low enough, such that $v^{COL}(\bar{n})/2 \geq v^{COM}(\bar{n}+1)$, i.e., if and only if $\alpha \leq \tilde{\alpha}(\bar{n})$. Finally, whenever the second equilibrium exists, it induces higher expected profits for the firms than the first equilibrium. In order to see this, note that the expected discounted profits of a firm in the first equilibrium are $v^{COM}(n) = \pi^{COM}(n)/(1-\delta) < v^{COM}(\bar{n}+1)$.

Proposition 1 has a powerful positive implication. An improvement in the probability of detecting collusion could lead to a wave of mergers that reduces the number of firms until collusion is sustainable again. Thus, using mergers firms could neutralize an improvement in antitrust enforcement that, otherwise, would have dissuaded firms from engaging in collusion.[15]

3.5 Welfare analysis

Proposition 1 has also an important normative implication. Mergers could be more harmful than expected if we do not properly take into account the impact that mergers may have on the competition regime. To illustrate this point, suppose that $\bar{\alpha}(n) < \alpha \leq \min\{\bar{\alpha}(n/2), \tilde{\alpha}(\bar{n})\}$. Then, from Proposition 1 Part 2, in period $t = 0$ there will be $m = n - \bar{n}$ mergers and the $\bar{n}$ new firms will collude until they are detected. Thereafter, there will be competition with $\bar{n}$ firms. The expected discounted total surplus from period $t = 0$ onward associated with the equilibrium path is:

$$W_0^E = \mathbf{E}_0\left[\sum\nolimits_{t=0}^{\infty} \delta^t TS_t\right] = \frac{TS^{COL} + \delta\alpha TS^{COM}(\bar{n})}{1-\delta(1-\alpha)} = \frac{\frac{3(a-c)^2}{8} + \delta\alpha\frac{(a-c)^2\bar{n}(\bar{n}+2)}{2(\bar{n}+1)^2}}{1-\delta(1-\alpha)}$$

The expected discounted welfare when n firms compete in each period is:

$$W_0^{COM}(n) = \mathbf{E}_0\left[\sum\nolimits_{t=0}^{\infty} \delta^t TS^{COM}(n)\right] = \frac{(a-c)^2 n(n+2)}{2(n+1)^2(1-\delta)}$$

Thus, the total welfare effect of the $m = n - \bar{n}$ mergers in period $t = 0$ is $W_0^E - W_0^{COM}(n) < 0$. Suppose for a moment that we do not take into account that once $\bar{n}$ firms remain, they will collude. That is, assume that $\bar{n}$ firms compete in each period. Then, the associated expected discounted welfare would be $W_0^{COM}(\bar{n})$, while the estimated welfare effect of the $m = n - \bar{n}$ mergers in period $t = 0$ would be $W_0^E - W_0^{COM}(\bar{n}) < 0$. It is easy to verify that $W_0^{COM}(n) > W_0^{COM}(\bar{n}) > W_0^E$, which implies $W_0^E - W_0^{COM}(\bar{n}) < W_0^E - W_0^{COM}(n) < 0$. Thus, if we do not properly consider the impact that mergers have on the competition regime, we underestimate the welfare effect of mergers.

The previous paragraph suggests the following decomposition of the welfare effects of mergers. Denote the unilateral welfare effect associated to a number of mergers as the change in W_0 due to the mergers, keeping the competition regime constant. When $\bar{\alpha}(n) < \alpha \leq \min\{\bar{\alpha}(n/2), \tilde{\alpha}(\bar{n})\}$, the unilateral welfare effect is given by:

$$UE = W_0^{COM}(\bar{n}) - W_0^{COM}(n) = \frac{(a-c)^2}{(1-\delta)}\left[\frac{\bar{n}(\bar{n}+2)}{2(\bar{n}+1)^2} - \frac{n(n+2)}{2(n+1)^2}\right] < 0$$

Denote the coordinated welfare effect associated to a number of mergers as the change in W_0 attributed to the change in the competition regime. When $\bar{\alpha}(n) < \alpha \leq \min\{\bar{\alpha}(n/2), \tilde{\alpha}(\bar{n})\}$, the coordinated welfare effect is given by:

$$CE = W_0^E - W_0^{COM}(\bar{n}) = \frac{(a-c)^2}{1-\delta(1-\alpha)}\left[\frac{3}{8} - \frac{\bar{n}(\bar{n}+2)}{2(\bar{n}+1)^2} - \frac{\delta^2\alpha\bar{n}(\bar{n}+2)}{(1-\delta)2(\bar{n}+1)^2}\right] < 0$$

Thus, when we do not take into account that once $\bar{n}$ firms remain, they will collude, we are only considering the unilateral welfare effect.

[15] If $\alpha > \bar{\alpha}(n/2)$, it is possible that an improvement in the probability of detecting collusion has only temporary effects on the sustainability of collusion. Since to merge firms are matched in pairs, it could take several periods until the industry reaches the number of firms required to sustain collusion. Thus, an increase in the probability of detecting collusion could induce firms to compete in the short run, but eventually, they switch to collusion.

4 Equilibrium and welfare with 4 firms and positive synergies ($s > 0$)

This section studies a setting with four firms in which mergers produce cost synergies. First, we study mergers under competition and show that if synergies are high enough, firms have an incentive to merge. Second, we explore how the number and cost-type of the firms operating in the market affects the sustainability of collusion. Third, we characterize merger decisions when a change in the number and cost-type of the firms affects the incentive to collude. Finally, we decompose the welfare change associated with mergers into an unilateral and a coordinated effect. The decomposition suggests important normative implications for merger policy.

4.1 Mergers under competition

Let N_t^H (N_t^L) denote the set of firms with unit cost c ($c(1-s)$) and n_t^H (n_t^L) the number of firms with unit cost c ($c(1-s)$) after merger decision have been made in period t (equivalently at the beginning of period $t+1$). Naturally, $N_t = N_t^H \cup N_t^L$ and $n_t = n_t^H + n_t^L$. Since, at the beginning of period $t = 0$, all firms have unit cost c, $N_{-1} = N_{-1}^H$, $N_{-1}^L = \varnothing$, $n_{-1}^H = n$ and $n_{-1}^L = 0$.

Suppose that in period t there is competition. Then, the quantity produced by firm $z \in N_t^L$ is $q_t^z = q_L^{COM}\left(n_t^L, n_t^H\right) = \frac{a-c+\left(n_t^H+1\right)cs}{n_t^H+n_t^L+1}$ and by firm $z \in N_t^H$ is $q_t^z = q_H^{COM}\left(n_t^L, n_t^H\right) = \frac{a-c-n_t^L cs}{n_t^H+n_t^L+1}$.[16] Therefore, the profits obtained by each firm are $\pi_t^z = \pi_L^{COM}\left(n_t^L, n_t^H\right) = \left(\frac{a-c+n_t^H cs+cs}{n_t^H+n_t^L+1}\right)^2$ for $z \in N_t^L$ and $\pi_t^z = \pi_H^{COM}\left(n_t^L, n_t^H\right) = \left(\frac{a-c-n_t^L cs}{n_t^H+n_t^L+1}\right)^2$ for $z \in N_t^H$.

While in the case of no synergies, under competition, there are no mergers, positive synergies could induce firms to merge. In order to see this, consider the situation when, initially, there are only 4 firms, i.e., $\left(n_{-1}^L, n_{-1}^H\right) = (0,4)$.[17] The following lemma characterizes the equilibrium.

Lemma 4 ***Mergers under competition***. *Suppose that $\left(n_{t-1}^L, n_{t-1}^H\right) = (0,4)$ and for all $\tau \geq t$ firms are forced to compete. Assume $\delta \geq 1/3$.*[18]

1. *If $0 \leq s < \bar{s}^1 = \left[\left(3-2\sqrt{2}\right) / \left(2\sqrt{2}+3\right)\right] d$, then the unique SPNE outcome path is $\left(n_\tau^L, n_\tau^H, (q_\tau^z)_{z\in N_\tau}\right) = \left(0, 4, \left(q_H^{COM}(0,4)\right)_{z\in N_\tau^H}\right)$ for all $\tau \geq t$.*

2. *If $\bar{s}^1 \leq s \leq d$, then the SPNE outcome path that induces the highest expected discounted profits is $\left(n_\tau^L, n_\tau^H, (q_\tau^z)_{z\in N_\tau}\right) = \left(2, 0, \left(q_L^{COM}(2,0)\right)_{z\in N_\tau^L}\right)$ for all $\tau \geq t$.* ***Proof***: *See Appendix A.2.* ■

The intuition behind Lemma 4 is as follows. When synergies are low ($s < \bar{s}^1$), firms do not have per se an incentive to merge for the same reasons we discusses when there are no synergies. Regardless of the

[16] Note that $d = \frac{a-c}{c} > n_t^L s$ assures that $q_H^{COM}\left(n_t^L, n_t^H\right) > 0$.

[17] From $\left(n_{-1}^L, n_{-1}^H\right) = (0,4)$ the industry can transition to $\left(n_0^L, n_0^H\right) = (0,4)$ (if none of the matches induces a merger), $\left(n_0^L, n_0^H\right) = (1,2)$ (if only one pair of firms merges) or $\left(n_0^L, n_0^H\right) = (2,0)$ (if each pair of firms merges). From $\left(n_0^L, n_0^H\right) = (1,2)$ the industry can transition to $\left(n_1^L, n_1^H\right) = (1,2)$ (if the pair of matched firms do not merge), $\left(n_1^L, n_1^H\right) = (2,0)$ (if the two firms with unit cost c are matched and they decide to merge); or $\left(n_1^L, n_1^H\right) = (1,1)$ (if the firm with unit cost $(1-s)c$ is matched with one of the firms with unit cost c and they decide to merge). Finally, recall that, once there are only two firms remaining, they are not allowed to merge.

[18] Indeed, we only need $\delta \geq \left(522\sqrt{2}-723\right) / \left(498\sqrt{2}-657\right) \approx 0.322$.

merging decisions of other firms, under competition, a firm that merges can always increase its profits by refusing to merge. More formally, $s < \bar{s}^1$ implies

$$\pi_H^{COM}(1,2) > \frac{\pi_L^{COM}(2,0)}{2} \text{ and } \pi_H^{COM}(0,4) > \frac{\pi_L^{COM}(1,2)}{2}$$

That is, a firm obtains higher profits when it is one of the firms with unit cost c and $\left(n_t^L, n_t^H\right) = (1,2)$ than when $\left(n_t^L, n_t^H\right) = (2,0)$. As a consequence, if $\left(n_{t-1}^L, n_{t-1}^H\right) = (0,4)$ and $\left(n_t^L, n_t^H\right) = (2,0)$, each firm prefers to deviate and refuse to merge. Analogously, a firm obtains higher profits if it is one of the firms with unit cost c and $\left(n_t^L, n_t^H\right) = (0,4)$ than if it merges and $\left(n_t^L, n_t^H\right) = (1,2)$. As a consequence, if $\left(n_{t-1}^L, n_{t-1}^H\right) = (0,4)$ and $\left(n_t^L, n_t^H\right) = (1,2)$, the firms that are merging prefer to deviate and refuse to merge.

For intermediate values of s ($\bar{s}^1 \leq s \leq d/5$), the cost reduction produced by a merger is high enough to induce firms to merge, but not high enough to make firms prefer that other matched firms do not merge. More formally, $\bar{s}^1 \leq s \leq d/5$ implies

$$\frac{\pi_L^{COM}(2,0)}{2} \geq \max\left\{\pi_H^{COM}(1,2), \frac{\pi_L^{COM}(1,2)}{2}\right\}$$

That is, if $s \geq \bar{s}^1$, then $\pi_L^{COM}(2,0)/2 \geq \pi_H^{COM}(1,2)$ and, hence, firms do not have an incentive to deviate when $\left(n_{t-1}^L, n_{t-1}^H\right) = (0,4)$ and $\left(n_t^L, n_t^H\right) = (1,2)$. Also, if $s \leq d/5$, then $\pi_L^{COM}(2,0)/2 \geq \pi_L^{COM}(1,2)/2$ and $\pi_L^{COM}(2,0)/2 \geq \pi_H^{COM}(1,2)$. Hence, firms prefer $\left(n_t^L, n_t^H\right) = (2,0)$ than $\left(n_t^L, n_t^H\right) = (1,2)$.

For high values of s ($s > d/5$), each pair of matched firms prefer to be the one that merges, while the other pair of firms does not merge. More formally, $s > d/5$ implies

$$\frac{\pi_L^{COM}(1,2)}{2} > \frac{\pi_L^{COM}(2,0)}{2} \geq \pi_H^{COM}(1,2)$$

That is, the firms that merge when $\left(n_t^L, n_t^H\right) = (1,2)$ get higher profits than when both pairs merge, but the firms that do not merge when $\left(n_t^L, n_t^H\right) = (1,2)$ get lower profits than when both pairs merge.[19] However, ex-ante, that is, before firms have been matched in period t, firm $z \in N_{t-1}$ does not know its role in equilibrium $\left(n_t^L, n_t^H\right) = (1,2)$. It could the case that $z \in \{x,z\} \in N_t^L$ or $z \in \{x,z\} \in N_t^H$. Thus, if firms decide to coordinate in equilibrium $\left(n_t^L, n_t^H\right) = (1,2)$, the expected profits of firm $z \in N_{t-1}$ are $\left[\pi_L^{COM}(1,2)/2 + \pi_H^{COM}(1,2)\right]/2$. (With probability 1/2, $z \in \{x,z\} \in N_t^L$ and, hence, z gets $\pi_L^{COM}(1,2)/2$, while with probability 1/2, $z \in \{x,z\} \in N_t^H$ and, hence, it gets $\pi_H^{COM}(1,2)$). On the other hand, when the equilibrium is $\left(n_t^L, n_t^H\right) = (2,0)$, the expected profits of firm z are $\pi_L^{COM}(2,0)/2$. In Appendix A.2 we show that it is always the case that $\pi_L^{COM}(2,0)/2 > \left[\pi_L^{COM}(1,2)/2 + \pi_H^{COM}(1,2)\right]/2$. Thus, ex-ante, firms prefer to coordinate in the equilibrium $\left(n_t^L, n_t^H\right) = (2,0)$.

[19] In other words, the merging game has a structure similar to that of the battle of the sexes in the following sense. In the battle of the sexes each player prefers one of the two pure strategy Nash equilibria to the correlated equilibrium in which they randomly play each pure strategy Nash equilibrium with probability 1/2, but each player prefers the correlated equilibrium rather than the other pure strategy Nash equilibrium. In the merging game under competition and $s > d/5$, each pair of matched firms prefer the equilibrium $\left(n_t^L, n_t^H\right) = (1,2)$ to the equilibrium $\left(n_t^L, n_t^H\right) = (2,0)$ when they are the pair that merge, but they prefer the equilibrium $\left(n_t^L, n_t^H\right) = (2,0)$ to the equilibrium $\left(n_t^L, n_t^H\right) = (1,2)$ when they are the pair that do not merge.

From a normative point of view, mergers under competition have a positive or negative effect depending on the value of s/d and the criteria employed for welfare comparisons. Firms as a group always obtain higher profits when the SPNE outcome path is $\left(n_\tau^L, n_\tau^H\right) = (0,2)$ for all $\tau \geq t$ rather than when it is $\left(n_\tau^L, n_\tau^H\right) = (0,4)$ for all $\tau \geq t$.[20] Intuitively, mergers reduce competition as well as the cost of the merging firms. Consumers are better off with mergers only when cost synergies are high enough. The consumer surplus is higher when the SPNE outcome path is $\left(n_\tau^L, n_\tau^H\right) = (0,2)$ for all $\tau \geq t$ than when it is $\left(n_\tau^L, n_\tau^H\right) = (0,4)$ for all $\tau \geq t$ if and only if $s > d/5$.[21] Thus, for consumers it is better that cost synergies are low (formally, $s < \bar{s}^1$) and, hence, firms do not merge or that cost synergies are high ($s > d/5$) and, hence, firms merge. Intermediate values of s (formally, $\bar{s}^1 \leq s < d/5$) are problematic because, in equilibrium, firms merge, but consumers would be better off if mergers are not allowed. The effect of mergers on total surplus also depend on s. For $s > \left(83 - 24\sqrt{11}\right) d/79 > \bar{s}^1$, mergers increase total surplus, while for $s < \left(8\sqrt{13} - 25\right) d/115$, they reduce total surplus.[22]

Summing up, cost synergies brings two important novelties for the analysis of mergers under competition. First, if cost synergies are high enough, there could be mergers in equilibrium. Second, mergers could increase total surplus and even the consumer surplus; in particular, when cost synergies are high enough. Next, we introduce the possibility that firms collude.

4.2 Sustainability of collusion

Consider the situation in period t after merger decisions have been made. Suppose that, among the remaining firms, n_t^H have unit cost c and n_t^L have unit cost $c(1-s)$. The following lemma characterizes the conditions under which firms can sustain collusion for any market structure that could emerge in period t if $\left(n_{-t}^L, n_{-t}^H\right) = (0,4)$, that is, when initially there are only 4 firms.

Lemma 5 ***Sustainability of collusion***. *Suppose that* $\left(n_{-t}^L, n_{-t}^H\right) = (0,4)$ *and before period t firms have not colluded. Assume that after merger decisions have been made in period t, there are* n_t^H *firms with unit cost c and* n_t^L *firms with unit cost* $c(1-s)$. *Then:*

1. *If* $\left(n_t^L, n_t^H\right) = (0,4)$, *then collusion can be sustained whenever* $\alpha \leq \bar{\alpha}(4) = 1 - (\delta)^{-1}(25/41)$.
2. *If* $\left(n_t^L, n_t^H\right) = (2,0)$, *then collusion can be sustained whenever* $\alpha \leq \bar{\alpha}(2) = 1 - (\delta)^{-1}(9/17)$.
3. *If* $\left(n_t^L, n_t^H\right) = (1,2)$ *and* $s \leq d$, *then collusion can be sustained whenever* $\alpha \leq \bar{\alpha}(1,2)(s) = 1 - \frac{4(d-s)^2(d+cs)^2}{\delta\left[16(d+s)^4 - (3d+s)^2(d+3s)^2\right]}$. *Moreover,* $\bar{\alpha}(1,2)(0) = \bar{\alpha}(3)$, $\bar{\alpha}(1,2)(d/5) = \bar{\alpha}(2)$ *and* $\bar{\alpha}(1,2)(s)$ *is strictly increasing in s.* ***Proof:*** *See Appendix A.2.* ■

[20]It is easy to prove that $\Pi_t^{COM}(2,0) > \Pi_t^{COM}(1,2) > \Pi_t^{COM}(0,4)$ for $s \in \left[0, \frac{(12\sqrt{6}+23)d}{67}\right)$; $\Pi_t^{COM}(2,0) = \Pi_t^{COM}(1,2) > \Pi_t^{COM}(0,4)$ for $s = \frac{(12\sqrt{6}+23)d}{67}$; and $\Pi_t^{COM}(1,2) > \Pi_t^{COM}(2,0) > \Pi_t^{COM}(0,4)$ for $s \in \left(\frac{(12\sqrt{6}+23)d}{67}, d\right]$.

[21]Indeed, it is easy to prove that $CS^{COM}(2,0) > CS^{COM}(1,2) > CS^{COM}(0,4)$ for $s \in (d/5, d]$; $CS^{COM}(2,0) = CS^{COM}(1,2) = CS^{COM}(0,4)$ for $s = d/5$; and $CS^{COM}(0,4) > CS^{COM}(1,2) > CS^{COM}(2,0)$ for $s \in [0, d/5)$.

[22]Indeed, it is easy to verify that t is easy to verify that if $s > \left(83 - 24\sqrt{11}\right) d/79$, then $TS^{COM}(2,0) > TS^{COM}(1,2) > TS^{COM}(0,4)$; if $\left(3\sqrt{3} - 5\right) d/5 < s < \left(83 - 24\sqrt{11}\right) d/79$, then $TS^{COM}(1,2) > TS^{COM}(2,0) > TS^{COM}(0,4)$; if $\left(8\sqrt{13} - 25\right) d/115 < s < \left(3\sqrt{3} - 5\right) d/5$, then $TS^{COM}(1,2) > TS^{COM}(0,4) > TS^{COM}(2,0)$; and if $s < \left(8\sqrt{13} - 25\right) d/115$, then $TS^{COM}(0,4) > TS^{COM}(1,2) > TS^{COM}(2,0)$, where $\left(83 - 24\sqrt{11}\right) d/79 > \left(3\sqrt{3} - 5\right) d/5 > \left(8\sqrt{13} - 25\right) d/115 > \bar{s}^1$.

Lemma 5 provides a very useful order of the sustainability of collusion given $\left(n_t^L, n_t^H\right)$, which we illustrate in Figure 2. For $0 \leq s < d/5$, we have $\bar{\alpha}(4) < \bar{\alpha}(1,2)(s) < \bar{\alpha}(1,2)(d/5) = \bar{\alpha}(2)$. Thus, for a given value of s, it is more complicated to sustain collusion when $\left(n_t^L, n_t^H\right) = (0,4)$ than when $\left(n_t^L, n_t^H\right) = (1,2)$, and it is more complicated to sustain collusion when $\left(n_t^L, n_t^H\right) = (1,2)$ than when $\left(n_t^L, n_t^H\right) = (2,0)$. For $s > d/5$, we have $\bar{\alpha}(4) < \bar{\alpha}(2) = \bar{\alpha}(1,2)(d/5) < \bar{\alpha}(1,2)(s)$. Thus, it is more complicated to sustain collusion when $\left(n_t^L, n_t^H\right) = (0,4)$ than when $\left(n_t^L, n_t^H\right) = (2,0)$, and it is more complicated to sustain collusion when $\left(n_t^L, n_t^H\right) = (2,0)$ than when $\left(n_t^L, n_t^H\right) = (1,2)$.

The intuition behind Lemma 5 is as follows. There are two forces that affect the sustainability of collusion: the total number of firms in the market and the cost heterogeneity of the firms. When the cost synergy s is below some threshold ($s < d/5$), the dominant force is the number of firms and cost heterogeneity only plays a role for a given number of firms. For example, it is easier to sustain collusion for $\left(n_t^L, n_t^H\right) = (2,0)$ than for $\left(n_t^L, n_t^H\right) = (0,4)$. When the cost synergy s is greater than $d/5$, cost heterogeneity becomes more important, making easier to sustain collusion when $\left(n_t^L, n_t^H\right) = (1,2)$ than when $\left(n_t^L, n_t^H\right) = (2,0)$.

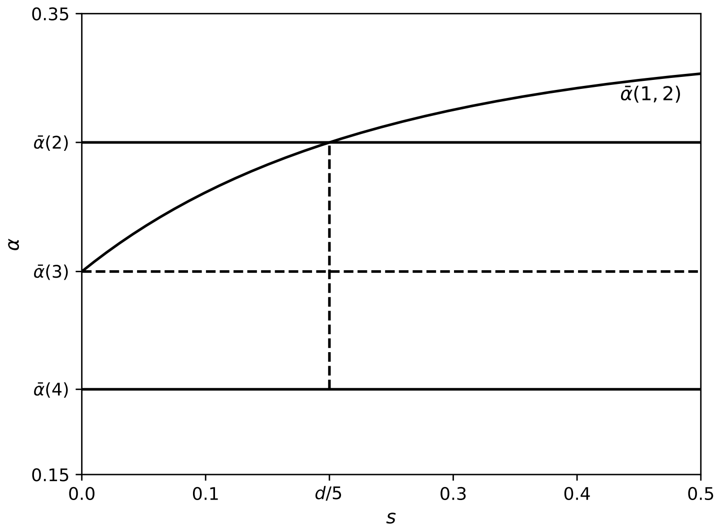

Figure 2: Sustainability of collusion. Note $\delta = 3/4$.

4.3 Mergers and collusion

Lemma 5 assumes that $\left(n_{t-1}^L, n_{t-1}^H\right) = (0,4)$ and characterizes the sustainability of collusion after merger decisions have been made in period t. No doubt, this is a necessary component to fully characterize the set of equilibrium outcome paths. The other key component is the merger decisions of the firms. For example, from Lemma 5, we know that if $\left(n_t^L, n_t^H\right) = (1,2)$, then collusion can be sustained whenever $\alpha \leq \bar{\alpha}(1,2)(s)$. However, we do not know if firms are willing to move from $\left(n_{t-1}^L, n_{t-1}^H\right) = (0,4)$ to $\left(n_t^L, n_t^H\right) = (1,2)$. From Lemmas 4 and 5, we must also consider the possibility that firms never collude, either because collusion is not sustainable or because they choose to do so. Overall, there are 7 possible SPNE outcome paths: 4 in which firms always compete and 3 in which collusion occurs, which we list below:

- $COM\,(0,4)$: $\left(n_\tau^L, n_\tau^H, (q_\tau^z)_{z\in N_\tau}\right) = \left(0, 4, \left(q_H^{COM}\,(0,4)\right)_{z\in N_\tau^H}\right)$ for all $\tau \geq t$.
- $COM\,(1,2)$: $\left(n_\tau^L, n_\tau^H, (q_\tau^z)_{z\in N_\tau}\right) = \left(1, 2, \left(q_L^{COM}\,(1,2)\right)_{z\in N_\tau^L}, \left(q_H^{COM}\,(1,2)\right)_{z\in N_\tau^H}\right)$ for $\tau \geq t$.
- $COM\,(2,0)$: $\left(n_\tau^L, n_\tau^H, (q_\tau^z)_{z\in N_\tau}\right) = \left(2, 0, \left(q_L^{COM}\,(2,0)\right)_{z\in N_\tau^L}\right)$ for $\tau \geq t$.
- $COM\,(1,1,2,0)$: $\left(n_\tau^L, n_\tau^H, (q_\tau^z)_{z\in N_\tau}\right) = \left(1, 2, \left(q_L^{COM}\,(1,2)\right)_{z\in N_\tau^L}, \left(q_H^{COM}\,(1,2)\right)_{z\in N_\tau^H}\right)$ for $\tau \geq t$ until the two remaining firms in N_τ^H are matched. Thereafter, $COM\,(2,0)$.
- $COL\,(0,4)$: $\left(n_\tau^L, n_\tau^H, (q_\tau^z)_{z\in N_\tau}\right) = \left(0, 4, \left(q_H^{COL}\,(0,4)\right)_{z\in N_\tau^H}\right)$ for $\tau \geq t$ until collusion is detected. Thereafter, $COM\,(0,4)$.
- $COL\,(1,2)$: $\left(n_\tau^L, n_\tau^H, (q_\tau^z)_{z\in N_\tau}\right) = \left(1, 2, \left(q_L^{COL}\,(1,2)\right)_{z\in N_\tau^L}, \left(q_H^{COL}\,(1,2)\right)_{z\in N_\tau^H}\right)$ for $\tau \geq t$ until collusion is detected. Thereafter, $COM\,(1,2)$.
- $COL\,(2,0)$: $\left(n_\tau^L, n_\tau^H, (q_\tau^z)_{z\in N_\tau}\right) = \left(2, 0, \left(q_L^{COL}\,(2,0)\right)_{z\in N_\tau^L}\right)$ $\tau \geq t$ until collusion is detected. Thereafter, $COM\,(2,0)$.

$COM\,(0,4)$ is an outcome path in which there are no mergers and the firms do not collude. $COM\,(1,2)$ is an outcome path in which only one pair of firms merge and there is competition. $COM\,(2,0)$ is an outcome path in which both pairs of matched firms merge and they always compete. $COM\,(1,1,2,0)$ is an outcome path in which $COM\,(2,0)$ is reached through two sequential mergers. $COL\,(0,4)$ is an outcome path in which there are no mergers and the firms collude until they are detected. $COL\,(1,2)$ is an outcome path in which only one pair of firms merge and there is collusion until it is detected. Finally, $COL\,(2,0)$ is an outcome path in which both pairs of matched firms merge and they collude until they are detected.

It is often the case that for the same set of parameters there are several SPNE outcome paths, in which case, we employ the following selection criteria. We allow the original 4 firms to select the SPNE outcome path that maximize its expected discounted profits. In the cases of $COM\,(1,2)$ and $COM\,(1,2)$, it is important to stress that if firms choose one of these equilibria, they do not know the role each particular firm will play in the equilibrium. For example, if $COM\,(1,2)$ is selected it is possible that $z \in \{x,z\} \in N_t^L$ and, hence, z gets $\pi_L^{COM}\,(1,2)/2$ or that $z \in \{x,z\} \in N_t^H$ and, hence, z gets $\pi_H^{COM}\,(1,2)$; with nature determining what is the role of each firm.[23] Using this selection criteria, let $\bar{E}$ denote the SPNE outcome path that maximizes the expected discounted profits of the firms. The following proposition characterizes $\bar{E}$.

Proposition 2 *Suppose that* $\left(n_{-t}^L, n_{-t}^H\right) = (0,4)$. *Let* $\hat{\alpha}\,(2,0) = \min\{\bar{\alpha}\,(2), \tilde{\alpha}\,(2,0)\,(s)\}$ *and* $\hat{\alpha}\,(1,2) = \min\{\bar{\alpha}\,(1,2)\,(s), \tilde{\alpha}\,(1,2)\,(s)\}$, *where*

$$\tilde{\alpha}\,(1,2)\,(s) = \left(\frac{1-\delta}{\delta}\right)\left[\frac{\frac{(d+3s)(d+s)^2}{4(3d+s)} - \frac{2(d)^2}{25}}{\frac{2(d)^2}{25} - \frac{(d+3s)^2}{16}}\right] \text{ and } \tilde{\alpha}\,(2,0)\,(s) = \left(\frac{1-\delta}{\delta}\right)\left[\frac{\frac{(d+s)^2}{8} - \frac{(d-s)^2}{8}}{\frac{(d-s)^2}{8} - \frac{(d+s)^2}{9}}\right].$$

[23]Moreover, we assume that nature assigns equal probability to each possible configuration of $COM\,(1,2)$ or $COL\,(1,2)$. As a consequence, $\Pr\left(z \in \{x,z\} \in N_t^L\right) = 1/2$ and $\Pr\left(z \in \{x,z\} \in N_t^H\right) = 1/2$.

1. *Suppose that* $0 \leq s < \bar{s}^1$.

   (a) *If* $\alpha \leq \hat{\alpha}(2,0)$, *then* $\bar{E} = COL(2,0)$.

   (b) *If* $\hat{\alpha}(2,0) < \alpha \leq \hat{\alpha}(1,2)$, *then* $\bar{E} = COL(1,2)$.

   (c) *If* $\max\{\hat{\alpha}(2,0), \hat{\alpha}(1,2)\} < \alpha \leq \bar{\alpha}(4)$, *then* $\bar{E} = COL(0,4)$.

   (d) *If* $\alpha > \max\{\hat{\alpha}(2,0), \hat{\alpha}(1,2), \bar{\alpha}(4)\}$, *then* $\bar{E} = COM(0,4)$.

2. *Suppose that* $\bar{s}^1 \leq s \leq d$.

   (a) *If* $\alpha \leq \bar{\alpha}(2)$, *then* $\bar{E} = COL(2,0)$.

   (b) *If* $\bar{\alpha}(2) < \alpha \leq \max\{\bar{\alpha}(2), \bar{\alpha}(1,2)(s)\}$, *then* $\bar{E} \in \{COL(1,2), COM(2,0)\}$.

   (c) *If* $\alpha > \max\{\bar{\alpha}(2), \bar{\alpha}(1,2)(s)\}$, *then* $\bar{E} = COM(2,0)$. ***Proof****: See Appendix A.2.* ■

Several remarks apply to Proposition 2, which is illustrated in Figure 3. First, as we have already mentioned, that collusion is sustainable for $\left(n_t^L, n_t^H\right)$ does not immediately imply that firms will have an incentive to transition from $\left(n_{t-1}^L, n_{t-1}^H\right)$ to $\left(n_t^L, n_t^H\right)$. Indeed, in Appendix A.2 we prove that $COL(1,2)$ is a SPNE outcome path if and only if [$s \in \left[0, \bar{s}^2\right)$ and $\alpha \leq \min\{\bar{\alpha}(1,2)(s), \tilde{\alpha}(1,2)(s)\}$] or [$s \geq \bar{s}^2$ and $\alpha \leq \bar{\alpha}(1,2)(s)$]; while $COL(2,0)$ is a SPNE outcome path if and only if [$s \in \left[0, \bar{s}^1\right)$ and $\alpha \leq \min\{\bar{\alpha}(2), \tilde{\alpha}(2,0)(s)\}$] or [$s \geq \bar{s}^1$ and $\alpha \leq \bar{\alpha}(2)$]. Indeed, $v_L^{COL}(1,2)/2 \geq v_L^{COM}(0,4)$ if and only if [$s \in \left[0, \bar{s}^2\right)$ $\alpha \leq \tilde{\alpha}(1,2)(s)$] or [$s \geq \bar{s}^2$], where

$$v_L^{COL}(1,2) = \frac{\pi_L^{COL}(1,2) + \alpha\delta\frac{\pi_L^{COM}(1,2)}{1-\delta}}{1-\delta(1-\alpha)} \text{ and } v_H^{COM}(0,4) = \frac{\pi_H^{COM}(0,4)}{(1-\delta)}.$$

That is, [$s \in \left[0, \bar{s}^2\right)$ $\alpha \leq \tilde{\alpha}(1,2)(s)$] or [$s \geq \bar{s}^2$] captures the extra condition required to make collusion not only sustainable for $\left(n_t^L, n_t^H\right) = (1,2)$, but also a SPNE outcome path. In other words, only when [$s \in \left[0, \bar{s}^2\right)$ $\alpha \leq \tilde{\alpha}(1,2)(s)$] or [$s \geq \bar{s}^2$], firms have an incentive to transition from $\left(n_{t-1}^L, n_{t-1}^H\right) = (0,4)$ to $\left(n_t^L, n_t^H\right) = (1,2)$ just before they begin colluding. Analogously, $v_L^{COL}(2,0)/2 \geq v_H^{COM}(1,2)$ if and only if [$s \in \left[0, \bar{s}^1\right)$ $\alpha \leq \tilde{\alpha}(2,0)(s)$] or [$s \geq \bar{s}^1$], where and $COL(2,0)$.

$$v_L^{COL}(2,0) = \frac{\pi_L^{COL}(2,0) + \alpha\delta\frac{\pi_L^{COM}(2,0)}{1-\delta}}{1-\delta(1-\alpha)} \text{ and } v_H^{COM}(1,2) = \frac{\pi_H^{COM}(1,2)}{(1-\delta)}$$

That is, [$s \in \left[0, \bar{s}^2\right)$ $\alpha \leq \tilde{\alpha}(1,2)(s)$] or [$s \geq \bar{s}^2$] captures the extra condition required to make collusion not only sustainable for $\left(n_t^L, n_t^H\right) = (1,2)$, but also a SPNE outcome path.

Second, while if firms are forced to compete, there are no mergers for $s < \bar{s}^1$, when firms can sustain some form of collusion, mergers could become the preferred SPNE outcome path. The reason is that firms need to merge to make collusion sustainable. In other words, the possibility of collusion makes otherwise unattractive mergers profitable enough to occur in equilibrum.

Third, mergers are crucial to expand the set of parameters for which collusion is sustainable. For example, if firms cannot merge, $\alpha > \bar{\alpha}(4)$ rules out collusion. However, when firms are allowed to merge, there is collusion even when $\alpha > \bar{\alpha}(4)$ (seem regions 1.a, 2.a and 2.b in Figure 3).

Finally, note how the probability of detecting collusion affects mergers and the equilibrium market structure for different values of s. For very low values of s, as α increases, we move from $COL\,(2,0)$ (region 1.a in Figure 3) to $COL\,(1,2)$ (region 1.b in Figure 3), then to $COL\,(0,4)$ (region 1.c in Figure 3) and, finally, to $COM\,(0,4)$ (region 1.d in Figure 3). Thus, even if an increase in α does not fully stop collusion, it could have an important effect on mergers. On the other hand, for high enough values of s, the probability of detecting collusion has no impact on mergers decisions. For example, for $\bar{s}^1 \leq s < d$, as α increases, we move from $COL\,(2,0)$ (region 2.a in Figure 3) to $COM\,(2,0)$ (region 2.c in Figure 3).[24] Thus, rising α eventually stop collusion, but it never affect merger decisions.

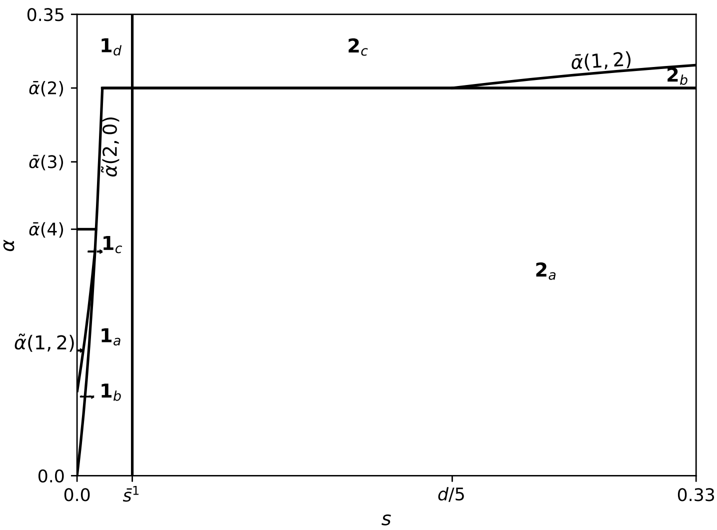

Figure 3: SPNE outcome paths. Note: $\delta = 3/4$.

4.4 Welfare Analysis and Optimal Merger Policy

Suppose that before the game is played, the competition authority can select the maximum number of mergers that will be allowed, which we denote by $\bar{m} \in \{0,1,2\}$. That is, $\bar{m} = 0$ indicates a complete ban on mergers, $\bar{m} = 1$ that only one merger will be authorized, and $\bar{m} = 2$ that up to two mergers will be accepted. Note that $\bar{m} = 2$ does not impose any further restriction beyond those in Proposition 2. Moreover, it is easy to follow the same analysis we employed in Proposition 2 for $\bar{m} = 0$ and $\bar{m} = 1$ (see Propositions 2.A.0 and 2.A.1 in Appendix A.2 for details). Figure 4 (panels a, b and c) illustrates the results. The figure shows the SPNE outcome path that maximizes the expected discounted profits of the firms for each merger policy chosen by the competition authority.

The following two propositions characterize the optimal merger policy of a competition authority with a mandate to maximize to the expected discounted consumer and total surplus, respectively. Formally, the optimal merger policy is given by $\bar{m}^*\,(CS) = \arg\max_{\bar{m}\in\{0,1,2\}} \mathbf{E}_0 \left[\sum_{t=0}^{\infty} \delta^t CS_t\right]$ when

[24]Indeed, in the Appendix we prove that if $\bar{s}^1 \leq s \leq \left(16+\sqrt{145}\right) d/37 \approx 0.75788d$ and $\bar{\alpha}\,(2) < \alpha \leq \max\{\bar{\alpha}\,(2), \bar{\alpha}\,(1,2)\,(s)\}$, then $\bar{E} = COM\,(2,0)$. Thus, for $\bar{s}^1 \leq s \leq \left(\frac{16+\sqrt{145}}{37}\right) d$, as α increases, we always move from $COL\,(2,0)$ (region 2.a in Figure 3) to $COM\,(2,0)$ (regions 2.b and 2.c in Figure 3).

the competition authority only takes into account the wellbeing of the consumers and $\bar{m}^*(TS) = \arg\max_{\bar{m}\in\{0,1,2\}} \mathbf{E}_0\left[\sum_{t=0}^{\infty}\delta^t TS_t\right]$ when it employs the total surplus criteria.

Proposition 3 ***Optimal merger policy (consumer surplus).*** *Suppose that* $s \in \left(\frac{d}{5}, \frac{(16+\sqrt{145})d}{37}\right]$. *Let* $\alpha^{CS}(s) = \left(\frac{1-\delta}{\delta}\right)\left[\frac{\left(\frac{d+s}{2}\right)^2-\left(\frac{4d}{5}\right)^2}{\left(\frac{4d}{5}\right)^2-\left(\frac{2(d+s)}{3}\right)^2}\right]$. *Then:*

1. *If* $\alpha \leq \bar{\alpha}(4)$, *then* $\bar{m}^*(CS) = 2$ *and* $\bar{E} = COL(2,0)$.
2. *If* $\bar{\alpha}(4) < \alpha \leq \bar{\alpha}(2)$.
   (a) *If* $\alpha < \alpha^{CS}(s)$, *then* $\bar{m}^*(CS) = 0$ *and* $\bar{E} = COM(0,4)$.
   (b) *If* $\alpha > \alpha^{CS}(s)$, *then* $\bar{m}^*(CS) = 2$ *and* $\bar{E} = COL(2,0)$.
3. *If* $\alpha > \bar{\alpha}(2)$, *then* $\bar{m}^*(CS) = 2$ *and* $\bar{E} = COM(2,0)$. ***Proof:*** *See Appendix A.2.* ■

Figure 4 illustrates Proposition 3. Note that we consider a region of the parameter space ($s > d/5$) in which, under competition, consumers are always better off if mergers are allowed (see the discussion at the end of Section 4.1). The problem is that firms may prefer to collude. Indeed, if $\alpha \leq \bar{\alpha}(4)$, then firms always collude regardless of the merger policy selected by the competition authority. Then, the optimal merger policy is to allow firms to merge, so at least consumers face a low-cost cartel. More importantly, in some cases, the competition authority can make collusion the preferred equilibrium for the firms or not depending on the merger policy it selects. Indeed, if $\bar{\alpha}(4) < \alpha \leq \bar{\alpha}(2)$., when the competition authority bans mergers, there is competition ($\bar{m} = 0$ leads to $COM(0,4)$), while when mergers are allowed, firms merge and collude ($\bar{m} = 1$ leads to $COL(1,2)$ and $\bar{m} = 2$ leads to $COL(2,0)$). Since for $s > d/5$ it is always the case that consumers are better off under $COL(2,0)$ than under $COL(1,2)$, the competition authority must decide between $\bar{m} = 0$, which induces $COM(0,4)$, and $\bar{m} = 2$, which induces $COL(2,0)$. As panel d in Figure 4 shows, when the cost synergy and the probability of detecting collusion are relatively low (s and α low), consumers are better off under $COM(0,4)$ than under $COL(2,0)$, while the opposite happens when the cost synergy and the probability of detecting collusion are relatively high.

For $\bar{\alpha}(2) < \alpha \leq \bar{\alpha}(1,2)$, if no merger is allowed, the equilibrium will be $COM(0,4)$; if only one merger is allowed, then the preferred equilibrium for the firms will be $COL(1,2)$; and, finally, if two mergers are allowed ($\bar{m} = 2$), $s \leq \left(16+\sqrt{145}\right)d/37$ implies that the preferred equilibrium for the firms will be $COM(2,0)$. We have already seen that for $s > d/5$, consumers always prefer $COM(2,0)$ to $COM(0,4)$ and $COM(1,2)$. Thus, the optimal merger policy is to set $\bar{m} = 2$. Finally, for $\alpha > \bar{\alpha}(2)$, if no mergers are allowed, the equilibrium will be $COM(0,4)$; if only one merger is allowed, then the preferred equilibrium for the firms will be $COM(1,2)$; and, finally, if two mergers are allowed ($\bar{m} = 2$), then the preferred equilibrium for the firms will be $COM(2,0)$. In some sense this is the opposite situation of $\alpha \leq \bar{\alpha}(4)$. While for $\alpha \leq \bar{\alpha}(4)$, firms will collude no matter the merger policy, for $\alpha > \bar{\alpha}(2)$, firms will compete regardless of the merger policy. Then, the competition authority will allow all mergers, to induce $COM(2,0)$, the best possible equilibrium for consumers.

To stress the importance of properly taking into account how merger policy affects collusion decisions, consider a myopic competition authority that does not internalize the possibility that firms will collude after they merge. If $s > d/5$, such a competition authority will allow all mergers on the incorrect belief that consumers will be worse off if mergers are restricted. However, for $\bar{\alpha}(4) < \alpha < \min\left\{\bar{\alpha}(2), \alpha^{CS}(s)\right\}$, consumers would be better off if the competition authority selects $\bar{m} = 0$ than if it chooses $\bar{m} = 2$ (see in Figure 4 panel d).

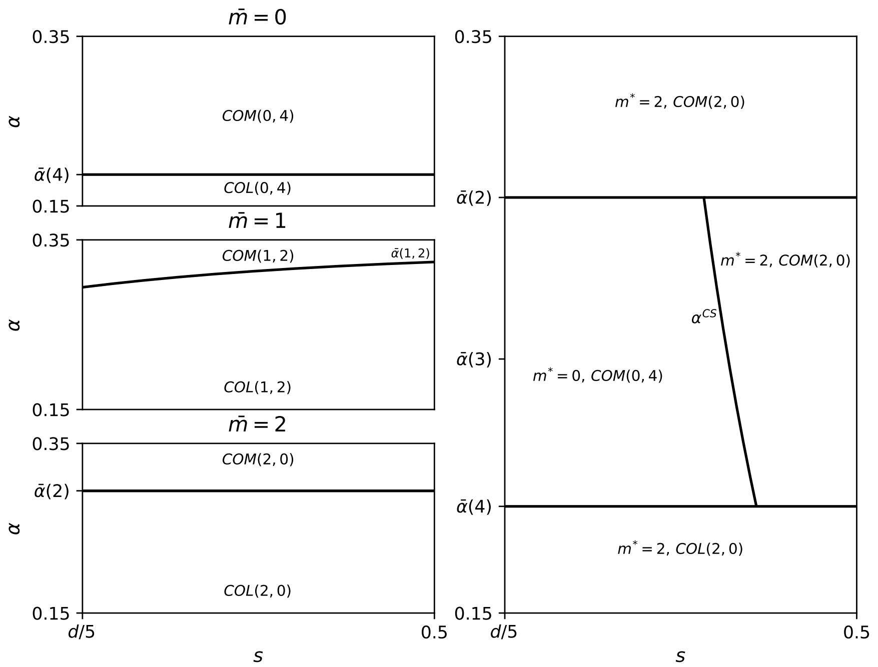

Figure 4: Optimal merger policy (consumer surplus). Note: $\delta = 3/4$.

Proposition 4 ***Optimal merger policy (total surplus)*** *Suppose that* $s \in \left(\frac{(83-24\sqrt{11})d}{79}, \frac{(16+\sqrt{145})d}{37}\right]$. *Then, there are two thresholds* $\alpha_1^{TS}(s)$ *and* $\alpha_2^{TS}(s)$ *such that*[25]*:*

1. *If* $\alpha \leq \bar{\alpha}(4)$, *then* $\bar{m}^*(TS) = 2$ *and* $\bar{E} = COL(2, 0)$.
2. *If* $\bar{\alpha}(4) < \alpha \leq \min\{\bar{\alpha}(1, 2), \bar{\alpha}(2)\}$.
   (a) *If* $\alpha < \alpha_1^{TS}(s)$, *then* $\bar{m}^*(TS) = 0$ *and* $\bar{E} = COM(0, 4)$
   (b) *If* $\alpha > \alpha_1^{TS}(s)$, *then* $\bar{m}^*(TS) = 2$ *and* $\bar{E} = COL(2, 0)$.
3. *If* $\bar{\alpha}(1, 2) < \alpha \leq \bar{\alpha}(2)$.
   (a) *If* $\alpha < \alpha_2^{TS}(s)$, *then* $\bar{m}^*(TS) = 1$ *and* $\bar{E} = COM(1, 2)$.
   (b) *If* $\alpha > \alpha_2^{TS}(s)$, *then* $\bar{m}^*(TS) = 2$ *and* $\bar{E} = COL(2, 0)$.

[25]See Appendix A.2 for the expressions of $\alpha_1^{TS}(s)$ and $\alpha_2^{TS}(s)$.

4. If $\alpha > \bar{\alpha}(2)$, then $\bar{m}^* (TS) = 2$ and $\bar{E} = COM\,(2,0)$. ***Proof:*** *See Appendix A.2.* ■

Figure 5 illustrates Proposition 4. Note that we consider a region of the parameter space ($s > \left(83 - 24\sqrt{11}\right) d/79$) in which $TS^{COM}\,(2,0) > TS^{COM}\,(1,2) > TS^{COM}\,(0,4)$, i.e., total surplus under $COM\,(2,0)$ ($COM\,(1,2)$) is higher than under $COM\,(1,2)$ ($COM\,(0,4)$). Therefore, if firms are forced to compete, a competition authority with a mandate to maximize the expected discounted total surplus will always allow mergers. Once again, the problem is that firms may prefer to collude. The interpretation of Proposition 4 is analogous to Proposition 3, with one important exception. In Proposition 3, $\bar{m} = 1$ is never the best optimal merger policy, while in Proposition 4, when $\bar{\alpha}\,(1,2) < \alpha < \min\left\{\alpha_2^{TS}\,(s)\,, \bar{\alpha}(2)\right\}$, we have $\bar{m}^*\,(TS) = 1$ (region 3.a in Figure 5). The reason for this is as follows. When, $\bar{\alpha}\,(1,2) < \alpha < \bar{\alpha}(2)$, if no merger is allowed, the equilibrium will be $COM\,(0,4)$; if only one merger is allowed, then the preferred equilibrium for the firms will be $COM\,(1,2)$; and, finally, if two mergers are allowed ($\bar{m} = 2$), then the preferred equilibrium for the firms will be $COL\,(2,0)$. For $s > \left(83 - 24\sqrt{11}\right) d/79$, the total surplus under $COM\,(1,2)$ is higher than under $COM\,(0,4)$. Thus, the optimal merger policy is to allow only one merger whenever the total surplus under $COM\,(1,2)$ is higher than under $COL\,(2,0)$, which occurs when the probability of detecting collusion is low enough (formally, $\alpha < \alpha_2^{TS}\,(s)$).

Proposition 4 assumes that the competition authority is fully aware of the effect that merger policy has on collusion decisions. In contrast, consider a myopic competition authority that does not internalize that relaxing merger policy could lead to collusion. If $s > \left(83 - 24\sqrt{11}\right) d/79$, such a competition authority will allow all mergers on the incorrect belief that mergers will increase total surplus. However, if $\bar{\alpha}\,(4) < \alpha \leq \min\left\{\bar{\alpha}\,(1,2)\,, \alpha_1^{TS}\,(s)\right\}$, total surplus will be higher if mergers are not allowed.

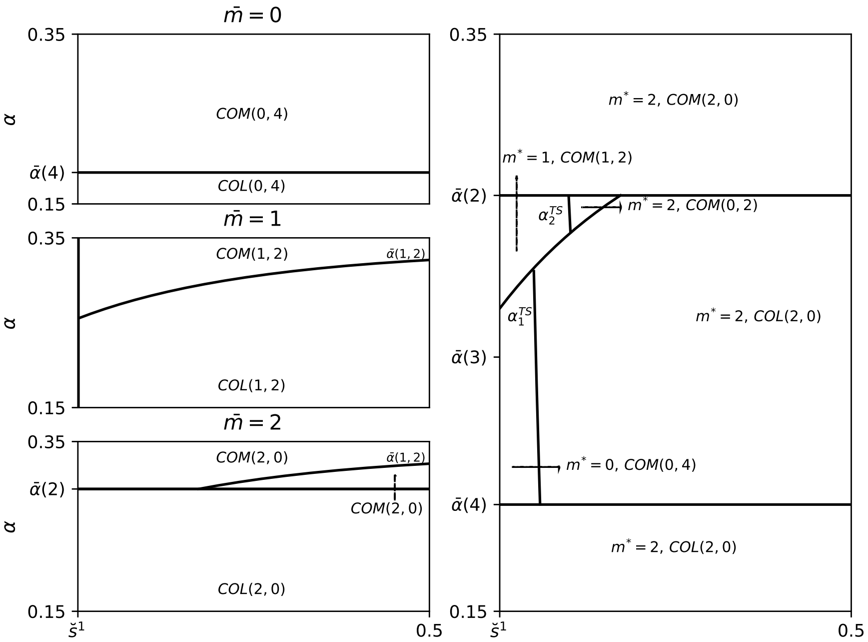

Figure 5: Optimal merger policy (total surplus). Note: $\delta = 3/4$.

4.4.1 Toward a formal definition of unilateral and coordinated effects

Propositions 3 and 4 suggest a decomposition of the welfare impact of mergers on two theoretically different effects, namely, a unilateral and a coordinated effect. Define the unilateral effect of a change in merger policy as the associated change in welfare, assuming that there will be no modification in the competition regime. Define the coordinated effect of a change in merger policy as the change in welfare exclusively attributed to the change in the competition regime. For example, suppose that a modification in the merger policy induces a change form $COM\,(0,4)$ to $COL\,(2,0)$. Then, the unilateral effect will be the change in welfare from $COM\,(0,4)$ to $COM\,(2,0)$, while the coordinated effect will be the change in welfare from $COM\,(2,0)$ to $COL\,(2,0)$.

More formally, suppose that the competition authority changes the maximum number of mergers allowed from $\bar{m}^I \in \{0,1,2\}$ to $\bar{m}^F \in \{0,1,2\}$. Let $\bar{E}\left(\bar{m}^I\right)$ $\left(\bar{E}\left(\bar{m}^F\right)\right)$ indicates the SPNE outcome path selected by the firms when the merger policy is $\bar{m}^I$ $\left(\bar{m}^F\right)$ and $W\left(\bar{E}\left(\bar{m}^I\right)\right)$ $\left(W\left(\bar{E}\left(\bar{m}^F\right)\right)\right)$ the expected discounted welfare induced by $\bar{E}\left(\bar{m}^I\right)$ $\left(\bar{E}\left(\bar{m}^F\right)\right)$. Let $\tilde{O}$ denote the outcome path when the competition regime is the one under $\bar{E}\left(\bar{m}^I\right)$ and the set of firms is the one under $\bar{E}\left(\bar{m}^F\right)$. Denote by $W\left(\tilde{O}\right)$ the expected discounted welfare induced by $\tilde{O}$. Then, the unilateral and coordinated effects associated with the change in merger policy from $\bar{m}^I$ to $\bar{m}^F$ are given by:

$$\begin{aligned} UE\left(\bar{m}^I,\bar{m}^F\right) &= W\left(\tilde{O}\right) - W\left(\bar{E}\left(\bar{m}^I\right)\right) \\ CE\left(\bar{m}^I,\bar{m}^F\right) &= W\left(\bar{E}\left(\bar{m}^F\right)\right) - W\left(\tilde{O}\right) \end{aligned}$$

Naturally, the total effect is given by $TE\left(\bar{m}^I,\bar{m}^F\right) = UE\left(\bar{m}^I,\bar{m}^F\right) + CE\left(\bar{m}^I,\bar{m}^F\right) = W\left(\bar{E}\left(\bar{m}^F\right)\right) - W\left(\bar{E}\left(\bar{m}^I\right)\right)$

To illustrate the computation of the unilateral and coordinated effects consider the following example. Suppose that the competition authority has a mandate to maximize the wellbeing of consumers and it is considering a change from $\bar{m}^I = 0$ to $\bar{m}^F = 2$. Assume that $s > d/5$ and $\bar{\alpha}(4) < \alpha \leq \min\left\{\bar{\alpha}(2), \alpha^{CS}(s)\right\}$, which implies that $\bar{E}(0) = COM\,(0,4)$ and $\bar{E}(2) = COL\,(2,0)$. Then, the unilateral, coordinated and total effects associated with this change in merger policy are given by:

$$\begin{aligned} UE\,(0,2) &= W\left(COM\,(2,0)\right) - W\left(COM\,(0,4)\right) = \frac{2\left(a-c+cs\right)^2}{9(1-\delta)} - \frac{8\left(a-c\right)^2}{25(1-\delta)} \\ CE\,(0,2) &= W\left(COL\,(2,0)\right) - W\left(COM\,(2,0)\right) = \frac{\left(\frac{1}{8}-\frac{2}{9}\right)\left(a-c+cs\right)^2}{\left(1-\delta+\alpha\delta\right)} \\ TE\,(0,2) &= W\left(COL\,(2,0)\right) - W\left(COM\,(0,4)\right) = \frac{\left[16\alpha\delta+9(1-\delta)\right]\left(a-c+cs\right)^2}{72(1-\delta)\left(1-\delta+\alpha\delta\right)} - \frac{8\left(a-c\right)^2}{25(1-\delta)} \end{aligned}$$

It is easy to verify that since $s > d/5$, then $UE\,(0,2) > 0$. Clearly, $CE\,(0,2) < 0$. It is more tedious, but simple to show that $\alpha < \alpha^{CS}(s)$ implies $TE\,(0,2) < 0$. Thus, a myopic competition authority that does not take into account that mergers could change the competition regime misses the coordinated effect and it could welcome mergers that reduce welfare even if there is ample evidence that synergies are high enough to generate positive unilateral effects.

5 Conclusions

We have developed a model that formalizes several connections between mergers and collusion and between antitrust enforcement and merger policy. One key policy implication is that competition authorities should seriously consider the possibility that mergers are a vehicle to make collusion sustainable. Thus, to approve a merger it should not be enough to show that cost synergies will more than compensate any reduction in the competitive pressure faced by the firms under oligopolistic competition. The competition authority should also check that a merger will not make collusion sustainable. A benevolent, but myopic competition authority that disregards this coordinated effect will end up accepting mergers that hurt consumers or even reduce aggregate welfare. This does not necessarily imply that if mergers will lead to collusion, they should never be allowed. It is still possible that cost synergies are high enough to make collusion with low-cost firms better than competition with high-cost firms. More importantly, the model also suggests an interesting connection between antitrust enforcement and merger policy. A competition authority that can easily detect and prosecute collusion should have a more lenient merger policy. Worst case scenario, if mergers lead to collusion, firms will be promptly detected and forced to compete.

There are several paths to extend our analysis. Here we will mention just two of them. First, we focus on horizontal mergers. It would be interesting to also explore the connections between vertical mergers and collusion. Second, we implicitly assume that firms, consumers and the competition authority belong to the same jurisdiction (e.g., an industry in a given country). It would be interesting to extend the analysis to firms that operate in different countries and face national competition authorities (for antitrust enforcement in a global economy, see Garcia et al. (2017)).

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Online Appendix to "Collusion, Mergers and Antitrust Policy"

This Appendix presents the proofs of all the lemmas and propositions in the main text.

A.1 No Synergies ($s = 0$)

We begin deducing equilibrium quantities, prices and profits under competition, collusion and deviation from collusion. Since, initially, all firms have the same cost function and when $s = 0$ mergers do not induce any cost reduction, all firms have always the same cost function, namely, $C(q) = cq$. Hence, the only relevant information about N_t is the number of firms in the set, which we denote by n_t.

Competition. Suppose that the n_t firms in N_t choose to compete. Then, the quantities produced by the firms in period t are given by the Cournot equilibrium with n_t homogeneous firms. Thus: $q^{z,COM}(N_t) = q^{COM}(n_t)(a-c)/(n_t+1)$, $\pi^{z,COM}(N_t) = \pi^{COM}(n_t) = (a-c)^2/(n_t+1)^2$ for all $z \in N_t$, $P^{COM}(n_t) = (a+cn_t)/(n_t+1)$, and $Q^{COM}(n_t) = n_t(a-c)/(n_t+1)$.

Collusion. Suppose that the n_t firms in N_t choose to collude. Then: $q^{z,COL}(N_t) = q^{COL}(n_t) = (a-c)/2n_t$, $\pi^{z,COL}(N_t) = \pi^{COL}(n_t) = (a-c)^2/4n_t$ for all $z \in N_t$, $P^{COL}(n_t) = (a+c)/2$, and $Q^{COL}(n_t) = (a-c)/2$.

Deviation from collusion. Suppose that all firms choose to collude except for firm z. Then, the best possible deviation for firm z is to produce $q^{z,DEV}(N_t) = \arg\max_q \left(a - \sum_{x\in N_t, x\neq z} q^{x,COL}(n_t) - q - c\right) q$. Therefore: $q^{z,DEV}(N_t) = q^{DEV}(n_t) = (a-c)(n_t+1)/4n_t$, $\pi^{z,DEV}(N_t) = \pi^{DEV}(n_t) = (a-c)^2(n_t+1)^2/(4n_t)^2$, $q^{x,COL}(N_t) = q^{COL}(n_t) = (a-c)/2n_t$, $\pi_t^x = 3(n_t+1)(a-c)^2/8(n_t)^2$ for $x \in N_t$ and $x \neq z$, $P^{DEV}(n_t) = 3a(n_t+1) + c(n_t-3)/4n_t$, and $Q^{DEV}(n_t) = (a-c)(2n_t-1)/4n_t$.

Lemma 1 ***No mergers under competition***. Suppose that for all $\tau \geq t$ firms are forced to compete. Then, the unique SPNE outcome path from period t onward is $\left(n_\tau, (q_\tau^z)_{z\in N_\tau}\right) = \left(n_{t-1}, \left(q^{COM}(n_{t-1})\right)_{z\in N_{t-1}}\right)$ for all $\tau \geq t$.

Proof: Suppose that for all $\tau \geq t$ firms will be forced to compete. Then, the profits obtained by firm z in period $\tau \geq t$ will be $\pi_\tau^z = \pi^{COM}(n_\tau) = (a-c)^2/(n_\tau+1)^2$. If $n_{t-1} = 3$, the proof is immediate because once only two firms remain in the market, mergers are not allowed. Then, if the pair of matched firms in period t decide to merge, the expected discounted profits of each of the merged firms will be $\pi^{COM}(2)/2(1-\delta)$, while if they do not merge, the expected discounted profits of each of the merged firms will be $\pi^{COM}(3)/(1-\delta)$. Since $\pi^{COM}(3) > \pi^{COM}(2)/2$, firms will prefer to do not merge. For $n_{t-1} > 3$, it is not difficult to prove that $\pi^{COM}(n_t+1) > \pi^{COM}(n_t)/2$ and the assumption that there are no mergers when the remaining number of firms is n_t imply that firms will not merge. Formally, assume that firms x and $y \in N_{t-1}$ are matched in period t and let n_t denote the number of firms in period t after merger decisions have been made when x and y merge. Moreover, assume for a moment that firms do not have an incentive to merge once the remaining number of firms is n_t. That is, if x and y decide to merge, from period $t+1$ onward there will be no more mergers. Then, the expected discounted profits of x and y will be given by $v^{COM}(n_t)/2 = \left(\pi^{COM}(n_t) + \delta v^{COM}(n_t)\right)/2$. Therefore, $v^{COM}(n_t)/2 = \pi^{COM}(n_t)/2(1-\delta)$. On the contrary, if x and y do not merge, then the expected discounted profits of x and y will be given by $v^{COM}(n_t+1) = \pi^{COM}(n_t+1)/(1-\delta)$. Since $n_t \geq 2$ implies $\pi^{COM}(n_t+1) > \pi^{COM}(n_t)/2$, firms will always prefer to do not merge. That is, if there are no mergers when the number of firms is n_t, there will never be enough mergers to reach n_t firms. Thus, if for all $\tau \geq t$ firms are forced to compete, in equilibrium, there will be no merger for all $\tau \geq t$. ■

Lemma 2 ***No mergers under collusion***. Suppose that for all $\tau \geq t$ firms are forced to collude until they are detected, whereupon they will not be allowed to merge anymore and will be forced to compete. Then, the unique SPNE outcome path from period t onward is $\left(n_\tau, (q_\tau^z)_{z \in N_\tau}\right) = \left(n_{t-1}, \left(q^{COL}(n_{t-1})\right)_{z \in N_{t-1}}\right)$ until collusion is detected and $\left(n_\tau, (q_\tau^z)_{z \in N_\tau}\right) = \left(n_{t-1}, \left(q^{COM}(n_{t-1})\right)_{z \in N_{t-1}}\right)$, thereafter.

Proof: Suppose that for all $\tau \geq t$ firms will have to collude until they are detected, whereupon they will not be allowed to merge anymore and will be forced to compete. In such a situation, a firm will get lower expected profits if it merges than if it does not merge. If $n_{t-1} = 3$, the proof is immediate because once only two firms remain in the market, mergers are not allowed. Then, if the pair of matched firms in period t decide to merge, the expected discounted profits of each of the merged firms will be $\pi^{COL}(2)/2(1-\delta)$, while if they do not merge, the expected discounted profits of each of the merged firms will be $\pi^{COL}(3)/(1-\delta)$. Since $\pi^{COL}(3) > \pi^{COL}(2)/2$, firms will prefer to do not merge. For $n_{t-1} > 3$, assume that firms x and y decide to merge. Then, the expected discounted profits of x and y will be

$$\frac{v^{COL}(n_t)}{2} = \frac{\frac{\pi^{COL}(n_t)}{2} + \frac{\delta\alpha\pi^{COM}(n_t)}{(1-\delta)2}}{1-\delta(1-\alpha)},$$

where $n_t \in [2, n)$ is the number of firms in period t after merger decisions have been made when x and y merge. Note that we have implicitly assumed that once there are n_t firms in the market there are no incentives to merge. On the contrary, if firms x and y do not merge, then the expected discounted profits of the firms will be

$$v^{COL}(n_t+1) = \frac{\pi^{COL}(n_t+1) + \frac{\delta\alpha\pi^{COM}(n_t+1)}{1-\delta}}{1-\delta(1-\alpha)},$$

where $n_t + 1 \leq n$ is the number of firms in period t after merger decisions have been made when x and y do not merge. Then:

$$v^{COL}(n_t+1) - \frac{v^{COL}(n_t)}{2} = \frac{\left[\pi^{COL}(n_t+1) - \frac{\pi^{COL}(n_t)}{2}\right] + \frac{\delta\alpha}{1-\delta}\left[\pi^{COM}(n_t+1) - \frac{\pi^{COM}(n_t)}{2}\right]}{1-\delta(1-\alpha)}.$$

The first square brackets captures the difference in collusion profits obtained by firms x and y between a situation in which x and y do not merge and a situation in which they merge. As we have already proved, $\pi^{COL}(n_t+1) > \pi^{COL}(n_t)/2$. That is, under collusion, a firm obtains a higher share of the industry monopoly profits if it does not merge.[26] The second square brackets captures the difference in competition profits obtained by firms x and y between a situation in which x and y do not merge and a situation in which they merge. Since in every period there is a probability $\alpha > 0$ that the competition authority detects collusion, eventually collusion is detected. When this happens, the competition authority bans mergers and forces firms to compete. As we have already proved, $\pi^{COM}(n_t+1) > \pi^{COM}(n_t)/2$. That is, under competition, a firm obtains higher profits if it refuses to merge. That is, if there are no mergers when the number of firms is n_t, there will never be enough mergers to reach n_t firms. Thus, if for all $\tau \geq t$ firms are forced to collude, in equilibrium, there will be no merger for all $\tau \geq t$. ■

Lemma 3 ***Sustainability of collusion.*** Suppose that before period t firms did not collude. Assume that after merger decisions have been made in period t, there are $n_t \geq 2$ firms. Then, for $\tau \geq t$, $\left(n_\tau, (q_\tau^z)_{z \in N_\tau}\right) =$

[26] Industry monopoly profits are $\pi^{COL} = (a-c)^2/4$. A firm obtains $1/(n_t+1)$ of π^{COL} if it does not merge and $1/2$ of $1/n_t$ of π^{COL} if it merges.

$\left(n_t, \left(q^{COL}(n_t)\right)_{z\in N_t}\right)$ until collusion is detected and $\left(n_\tau, (q^z_\tau)_{z\in N_\tau}\right) = \left(n_t, \left(q^{COM}(n_t)\right)_{z\in N_t}\right)$ thereafter is an SPNE outcome path from t onward if and only if $\alpha \le \bar{\alpha}(n_t)$, where:

$$\bar{\alpha}(n_t) = 1 - \frac{(n_t-1)^2 (n_t+1)^2}{\delta\left[(n_t+1)^4 - 16(n_t)^2\right]}.$$

Moreover, $\bar{\alpha}(n_t)$ is decreasing in n_t.

Proof. Due to Lemma 1, there are no mergers along the punishment path. Due to Lemma 2, there are no mergers while firms are colluding. Then, the expected discounted profits of firm $z \in N_t$ under collusion is given by:

$$v^{COL}(n_t) = \frac{\pi^{COL}(n_t) + \frac{\delta\alpha}{1-\delta}\pi^{COM}(n_t)}{1-\delta(1-\alpha)} = \frac{(a-c)^2\left[\frac{1}{4n_t} + \frac{\delta\alpha}{(n_t+1)^2(1-\delta)}\right]}{1-\delta(1-\alpha)},$$

while the expected discounted profits if firm z deviates from collusion is:

$$v^{DEV}(n_t) = \pi^{DEV}(n_t) + \frac{\delta}{1-\delta}\pi^{COM}(n_t).$$

Therefore, n_t firms can sustain collusion if and only if $v^{COL}(n_t) \ge v^{DEV}(n_t)$ or, which is equivalent,

$$\alpha \le \bar{\alpha}(n_t) = 1 - \frac{(n_t-1)^2 (n_t+1)^2}{\delta\left[(n_t+1)^4 - 16(n_t)^2\right]}$$

Finally, taking the derivative of $\bar{\alpha}(n_t)$ with respect to n_t, it is tedious but simple to prove that $\bar{\alpha}(n_t)$ is strictly decreasing in n_t for any δ and $n_t \ge 2$. ■

Proposition 1. Assume that $\delta > \frac{(n-1)^2(n+1)^2}{(n+1)^4-16n^2}$. Let $\bar{n} = \lceil \bar{\alpha}^{-1}(\alpha) \rceil$ and $\tilde{\alpha}(n_t) = \left(\frac{1-\delta}{\delta}\right)\frac{(n_t+1)^2(n_t-2)^2}{4n_t\left[(n_t)^2-2\right]}$, where $\lceil x \rceil$ indicates the integer part of x.

1. Suppose that $\alpha \le \bar{\alpha}(n)$. Then, the n original firms collude from $t = 0$ until they are detected. Thereafter, there is competition with n firms operating in the market.
2. Suppose that $\bar{\alpha}(n) < \alpha \le \bar{\alpha}(n/2)$.
   (a) If $\alpha \le \tilde{\alpha}(\bar{n})$, in period $t = 0$ there are $m = n - \bar{n}$ mergers and the $\bar{n}$ new firms collude until they are detected. Thereafter, there is competition with $\bar{n}$ firms operating in the market.
   (b) If $\alpha > \tilde{\alpha}(\bar{n})$, in every period there is competition with n firms.

Proof of Part 1. Suppose that $\alpha \le \bar{\alpha}(n)$. Then, from Lemma 3, firms can sustain collusion regardless of the merger decisions, while from Lemma 2, if firms will end up colluding, no firm has an incentive to merge. Therefore, if $\alpha \le \bar{\alpha}(n)$, in equilibrium, there will be no mergers and the n original firms will begin collusion immediately. That is, in equilibrium, for all $t \ge 0$, $m^x_t(y) = 0$ for $x, y \in N_{t-1}$, $q^z_t = q^{COL}(n) = (a-c)/2n$ until firms are detected and, thereafter, $q^z_t = q^{COM}(n) = (a-c)/(n+1)$ for $z \in N_t$.

Proof of Part 2. Suppose that $\bar{\alpha}(n) < \alpha \le \bar{\alpha}(n/2)$. Then, from Lemma 3, firms cannot sustain collusion if the number of firms is $n_0 \in \left(\bar{\alpha}^{-1}(\alpha), n\right]$, but they can sustain collusion if $n_0 \in \left[n/2, \bar{\alpha}^{-1}(\alpha)\right]$. Thus, collusion can be sustained in period $t = 0$ if the number of mergers is $m_0 \geq n - \lceil \bar{\alpha}^{-1}(\alpha) \rceil$. We must consider four possible cases.

Case 1 ($n_0 < \lceil \bar{\alpha}^{-1}(\alpha) \rceil$). Suppose that, after merger decisions have been made in period $t = 0$, the number of firms is $n_0 < \lceil \bar{\alpha}^{-1}(\alpha) \rceil$. Then, from Lemma 3, firms will begin colluding in period $t = 0$. Thus, for $t \geq 0$, $q_t^z = q^{COL}(n_0) = (a - c)/2n_0$ until firms are detected and, thereafter, $q_t^z = q^{COM}(n_0) = (a - c)/(n_0 + 1)$ for $z \in N_t$. Moreover, $m_t^x(y) = 0$ for $x, y \in N_{t-1}$ and $t > 0$. To prove that this is not an equilibrium, assume that firms x and y are matched in period $t = 0$ and they are one of the pairs of matched firms that decide to merge. Then, the expected discounted profits of firms x and y are

$$\frac{v^{COL}(n_0)}{2} = \frac{\pi^{COL}(n_0) + \frac{\delta\alpha}{1-\delta}\pi^{COM}(n_0)}{2[1 - \delta(1 - \alpha)]} = \frac{(a - c)^2 \left[\frac{1}{4n_0} + \frac{\delta\alpha}{(n_0+1)^2(1-\delta)}\right]}{2[1 - \delta(1 - \alpha)]}$$

On the contrary, if either x or y chooses to deviate and rejects the merger, then the expected discounted profits of x and y would be

$$v^{COL}(n_0 + 1) = \frac{\pi^{COL}(n_0 + 1) + \frac{\delta\alpha}{1-\delta}\pi^{COM}(n_0 + 1)}{1 - \delta(1 - \alpha)} = \frac{(a - c)^2 \left[\frac{1}{4(n_0+1)} + \frac{\delta\alpha}{(n_0+2)^2(1-\delta)}\right]}{[1 - \delta(1 - \alpha)]}$$

where $n_0 + 1 \leq \lceil \bar{\alpha}^{-1}(\alpha) \rceil$. $v^{COL}(n_0 + 1) > v^{COL}(n_0)/2$ if and only $(n_0 - 1)/4n_0 + \delta\alpha \left[(n_0)^2 - 2\right]/(1 - \delta)(n_0 + 2)^2(n_0 + 1) > 0$, which always holds. Thus, when $n_0 < \lceil \bar{\alpha}^{-1}(\alpha) \rceil$, firms x and y prefer to do not merge.

Case 2 ($n_0 = \bar{n} = \lceil \bar{\alpha}^{-1}(\alpha) \rceil$). Suppose that, after merger decisions have been made in period $t = 0$, the number of firms is $n_0 = \bar{n} = \lceil \bar{\alpha}^{-1}(\alpha) \rceil$. Then, from Lemma 3, firms will begin colluding in period $t = 0$. Thus, for $t \geq 0$, $q_t^z = q^{COL}(\bar{n}) = (a - c)/2\bar{n}$ until firms are detected and, thereafter, $q_t^z = q^{COM}(\bar{n}) = (a - c)/(\bar{n} + 1)$ for $z \in N_t$. Moreover, $m_t^x(y) = 0$ for $x, y \in N_{t-1}$ and $t > 0$. For this to be an equilibrium, we must verify that none of the matched firms in period $t = 0$ that are choosing to merge prefer to do not merge. In order to do so, assume that firms x and y are matched in period $t = 0$ and they are one of the pairs of matched firms that decide to merge. Then, the expected discounted profits of firms x and y are

$$\frac{v^{COL}(\bar{n})}{2} = \frac{\pi^{COL}(\bar{n}) + \frac{\delta\alpha}{1-\delta}\pi^{COM}(\bar{n})}{2[1 - \delta(1 - \alpha)]} = \frac{(a - c)^2 \left[\frac{1}{4\bar{n}} + \frac{\delta\alpha}{(\bar{n}+1)^2(1-\delta)}\right]}{2[1 - \delta(1 - \alpha)]}$$

On the contrary, if either x or y chooses to deviate and rejects the merger, then the expected discounted profits of firms x and y would be

$$v^{COM}(\bar{n} + 1) = \frac{\pi^{COM}(\bar{n} + 1)}{1 - \delta} = \frac{(a - c)^2}{(1 - \delta)(\bar{n} + 2)^2}$$

$v^{COL}(\bar{n})/2 \geq v^{COM}(\bar{n} + 1)$ if and only if $\alpha \leq \tilde{\alpha}(\bar{n}) = \left(\frac{1-\delta}{\delta}\right)\frac{(\bar{n}-2)^2(\bar{n}+1)^2}{4\bar{n}\left[(\bar{n})^2-2\right]}$. Therefore, if $\alpha \leq \tilde{\alpha}(\bar{n})$, in period $t = 0$ there are $m_0 = n - \lceil \bar{\alpha}^{-1}(\alpha) \rceil$ mergers and, thereafter, there are no more mergers. The remaining firms ($\bar{n} = \lceil \bar{\alpha}^{-1}(\alpha) \rceil$) collude until they are detected and forced to compete.

Case 3 ($\lceil \bar{\alpha}^{-1}(\alpha) \rceil < n_0 < n$). Suppose that, after merger decisions have been made in period $t = 0$, $n_0 \in \left(\lceil \bar{\alpha}^{-1}(\alpha) \rceil, n\right)$. Then, from Lemma 3, firms cannot sustain collusion in period $t = 0$ and, therefore, they will compete for all $t \geq 0$. Thus, for $t \geq 0$, $q_t^z = q^{COM}(n_0) = (a - c)/(n_0 + 1)$ for $z \in N_t$. Moreover,

$m_t^x(y) = 0$ for $x, y \in N_{t-1}$ and $t > 0$. To prove that this is not an equilibrium, assume that firms x and $y \in N_{t-1}$ are matched in period t and they are one of the pairs of matched firms that decide to merge. If firms x and y merge, then the expected discounted profits of x and y will be

$$\frac{v^{COM}(n_0)}{2} = \frac{\pi^{COM}(n_0)}{2(1-\delta)} = \frac{(a-c)^2}{2(1-\delta)(n_0+1)^2},$$

On the contrary, if either x or y chooses to deviate and rejects the merger, then the expected discounted profits of x and y would be

$$v^{COM}(n_0+1) = \frac{\pi^{COM}(n_0+1)}{1-\delta} = \frac{(a-c)^2}{(1-\delta)(n_0+2)^2},$$

Clearly, $v^{COM}(n_0+1) > v^{COM}(n_0)/2$. Thus, when $\lceil \bar{\alpha}^{-1}(\alpha) \rceil < n_0 < n$, firms x and y prefer to do not merge.

Case 4 ($n_0 = n$). If all firms but x refuse to merge, then, there is nothing that firm x can do to induce a merger. Thus, no merger is an equilibrium. Moreover, in this equilibrium, the expected discounted profits of a firm are

$$v^{COM}(n) = \frac{\pi^{COM}(n)}{1-\delta} = \frac{(a-c)^2}{(1-\delta)(n+1)^2}$$

Equilibrium comparison. If $\alpha > \tilde{\alpha}(\bar{n})$, there is only one equilibrium ($n_0 = n$), while if $\alpha \leq \tilde{\alpha}(\bar{n})$, there are two equilibria ($n_0 = n$ and $n_0 = \bar{n} = \lceil \bar{\alpha}^{-1}(\alpha) \rceil$). Next, we prove that $n_0 = \bar{n} = \lceil \bar{\alpha}^{-1}(\alpha) \rceil$ induces higher expected discounted profits to each firm than $n_0 = n$. If $n_0 = n$, the expected discounted profits of each firm are $v^{COM}(n)$. If $n_0 = \bar{n} = \lceil \bar{\alpha}^{-1}(\alpha) \rceil$, the expected discounted profits of the firms that did not merge are $v^{COL}(\bar{n})$, while the expected discounted profits of the firms that merged are $v^{COL}(\bar{n})/2$. Note that $v^{COM}(\bar{n}+1) \geq v^{COM}(n)$. Also $\alpha \leq \tilde{\alpha}(\bar{n})$ if and only if $v^{COL}(\bar{n})/2 \geq v^{COM}(\bar{n}+1)$. Thus, whenever $\alpha \leq \tilde{\alpha}(\bar{n})$, $v^{COL}(\bar{n})/2 \geq v^{COM}(n)$.

This completes the proof of Proposition 1. ■

A.2 Positive Synergies ($s > 0$)

We begin deducing equilibrium quantities, prices and profits under competition, collusion and deviation from collusion.

Competition. Suppose that, after merger decisions have been made in period t, there n_t^H firms with unit cost c and n_t^L firms with unit cost $c(1-s)$, with $n_t^L s < d = \frac{a-c}{c}$. Assume that all firms choose to compete. Then, the quantities produced by the firms in period t are given by the Cournot equilibrium with n_t^H firms with unit cost c and n_t^L firms with unit cost $c(1-s)$. Thus:

- $q^{z,COM}(N_t) = q_L^{COM}(n_t^L, n_t^H) = \frac{a-c+(n_t^H+1)cs}{n_t^L+n_t^H+1}$ for $z \in N_t^L$ and $q^{z,COM}(N_t) = q_H^{COM}(n_t^L, n_t^H) = \frac{a-c-n_t^L cs}{n_t^L+n_t^H+1}$ for $z \in N_t^H$;

- $\pi^{z,COM}(N_t) = \pi_L^{COM}(n_t^L, n_t^H) = \left(\frac{a-c+n_t^H cs+cs}{n_t^L+n_t^H+1}\right)^2$ for $z \in N_t^L$ and $\pi^{z,COM}(N_t) = \pi_H^{COM}(n_t^L, n_t^H) = \left(\frac{a-c-n_t^L cs}{n_t^L+n_t^H+1}\right)^2$ for $z \in N_t^H$;

- $P^{COM}(N_t) = P^{COM}\left(n_t^L, n_t^H\right) = \frac{a+c\left(n_t^L+n_t^H\right)-n_t^L cs}{n_t^L+n_t^H+1}$ and $Q^{COM}(N_t) = Q^{COM}\left(n_t^L, n_t^H\right) = \frac{(a-c)\left(n_t^L+n_t^H\right)+n_t^L cs}{\left(n_t^L+n_t^H\right)+1}$.

- The market share of firm $z \in N_t^L$ is

$$s_L^{COM}\left(n_t^L, n_t^H\right) = \frac{q_L^{COM}\left(n_t^L, n_t^H\right)}{Q^{COM}\left(n_t^L, n_t^H\right)} = \frac{a-c+\left(n_t^H+1\right)cs}{(a-c)\left(n_t^L+n_t^H\right)+n_t^L cs}$$

  while the market share of firm $z \in N_t^H$ is

$$s_H^{COM}\left(n_t^L, n_t^H\right) = \frac{q_L^{COM}\left(n_t^L, n_t^H\right)}{Q^{COM}\left(n_t^L, n_t^H\right)} = \frac{a-c-n_t^L cs}{(a-c)\left(n_t^L+n_t^H\right)+n_t^L cs}$$

Collusion. Suppose that, after merger decisions have been made in period t, there n_t^H firms with unit cost c and n_t^L firms with unit cost $c(1-s)$. Assume that firms in N_t choose to collude. If $n_t^L = 0$, then the aggregate production is the quantity produced by a monopolist with unit cost c. If $n_t^L \geq 1$, then the aggregate production is the quantity produced by a monopolist with unit cost $c(1-s)$. Thus:

- $P^{COL}(N_t) = P^{COL}\left(n_t^L, n_t^H\right) = \frac{a+\left(1-s\left(n_t^L\right)\right)c}{2}$ and $Q^{COL}(N_t) = Q^{COL}\left(n_t^L, n_t^H\right) = \frac{a-\left(1-s\left(n_t^L\right)\right)c}{2}$, where $s(0) = 0$ and $s\left(n_t^L\right) = s$ if $n_t^L > 0$.

Since under collusion, all firms reduce production to induce the monopoly quantity, but keeping the market shares that firms would obtain under competition, we have:

- $q^{z,COL}(N_t) = q_L^{COL}\left(n_t^L, n_t^H\right) = s_L^{COM}\left(n_t^L, n_t^H\right) Q^{COL}\left(n_t^L, n_t^H\right)$ for $z \in N_t^L$ and $q^{z,COL}(N_t) = q^{z,COL}\left(n_t^L, n_t^H\right) = s_H^{COM}\left(n_t^L, n_t^H\right) Q^{COL}\left(n_t^L, n_t^H\right)$ for $z \in N_t^H$;

- $\pi^{z,COL}(N_t) = \pi_L^{COL}\left(n_t^L, n_t^H\right) = s_L^{COM}\left(n_t^L, n_t^H\right)\left[Q^{COL}\left(n_t^L, n_t^H\right)\right]^2$ for $z \in N_t^L$ and $\pi^{z,COL}(N_t) = \pi_H^{COL}\left(n_t^L, n_t^H\right) = s_H^{COM}\left(n_t^L, n_t^H\right)\left[Q^{COL}\left(n_t^L, n_t^H\right) - s\left(n_t^L\right)c\right] Q^{COL}\left(n_t^L, n_t^H\right)$ for $z \in N_t^H$.

Deviation from collusion. Suppose that all firms choose to collude except for firm z. We must consider two possible cases.

a. Suppose that $z \in N_t^L$. Then, $q^{z,DEV}(N_t) = \arg\max_q \left(a - \sum_{x \in N_t, x \neq z} q^{z,COL}\left(n_t^L, n_t^H\right) - q - c(1-s)\right) q$. Therefore:

- $q^{z,DEV}(N_t) = q^{DEV-L}\left(n_t^A, n_t^H\right) = \frac{\left[1+s_L^{COM}\left(n_t^L, n_t^H\right)\right] Q^{COL}\left(n_t^L, n_t^H\right)}{2}$;

- $\pi^{z,DEV}(N_t) = \pi^{DEV-L}\left(n_t^L, n_t^H\right) = \left[\frac{1+s_L^{COM}\left(n_t^L, n_t^H\right)}{2}\right]^2 \left[Q^{COL}\left(n_t^L, n_t^H\right)\right]^2$, $\pi^x(N_t) = \left[\frac{1+s_L^{COM}\left(n_t^L, n_t^H\right)}{2}\right] s_L^{COM}\left(n_t^L, n_t^H\right)\left[Q^{COL}\left(n_t^L, n_t^H\right)\right]^2$ for $x \in N_t^L$ and $x \neq z$, $\pi^x(N_t) = \left[\frac{1+s_L^{COM}\left(n_t^L, n_t^H\right)}{2} - s\left(n_t^L\right)c\right] s_H^{COM}\left(n_t^L, n_t^H\right)\left[Q^{COL}\left(n_t^L, n_t^H\right)\right]^2$ for $x \in N_t^H$;

- $Q^{DEV}(N_t) = Q^{DEV-L}(n_t^L, n_t^H) = \left[\frac{3-s_L^{COM}(n_t^L,n_t^H)}{2}\right] Q^{COL}(n_t^L, n_t^H)$ and $P^{DEV}(N_t) = P^{DEV-L}(n_t^L, n_t^H) = a - \left[\frac{3-s_L^{COM}(n_t^L,n_t^H)}{2}\right] Q^{COL}(n_t^L, n_t^H)$.

b. Suppose that $z \in N_t^H$. Suppose that $z \in N_t^H$. Then, $q^{z,DEV}(N_t) = \arg\max_q \left(a - \sum_{x \in N_t, x \neq z} q^{z,COL}(n_t^L, n_t^H) - q - c\right) q$. Therefore:

- $q^{z,DEV}(N_t) = q^{DEV-H}(n_t^A, n_t^H) = \frac{[1+s_H^{COM}(n_t^L,n_t^H)]Q^{COL}(n_t^L,n_t^H)-s(n_t^L)c}{2}$;
- $\pi^{z,DEV}(N_t) = \pi^{DEV-H}(n_t^L, n_t^H) = \left[\frac{[1+s_H^{COM}(n_t^L,n_t^H)]Q^{COL}(n_t^L,n_t^H)-s(n_t^L)c}{2}\right]^2$, $\pi^x(N_t) = \left[\frac{[1+s_H^{COM}(n_t^L,n_t^H)]Q^{COL}(n_t^L,n_t^H)}{2} - s(n_t^L)c\right] s_H^{COM}(n_t^L, n_t^H) Q^{COL}(n_t^L, n_t^H)$ for $x \in N_t^H$ and $x \neq z$, $\pi^x(N_t) = \left[\frac{[1+s_H^{COM}(n_t^L,n_t^H)]Q^{COL}(n_t^L,n_t^H)}{2} + \frac{s(n_t^L)c}{2}\right] s_L^{COM}(n_t^L, n_t^H) Q^{COL}(n_t^L, n_t^H)$ for $x \in N_t^L$.
- $Q^{DEV}(N_t) = Q^{DEV-H}(n_t^L, n_t^H) = \left[\frac{3-s_H^{COM}(n_t^L,n_t^H)}{2}\right] Q^{COL}(n_t^L, n_t^H) - \frac{s(n_t^L)c}{2}$, and $P^{DEV}(N_t) = P^{DEV-H}(n_t^L, n_t^H) = a + \frac{s(n_t^L)c}{2} - \left[\frac{3-s_H^{COM}(n_t^L,n_t^H)}{2}\right] Q^{COL}(n_t^L, n_t^H)$.

Lemma 4 *Mergers* ***under competition.*** Suppose that $s \leq d = (a-c)/c$, $\delta \geq \bar{\delta} = (522\sqrt{2} - 723) / (498\sqrt{2} - 657) \approx 0.32191$, $(n_{t-1}^L, n_{t-1}^H) = (0, 4)$ and for all $\tau \geq t$ firms are forced to compete.

1. If $s < \bar{s}^1 = \left[(3 - 2\sqrt{2}) / (2\sqrt{2} + 3)\right] d$, then the unique SPNE is $(n_\tau^L, n_\tau^H) = (0, 4)$ for all $\tau \geq t$.
2. If $\bar{s}^1 \leq s < d/5$, then the SPNE that induces the highest profits for each firm is $(n_\tau^L, n_\tau^H) = (2, 0)$ for all $\tau \geq t$.
3. If $s \geq d/5$, then $(n_\tau^L, n_\tau^H) = (1, 2)$ for all $\tau \geq t$ is the preferred SPNE for the pair of firms that merge when the equilibrium is $(n_\tau^L, n_\tau^H) = (1, 2)$, while $(n_\tau^L, n_\tau^H) = (2, 0)$ for all $\tau \geq t$ is the preferred SPNE for other firms.

Proof: In order to find all the SPNE from period t onward, we must consider three possible types of subgames that could emerge from $(n_{t-1}^L, n_{t-1}^H) = (0, 4)$: (i) two firms remain in the industry, i.e., $(n_{u-1}^L, n_{u-1}^H) = (1, 1)$ or $(n_{u-1}^L, n_{u-1}^H) = (2, 0)$ for $u > t$; (ii) three firms remain in the industry, i.e., $(n_{u-1}^L, n_{u-1}^H) = (1, 2)$ for $u > t$; and (iii) four firms remain in the industry, i.e., $(n_{u-1}^L, n_{u-1}^H) = (0, 4)$ for $u \geq t$.

(i) Suppose that only **two firms remain in the industry**, i.e., $(n_{u-1}^L, n_{u-1}^H) = (1, 1)$ or $(n_{u-1}^L, n_{u-1}^H) = (2, 0)$. Then the unique SPNE from period u onward is $(n_\tau^L, n_\tau^H) = (n_{u-1}^L, n_{u-1}^H)$ for all $\tau \geq u$. The reason is that once only two firms remain, mergers are not allowed.

(ii) Suppose that **three firms remain in the industry**, i.e., $(n_{u-1}^L, n_{u-1}^H) = (1, 2)$. Without loss of generality, assume that $N_{u-1} = \{\{1, 2\}, \{3\}, \{4\}\}$. Then, the set of possible matches in period u is $M_u = \{M^1, M^2, M^3\}$, where $M^1 = \{\{1, 2, 3\}, \{4\}\}$, $M^2 = \{\{1, 2, 4\}, \{3\}\}$, and $M^3 = \{\{1, 2\}, \{3, 4\}\}$. There are four possible equilibria that must be considered.

Case 1: $\left(n_u^L, n_u^H\right) = (1,2)$ for all $M_u \in M_u$. This is always a SPNE from period u onward, because if all firms but x refuse to merge, then, there is nothing that firm x can do to induce a merger.

Case 2: $\left(n_u^L, n_u^H\right) = (1,2)$ when $M_u = M^1, M^2$ and $\left(n_u^L, n_u^H\right) = (2,0)$ when $M_u = M^3$. Then, the expected discounted profits of firms $\{3\}$ and $\{4\}$ when $M_t = M^3$ are $v^{\{3\}}\left(M^3\right)(1) = v^{\{4\}}\left(M^3\right)(1) = \pi_L^{COM}(2,0)/2(1-\delta)$. Now, suppose that when $M_u = M^3$ firms $\{3\}$ or $\{4\}$ deviate and, hence, they do not merge. Then, the expected discounted profits of firms $\{3\}$ and $\{4\}$ would be $v^{\{3\}}\left(M^3\right)(0) = v^{\{4\}}\left(M^3\right)(0) = \pi_H^{COM}(1,2)/(1-\delta)$. Firms $\{3\}$ or $\{4\}$ do not have an incentive to deviate if and only if $v^{\{3\}}\left(M^3\right)(1) \geq v^{\{3\}}\left(M^3\right)(0)$ or, which is equivalent, $\pi_L^{COM}(2,0)/2 \geq \pi_H^{COM}(1,2)$. $\pi_L^{COM}(2,0)/2 \geq \pi_H^{COM}(1,2)$ if and only if $s \geq \bar{s}^1 = \left[\left(3-2\sqrt{2}\right)/\left(2\sqrt{2}+3\right)\right]\left(\frac{a-c}{c}\right)$. Thus, if $s \geq \bar{s}^1$ and $\left(n_{u-1}^L, n_{u-1}^H\right) = (1,2)$, then $\left(n_\tau^L, n_\tau^H\right) = (1,2)$ for all $\tau \geq u$ until the first time that $M_\tau = M^3$ and $\left(n_\tau^L, n_\tau^H\right) = (2,0)$, thereafter, is a SPNE from period u onward.

Case 3: $\left(n_u^L, n_u^H\right) = (1,1)$ when $M_u = M^1, M^2$ and $\left(n_u^L, n_u^H\right) = (1,2)$ when $M_u = M^3$. Then, the expected discounted profits of firm $\{1,2\}$ are $v^{\{1,2\}}\left(M^1\right)(1) = v^{\{1,2\}}\left(M^2\right)(1) = \pi_L^{COM}(1,1)/2(1-\delta)$. Now, suppose that when $M_u = M^1, M^2$, firm $\{1,2\}$ deviates and, hence, there is no merge. Then, the expected discounted profits of firm $\{1,2\}$ would be $v^{\{1,2\}}\left(M^1\right)(0) = \pi_L^{COM}(1,2)/(1-\delta)$. Firm $\{1,2\}$ does not have an incentive to deviate if and only if $v^{\{1,2\}}\left(M^1\right)(1) \geq v^{\{1,2\}}\left(M^1\right)(0)$ or, which is equivalent, $\pi_L^{COM}(1,1)/2 \geq \pi_L^{COM}(1,2)$. Since this inequality never holds, if $\left(n_{u-1}^L, n_{u-1}^H\right) = (1,2)$, $\left(n_\tau^L, n_\tau^H\right) = (1,2)$ for all $\tau \geq u$ until the first time that $M_\tau = M^1, M^2$ and $\left(n_\tau^L, n_\tau^H\right) = (1,1)$, thereafter, is never a SPNE from period u onward.

Case 4: $\left(n_u^L, n_u^H\right) = (1,1)$ when $M_u = M^1, M^2$ and $\left(n_u^L, n_u^H\right) = (2,0)$ when $M_u = M^3$. Then, the expected discounted profits of firm $\{1,2\}$ are $v^{\{1,2\}}\left(M^1\right)(1) = \pi_L^{COM}(1,1)/2(1-\delta)$. Now, suppose that when $M_u = M_1, M_2$, firm $\{1,2\}$ deviates. Then, the expected discounted profits of firm $\{1,2\}$ would be $v^{\{1,2\}}\left(M^1\right)(0) = \pi_L^{COM}(1,2)/(1-\delta)$. Firm $\{1,2\}$ does not have an incentive to deviate if and only if $v^{\{1,2\}}\left(M^1\right)(1) \geq v^{\{1,2\}}\left(M^1\right)(0)$ or, which is equivalent, $\pi_L^{COM}(1,1)/2 \geq \pi_L^{COM}(1,2)$. Since this inequality never holds, if $\left(n_{u-1}^L, n_{u-1}^H\right) = (1,2)$, $\left(n_\tau^L, n_\tau^H\right) = (1,1)$ for all $\tau \geq u$ if $M_u = M^1, M^2$ and $\left(n_\tau^L, n_\tau^H\right) = (2,0)$ for all $\tau \geq u$ when $M_u = M^3$ is never a SPNE from u onward.

(iii) Suppose that **four firms remain in the industry**, i.e., $\left(n_{u-1}^L, n_{u-1}^H\right) = (0,4)$. Then, the set of possible matches in period u is $\tilde{M}_u = \left\{\tilde{M}^1, \tilde{M}^2, \tilde{M}^3\right\}$, where $\tilde{M}^1 = \{\{1,2\},\{3,4\}\}$, $\tilde{M}^2 = \{\{1,3\},\{2,4\}\}$, and $\tilde{M}^3 = \{\{1,4\},\{2,3\}\}$. There are four possible equilibria that must be considered.

Case 1: $\left(n_u^L, n_u^H\right) = (0,4)$ for all $M_u \in \tilde{M}_u$. This is always a SPNE from period u onward, because if all firms but x refuse to merge, then, there is nothing that firm x can do to induce a merger.

Case 2: $\left(n_u^L, n_u^H\right) = (2,0)$ for all $M_u \in \tilde{M}_u$. Then, the expected discounted profits of firm z are $v^z(M_u)(1) = \pi_L^{COM}(2,0)/2(1-\delta)$ for $z \in \{1,2,3,4\}$. Now, suppose that one of the firms (say, firm $\{1\}$) deviates and, hence, the merger between firm $\{1\}$ and its match fails. Then, the expected discounted profits of firm $\{1\}$ would be $v^{\{1\}}(M_u)(0) = \pi_H^{COM}(1,2)/(1-\delta)$. Firm $\{1\}$ does not have an incentive to deviate if and only if $v^{\{1\}}(M_u)(1) \geq v^{\{1\}}(M_u)(0)$ or, which is equivalent, $\pi_L^{COM}(2,0)/2 \geq \pi_H^{COM}(1,2)$. $\pi_L^{COM}(2,0)/2 \geq \pi_H^{COM}(1,2)$ if and only if $s \geq \bar{s}^1 = \left[\left(3-2\sqrt{2}\right)/\left(2\sqrt{2}+3\right)\right]\left(\frac{a-c}{c}\right)$. Thus, if $s \geq \bar{s}^1$ and $\left(n_{u-1}^L, n_{u-1}^H\right) = (0,4)$, $\left(n_\tau^L, n_\tau^H\right) = (2,0)$ for all $\tau \geq u$ is a SPNE from period u onward.

Case 3: $\left(n_u^L, n_u^H\right) = (1,2)$ for all $M_u \in \tilde{M}_u$ and $\left(n_\tau^L, n_\tau^H\right) = (1,2)$ for all $M_\tau \in M_\tau$ and $\tau > u$. Note that we are implicitly assuming the equilibrium studied in case 1 when $\left(n_{\tau-1}^L, n_{\tau-1}^H\right) = (1,2)$. In other words, $\left(n_\tau^L, n_\tau^H\right) = (1,2)$ for all $\tau \geq u$. Then, the expected discounted profits of firm z are $v^z(M_u)(1) = \pi_L^{COM}(1,2)/2(1-\delta)$. Now, suppose that one of the firms (say, firm $\{1\}$) deviates and, hence, the merger of firm $\{1\}$ with its match fails. Then, the expected discounted profits of firm $\{1\}$ would be $v^{\{1\}}(M_u)(0) = \pi_H^{COM}(4,0)/(1-\delta)$. Firm

$\{1\}$ does not have an incentive to deviate if and only if $v^z(M_u)(1) \geq v^{\{1\}}(M_u)(0)$ or, which is equivalent, $\pi_L^{COM}(1,2)/2 \geq \pi_H^{COM}(4,0)$. $\pi_L^{COM}(1,2)/2 \geq \pi_H^{COM}(4,0)$ if and only if $s \geq \bar{s}^2 = \left[\left(4\sqrt{2}-5\right)/15\right]\left(\frac{a-c}{c}\right)$. Thus, if $s \geq \bar{s}^2$ and $\left(n_{u-1}^L, n_{u-1}^H\right) = (0,4)$, $\left(n_\tau^L, n_\tau^H\right) = (1,2)$ for all $\tau \geq u$ is a SPNE from period t onward.

Case 4: $\left(n_u^L, n_u^H\right) = (1,2)$ for all $M_u \in \tilde{M}_u$ and $\left(n_\tau^L, n_\tau^H\right) = (1,2)$ when $M_\tau = M^1, M^2$ and $\left(n_\tau^L, n_\tau^H\right) = (2,0)$ when $M_\tau = M^3$ for $\tau > u$. In other words, $\left(n_\tau^L, n_\tau^H\right) = (1,2)$ for all $\tau \geq u$ until the first time the two firms with cost c are matched and $\left(n_\tau^L, n_\tau^H\right) = (2,0)$, thereafter. Note that we are implicitly assuming the equilibrium studied in case 2 when $\left(n_{\tau-1}^L, n_{\tau-1}^H\right) = (1,2)$. Moreover, for this to be an equilibrium path, it must be the case that $s \geq \bar{s}^1$. Without loss of generality assume that $M_u = \tilde{M}^1$ and firms $\{1\}$ and $\{2\}$ choose to merger, while firms $\{3\}$ and $\{4\}$ do not merge. Then, the expected discounted profits of firm $\{1\}$ are

$$\begin{aligned} v^{\{1\}}\left(\tilde{M}^1\right)(1) &= \frac{\pi_L^{COM}(1,2)}{2} + \delta\left[\frac{1}{3}v^{\{1\}}\left(M^1\right)(0) + \frac{1}{3}v^{\{1\}}\left(M^2\right)(0) + \frac{1}{3}\frac{\pi_L^{COM}(2,0)}{2(1-\delta)}\right] \\ v^{\{1\}}\left(M^1\right)(0) &= v^{\{1\}}\left(M^2\right)(0) = v^{\{1\}}\left(\tilde{M}^1\right)(1) \end{aligned}$$

That is, when $M_u = \tilde{M}^1$, if firms $\{1\}$ and $\{2\}$ merge, in period u, each firm obtains $\pi_L^{COM}(1,2)/2$. In the following period, with probability $1/3$, the match is $M^1 = \{\{1,2,3\},\{4\}\}$ and, hence, there are no mergers; with probability $1/3$, the match is $M^2 = \{\{1,2,4\},\{3\}\}$ and, hence, there are no merger; while with probability $1/3$, the match is $M^3 = \{\{1,2\},\{3,4\}\}$ and, hence, firms $\{1\}$ and $\{2\}$ merge, in which case firms $\{1\}$ gets $\pi_L^{COM}(2,0)/2$ in each subsequent period. Now, suppose that firm $\{1\}$ deviates and, hence, the merger of firms $\{1\}$ and $\{2\}$ fails. Then, the expected discounted profits of firm $\{1\}$ would be $v^{\{1\}}\left(\tilde{M}^1\right)(0) = \pi_H^{COM}(0,4)/(1-\delta)$. Firm $\{1\}$ does not have an incentive to deviate if and only if $v^{\{1\}}\left(\tilde{M}^1\right)(1) \geq v^{\{1\}}\left(\tilde{M}^1\right)(0)$ or, which is equivalent, $3\pi_L^{COM}(1,2)(1-\delta) + \delta\pi_L^{COM}(2,0) \geq 2(3-2\delta)\pi_H^{COM}(0,4)$. This inequality holds if and only if $F(s) = 3\left(\frac{a-c+3cs}{4}\right)^2(1-\delta) + \delta\left(\frac{a-c+cs}{3}\right)^2 - 2\left(\frac{a-c}{5}\right)^2(3-2\delta) \geq 0$. Note that $F\left(\bar{s}^1\right) \geq 0$ if and only $\delta \geq \bar{\delta} = \left(522\sqrt{2}-723\right)/\left(498\sqrt{2}-657\right) \approx 0.32191$, and $F'(s) > 0$ for all $s > 0$. Hence, if $\delta \geq \bar{\delta}$, $F(s) \geq 0$ for all $s \geq \bar{s}^1$. Therefore, if $\delta \geq \bar{\delta}$, $s \geq \bar{s}^1$, and $\left(n_{u-1}^L, n_{u-1}^H\right) = (0,4)$, then $\left(n_\tau^L, n_\tau^H\right) = (1,2)$ for all $\tau \geq u$ until the first time the two firms with cost c are matched and $\left(n_\tau^L, n_\tau^H\right) = (2,0)$, thereafter, is a SPNE.

Summary of possible equilibria.

- $COM(0,4)$: $\left(n_\tau^L, n_\tau^H, (q_\tau^z)_{z\in N_\tau}\right) = \left(0, 4, \left(q_H^{COM}(0,4)\right)_{z\in N_\tau^H}\right)$ for all $\tau \geq t$ is always a SPNE outcome path.

- $COM(1,2)$: $\left(n_\tau^L, n_\tau^H, (q_\tau^z)_{z\in N_\tau}\right) = \left(1, 2, \left(q_L^{COM}(1,2)\right)_{z\in N_\tau^L}, \left(q_H^{COM}(1,2)\right)_{z\in N_\tau^H}\right)$ for $\tau \geq t$ is a SPNE outcome path whenever $s \geq \bar{s}^2$.

- $COM(2,0)$: $\left(n_\tau^L, n_\tau^H, (q_\tau^z)_{z\in N_\tau}\right) = \left(2, 0, \left(q_L^{COM}(2,0)\right)_{z\in N_\tau^L}\right)$ for all $\tau \geq t$ is a SPNE outcome path whenever $s \geq \bar{s}^1$.

- $COM(1,2-2,0)$: $\left(n_\tau^L, n_\tau^H\right) = \left(1, 2, \left(q_L^{COM}(1,2)\right)_{z\in N_\tau^L}, \left(q_H^{COM}(1,2)\right)_{z\in N_\tau^H}\right)$ for all $\tau \geq t$ until the first time the two firms with cost c are matched and $\left(n_\tau^L, n_\tau^H\right) = \left(2, 0, \left(q_L^{COM}(2,0)\right)_{z\in N_\tau^L}\right)$ thereafter is a SPNE outcome path whenever $s \geq \bar{s}^1$ (provided that $\delta \geq \bar{\delta}$).

Equilibrium selection. Note that: $0 < \bar{s}^1 < \bar{s}^2 < d/5$; $\pi_L^{COM}(2,0)/2 > \pi_H^{COM}(0,4)$; $\pi_L^{COM}(2,0)/2 > \pi_H^{COM}(1,2)$ if and only if $s > \bar{s}^1$; $\pi_L^{COM}(1,2)/2 > \pi_H^{COM}(0,4)$ if and only if $s > \bar{s}^2$; $\pi_L^{COM}(2,0) > \pi_L^{COM}(1,2)$ if and only if $s < d/5$; and $\pi_L^{COM}(2,0)/2 > \left[\pi_L^{COM}(1,2)/2 + \pi_H^{COM}(1,2)\right]/2$.

Suppose that $0 \leq s < \bar{s}^1$. Then, the unique SPNE is $COM(0,4)$.

Suppose that $\bar{s}^1 \leq s < \bar{s}^2$. Then, there are three possible equilibria: $COM(0,4)$, $COM(2,0)$, and $COM(1,2-2,0)$. Since $\pi_L^{COM}(2,0)/2 > \pi_H^{COM}(0,4)$, all firms obtains higher profits under $COM(2,0)$ than under $COM(0,4)$. Under $COM(1,2-2,0)$, the expected discounted profits of each of the firms that merge first are (say firms $\{1\}$ and $\{2\}$):

$$v^{\{1\}}\left(\tilde{M}^1\right)(1) = v^{\{2\}}\left(\tilde{M}^1\right)(1) = \frac{\frac{3\pi_L^{COM}(1,2)}{2} + \delta\frac{\pi_L^{COM}(2,0)}{2(1-\delta)}}{3-2\delta}$$

while the expected discounted profits of each of the firms that merge second are (say firms $\{3\}$ and $\{4\}$):

$$v^{\{3\}}\left(\tilde{M}^1\right)(1) = v^{\{4\}}\left(\tilde{M}^1\right)(1) = \frac{3\pi_H^{COM}(1,2) + \delta\frac{\pi_L^{COM}(2,0)}{2(1-\delta)}}{3-2\delta}$$

$\pi_L^{COM}(2,0)/2(1-\delta) > v^{\{1\}}\left(\tilde{M}^1\right)(1)$ if and only if $\pi_L^{COM}(2,0) > \pi_L^{COM}(1,2)$, which holds for all $0 < s < d/5$. $\pi_L^{COM}(2,0)/2(1-\delta) \geq v^{\{3\}}\left(\tilde{M}^1\right)(1)$ if and only if $\pi_L^{COM}(2,0)/2 \geq \pi_H^{COM}(1,2)$, which holds for all $s \geq \bar{s}^1$. Thus, $COM(2,0)$ is the SPNE that induces the highest profits for each firm.

Suppose that $\bar{s}^2 \leq s < d/5$. Then, there are four possible equilibria: $COM(0,4)$, $COM(2,0)$, $COM(1,2)$ and $COM(1,2-2,0)$. Since $\pi_L^{COM}(2,0)/2 > \pi_H^{COM}(0,4)$, all firms obtains higher profits under $COM(2,0)$ than under $COM(0,4)$. Since $\pi_L^{COM}(2,0) > \pi_L^{COM}(1,2)$ for all $0 < s < d/5$, $COM(2,0)$ induces higher profits than $COM(1,2)$ ($COM(1,2-2,0)$) for the firms that merge (first). Since $\pi_L^{COM}(2,0)/2 \geq \pi_H^{COM}(1,2)$ for all $s \geq \bar{s}^1$, $COM(2,0)$ induces higher profits than $COM(1,2)$ ($COM(1,2-2,0)$) for the firms that do not merge (merge second) under $COM(1,2)$ ($COM(1,2-2,0)$). Thus, $COM(2,0)$ is the SPNE that induces the highest profits for each firm.

Suppose that $d/5 \leq s < d$. Then, there are four possible equilibria: $COM(0,4)$, $COM(2,0)$, $COM(1,2)$ and $COM(1,2-2,0)$. Since $\pi_L^{COM}(2,0)/2 > \pi_H^{COM}(0,4)$, all firms obtains higher profits under $COM(2,0)$ than under $COM(0,4)$. Since $\pi_L^{COM}(1,2) \geq \pi_L^{COM}(2,0)$ for $s \geq d/5$, the firms that merge under $COM(1,2)$ obtains higher profits under $COM(1,2)$ than under $COM(2,0)$ or $COM(1,2-2,0)$. Thus, for the firms that merge under $COM(1,2)$, $COM(1,2)$ is the SPNE the induces the highest profits. Since $\pi_L^{COM}(2,0)/2 \geq \pi_H^{COM}(1,2)$ for all $s \geq \bar{s}^1$, $COM(2,0)$ induces higher profits than $COM(1,2)$ for the firms that do not merge (merge second) under $COM(1,2)$ ($COM(1,2-2,0)$). Thus, for the firms that do not merge (merge second) under $COM(1,2)$ ($E(1,2-2,0)$), $COM(2,0)$ is the SPNE the induces the highest profits. Finally, note that, before firms are matched, the expected profits of a firm under $COM(1,2)$ are given by $v^{COM}(1,2) = (1/2)\left(\pi_L^{COM}(1,2)/2 + \pi_L^{COM}(1,2)\right)$ (with probability $1/2$ in equilibrium $COM(1,2)$, a firm is part of a merge and with probability $1/2$, it is not). Under, $COM(2,0)$ each firm always obtains $\pi_L^{COM}(2,0)/2$. Since $\pi_L^{COM}(2,0)/2 > \left[\pi_L^{COM}(1,2)/2 + \pi_H^{COM}(1,2)\right]/2$, under $COM(2,0)$ a firm obtains higher expected profits than under $COM(1,2)$.

This completes the proof of Lemma 5. ■

Lemma 5 ***Sustainability of collusion.*** Suppose that $s \leq d$ and before period t firms did not collude. Assume that after merger decisions have been made in period t, there are n_t^H firms with unit cost c and n_t^L firms with unit cost $c(1-s)$. Then:

1. If $\left(n_t^L, n_t^H\right) = (0,4)$, then collusion can be sustained whenever $\alpha \leq \bar{\alpha}(4) = 1 - \frac{25}{\delta 41}$.

2. If $\left(n_t^L, n_t^H\right) = (2,0)$, then collusion can be sustained whenever $\alpha \leq \bar{\alpha}(2) = 1 - \frac{9}{\delta 17}$.

3. If $\left(n_t^L, n_t^H\right) = (1,2)$, then collusion can be sustained whenever $\alpha \leq \bar{\alpha}(1,2) = 1 - \frac{4(d-s)^2(d+cs)^2}{\delta\left[16(d+s)^4-(3d+s)^2(d+3s)^2\right]}$. Moreover, $\bar{\alpha}(1,2) = \bar{\alpha}(3)$ when $s=0$, $\bar{\alpha}(1,2) = \bar{\alpha}(2)$ when $s = d/5$, and $\bar{\alpha}(1,2)$ is strictly increasing in s.

4. If $\left(n_t^L, n_t^H\right) = (1,1)$, then collusion can be sustained whenever $\alpha \leq \bar{\alpha}(1,1) = 1 - \frac{(d-s)^2(d+s)^2}{\delta\left[9(d+s)^4-\frac{16}{9}(2d+s)^2(d+2s)^2\right]}$. Moreover, $\bar{\alpha}(1,1) = \bar{\alpha}(2)$ when $s=0$ and $\bar{\alpha}(1,1)$ is strictly increasing in s.

Proof: First we prove a relatively general result on the sustainability of collusion. Suppose that before period t firms did not collude. Assume that after merger decisions have been made in period t, there are n_t^H firms with unit cost c and n_t^L firms with unit cost $c(1-s)$, with $sn_t^L \leq d$.

The expected discounted profits of firm $z \in N_t^L$ under collusion is given by:

$$v_L^{COL}\left(n_t^L, n_t^H\right) = \frac{\pi_L^{COL}\left(n_t^L, n_t^H\right) + \delta\alpha\frac{\pi_L^{COM}\left(n_t^L, n_t^H\right)}{1-\delta}}{1-\delta(1-\alpha)},$$

where $\pi_L^{COL}\left(n_t^L, n_t^H\right) = \left[\frac{a-c+\left(n_t^H+1\right)cs}{(a-c)\left(n_t^L+n_t^H\right)+n_t^L cs}\right]\left[\frac{a-c+cs}{2}\right]^2$ and $\pi_L^{COM}\left(n_t^L, n_t^H\right) = \left[\frac{a-c+\left(n_t^H+1\right)cs}{n_t^L+n_t^H+1}\right]^2$. The expected discounted profits if firm $z \in N_t^L$ deviates from collusion is:

$$v_L^{DEV}\left(n_t^L, n_t^H\right) = \pi^{DEV-L}\left(n_t^L, n_t^H\right) + \frac{\delta}{1-\delta}\pi_L^{COM}\left(n_t^L, n_t^H\right).$$

where $\pi^{DEV-L}\left(n_t^L, n_t^H\right) = \left[\frac{\left(n_t^L+n_t^H+1\right)(a-c+cs)}{(a-c)\left(n_t^L+n_t^H\right)+n_t^L cs}\frac{1}{2}\right]^2\left[\frac{a-c+cs}{2}\right]^2$. Firm z has no incentive to deviate if and only if

$$\begin{aligned}\alpha &\leq \bar{\alpha}^L\left(n_t^L, n_t^H\right) = 1 - \frac{\left[\pi^{DEV-L}\left(n_t^L, n_t^H\right) - \pi_L^{COL}\left(n_t^L, n_t^H\right)\right]}{\delta\left[\pi^{DEV-L}\left(n_t^L, n_t^H\right) - \pi_L^{COM}\left(n_t^L, n_t^H\right)\right]} \\ &= 1 - \frac{(\delta)^{-1}\left[\frac{(n_t-1)d-\left(n_t^H-n_t^L+1\right)s\left(n_t^L\right)}{dn_t+n_t^L s}\right]^2}{\left[\frac{(n_t+1)\left(d+s\left(n_t^L\right)\right)}{dn_t+n_t^L s\left(n_t^L\right)}\right]^2 - \left[\frac{4}{d+s\left(n_t^L\right)}\right]^2\left[\frac{d+\left(n_t^H+1\right)s\left(n_t^L\right)}{n_t+1}\right]^2},\end{aligned}$$

where $n_t = n_t^L + n_t^L$, $s(0) = 0$ and $s\left(n_t^L\right) = s$ if $n_t^L > 0$.

The expected discounted profits of firm $z \in N_t^H$ under collusion is given by:

$$v_H^{COL}\left(n_t^L, n_t^H\right) = \frac{\pi_H^{COL}\left(n_t^L, n_t^H\right) + \delta\alpha\frac{\pi_H^{COM}\left(n_t^L, n_t^H\right)}{1-\delta}}{1-\delta(1-\alpha)},$$

where $\pi_H^{COL}\left(n_t^L, n_t^H\right) = \left[\frac{a-c-n_t^L cs}{(a-c)\left(n_t^L+n_t^H\right)+n_t^L cs}\right]\left[\frac{a-\left(1-s\left(n_t^L\right)\right)c}{2} - s\left(n_t^L\right)c\right]\frac{a-\left(1-s\left(n_t^L\right)\right)c}{2}$ and $\pi_H^{COM}\left(n_t^L, n_t^H\right) = \left(\frac{a-c-n_t^L cs}{n_t^L+n_t^H+1}\right)^2$. The expected discounted profits if firm $z \in N_t^H$ deviates from collusion is:

$$v_H^{DEV}\left(n_t^L, n_t^H\right) = \pi^{DEV-H}\left(n_t^L, n_t^H\right) + \frac{\delta}{1-\delta}\pi_H^{COM}\left(n_t^L, n_t^H\right).$$

where $\pi^{DEV-H}\left(n_t^L, n_t^H\right) = \left\{\left[\frac{(a-c)\left(n_t^L+n_t^H+1\right)}{(a-c)\left(n_t^L+n_t^H\right)+n_t^L cs}\right]\frac{a-\left(1-s\left(n_t^L\right)\right)c}{2} - s\left(n_t^L\right)c\right\}^2/4$. Therefore, firm z has no incentive to deviate if and only if

$$\begin{aligned}
\alpha &\leq \bar{\alpha}^H\left(n_t^L, n_t^H\right) = 1 - \frac{\left[\pi^{DEV-H}\left(n_t^L, n_t^H\right) - \pi_H^{COL}\left(n_t^L, n_t^H\right)\right]}{\delta\left[\pi^{DEV-H}\left(n_t^L, n_t^H\right) - \pi_H^{COM}\left(n_t^L, n_t^H\right)\right]} \\
&= 1 - \frac{(\delta)^{-1}\left[\frac{d-s\left(n_t^L\right)}{d+s\left(n_t^L\right)} - \frac{d-n_t^L s\left(n_t^L\right)}{dn_t+n_t^L s\left(n_t^L\right)}\right]^2}{\left[\frac{d-s\left(n_t^L\right)}{d+s\left(n_t^L\right)} + \frac{d-n_t^L s\left(n_t^L\right)}{dn_t+n_t^L s\left(n_t^L\right)}\right]^2 - \left(\frac{4}{d+s\left(n_t^L\right)}\right)^2\left[\frac{d-n_t^L s\left(n_t^L\right)}{n_t+1}\right]^2},
\end{aligned}$$

where $n_t = n_t^L + n_t^L$, $s(0) = 0$ and $s\left(n_t^L\right) = s$ if $n_t^L > 0$.

Thus, if after merger decision have been made in period t, collusion can be sustained if and only if $\alpha \leq \bar{\alpha}\left(n_t^L, n_t^H\right) = \min\left\{\bar{\alpha}^L\left(n_t^L, n_t^H\right), \bar{\alpha}^H\left(n_t^L, n_t^H\right)\right\}$. Note that this is a generalization of Lemma 3. Indeed, if $\left(n_t^L, n_t^H\right) = (0, n_t)$ or $\left(n_t^L, n_t^H\right) = (n_t, 0)$, then $\bar{\alpha}^L(0, n_t) = \bar{\alpha}^H(0, n_t) = \bar{\alpha}^H(n_t, 0) = \bar{\alpha}^H(n_t, 0) = \bar{\alpha}(n_t)$.

Suppose that $n_t = 4$.

Proofs of Parts 1 and 2. Part 1 (2) is a direct application of Lemma 3 for $n_t^L + n_t^H = n_t = 4$ ($n_t^L + n_t^H = n_t = 2$).

Proof of Part 3. Suppose that $\left(n_t^L, n_t^H\right) = (1, 2)$. Then, collusion can be sustained if and only if $\alpha \leq \min\left\{\bar{\alpha}^L(1,2), \bar{\alpha}^H(1,2)\right\}$, where:

$$\begin{aligned}
\bar{\alpha}^L(1,2) &= 1 - \frac{4(d-s)^2(d+s)^2}{\delta\left[16(d+s)^4 - (3d+s)^2(d+3s)^2\right]} \\
\bar{\alpha}^H(1,2) &= 1 - \frac{4(d)^2}{\delta(7d+3s)(d+s)}
\end{aligned}$$

If $s = 0$, then $\bar{\alpha}^L(1,2) = \bar{\alpha}^H(1,2) = \bar{\alpha}(3) = 1 - \frac{4}{\delta 7}$. Furthermore, $\bar{\alpha}^L(1,2) < \bar{\alpha}^H(1,2)$ if and only if

$$\begin{gathered}
\frac{4(d-s)^2(d+s)^2}{16(d+s)^4 - (3d+s)^2(d+3s)^2} > \frac{4(d)^2}{(7d+3s)(d+s)} \Leftrightarrow \\
s > 0 \text{ and } (d-s)^2\left(3s^3 + 16s^2 d + 23sd^2 + 6d^3\right) > 0
\end{gathered}$$

Therefore, $\bar{\alpha}^L(1,2) < \bar{\alpha}^H(1,2)$ for all $s > 0$.

Finally, taking the derivative of $\bar{\alpha}^L(1,2)$ with respect to s with obtain:

$$\begin{aligned}
\frac{\partial \bar{\alpha}^L(1,2)}{\partial s} &= \frac{16d\left(d^2-s^2\right)}{\delta} \frac{d^4+6s^2d^2+s^4-4ds}{\left[16(d+s)^4-(3d+s)^2(d+3s)^2\right]^2} \\
&\geq \frac{16d\left(d^2-s^2\right)}{\delta} \frac{d^4+6s^2d^2-4ds}{\left[16(d+s)^4-(3d+s)^2(d+3s)^2\right]^2} \\
&> \frac{16d\left(d^2-s^2\right)}{\delta} \frac{d^4-\frac{2}{3}}{\left[16(d+s)^4-(3d+s)^2(d+3s)^2\right]^2} > 0
\end{aligned}$$

The first inequality holds because $s \geq 0$. The second inequality holds because $\arg\min_s \left\{d^4+6s^2d^2-4ds\right\} = (3d)^{-1}$. The last inequality holds because $d > 1$. It only rest to prove that $\bar{\alpha}^L(1,2) = \bar{\alpha}(2)$ if and only if $s = d/5$. Since $\bar{\alpha}^L(1,2)$ is strictly increasing in s and $\bar{\alpha}(2)$ does not depend on s, the equation $\bar{\alpha}^L(1,2) = \bar{\alpha}(2)$ has at most one solution. Moreover, it is easy to verify that $s = d/5$ is a solution to $\bar{\alpha}^L(1,2) = \bar{\alpha}(2)$.

Proof of Part 4. Suppose that $\left(n_t^L, n_t^H\right) = (1,1)$. Then, collusion can be sustained if and only if $\alpha \leq \min\left\{\bar{\alpha}^L(1,1), \bar{\alpha}^H(1,1)\right\}$, where:

$$\begin{aligned}
\bar{\alpha}^L(1,1) &= 1 - \frac{(d-s)^2(d+s)^2}{\delta\left[9(d+s)^4-\frac{16}{9}(2d+s)^2(d+2s)^2\right]} \\
\bar{\alpha}^H(1,1) &= 1 - \frac{(d)^2}{\delta\left[(3d+2s)^2-\frac{16}{9}(2d+s)^2\right]}
\end{aligned}$$

If $s = 0$, then $\bar{\alpha}^L(1,1) = \bar{\alpha}^H(1,1) = \bar{\alpha}(2) = 1 - \frac{9}{\delta 17}$. Furthermore, $\bar{\alpha}^L(1,1) < \bar{\alpha}^H(1,1)$ if and only if

$$\begin{gathered}
\frac{(d-s)^2(d+s)^2}{9(d+s)^4-\frac{16}{9}(2d+s)^2(d+2s)^2} > \frac{(d)^2}{(3d+2s)^2-\frac{16}{9}(2d+s)^2} \Leftrightarrow \\
s > 0 \text{ and } (d-s)^2\left(5s^3+21s^2d+27sd^2+10d^3\right) > 0.
\end{gathered}$$

Therefore, $\bar{\alpha}^L(1,1) < \bar{\alpha}^H(1,1)$ for all $s > 0$.

$$\begin{aligned}
\frac{\partial \bar{\alpha}^L(1,1)}{\partial s} &= 1 - \frac{(d-s)^2(d+s)^2}{\delta\left[9(d+s)^4-\frac{16}{9}(2d+s)^2(d+2s)^2\right]} \\
&= \frac{1-\bar{\alpha}^L(1,1)}{(d+s)(d-s)}\left\{4s + \frac{\left[36(d+s)^3-\frac{32}{9}(2d+s)(d+2s)(5d+4s)\right](d+s)(d-s)}{9(d+s)^4-\frac{16}{9}(2d+s)^2(d+2s)^2}\right\} > 0
\end{aligned}$$

This completes the proof of Lemma 5. ■

Let $\bar{E}$ denote the SPNE outcome path that induces the highest discounted expected profits.

Proposition 2. Let

$$\tilde{\alpha}(1,2)(s) = \left(\frac{1-\delta}{\delta}\right)\left[\frac{\frac{(d+3s)(d+s)^2}{4(3d+s)} - \frac{2(d)^2}{25}}{\frac{2(d)^2}{25} - \frac{(d+3s)^2}{16}}\right]$$

$$\tilde{\alpha}(2,0)(s) = \left(\frac{1-\delta}{\delta}\right)\left[\frac{\frac{(d+s)^2}{8} - \frac{(d-s)^2}{8}}{\frac{(d-s)^2}{8} - \frac{(d+s)^2}{9}}\right]$$

1. Suppose that $0 \leq s < \bar{s}^1$.

   (a) If $\alpha \leq \min\{\bar{\alpha}(2), \tilde{\alpha}(2,0)(s)\}$, then $\bar{E} = COL(2,0)$.

   (b) If $\min\{\bar{\alpha}(2), \tilde{\alpha}(2,0)(s)\} < \alpha \leq \min\{\bar{\alpha}(1,2)(s), \tilde{\alpha}(1,2)(s)\}$, then $\bar{E} = COL(1,2)$.

   (c) If $\max\{\min\{\bar{\alpha}(2), \tilde{\alpha}(2,0)(s)\}, \min\{\bar{\alpha}(1,2)(s), \tilde{\alpha}(1,2)(s)\}\} < \alpha \leq \bar{\alpha}(4)$, then $\bar{E} = COL(0,4)$.

   (d) If $\alpha > \max\{\min\{\bar{\alpha}(2), \tilde{\alpha}(2,0)(s)\}, \min\{\bar{\alpha}(1,2)(s), \tilde{\alpha}(1,2)(s)\}, \bar{\alpha}(4)\}$, then $\bar{E} = COM(0,4)$.

2. Suppose that $\bar{s}^1 \leq s \leq d$.

   (a) If $\alpha \leq \bar{\alpha}(2)$, then $\bar{E} = COL(2,0)$.

   (b) If $\bar{\alpha}(2) < \alpha \leq \max\{\bar{\alpha}(2), \bar{\alpha}(1,2)(s)\}$, then $\bar{E} \in \{COL(1,2), COM(2,0)\}$.

   (c) If $\alpha > \max\{\bar{\alpha}(2), \bar{\alpha}(1,2)(s)\}$, then $\bar{E} = COM(2,0)$.

Proof. Lemma 4 characterizes the SPNE outcome paths under competition. Clearly, these outcome paths are still SPNE outcome paths when firms can choose to collude. Next, we characterize the SPNE outcome paths in which firms collude.

Equilibria with collusion. Assume that $\left(n_{t-1}^L, n_{t-1}^H\right) = (0,4)$. Then, from Lemma 5, depending on the merger decisions of the firms in period t, there are 4 possible types of outcome paths in which firms can sustain collusion. Moreover, within each type we consider the outcome path in which collusion occurs as soon as possible.

Type 1. Suppose that in period t there are no mergers and, hence, $\left(n_t^L, n_t^H\right) = (0,4)$. Then, from Lemma 5, firms can sustain collusion whenever $\alpha \leq \bar{\alpha}(4)$. Thus, if $\alpha \leq \bar{\alpha}(4)$, then $\left(n_\tau^L, n_\tau^H, (q_\tau^z)_{z\in N_\tau}\right) = \left(0, 4, \left(q_H^{COL}(0,4)\right)_{z\in N_\tau^H}\right)$ for all $\tau \geq t$ until collusion is detected and $\left(n_\tau^L, n_\tau^H, (q_\tau^z)_{z\in N_\tau}\right) = \left(0, 4, \left(q_H^{COM}(0,4)\right)_{z\in N_\tau^H}\right)$ thereafter is a SPNE outcome path.

Type 2. Suppose that in period t only one pair of firms decide to merge and, hence, $\left(n_t^L, n_t^H\right) = (1,2)$. Then, from Lemma 5, firms can sustain collusion whenever $\alpha \leq \bar{\alpha}(1,2)(s)$. Assume that firms x and $y \in N_{t-1}^H$ are matched in period t. If x and y decide to merge, then the expected discounted profits of the new firm $z = x \cup y$ conditional on the other pair of matched firms not merging, is given by:

$$v_L^{COL}(1,2) = \frac{\pi_L^{COL}(1,2) + \alpha\delta\frac{\pi_L^{COM}(1,2)}{1-\delta}}{1-\delta(1-\alpha)},$$

where $\pi_L^{COL}(1,2)$ and $\pi_L^{COM}(1,2)$ are the collusion and competition profits of the merged firm z when $\left(n_t^L, n_t^H\right) = (1,2)$, respectively. Therefore, $v_t^x = v_t^y = v_L^{COL}(1,2)/2$. On the other hand, if firms x and y do not merge, then the expected discounted profits of x and y conditional on the other pair of matched firms not merging, is given by:

$$v_t^x = v_t^y = v_H^{COM}(0,4) = \frac{\pi_H^{COM}(0,4)}{1-\delta},$$

where $\pi_H^{COM}(0,4)$ is the competition profits when $\left(n_t^L, n_t^H\right) = (0,4)$. Thus, there are incentives to merge if and only if $v_L^{COL}(1,2)/2 \geq v_L^{COM}(0,4)$ or, which is equivalent, if and only if $(1-\delta)\left[\pi_L^{COL}(1,2) - 2\pi_H^{COM}(0,4)\right] \geq \delta\alpha\left[2\pi_H^{COM}(0,4) - \pi_L^{COM}(1,2)\right]$. Therefore, we must consider two possible cases:

a. Suppose that $s < \bar{s}^2 = \left(4\sqrt{2}-5\right)d/15$. Then, $\pi_L^{COL}(1,2) > 2\pi_H^{COM}(0,4) > \pi_L^{COM}(1,2)$. Hence, firms have incentives to merge if and only if $\alpha \leq \tilde{\alpha}(1,2)(s)$, where:

$$\begin{aligned} \tilde{\alpha}(1,2)(s) &= \left(\frac{1-\delta}{\delta}\right)\left[\frac{\pi_L^{COL}(1,2) - 2\pi_H^{COM}(0,4)}{2\pi_H^{COM}(0,4) - \pi_L^{COM}(1,2)}\right] \\ &= \left(\frac{1-\delta}{\delta}\right)\left[\frac{\left(\frac{d+3s}{3d+s}\right)\left(\frac{d+s}{2}\right)^2 - 2\left(\frac{d}{5}\right)^2}{2\left(\frac{d}{5}\right)^2 - \left(\frac{d+3s}{4}\right)^2}\right] \end{aligned}$$

Moreover, note that: (i) $\tilde{\alpha}(1,2)(0) = \left(\frac{1-\delta}{\delta}\right)\left(\frac{25/12-2}{2-25/16}\right) < \bar{\alpha}(1,2)(0) = \bar{\alpha}(3)$; (ii) $\lim_{s\to(\bar{s}^2)^-}\tilde{\alpha}(1,2)(s) = \infty$; and (iii) $\tilde{\alpha}(1,2)(s)$ is increasing for all $s \in \left[0, \bar{s}^2\right)$. Hence, there exists a unique $s^* \in \left(0, \bar{s}^2\right)$ such that $\tilde{\alpha}(1,2)(s^*) = \bar{\alpha}(1,2)(s^*)$. Also note that $\tilde{\alpha}(1,2)\left(\bar{s}^1\right) > \bar{\alpha}(4)$.

b. Suppose that $s \geq \bar{s}^2$. Then, $\pi_L^{COL}(1,2) > \pi_L^{COM}(1,2) \geq 2\pi_H^{COM}(0,4)$. Hence, firms always have incentives to merge.

Thus, $\left(n_\tau^L, n_\tau^H, (q_\tau^z)_{z\in N_\tau}\right) = \left(1, 2, \left(q_L^{COL}(1,2)\right)_{z\in N_\tau^L}, \left(q_H^{COL}(1,2)\right)_{z\in N_\tau^H}\right)$ for all $\tau \geq t$ until collusion is detected and $\left(n_\tau^L, n_\tau^H, (q_\tau^z)_{z\in N_\tau}\right) = \left(1, 2, \left(q_L^{COM}(1,2)\right)_{z\in N_\tau^L}, \left(q_H^{COM}(1,2)\right)_{z\in N_\tau^H}\right)$ thereafter, is a SPNE outcome path if and only if [$s \in \left[0, \bar{s}^2\right)$ and $\alpha \leq \min\left\{\bar{\alpha}(1,2)(s), \tilde{\alpha}(1,2)(s)\right\}$] or [$s \geq \bar{s}^2$ and $\alpha \leq \bar{\alpha}(1,2)(s)$].

Type 3. Suppose that in period t only one pair of firms decide to merge and, hence, $\left(n_t^L, n_t^H\right) = (1,2)$. Moreover, suppose that firms do not begin colluding in period t, but rather wait until one of the high cost firms is matched with the low cost firm, when they merge. Say that this occurs in period τ. Then, $\left(n_\tau^L, n_\tau^H\right) = (1,1)$ and, from Lemma 5, in period τ firms can sustain collusion whenever $\alpha \leq \bar{\alpha}(1,1)$. For this outcome path to be a SPNE we must verify that firms do not have an incentive to deviate in period τ. Assume that firms $x \in N_{\tau-1}^H$ and $y \in N_{\tau-1}^L$ are matched in period τ. If x and y decide to merge, then the expected discounted profits of the new firm $z = x \cup y$ are given by:

$$v_L^{COL}(1,1) = \frac{\pi_L^{COL}(1,1) + \delta\alpha\frac{\pi_L^{COM}(1,1)}{1-\delta}}{1-\delta(1-\alpha)},$$

where $\pi_L^{COL}(1,1)$ and $\pi_L^{COM}(1,1)$ are the collusion and competition profits of the merged firm z when $\left(n_t^L, n_t^H\right) = (1,1)$, respectively. Therefore, $v_t^x = v_t^y = v_L^{COL}(1,1)/2$. On the other hand, if firms x and y do not merge, then the expected discounted profits of x and y are given by:

$$v_H^{COM}(1,2) = \frac{\pi_H^{COM}(1,2)}{1-\delta} \text{ and } v_L^{COM}(1,2) = \frac{\pi_L^{COM}(1,2)}{1-\delta},$$

respectively, where $\pi_H^{COM}(1,2)$ ($\pi_L^{COM}(1,2)$) is the competition profits for a high (low) cost firm when $\left(n_t^L, n_t^H\right) = (1,2)$. Thus, there are incentives to merge if and only if $v_L^{COL}(1,1)/2 \geq v_L^{COM}(1,2)$ and $v_L^{COL}(1,1)/2 \geq v_H^{COM}(1,2)$ or, which is equivalent, if and only if $(1-\delta)\left[\pi_L^{COL}(1,1) - 2\pi_L^{COM}(1,2)\right] \geq \left[2\pi_L^{COM}(1,2) - \pi_L^{COM}(1,1)\right]\delta\alpha$. However, this inequality never holds because $2\pi_L^{COM}(1,2) \geq \pi_L^{COL}(1,1)$ and $2\pi_L^{COM}(1,2) > \pi_L^{COM}(1,1)$.

Type 4. Suppose that in period t there are two mergers and, hence, $\left(n_t^L, n_t^H\right) = (2,0)$. Then, from Lemma 5, firms can sustain collusion whenever $\alpha \leq \bar{\alpha}(2)$. Assume that firms x and $y \in N_{t-1}^H$ are matched in period t. If x and y decide to merge, then the expected discounted profits of the new firm $z = x \cup y$ conditional on the other pair of matched firms merging, is given by:

$$v_L^{COL}(2,0) = \frac{\pi_L^{COL}(2,0) + \alpha\delta\frac{\pi_L^{COM}(2,0)}{1-\delta}}{1-\delta(1-\alpha)},$$

where $\pi_L^{COL}(2,0)$ and $\pi_L^{COM}(2,0)$ are the collusion and competition profits of the merged firm z when $\left(n_t^L, n_t^H\right) = (2,0)$, respectively. Therefore, $v_t^x = v_t^y = v_L^{COL}(2,0)/2$. On the other hand, if firms x and y do not merge, then the expected discounted profits of x and y conditional on the other pair of matched firms merging, is given by

$$v_t^x = v_t^y = v_H^{COM}(1,2) = \frac{\pi_H^{COM}(1,2)}{1-\delta},$$

where $\pi_H^{COM}(1,2)$ is the competition profits when $\left(n_t^L, n_t^H\right) = (1,2)$. Thus, there are incentives to merge if and only if $v_L^{COL}(2,0)/2 \geq v_H^{COM}(1,2)$ or, which is equivalent, if and only if $(1-\delta)\left[\pi_L^{COL}(2,0) - 2\pi_H^{COM}(1,2)\right] \geq \delta\alpha\left[2\pi_H^{COM}(1,2) - \pi_L^{COM}(2,0)\right]$. Therefore, we must consider two possible cases:

a. Suppose that $s < \bar{s}^1 = \left(3\sqrt{2}-4\right)d/\left(4+3\sqrt{2}\right)$. Then, $\pi_L^{COL}(2,0) > 2\pi_H^{COM}(1,2) > \pi_L^{COM}(2,0)$. Hence, firms have incentives to merge if and only if $\alpha \leq \tilde{\alpha}(2,0)(s)$, where:

$$\begin{aligned}\tilde{\alpha}(2,0)(s) &= \left(\frac{1-\delta}{\delta}\right)\left[\frac{\pi_L^{COL}(2,0) - 2\pi_H^{COM}(1,2)}{2\pi_H^{COM}(1,2) - \pi_L^{COM}(2,0)}\right] \\ &= \left(\frac{1-\delta}{\delta}\right)\left[\frac{\frac{1}{2}\left(\frac{d+s}{2}\right)^2 - 2\left(\frac{d-s}{4}\right)^2}{2\left(\frac{d-s}{4}\right)^2 - \left(\frac{d+s}{3}\right)^2}\right]\end{aligned}$$

Moreover, note that: (i) $\tilde{\alpha}(2,0)(0) = 0 < \bar{\alpha}(4)$; (ii) $\lim_{s\to(\bar{s}^1)^-\_}\tilde{\alpha}(2,0)(s) = \infty$; and (iii) $\tilde{\alpha}(2,0)(0)$ is increasing for all $s \in \left[0, \bar{s}^1\right)$. Hence, there exist unique $\tilde{s}^1$, $\tilde{s}^2$, $\tilde{s}^3 \in \left(0, \bar{s}^1\right)$ such that $\tilde{\alpha}(2,0)\left(\tilde{s}^1\right) = \bar{\alpha}(4)$, $\tilde{\alpha}(2,0)\left(\tilde{s}^2\right) = \bar{\alpha}(1,2)\left(\tilde{s}^2\right)$, $\tilde{\alpha}(2,0)\left(\hat{s}^3\right) = \bar{\alpha}(2)$, and $\tilde{s}^1 < \tilde{s}^2 < \tilde{s}^3$.

b. Suppose that $s \geq \bar{s}^1$. Then, $\pi_L^{COL}(2,0) > \pi_L^{COM}(2,0) \geq 2\pi_H^{COM}(1,2)$. Hence, firms always have incentives to merge.

Thus, $\left(n_\tau^L, n_\tau^H, (q_\tau^z)_{z\in N_\tau}\right) = \left(2, 0, \left(q_L^{COL}(2,0)\right)_{z\in N_\tau^L}\right)$ for all $\tau \geq t$ until collusion is detected and $\left(n_\tau^L, n_\tau^H, (q_\tau^z)_{z\in N_\tau}\right) = 1, 2, \left(q_L^{COM}(1,2)\right)_{z\in N_\tau^L}$, thereafter, is a SPNE outcome path if and only if $[s \in \left[0, \bar{s}^1\right)$ and $\alpha \leq \min\left\{\bar{\alpha}(2), \tilde{\alpha}(2,0)(s)\right\}]$ or $[s \geq \bar{s}^1$ and $\alpha \leq \bar{\alpha}(2)]$

Summary of possible equilibria in which collusion occurs.

- $COL(0,4)$: $\left(n_\tau^L, n_\tau^H, (q_\tau^z)_{z\in N_\tau}\right) = \left(0, 4, \left(q_H^{COL}(0,4)\right)_{z\in N_\tau^H}\right)$ for all $\tau \geq t$ until collusion is detected and $\left(n_\tau^L, n_\tau^H, (q_\tau^z)_{z\in N_\tau}\right) = \left(0, 4, \left(q_H^{COM}(0,4)\right)_{z\in N_\tau^H}\right)$ thereafter, is a SPNE outcome path if and only if $\alpha \leq \bar{\alpha}(4)$.

- $COL\left(1,2\right)$: $\left(n_{\tau}^{L}, n_{\tau}^{H}, \left(q_{\tau}^{z}\right)_{z\in N_{\tau}}\right) = \left(1, 2, \left(q_{L}^{COL}\left(1,2\right)\right)_{z\in N_{\tau}^{L}}, \left(q_{H}^{COL}\left(1,2\right)\right)_{z\in N_{\tau}^{H}}\right)$ for all $\tau \geq t$ until collusion is detected and $\left(n_{\tau}^{L}, n_{\tau}^{H}, \left(q_{\tau}^{z}\right)_{z\in N_{\tau}}\right) = \left(1, 2, \left(q_{L}^{COM}\left(1,2\right)\right)_{z\in N_{\tau}^{L}}, \left(q_{H}^{COM}\left(1,2\right)\right)_{z\in N_{\tau}^{H}}\right)$ thereafter, is a SPNE outcome path if and only if [$s \in \left[0, \bar{s}^{2}\right)$ and $\alpha \leq \min\left\{\bar{\alpha}\left(1,2\right)\left(s\right), \tilde{\alpha}\left(1,2\right)\left(s\right)\right\}$] or [$s \geq \bar{s}^{2}$ and $\alpha \leq \bar{\alpha}\left(1,2\right)\left(s\right)$].

- $COL\left(2,0\right)$: $\left(n_{\tau}^{L}, n_{\tau}^{H}, \left(q_{\tau}^{z}\right)_{z\in N_{\tau}}\right) = \left(2, 0, \left(q_{L}^{COL}\left(2,0\right)\right)_{z\in N_{\tau}^{L}}\right)$ for all $\tau \geq t$ until collusion is detected and $\left(n_{\tau}^{L}, n_{\tau}^{H}, \left(q_{\tau}^{z}\right)_{z\in N_{\tau}}\right) = 1, 2, \left(q_{L}^{COM}\left(1,2\right)\right)_{z\in N_{\tau}^{L}}$, thereafter, is a SPNE outcome path if and only if [$s \in \left[0, \bar{s}^{1}\right)$ and $\alpha \leq \min\left\{\bar{\alpha}\left(2\right), \tilde{\alpha}\left(2,0\right)\left(s\right)\right\}$] or [$s \geq \bar{s}^{1}$ and $\alpha \leq \bar{\alpha}\left(2\right)$]

Equilibrium selection.

- $COL\left(2,0\right)$ versus $COL\left(0,4\right)$. Note that $\pi_{L}^{COM}\left(2,0\right) > 2\pi_{H}^{COM}\left(0,4\right)$ and $\pi_{L}^{COL}(2,0) > 2\pi_{H}^{COL}(0,4)$, which implies that $v_{L}^{COL}(2,0) > 2v_{H}^{COL}(0,4)$, where $v_{L}^{COL}(2,0) = \frac{\pi_{L}^{COL}(2,0)+\alpha\delta\frac{\pi_{L}^{COM}(2,0)}{1-\delta}}{1-\delta(1-\alpha)}$ and $v_{H}^{COL}(0,4) = \frac{\pi_{H}^{COL}(0,4)+\alpha\delta\frac{\pi_{H}^{COM}(0,4)}{1-\delta}}{1-\delta(1-\alpha)}$. Thus, each firm obtains higher expected discounted profits under $COL\left(2,0\right)$ that under $COL\left(0,4\right)$.

- $COL\left(2,0\right)$ versus $COL\left(1,2\right)$. The expected discounted profits of a firm under $COL\left(2,0\right)$ are given by $v_{L}^{COL}(2,0)/2$, where $v_{L}^{COL}(2,0) = \frac{\pi_{L}^{COL}(2,0)+\alpha\delta\frac{\pi_{L}^{COM}(2,0)}{1-\delta}}{1-\delta(1-\alpha)}$. The expected discounted profits of a firm under $COL\left(1,2\right)$ are given by $v_{L}^{COL}(1,2)/4 + v_{H}^{COL}(1,2)/2$, where $v_{L}^{COL}(1,2) = \frac{\pi_{L}^{COL}(1,2)+\alpha\delta\frac{\pi_{L}^{COM}(1,2)}{1-\delta}}{1-\delta(1-\alpha)}$ and $v_{H}^{COL}\left(1,2\right) = \frac{\pi_{H}^{COL}(1,2)+\alpha\delta\frac{\pi_{H}^{COM}(1,2)}{1-\delta}}{1-\delta(1-\alpha)}$. That is, with probability $1/2$, a firm plays is one of the firms that merge in the equilibrium $COL\left(1,2\right)$ and, hence, it obtains $v_{L}^{COL}(1,2)/2$, while with probability $1/2$, the firms is one of the firms that do not merge in the equilibrium $COL\left(1,2\right)$ and, hence, it obtains $v_{H}^{COL}(1,2)$. Note that $\pi_{L}^{COL}\left(2,0\right)/2 \geq \pi_{L}^{COL}\left(1,2\right)/4 + \pi_{H}^{COL}\left(1,2\right)/2$ if and only if $s \geq 0$. Also note that $\pi_{L}^{COM}\left(2,0\right)/2 > \pi_{L}^{COM}\left(1,2\right)/4 + \pi_{H}^{COM}\left(1,2\right)/2$ if and only if $5 + 64\left(s/d\right) - 67\left(s/d\right)^{2} > 0$, which always holds for $0 \leq s \leq d$. Thus, it is always the case that $\pi_{L}^{COM}\left(2,0\right)/2 > \pi_{L}^{COM}\left(1,2\right)/4 + \pi_{H}^{COM}\left(1,2\right)/2$. Thus, each firm obtains higher expected discounted profits under $COL\left(2,0\right)$ that under $COL\left(1,2\right)$.

- $COL\left(1,2\right)$ versus $COL\left(0,4\right)$. The expected discounted profits of a firm under $COL\left(0,4\right)$ are given by $v_{H}^{COL}(0,4) = \frac{\pi_{H}^{COL}(0,4)+\alpha\delta\frac{\pi_{H}^{COM}(0,4)}{1-\delta}}{1-\delta(1-\alpha)}$. The expected discounted profits of a firm under $COL\left(1,2\right)$ are given by $v_{L}^{COL}(1,2)/4 + v_{H}^{COL}(1,2)/2$. Note that $\pi_{L}^{COL}\left(1,2\right)/4 + \pi_{H}^{COL}\left(1,2\right)/2 \geq \pi_{H}^{COL}\left(0,4\right)$ if and only if $2s + 5s^{2}\left(1+s\right) \geq 0$, which always holds for $s \geq 0$. Also note that $\pi_{L}^{COM}\left(1,2\right)/4 + \pi_{H}^{COM}\left(1,2\right)/2 > \pi_{H}^{COM}\left(0,4\right)$ if and only if $11\left(1+25s^{2}\right) + 50s > 0$, which always holds for $s \geq 0$. Thus, each firm obtains higher expected discounted profits under $COL\left(1,2\right)$ that under $COL\left(0,4\right)$.

Recall from Lemma 4 that, under competition, if $0 \leq s < \bar{s}^{1}$, then the unique SPNE is $COM\left(0,4\right)$, while if $s \geq \bar{s}^{1}$, then $COM\left(2,0\right)$ is the preferred SPNE outcome path for the firms, where:

- $COM\left(0,4\right)$: $\left(n_{\tau}^{L}, n_{\tau}^{H}, \left(q_{\tau}^{z}\right)_{z\in N_{\tau}}\right) = \left(0, 4, \left(q_{H}^{COM}\left(0,4\right)\right)_{z\in N_{\tau}^{H}}\right)$ for all $\tau \geq t$.

- $COM\,(2,0)$: $\left(n_\tau^L, n_\tau^H, (q_\tau^z)_{z\in N_\tau}\right) = \left(2, 0, \left(q_L^{COM}\,(2,0)\right)_{z\in N_\tau^L}\right)$ for all $\tau \geq t$.

To complete the comparison of the SPNE outcome paths we must consider several cases.

Case 1. Suppose that $0 \leq s < \bar{s}^1$.

Case 1.a. If $\alpha \leq \min\{\bar{\alpha}\,(2), \tilde{\alpha}\,(2,0)\,(s)\}$, then $COL\,(2,0)$ is a SPNE outcome path and, hence, for the firms it is the preferred equilibrium outcome path among those in which firms collude. From Lemma 4, under competition, $COM\,(0,4)$ is the unique SPNE outcome path. Firms obtain higher expected discounted profits under $COL\,(2,0)$ than under $COM\,(2,0)$. Firms obtains higher expected discounted profits under $COM\,(2,0)$ than under $COM\,(0,4)$. Thus, firms obtain higher expected discounted profits under $COL\,(2,0)$ than under $COM\,(0,4)$.

Case 1.b. $\min\{\bar{\alpha}\,(2), \tilde{\alpha}\,(2,0)\,(s)\} < \alpha \leq \min\{\bar{\alpha}\,(1,2)\,(s), \tilde{\alpha}\,(1,2)\,(s)\}$, then $COL\,(1,2)$ is a SPNE outcome path, but $COL\,(2,0)$ is not SPNE outcome path. Therefore, $COL\,(1,2)$ is the preferred equilibrium outcome path among those in which firms collude. From Lemma 4, under competition, $COM\,(0,4)$ is the unique SPNE outcome path. All firms obtain higher expected discounted profits under $COL\,(1,2)$ than under $COL\,(0,4)$. Firms obtain higher expected discounted profits under $COL\,(0,4)$ than under $COM\,(0,4)$. Thus, firms obtain higher expected discounted profits under $COL\,(1,2)$ than under $COM\,(0,4)$.

Case 1.c. If $\max\{\min\{\bar{\alpha}\,(2), \tilde{\alpha}\,(2,0)\,(s)\}, \min\{\bar{\alpha}\,(1,2)\,(s), \tilde{\alpha}\,(1,2)\,(s)\}\} < \alpha \leq \bar{\alpha}\,(4)$, then $COL\,(0,4)$ is the only SPNE outcome path in which firms collude. Firms obtain higher expected discounted profits under $COL\,(0,4)$ than under $COM\,(0,4)$.

Case 1.d. If $\max\{\min\{\bar{\alpha}\,(2), \tilde{\alpha}\,(2,0)\,(s)\}, \min\{\bar{\alpha}\,(1,2)\,(s), \tilde{\alpha}\,(1,2)\,(s)\}, \bar{\alpha}\,(4)\} < \alpha$, then there is no SPNE outcome path in which firms collude. Then, from Lemma 4, $COM\,(0,4)$ is the only equilibrium outcome path for the firms.

Case 2. Suppose that $d/5 \leq s < d/5$.

Case 2.a. If $\alpha \leq \bar{\alpha}\,(2)$, then $COL\,(2,0)$ is a SPNE outcome path and, hence, for the firms it is the preferred equilibrium outcome path among those in which firms collude. From Lemma 4, under competition, $COM\,(2,0)$ is the preferred equilibrium outcome path for the firms among those in which firms never collude. Clearly, firms obtain higher expected discounted profits under $COL\,(2,0)$ than under $COM\,(2,0)$.

Case 2.b. If $\alpha > \bar{\alpha}\,(2)$, then there is no SPNE outcome path in which firms collude. Then, from Lemma 4, $COM\,(2,0)$ is the preferred equilibrium outcome path for the firms.

Case 3. Suppose that $d/5 \leq s \leq d$.

Case 3.a. If $\alpha \leq \bar{\alpha}\,(2)$, then $COL\,(2,0)$ is a SPNE outcome path and, hence, for the firms it is the preferred equilibrium outcome path among those in which firms collude. From Lemma 4, under competition, $COM\,(2,0)$ is the preferred equilibrium outcome path for the firms among those in which firms never collude. Clearly, firms obtain higher expected discounted profits under $COL\,(2,0)$ than under $COM\,(2,0)$.

Case 3.b. If $\bar{\alpha}\,(2) < \alpha \leq \bar{\alpha}\,(1,2)\,(s)$, then $COL\,(1,2)$ is the only SPNE outcome path in which firms collude. From Lemma 4, under competition, $COM\,(2,0)$ is the preferred equilibrium outcome path for the firms among those in which firms never collude. The expected discounted profits of a firm under $COL\,(1,2)$ are given by $v_L^{COL}(1,2)/4 + v_H^{COL}(1,2)/2$, where $v_L^{COL}(1,2) = \frac{\pi_L^{COL}(1,2)+\alpha\delta\frac{\pi_L^{COM}(1,2)}{1-\delta}}{1-\delta(1-\alpha)}$ and $v_H^{COL}\,(1,2) = \frac{\pi_H^{COL}(1,2)+\alpha\delta\frac{\pi_H^{COM}(1,2)}{1-\delta}}{1-\delta(1-\alpha)}$. The expected discounted profits of a firm under $COM\,(2,0)$ is $v_L^{COM}(2,0)/2$, where

$v_L^{COM}(2,0) = \frac{\pi_L^{COM}(2,0)}{1-\delta}$. Thus, $v_L^{COM}(2,0)/2 > v_L^{COL}(1,2)/4 + v_H^{COL}(1,2)/2$ if and only if

$$\frac{\pi_L^{COM}(2,0)}{2(1-\delta)} > \frac{\pi_L^{COL}(1,2) + \alpha\delta\frac{\pi_L^{COM}(1,2)}{1-\delta}}{4[1-\delta(1-\alpha)]} + \frac{\pi_H^{COL}(1,2) + \alpha\delta\frac{\pi_H^{COM}(1,2)}{1-\delta}}{2[1-\delta(1-\alpha)]}$$

We have already proved that $2\pi_L^{COM}(2,0) - 2\pi_H^{COM}(1,2) - \pi_L^{COM}(1,2) > 0$ always holds for $0 \leq s \leq d$. Hence, $v_L^{COM}(2,0)/2 > v_L^{COL}(1,2)/4 + v_H^{COL}(1,2)/2$ if and only if $\alpha > \hat{\alpha}(s)$ where

$$\begin{aligned} \hat{\alpha}(s) &= \left(\frac{1-\delta}{\delta}\right)\left[\frac{\pi_L^{COL}(1,2) + 2\pi_H^{COL}(1,2) - 2\pi_L^{COM}(2,0)}{2\pi_L^{COM}(2,0) - 2\pi_H^{COM}(1,2) - \pi_L^{COM}(1,2)}\right] \\ &= \left(\frac{1-\delta}{\delta}\right)\left[\frac{\frac{(d+3s)(d-s)^2}{4(3d+s)} + \frac{(d-s)^2(d+s)}{2(3d+s)} - 2\left(\frac{d+s}{3}\right)^2}{2\frac{(d+s)^2}{9} - \frac{(d-s)^2}{8} - \frac{(d+3cs)^2}{16}}\right] \end{aligned}$$

Also note that if $d/5 \leq s \leq \left(\frac{16+\sqrt{145}}{37}\right) d \approx 0.75788d$, then $\pi_L^{COL}(1,2) + 2\pi_H^{COL}(1,2) - 2\pi_L^{COM}(2,0) \leq 0$, while if $s > \left(\frac{16+\sqrt{145}}{37}\right) d$, then $\pi_L^{COL}(1,2) + 2\pi_H^{COL}(1,2) - 2\pi_L^{COM}(2,0) > 0$. Therefore, if $d/5 \leq s \leq \left(\frac{16+\sqrt{145}}{37}\right) d \approx 0.75788d$ or $s > \left(\frac{16-\sqrt{145}}{37}\right) d$ and $\alpha > \hat{\alpha}(s)$ each firm obtains higher expected discounted profits under $COM(2,0)$ that under $COL(1,2)$, while if $s > \left(\frac{16+\sqrt{145}}{37}\right) d$ and $\alpha < \hat{\alpha}(s)$, each firm obtains higher expected discounted profits under $COL(1,2)$ than under $COM(2,0)$.

Case 3.c. If $\alpha > \bar{\alpha}(1,2)(s)$, there is no SPNE outcome path in which firms collude. Then, from Lemma 4, $COM(2,0)$ is the preferred equilibrium outcome path for the firms.

This complete the proof of Proposition 2. Indeed, note we have proved a stronger result than Proposition 2 since we have also characterized the preferred SPNE when $d/5 \leq s \leq d$ and f $\bar{\alpha}(2) < \alpha \leq \bar{\alpha}(1,2)(s)$. ■

Let $\bar{E}(\bar{m}=0)$ denote the SPNE outcome path that induces the highest discounted expected profits when the competition authority does not allow mergers.

Proposition 2.A.0. Suppose that the competition authority does not allow mergers. Then:

1. If $\alpha \leq \bar{\alpha}(4)$, then $\bar{E}(\bar{m}=0) = COL(0,4)$.
2. If $\alpha > \bar{\alpha}(4)$, then $\bar{E}(\bar{m}=0) = COM(0,4)$.

Proof. Immediate from Lemma 5 since for $(n_t^L, n_t^H) = (0,4)$ collusion can be sustained whenever $\alpha \leq \bar{\alpha}(4) = 1 - \frac{25}{\delta 41}$. ■

Let $\bar{E}(\bar{m}=1)$ denote the SPNE outcome path that induces the highest discounted expected profits when the competition authority allows only one merger.

Proposition 2.A.1. Suppose that the maximum number of mergers that the competition authority allows is $\bar{m} = 1$.

1. Suppose that $0 \leq s < \bar{s}^2$.
   (a) If $\alpha \leq \min\{\bar{\alpha}(1,2)(s), \tilde{\alpha}(1,2)(s)\}$, then $\bar{E}(\bar{m}=1) = COL(1,2)$.
   (b) If $\min\{\bar{\alpha}(1,2)(s), \tilde{\alpha}(1,2)(s)\} < \alpha \leq \bar{\alpha}(4)$, then $\bar{E}(\bar{m}=1) = COL(0,4)$.

(c) If $\alpha > \max\{\min\{\bar{\alpha}(1,2)(s), \tilde{\alpha}(1,2)(s)\}, \bar{\alpha}(4)\}$, then $\bar{E}(\bar{m}=1) = COM(0,4)$.

2. Suppose that $\bar{s}^2 \leq s \leq d$.

   (a) If $\alpha \leq \bar{\alpha}(1,2)$, then $\bar{E}(\bar{m}=1) = COL(1,2)$.

   (b) If $\alpha > \bar{\alpha}(1,2)$, then $\bar{E}(\bar{m}=1) = COM(1,2)$.

Proof. Suppose that the competition authority allows only one merger.

Summary of possible equilibria in which collusion does not occur. Assume that $\left(n_{t-1}^L, n_{t-1}^H\right) = (0,4)$. Then, following the same steps we use in the proof of Lemma 4 but restricting $\left(n_t^L, n_t^H\right) \in \{(0,4),(1,2)\}$ we obtain that there are 2 possible types of outcome paths in which firms never collude. As in Proposition 2, within each type we consider. As in Lemma 4, within each type, we consider the outcome path in which mergers occur as soon as possible.

- $COM(0,4)$: $\left(n_\tau^L, n_\tau^H, (q_\tau^z)_{z\in N_\tau}\right) = \left(0, 4, \left(q_H^{COM}(0,4)\right)_{z\in N_\tau^H}\right)$ for all $\tau \geq t$ is always a SPNE outcome path.
- $COM(1,2)$: $\left(n_\tau^L, n_\tau^H, (q_\tau^z)_{z\in N_\tau}\right) = \left(1, 2, \left(q_L^{COM}(1,2)\right)_{z\in N_\tau^L}, \left(q_H^{COM}(1,2)\right)_{z\in N_\tau^H}\right)$ for $\tau \geq t$ is a SPNE outcome path whenever $s \geq \bar{s}^2$.

Summary of possible equilibria in which collusion occurs. Assume that $\left(n_{t-1}^L, n_{t-1}^H\right) = (0,4)$. Then, following the same steps we use in the proof of Proposition 2 but restricting $\left(n_t^L, n_t^H\right) \in \{(0,4),(1,2)\}$ we obtain that there are 2 possible types of outcome paths in which firms can sustain collusion. As in Proposition 2, within each type we consider the outcome path in which collusion occurs as soon as possible.

- $COL(0,4)$: $\left(n_\tau^L, n_\tau^H, (q_\tau^z)_{z\in N_\tau}\right) = \left(0, 4, \left(q_H^{COL}(0,4)\right)_{z\in N_\tau^H}\right)$ for all $\tau \geq t$ until collusion is detected and $\left(n_\tau^L, n_\tau^H, (q_\tau^z)_{z\in N_\tau}\right) = \left(0, 4, \left(q_H^{COM}(0,4)\right)_{z\in N_\tau^H}\right)$ thereafter, is a SPNE outcome path if and only if $\alpha \leq \bar{\alpha}(4)$.
- $COL(1,2)$: $\left(n_\tau^L, n_\tau^H, (q_\tau^z)_{z\in N_\tau}\right) = \left(1, 2, \left(q_L^{COL}(1,2)\right)_{z\in N_\tau^L}, \left(q_H^{COL}(1,2)\right)_{z\in N_\tau^H}\right)$ for all $\tau \geq t$ until collusion is detected and $\left(n_\tau^L, n_\tau^H, (q_\tau^z)_{z\in N_\tau}\right) = \left(1, 2, \left(q_L^{COM}(1,2)\right)_{z\in N_\tau^L}, \left(q_H^{COM}(1,2)\right)_{z\in N_\tau^H}\right)$ thereafter, is a SPNE outcome path if and only if [$s \in \left[0, \bar{s}^2\right)$ and $\alpha \leq \min\{\bar{\alpha}(1,2)(s), \tilde{\alpha}(1,2)(s)\}$] or [$s \geq \bar{s}^2$ and $\alpha \leq \bar{\alpha}(1,2)(s)$].

Equilibrium selection. In Proposition 2 we have already proved that firms obtain higher expected discounted profits under $COL(1,2)$ than under $COM(0,4)$. We have also proved that $\pi_L^{COM}(1,2)/4 + \pi_H^{COM}(1,2)/2 > \pi_H^{COM}(0,4)$ for all $s \geq 0$, which implies that firms obtain higher expected discounted profits under $COM(1,2)$ than under $COM(0,4)$.

To complete the comparison of the SPNE outcome paths we must consider several cases.

Case 1. Suppose that $0 \leq s < \bar{s}^2$.

Case 1.a. If $\alpha \leq \min\{\bar{\alpha}(1,2)(s), \tilde{\alpha}(1,2)(s)\}$, then $COL(1,2)$ is a SPNE outcome path and, hence, for the firms it is the preferred equilibrium outcome path among those in which firms collude. Under competition, $COM(0,4)$ is the unique SPNE outcome path. Firms obtain higher expected discounted profits under $COL(1,2)$ than under $COM(1,2)$. Firms obtains higher expected discounted profits under $COM(1,2)$

than under $COM\,(0,4)$. Thus, firms obtain higher expected discounted profits under $COL\,(1,2)$ than under $COM\,(0,4)$.

Case 1.b. If $\min\{\bar{\alpha}\,(1,2)\,(s)\,,\tilde{\alpha}\,(1,2)\,(s)\} < \alpha \leq \bar{\alpha}\,(4)$, then $COL\,(0,4)$ is the only SPNE outcome path in which firms collude. Under competition, $COM\,(0,4)$ is the unique SPNE outcome path. All firms obtain higher expected discounted profits under $COL\,(0,4)$ than under $COM\,(0,4)$.

Case 1.c. If $\alpha > \max\{\min\{\bar{\alpha}\,(1,2)\,(s)\,,\tilde{\alpha}\,(1,2)\,(s)\}\,,\bar{\alpha}\,(4)\}$, then there is no SPNE outcome path in which firms collude. Under competition, $COM\,(0,4)$ is the only equilibrium outcome path for the firms.

Case 2. Suppose that $\bar{s}^2 \leq s \leq d$

Case 2.a. If $\alpha \leq \bar{\alpha}\,(1,2)$, then $COL\,(1,2)$ is a SPNE outcome path and, hence, for the firms it is the preferred equilibrium outcome path among those in which firms collude. Under competition, the SPNE outcome paths are $COM\,(1,2)$ and $COM\,(0,4)$. All firms obtain higher expected discounted profits under $COL\,(1,2)$ than under $COM\,(1,2)$ or $COM\,(0,4)$.

Case 2.b. If $\alpha > \bar{\alpha}\,(1,2)$, then there is no SPNE outcome path in which firms collude. Under competition, the SPNE outcome paths are $COM\,(1,2)$ and $COM\,(0,4)$. All firms obtain higher expected discounted profits under $COM\,(1,2)$ than under $COM\,(0,4)$.

This complete the proof of Proposition 2.A.1. ■

Proposition 3 Optimal merger policy (consumer surplus). Suppose that $s \in \left(\frac{d}{5}, \frac{\left(16+\sqrt{145}\right)d}{37}\right]$. Let $\alpha^{CS}\,(s) = \left(\frac{1-\delta}{\delta}\right)\left[\frac{\left(\frac{d+s}{2}\right)^2-\left(\frac{4d}{5}\right)^2}{\left(\frac{4d}{5}\right)^2-\left(\frac{2(d+s)}{3}\right)^2}\right]$. Then:

1. If $\alpha \leq \bar{\alpha}\,(4)$, then $\bar{m}^* = 2$ and $\bar{E} = COL\,(2,0)$.
2. If $\bar{\alpha}\,(2) \leq \alpha < \bar{\alpha}\,(4)$.
   (a) If $\alpha < \alpha^{CS}\,(s)$, then $\bar{m}^* = 0$ and $\bar{E} = COM\,(0,4)$.
   (b) If $\alpha > \alpha^{CS}\,(s)$, then $\bar{m}^* = 2$ and $\bar{E} = COL\,(2,0)$.
3. If $\alpha > \bar{\alpha}\,(2)$, then $m^* = 2$ and $\bar{E} = COM\,(2,0)$.

Proof. Let $\bar{m} \in \{0,1,2\}$ denote the maximum number of mergers allowed by the competition authority.

Equilibrium. Let $\bar{E}\,(\bar{m})$ denote the SPNE outcome path that induces the highest discounted expected profits when the maximum number of mergers allowed is $\bar{m}$.

Case 1. Suppose that $\bar{m} = 0$ (no mergers are allowed). Then, from Proposition 2.A.0, if $\alpha < \bar{\alpha}(4)$, then $\bar{E}\,(0) = COM(0,4)$, whie if $\alpha \geq \bar{\alpha}(4)$, then $\bar{E}\,(0) = COM(0,4)$.

Case 2. Suppose that $\bar{m} = 1$ (at most one merger is allowed). Since $s \in \left(d/5, \left(16+\sqrt{145}\right)d/37\right]$, from Proposition 2.A.1, if $\alpha \leq \bar{\alpha}(1,2)$, then $\bar{E}\,(1) = COL(1,2)$, while if $\alpha > \bar{\alpha}(1,2)$, then $\bar{E}\,(1) = COM(1,2)$.

Case 3. Suppose that $\bar{m} = 2$ (at most 2 mergers are allowed). Since $s \in \left(d/5, \left(16+\sqrt{145}\right)d/37\right]$, from Proposition 2, if $\alpha \leq \bar{\alpha}(2)$, $\bar{E}\,(2) = COL(2,0)$, while if $\alpha > \bar{\alpha}(2)$, then $\bar{E}\,(2) = COM(2,0)$.

Welfare Comparisons. Let $W^{CS}\,(O) = \mathbf{E}_0\left[\sum_{t=0}^{\infty}\delta^t CS_t\right]$ denote the expected discounted consumer surplus under the outcome path $O \in \{COM(0,4), COM(1,2), COM(2,0), COL(0,4), COL(1,2), COL(2,0)\}$.

- $W^{CS}\left(COM(2,0)\right) > W^{CS}\left(COM(1,2)\right) > W^{CS}\left(COM(0,4)\right)$. To prove this note that

$$W^{CS}\left(COM(2,0)\right) - W^{CS}\left(COM(1,2)\right) = \frac{CS^{COM}(2,0) - CS^{COM}(1,2)}{1-\delta}$$

Moreover, $CS^{COM}(2,0) > CS^{COM}(1,2)$ if and only if $Q^{COM}(2,0) > Q^{COM}(1,2)$, which always if and only if $s > d/5$. Also note that

$$W^{CS}\left(COM(1,2)\right) - W^{CS}\left(COM(0,4)\right) = \frac{CS^{COM}(1,2) - CS^{COM}(0,4)}{1-\delta}$$

Moreover, $CS^{COM}(1,2) > CS^{COM}(0,4)$ if and only if $Q^{COM}(1,2) > Q^{COM}(0,4)$, which always if and only if $s > d/5$.

- $W^{CS}\left(COL(2,0)\right) > W^{CS}\left(COL(1,2)\right) > W^{CS}\left(COL(0,4)\right)$. To prove this note that

$$W^{CS}\left(COL(2,0)\right) - W^{CS}\left(COL\left(1,2\right)\right) = \frac{CS^{COL}\left(2,0\right) - CS^{COL}(1,2) + \alpha\delta\frac{CS^{COM}(2,0) - CS^{COM}(1,2)}{1-\delta}}{1-\delta\left(1-\alpha\right)}$$

We have already proved that $CS^{COM}\left(2,0\right) > CS^{COM}(1,2)$ if and only if $s > d/5$. Moreover, $CS^{COL}\left(2,0\right) = CS^{COL}(1,2)$ because $Q^{COL}(2,0) = Q^{COL}(1,2)$. Also note that

$$W^{CS}\left(COL\left(1,2\right)\right) - W^{CS}\left(COL\left(0,4\right)\right) = \frac{CS^{COL}\left(1,2\right) - CS^{COL}(0,4) + \alpha\delta\frac{CS^{COM}(1,2) - CS^{COM}(0,4)}{1-\delta}}{1-\delta\left(1-\alpha\right)}$$

We have already proved that $CS^{COM}\left(1,2\right) > CS^{COM}(0,4)$ if and only if $s > d/5$. Moreover, $CS^{COL}\left(1,2\right) > CS^{COL}(0,4)$ because $Q^{COL}(1,2) > Q^{COL}(0,4)$.

- $W^{CS}\left(COL\left(2,0\right)\right) > W^{CS}\left(COM\left(0,4\right)\right)$ if and only if $[s > 3d/5]$ or $[d/5 < s < 3d/5$ and $\alpha > \alpha^{CS}\left(s\right)]$, where $\alpha^{CS}\left(s\right) = \left(\frac{1-\delta}{\delta}\right)\left[\frac{\left(\frac{d+s}{2}\right)^2 - \left(\frac{4d}{5}\right)^2}{\left(\frac{4d}{5}\right)^2 - \left(\frac{2(d+s)}{3}\right)^2}\right]$. To prove this, note that:

$$W^{CS}\left(COL(2,0) - W^{CS}\left(COM(0,4\right) = \frac{CS^{COL}\left(2,0\right) + \alpha\delta\frac{CS^{COM}(2,0)}{1-\delta}}{1-\delta\left(1-\alpha\right)} - \frac{CS^{COM}\left(0,4\right)}{1-\delta}$$

Therefore, $W^{CS}\left(COL(2,0\right) > W^{CS}\left(COM(0,4\right)$ if and only if

$$\alpha\delta\left[CS^{COM}\left(2,0\right) - CS^{COM}\left(0,4\right)\right] > (1-\delta)\left[CS^{COM}\left(0,4\right) - CS^{COL}\left(2,0\right)\right]$$

We have already proved that $CS^{COM}\left(2,0\right) - CS^{COM}\left(0,4\right)$ if and only if $s > d/5$.. Moreover, $CS^{COM}\left(0,4\right) > CS^{COL}\left(2,0\right)$ if and only if $s < 3d/5$. Thus, for $s \geq 3d/5$, it is always the case that $W^{CS}\left(COL(2,0\right) > W^{CS}\left(COM(0,4\right)$. For $d/5 < s < 3d/5$. $W^{CS}\left(COL(2,0\right) > W^{CS}\left(COM(0,4\right)$ if and only if $\alpha > \alpha^{CS}\left(s\right)$, where

$$\begin{aligned}\alpha^{CS}\left(s\right) &= \left(\frac{1-\delta}{\delta}\right)\left[\frac{CS^{COM}\left(0,4\right) - CS^{COL}\left(2,0\right)}{CS^{COM}\left(2,0\right) - CS^{COM}\left(0,4\right)}\right] \\ &= \left(\frac{1-\delta}{\delta}\right)\left[\frac{\left(\frac{d+s}{2}\right)^2 - \left(\frac{4d}{5}\right)^2}{\left(\frac{4d}{5}\right)^2 - \left(\frac{2(d+s)}{3}\right)^2}\right]\end{aligned}$$

Moreover, note that: (i) $\alpha^{CS}(3d/5) = 0$, (ii) $\lim_{s \to (d/5)^-} \alpha^{CS}(s) = \infty$, and (iii) $\alpha^{CS}(s)$ is decreasing for all $s \in (d/5, 3d/5]$. Hence, there exist unique $s^{*,1}$, s^{*2}, $s^{*,3} \in (d/5, 3d/5)$ such that $\alpha^{CS}\left(s^{*,1}\right) = \bar{\alpha}(1,2)\left(s^{*,1}\right)$, $\alpha^{CS}\left(s^{*,2}\right) = \bar{\alpha}(2)$, and $\alpha^{CS}\left(s^{*,3}\right) = \bar{\alpha}(4)$, and $s^{*,1} < s^{*,2} < s^{*,3}$.

Optimal Merger Policy.

Case 1. Suppose $\alpha \leq \bar{\alpha}(4)$. Then, $\bar{E}(0) = COL(0,4)$, $\bar{E}(1) = COL(1,2)$ and $\bar{E}(2) = COL(2,0)$. Since $W^{CS}(COL(2,0)) > W^{CS}(COL(1,2)) > W^{CS}(COL(0,4))$, $\bar{m} = 2$ induces the highest welfare. Moreover, under the optimal policy, $\bar{E} = COL(2,0)$.

Case 2. Suppose $\bar{\alpha}(4) < \alpha \leq \bar{\alpha}(2)$. Then, $\bar{E}(0) = COM(0,4)$, $\bar{E}(1) = COL(1,2)$ and $\bar{E}(2) = COL(2,0)$. $W^{CS}(COL(2,0)) > W^{CS}(COL(1,2))$ and $W^{CS}(COL(2,0)) > W^{CS}(COM(0,4))$ if and only [$s > 3d/5$] or [$d/5 < s < 3d/5$ and $\alpha > \alpha^{CS}(s)$]. Therefore, [$s > 3d/5$] or [$d/5 < s < 3d/5$ and $\alpha > \alpha^{CS}(s)$], then $\bar{m} = 2$ induces the highest welfare and, hence, $\bar{E} = COL(2,0)$, while if $d/5 < s < 3d/5$ and $\alpha < \alpha^{CS}(s)$, $\bar{m} = 0$ induces the highest welfare and, hence, $\bar{E} = COM(0,4)$.

Case 3.a. Suppose $\bar{\alpha}(2) < \alpha \leq \bar{\alpha}(1,2)$. Then, $\bar{E}(0) = COM(0,4)$, $\bar{E}(1) = COL(1,2)$ and $\bar{E}(2) = COM(2,0)$. Since $W^{CS}(COM(2,0)) > W^{CS}(COM(0,4))$ and $W^{CS}(COM(2,0)) > W^{CS}(COM(1,2)) > W^{CS}(COL(1,2))$, $\bar{m} = 2$ induces the highest welfare. Moreover, under the optimal policy, $\bar{E} = COM(2,0)$.

Case 3.b. Suppose $\alpha > \bar{\alpha}(1,2)$. Then, $\bar{E}(0) = COM(0,4)$, $\bar{E}(1) = COM(1,2)$ and $\bar{E}(2) = COM(2,0)$. Since $W^{CS}(COM(2,0)) >$ and $W^{CS}(COM(2,0)) > W^{CS}(COM(1,2)) > W^{CS}(COM(0,4))$, $\bar{m} = 2$ induces the highest welfare. Moreover, under the optimal policy, $\bar{E} = COM(2,0)$.

This complete the proof of Proposition 3. ■

Proposition 4 Optimal merger policy (total surplus). Suppose that $s \in \left(\left(\frac{83-24\sqrt{11}}{79}\right)d, \left(\frac{16+\sqrt{145}}{37}\right)d\right]$. Let $\alpha_1^{TS}(s) = \left(\frac{1-\delta}{\delta}\right)\left[\frac{\left(\frac{3}{4}\right)\left(\frac{4d}{5}\right)^2 - \left(\frac{3}{2}\right)\left(\frac{d+s}{2}\right)^2}{\left(\frac{2(d+s)}{3}\right)^2 - \left(\frac{3}{4}\right)\left(\frac{4d}{5}\right)^2}\right]$ and $\alpha_2^{TS}(s) = \left(\frac{1-\delta}{\delta}\right)\left[\frac{\left(\frac{3}{2}\right)\left(\frac{a-c+cs}{2}\right)^2 - \left(1/2+s_L(1,2)^2+2s_H(1,2)^2\right)\left(\frac{3(a-c)+cs}{4}\right)^2}{\left(1/2+s_L(1,2)^2+2s_H(1,2)^2\right)\left(\frac{3(a-c)+cs}{4}\right)^2 - \left(\frac{2(a-c+cs)}{3}\right)^2}\right]$. Then:

1. If $\alpha \leq \bar{\alpha}(4)$, then $\bar{m}^*(TS) = 2$ and $\bar{E} = COL(2,0)$.
2. If $\bar{\alpha}(4) < \alpha \leq \bar{\alpha}(1,2)$.
   (a) If $\alpha < \alpha^{TS}(s)$, then $\bar{m}^*(TS) = 0$ and $\bar{E} = COM(0,4)$.
   (b) If $\alpha > \alpha^{TS}(s)$, then $\bar{m}^*(TS) = 2$ and $\bar{E} = COL(2,0)$.
3. If $\bar{\alpha}(1,2) < \alpha \leq \bar{\alpha}(2)$.
   (a) If $\alpha < \alpha^{TS*}(s)$, then $\bar{m}^*(TS) = 1$ and $\bar{E} = COM(1,2)$.
   (b) If $\alpha > \alpha^{TS*}(s)$, then $\bar{m}^*(TS) = 2$ and $\bar{E} = COL(2,0)$.
4. If $\alpha > \bar{\alpha}(2)$, then $\bar{m}^*(TS) = 2$ and $\bar{E} = COM(2,0)$.

Proof. Let $\bar{m} \in \{0,1,2\}$ denote the maximum number of mergers allowed by the competition authority.

Equilibrium. Let $\bar{E}(\bar{m})$ denote the SPNE outcome path that induces the highest discounted expected profits when the maximum number of mergers allowed is $\bar{m}$.

Case 1. Suppose that $\bar{m} = 0$ (no mergers are allowed). Then, from Proposition 2.A.0, if $\alpha < \bar{\alpha}(4)$, then $\bar{E}(0) = COM(0,4)$, whie if $\alpha \geq \bar{\alpha}(4)$, then $\bar{E}(0) = COM(0,4)$.

Case 2. Suppose that $\bar{m} = 1$ (at most one merger is allowed). Let $\check{s}^1 = \left(83 - 24\sqrt{11}\right) d/79$ and $\check{s}^2 = \left(16 + \sqrt{145}\right) d/37$. Since $s \in (\check{s}^1, \check{s}^2]$, from Proposition 2.A.1, if $\alpha \leq \bar{\alpha}(1,2)$, then $\bar{E}(1) = COL(1,2)$, while if $\alpha > \bar{\alpha}(1,2)$, then $\bar{E}(1) = COM(0,4)$ for $\check{s}^1 < s < \bar{s}^2$ and $\bar{E}(1) = COM(1,2)$ for $\bar{s}^2 \leq s \leq \check{s}^2$.

Case 3. Suppose that $\bar{m} = 2$ (at most 2 mergers are allowed). Since $s \in (\check{s}^1, \check{s}^2]$, from Proposition 2, if $\alpha \leq \bar{\alpha}(2)$, $\bar{E}(2) = COL(2,0)$, while if $\alpha > \bar{\alpha}(2)$, then $\bar{E}(2) = COM(2,0)$.

Welfare Comparisons. Let $W^{TS}(O) = \mathbf{E}_0\left[\sum_{t=0}^{\infty} \delta^t TS_t\right]$ denote the expected discounted consumer surplus under the outcome path $O \in \{COM(0,4), COM(1,2), COM(2,0), COL(0,4), COL(1,2), COL(2,0)\}$.

- $W^{TS}(COM(2,0)) > W^{TS}(COM(1,2)) > W^{CS}(COM(0,4))$. To prove this note that

$$W^{TS}(COM(2,0)) - W^{TS}(COM(1,2)) = \frac{TS^{COM}(2,0) - TS^{COM}(1,2)}{1-\delta}$$

  Moreover, $TS^{COM}(2,0) > TS^{COM}(1,2)$ if and only if $s > \check{s}^1$. Also note that

$$W^{TS}(COM(1,2)) - W^{TS}(COM(0,4)) = \frac{TS^{COM}(1,2) - TS^{COM}(0,4)}{1-\delta}$$

  Moreover, $TS^{COM}(1,2) - TS^{COM}(0,4)$ if and only if $s > \left(8\sqrt{13} - 25\right) d/115$, which always holds because $\check{s}^1 > \left(8\sqrt{13} - 25\right) d/115$.

- $W^{TS}(COL(2,0)) > W^{TS}(COL(1,2)) > W^{TS}(COL(0,4))$. To prove this note that

$$W^{TS}(COL(2,0)) - W^{TS}(COL(1,2)) = \frac{TS^{COL}(2,0) - TS^{COL}(1,2) + \alpha\delta\frac{TS^{COM}(2,0) - TS^{COM}(1,2)}{1-\delta}}{1-\delta(1-\alpha)}$$

  We have already proved that $TS^{COM}(2,0) > TS^{COM}(1,2)$ if and only if $s > \check{s}^1$. Moreover, $TS^{COL}(2,0) = TS^{COL}(1,2)$. Therefore, $W^{TS}(COL(2,0)) > W^{TS}(COL(1,2))$ for $s > \check{s}^1$. Also note that

$$W^{TS}(COL(1,2)) - W^{TS}(COL(0,4)) = \frac{TS^{COL}(1,2) - TS^{COL}(0,4) + \alpha\delta\frac{TS^{COM}(1,2) - TS^{COM}(0,4)}{1-\delta}}{1-\delta(1-\alpha)}$$

  We have already proved that for $s > \check{s}^1$ it is always the case that $TS^{COM}(1,2) > TS^{COM}(0,4)$. Moreover, for $s > 0$ we have $TS^{COL}(1,2) > TS^{COL}(0,4)$. Therefore, for $s > \check{s}^1$ we have $W^{TS}(COL(1,2)) > W^{TS}(COL(0,4))$.

- $W^{TS}(COL(2,0)) > W^{TS}(COM(0,4))$ if and only if [$s > \left(8 - 5\sqrt{2}\right) d/5\sqrt{2}$] or [$\left(3\sqrt{3} - 5\right) d/5 < s < \left(8 - 5\sqrt{2}\right) d/5\sqrt{2}$ and $\alpha > \alpha_1^{TS}(s)$], where $\alpha_1^{TS}(s) = \left(\frac{1-\delta}{\delta}\right)\left[\frac{\left(\frac{d+s}{2}\right)^2 - \left(\frac{4d}{5}\right)^2}{\left(\frac{4d}{5}\right)^2 - 4\left(\frac{d+s}{3}\right)^2}\right]$. To prove this, note that

$$W^{TS}(COL(2,0)) - W^{TS}(COM(0,4)) = \frac{TS^{COL}(2,0) + \alpha\delta\frac{TS^{COM}(2,0)}{1-\delta}}{1-\delta(1-\alpha)} - \frac{TS^{COM}(0,4)}{1-\delta}$$

Therefore, $W^{TS}(COL(2,0)) > W^{TS}(COM(0,4))$ if and only if

$$\alpha\delta\left[TS^{COM}(2,0) - TS^{COM}(0,4)\right] > (1-\delta)\left[TS^{COM}(0,4) - TS^{COL}(2,0)\right]$$

We have already proved that for $s > \check{s}^1$ we have $TS^{COM}(2,0) > TS^{COM}(0,4)$. Moreover, $TS^{COM}(0,4) > TS^{COL}(2,0)$ if and only if $s < \left(8-5\sqrt{2}\right)d/5\sqrt{2}$. Thus, for $s \geq \left(8-5\sqrt{2}\right)d/5\sqrt{2}$, it is always the case that $W^{TS}(COL(2,0)) > W^{TS}(COM(0,4))$. For $\left(3\sqrt{3}-5\right)d/5 < s < \left(8-5\sqrt{2}\right)d/5\sqrt{2}$, $W^{TS}(COL(2,0)) > W^{TS}(COM(0,4))$ if and only if $\alpha > \alpha_1^{TS}(s)$, where

$$\begin{aligned}\alpha_1^{TS}(s) &= \left(\frac{1-\delta}{\delta}\right)\left[\frac{TS^{COM}(0,4) - TS^{COL}(2,0)}{TS^{COM}(2,0) - TS^{COM}(0,4)}\right] \\ &= \left(\frac{1-\delta}{\delta}\right)\left[\frac{\left(\frac{3}{4}\right)\left(\frac{4d}{5}\right)^2 - \left(\frac{3}{2}\right)\left(\frac{d+s}{2}\right)^2}{\left(\frac{2(d+s)}{3}\right)^2 - \left(\frac{3}{4}\right)\left(\frac{4d}{5}\right)^2}\right]\end{aligned}$$

Moreover, note that: (i) $\alpha_1^{TS}\left(\left(8-5\sqrt{2}\right)d/5\sqrt{2}\right) = 0$, (ii) $\lim_{s\to\left(3\sqrt{3}-5\right)d/5^-}\alpha_1^{TS}(s) = \infty$, and (iii) $\alpha_1^{TS}(s)$ is decreasing for all $\left(3\sqrt{3}-5\right)d/5 < s < \left(8-5\sqrt{2}\right)d/5\sqrt{2}$. Hence, there exist unique $s^{*,1}$, s^{*2}, $s^{*,3} \in \left(\left(3\sqrt{3}-5\right)d/5, \left(8-5\sqrt{2}\right)d/5\sqrt{2}\right)$ such that $\alpha_1^{TS}\left(s^{*,1}\right) = \bar{\alpha}(1,2)\left(s^{*,1}\right)$, $\alpha_1^{TS}\left(s^{*,2}\right) = \bar{\alpha}(2)$, and $\alpha_1^{TS}\left(s^{*,3}\right) = \bar{\alpha}(4)$, and $s^{*,1} < s^{*,2} < s^{*,3}$.

- $W^{TS}(COL(2,0)) > W^{TS}(COM(1,2))$ if and only if $[s > 3d/11]$ or $[\check{s}^1 < s < 3d/11$ and $\alpha > \alpha_2^{TS}(s)]$, where $\alpha_2^{TS}(s) = \left(\frac{1-\delta}{\delta}\right)\left[\frac{\left(\frac{3}{2}\right)\left(\frac{a-c+cs}{2}\right)^2 - \left(1/2+s_L(1,2)^2+2s_H(1,2)^2\right)\left(\frac{3(a-c)+cs}{4}\right)^2}{\left(1/2+s_L(1,2)^2+2s_H(1,2)^2\right)\left(\frac{3(a-c)+cs}{4}\right)^2 - \left(\frac{2(a-c+cs)}{3}\right)^2}\right]$. To prove this, note that:

$$W^{TS}(COL(2,0)) - W^{TS}(COM(1,2)) = \frac{TS^{COL}(2,0) + \alpha\delta\frac{TS^{COM}(2,0)}{1-\delta}}{1-\delta(1-\alpha)} - \frac{TS^{COM}(1,2)}{1-\delta}$$

Therefore, $W^{TS}(COL(2,0)) > W^{TS}(COM(1,2))$ if and only if

$$\alpha\delta\left[TS^{COM}(2,0) - TS^{COM}(1,2)\right] > (1-\delta)\left[TS^{COM}(1,2) - TS^{COL}(2,0)\right]$$

We have already proved that $TS^{COM}(2,0) > TS^{COM}(1,2)$ if and only if $s > \check{s}^1$. Moreover, $TS^{COM}(1,2) > TS^{COL}(2,0)$ if and only if $s < 3d/11$. Thus, for $s \geq 3d/11$, it is always the case that $W^{TS}(COL(2,0)) > W^{TS}(COM(1,2))$. For $\check{s}^1 < s < 3d/11$, $W^{TS}(COL(2,0)) > W^{TS}(COM(1,2))$ if and only if $\alpha > \hat{\alpha}_2^{TS}(s)$, where

$$\begin{aligned}\alpha_2^{TS}(s) &= \left(\frac{1-\delta}{\delta}\right)\left[\frac{TS^{COM}(1,2) - TS^{COL}(2,0)}{TS^{COM}(2,0) - TS^{COM}(1,2)}\right] \\ &= \left(\frac{1-\delta}{\delta}\right)\left[\frac{\left(\frac{3}{2}\right)\left(\frac{a-c+cs}{2}\right)^2 - \left(1/2+s_L(1,2)^2+2s_H(1,2)^2\right)\left(\frac{3(a-c)+cs}{4}\right)^2}{\left(1/2+s_L(1,2)^2+2s_H(1,2)^2\right)\left(\frac{3(a-c)+cs}{4}\right)^2 - \left(\frac{2(a-c+cs)}{3}\right)^2}\right]\end{aligned}$$

Moreover, note that: (i) $\alpha_2^{TS}(s)(3d/11) = 0$, (ii) $\lim_{s\to(\check{s}^1)^-}\alpha_2^{TS}(s) = \infty$, and (iii) $\alpha_2^{TS}(s)$ is decreasing for all $s \in (\check{s}^1, 3d/11]$. Hence, there exist unique $s^{**,1}$, s^{**2}, $s^{**,3} \in \left(\check{s}^1, 3d/11\right)$ such that $\alpha_2^{TS}\left(s^{**,1}\right) = \bar{\alpha}(1,2)\left(s^{**,1}\right)$, $\alpha_2^{TS}\left(s^{**,2}\right) = \bar{\alpha}(2)$, and $\alpha_2^{TS}\left(s^{**,3}\right) = \bar{\alpha}(4)$, and $s^{**,1} < s^{**,2} < s^{**,3}$.

Threshold Comparisons. The following inequalities hold $\bar{s}^1 < \check{s}^1 < \bar{s}^2 < 3d/11$. Additionally, $\left(3\sqrt{3}-5\right)d/5 < \check{s}^1 < \bar{s}^2$.

Optimal Merger Policy.

Case 1. Suppose $\alpha \leq \bar{\alpha}(4)$. Then, $\bar{E}(0) = COL(0,4)$, $\bar{E}(1) = COL(1,2)$ and $\bar{E}(2) = COL(2,0)$. Since $W^{TS}(COL(2,0)) > W^{TS}(COL(1,2)) > W^{TS}(COL(0,4))$, $\bar{m} = 2$ induces the highest welfare. Moreover, under the optimal policy, $\bar{E} = COL(2,0)$.

Case 2. Suppose $\bar{\alpha}(4) < \alpha \leq \min\{\bar{\alpha}(1,2), \bar{\alpha}(2)\}$. Then, $\bar{E}(0) = COM(0,4)$, $\bar{E}(1) = COL(1,2)$ and $\bar{E}(2) = COL(2,0)$. $W^{TS}(COM(0,4)) > W^{TS}(COL(2,0))$ and $W^{TS}(COM(0,4)) > W^{TS}(COL(1,2))$ if and only $[\check{s}^1 < s < \left(8-5\sqrt{2}\right)d/5\sqrt{2}$ and $\alpha < \hat{\alpha}_1^{TS}(s)]$. Therefore, $[\check{s}^1 < s < \left(8-5\sqrt{2}\right)d/5\sqrt{2}$ and $\alpha < \alpha_1^{TS}(s)]$, then $\bar{m} = 0$ induces the highest welfare and, hence, $\bar{E} = COM(0,4)$, while if $[\check{s}^1 < s < \left(8-5\sqrt{2}\right)d/5\sqrt{2}$ and $\alpha > \alpha_1^{TS}(s)]$ or $[s \geq \left(8-5\sqrt{2}\right)d/5\sqrt{2}]$, then $\bar{m} = 2$ induces the highest welfare and, hence, $\bar{E} = COL(2,0)$.

Case 3. Suppose $\bar{\alpha}(1,2) < \alpha \leq \bar{\alpha}(2)$. Then, $\bar{E}(0) = COM(0,4)$, $\bar{E}(1) = COM(1,2)$ and $\bar{E}(2) = COL(2,0)$. Since $W^{TS}(COM(1,2)) > W^{TS}(COL(2,0))$ and $W^{TS}(COM(1,2)) > W^{TS}(COM(0,4))$, if and only if $[\check{s}^1 < s < 3d/11$ and $\alpha < \alpha_2^{TS}(s)]$. Therefore, $[\check{s}^1 < s < 3d/11$ and $\alpha < \alpha_2^{TS}(s)]$, then $\bar{m} = 1$ induces the highest welfare and, hence, $\bar{E} = COM(1,2)$, while if $[\check{s}^1 < s < 3d/11$ and $\alpha > \alpha_2^{TS}(s)]$ or $[s \geq 3d/11]$, then $\bar{m} = 2$ induces the highest welfare and, hence, $\bar{E} = COL(2,0)$.

Case 4. Suppose $\alpha > \bar{\alpha}(2)$

Case 4.a. Suppose $\bar{\alpha}(2) < \alpha \leq \bar{\alpha}(1,2)$. Then, $\bar{E}(0) = COM(0,4)$, $\bar{E}(1) = COL(1,2)$ and $\bar{E}(2) = COM(2,0)$. Since $W^{TS}(COM(2,0)) > W^{TS}(COM(0,4))$ and $W^{TS}(COM(2,0)) > W^{TS}(COM(1,2)) > W^{TS}(COL(1,2))$, $\bar{m} = 2$ induces the highest welfare. Moreover, under the optimal policy, $\bar{E} = COM(2,0)$.

Case 4.b. Suppose $\check{s}^1 < s \leq \bar{s}^2$ and $\alpha > \max\{\bar{\alpha}(2), \bar{\alpha}(1,2)\}$. Then, $\bar{E}(0) = COM(0,4)$, $\bar{E}(1) = COM(0,4)$ and $\bar{E}(2) = COM(2,0)$. Since $W^{TS}(COM(2,0)) > W^{TS}(COM(1,2)) > W^{TS}(COM(0,4))$, $\bar{m} = 2$ induces the highest welfare. Moreover, under the optimal policy, $\bar{E} = COM(2,0)$.

Case 4.c. Suppose that $s > \bar{s}^2$ and $\alpha > \max\{\bar{\alpha}(2), \bar{\alpha}(1,2)\}$. Then, $\bar{E}(0) = COM(0,4)$, $\bar{E}(1) = COM(1,2)$ and $\bar{E}(2) = COM(2,0)$. Since $W^{TS}(COM(2,0)) > W^{TS}(COM(1,2)) > W^{TS}(COM(0,4))$, $\bar{m} = 2$ induces the highest welfare. Moreover, under the optimal policy, $\bar{E} = COM(2,0)$.

This complete the proof of Proposition 4. ■