Nonlinear Pricing under Asymmetric Competition with Complete Information*

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Abstract

Motivated by several recent antitrust cases, we study a strategic model of competition in intermediate-goods markets. Our model is a three-stage game with complete information in which a dominant firm offers a general tariff first and then a rival firm responds with a per-unit price, followed by a buyer making her decision to purchase from one or both firms. We characterize subgame perfect equilibria of the game and study the implications of the equilibrium outcome.

Our paper makes three main contributions. First, it provides a novel explanation for the prevalence of nonlinear pricing (a menu of offers conditional on volumes) under duopoly in the absence of private information: The dominant firm can use a menu of offers to constrain its rival’s choices and extract surplus from the buyer. Second, it shows that when the capacity of the rival firm is constrained, as compared to linear pricing schemes, the nonlinear pricing tariff

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adopted by the dominant firm reduces the price, sales, and profits of the rival
firm as well as the buyer’s surplus. In other words, nonlinear pricing may
have antitrust implications in the sense that it can lead to partial foreclosure
and harm consumer welfare. Third, we establish an equivalence between a
subgame perfect equilibrium of the game and an optimal mechanism in a
“virtual” principal-agent model with hidden action and hidden information.
This involves treating the rival firm’s (an agent’s) price as its hidden action
meanwhile letting the buyer (another agent) to report the rival firm’s price as
her private information to the dominant firm (the principal). As a result of
such an equivalence, we can apply mechanism design techniques to solve for
subgame perfect equilibria of the game.

Keywords: Nonlinear Pricing, Asymmetric Competition, Capacity Constraint,
Complete Information, Subgame Perfect Equilibrium, Principal-Agent Model,
and Partial Foreclosure.

JEL Code: L13, L42, K21

1 Introduction

Nonlinear pricing is often observed in intermediate-goods markets. It takes the form
of various rebates and discounts conditional on volumes (or share of the volumes
among competitors) purchased by a buyer. An example is all-units discount pricing
scheme that lowers a buyer’s marginal price on every unit purchased when the buyer’s
purchase exceeds or is equal to a pre-specified volume threshold. The adoption of
such conditional rebates and discounts by dominant firms has become a prominent
antitrust issue. Indeed, in a number of recent antitrust cases in the U.S., E.U.,
Canada, and China, a plaintiff (a government antitrust agency or a rival firm) alleged
that a dominant firm used pricing schemes such as conditional rebates/discounts to
its downstream buyers to fully or partially exclude its rival firm(s) and that such
an exclusion had harmed competition and consumer welfare. Those antitrust cases
share some common features: First, there is a firm that is considered as “dominant”
in market share, capacity, product lines, profits, and so on. Second, there is one or
several smaller firms (or recent entrants) that have limited capacity, narrower product lines, or limited distribution channels. Third, the “dominant” firm typically offers more complex pricing schemes (e.g., rebates/discounts conditional on volumes) than its rival(s). What explains the observed practices of various nonlinear pricing schemes in intermediate-goods markets and what are the implications of those practices? The main objective of this paper is to provide an explanation for nonlinear pricing in the presence of asymmetric competition and in the absence of private information.

Motivated by recent antitrust cases, we study a stylized model of asymmetric competition. In the model, there are two firms, a dominant firm (Firm 1) and a rival firm (Firm 2). Both firms can produce a homogeneous product at constant marginal cost. However, the rival firm is capacity constrained. There is a representative downstream buyer who may purchase the product from one or both firms. We consider a three-stage game with complete information in which the dominant firm offers a general tariff first and then its rival firm responds with a per-unit price, followed by the buyer making her decision to purchase from one or both firms. We characterize subgame perfect equilibria of the game and study the implications of the equilibrium outcome.

Our model involves three kinds of asymmetries between the two firms. The first is concerned with pricing schemes: The dominant firm is able to make nonlinear tariff schedules, i.e., payments conditional on volumes, while the rival firm can only choose linear pricing schemes. This assumption appears to be consistent with the observations from the major antitrust cases, and is perhaps due to the fact that the dominant firm is more experienced in dealing with downstream buyers than new entrants to the market. The second asymmetry concerns the timing of the game: The dominant firm commits to offering tariffs before its rival. This might be related to the dominant firm’s bargaining power and its willingness to commit its offers when dealing with the buyer. Another asymmetry is about capacity levels of the firms. That is, relatively to the demand size the dominant firm has no capacity limit while its rival is capacity-constrained. Our analysis suggests that the asymmetry in capacity is not crucial for the equilibrium adoption of nonlinear pricing by the dominant firm, but is important for the results of partial foreclosure and harming
the buyer welfare.

Our paper makes several major contributions. First, it provides a novel explanation for the prevalence of nonlinear pricing (a menu of offers conditional on volumes) under duopoly in the absence of private information: The dominant firm can use a menu of offers to constrain its rival’s choices and extract surplus from the buyer. Second, it shows that when the capacity of the rival firm is relatively small, as compared to linear pricing schemes, the nonlinear pricing tariff adopted by the dominant firm reduces the price, sales, and profits of the rival firm as well as the buyer’s surplus. In other words, nonlinear pricing in this context can lead to partial foreclosure and harm consumer welfare, which may have antitrust implications. Third, we establish an equivalence between a subgame perfect equilibrium of the game and an optimal mechanism in a “virtual” principal-agent model with hidden action and hidden information. This involves treating the rival firm’s (an agent’s) price as its hidden action meanwhile letting the buyer (another agent) to report the rival firm’s price as her private information to the dominant firm (the principal). As a result of such an equivalence, we can apply mechanism design techniques to characterize subgame perfect equilibria of the game. Other properties of the equilibrium tariffs are also discussed in the paper.

[Link to the literature to be added.]

The remainder of the paper is organized as follows. In Section 2, we set up our model of asymmetric competition in intermediate-goods markets. Section 3 offers some preliminary analysis. Section 4 characterizes the equilibrium outcome of the game. In particular, Subsection 4.1 describes the dominant firm’s optimization problem and establishes an equivalence between a subgame perfect equilibrium of the game and an optimal mechanism in a “virtual” principal-agent model. Subsection 4.2 characterizes the buyer’s incentive compatibility and individual rationality constraints. Subsection 4.3 provides a complete characterization of the equilibrium outcome. Other properties and implications of the equilibrium are discussed in Section 5. Section 6 contains concluding remarks.
2 Model

There are two firms, producing a homogeneous product, and one buyer. Firm 1, also known as the dominant firm (D), can produce any quantity at a unit cost \( c \geq 0 \). Firm 2, also known as the rival firm (R), has a capacity \( k \in (0, \infty) \], up to which it can produce any quantity at the same unit cost \( c \geq 0 \). If the buyer (B) chooses to buy \( Q \geq 0 \) units from firm 1 and \( q \in [0, k] \) units from firm 2, his payoff is the gross utility given by \( u(Q + q) \), less the payments to the two firms.

Assumption 1. The utility function \( u : \mathbb{R}^+ \to \mathbb{R} \) is twice continuously differentiable, satisfies \( u(0) = 0 \), \( u''(\cdot) < 0 \), \( u'(0) > c \), and there exists a unique \( q^e > 0 \) such that \( u'(q^e) = c \). This \( q^e \) is called the efficient quantity.

The game is as follows. First, firm 1 offers a nonlinear tariff \( \tau(\cdot) \), which specifies the payment \( \tau(Q) \in \mathbb{R} \cup \{\infty\} \) that the buyer has to make if the buyer chooses to buy \( Q \geq 0 \) units from firm 1, with the restriction that \( \tau(0) \leq 0 \). (\( \tau(Q) = \infty \) means that purchasing \( Q \) units is not allowed.) Second, after observing \( \tau(\cdot) \), firm 2 offers a unit price \( p \) (which is meant to apply up to \( k \) units). Third, after observing \( \tau(\cdot) \) and \( p \), the buyer chooses the quantities he buys from firm 1 and from firm 2. (Dual sourcing is allowed.)

To avoid expositional complications, we assume that the set of feasible unit prices firm 2 can choose is

\[
P \equiv [c, u'(0)].
\]

Note that firm 2 always make nonpositive profit if it chooses a unit price below \( c \) or above \( u'(0) \).

Note that our game is a complete information and perfect information one. Our equilibrium concept is pure strategy subgame perfect equilibrium, or simply called subgame perfect equilibrium (SPE).

We say a tariff \( \tau : \mathbb{R}^+ \to \mathbb{R} \cup \{\infty\} \) is regular if the subgame after firm 1 offers \( \tau \) has some SPE. Clearly, if we allow firm 1 to choose irregular tariff, the whole game
has no SPE. Therefore, we assume that the set of feasible tariffs firm 1 can choose is

\[ T \equiv \left\{ \tau \in (\mathbb{R} \cup \{\infty\})^{\mathbb{R}^+} : \tau \text{ is regular and } \tau(0) \leq 0 \right\}. \]

A SPE is composed of a firm 1’s strategy \( \tau^* \in T \), a firm 2’s strategy \( p^* : T \to \mathcal{P} \), and a buyer’s strategy \( q^* : T \times \mathcal{P} \to \mathbb{R}_+ \times [0,k] \), such that

(BSR): \( q^*(\tau,p) \in \operatorname{argmax}_{(Q,q) \in \mathbb{R}_+ \times [0,k]} \{u(Q + q) - pq - \tau(Q)\} \quad \forall (\tau,p) \in T \times \mathcal{P} \),

(RSR): \( p^*(\tau) \in \operatorname{argmax}_{p \in \mathcal{P}} (p - c)q^*_2(\tau,p) \quad \forall \tau \in T \),

(DSR): \( \tau^* \in \operatorname{argmax}_{\tau \in T} \{\tau(q^*_1(\tau,p^*(\tau))) - cq^*_1(\tau,p^*(\tau))\} \).

Let \( D(\cdot) \) denote the buyer’s demand function and \( \pi(\cdot) \) the monopoly profit function, i.e.,

\[ D(p) \equiv \operatorname{argmax}_{\tilde{q} \geq 0} \{u(\tilde{q}) - p\tilde{q}\}, \]

\[ \pi(p) \equiv (p - c)D(p). \]

Assumption 1 implies that \( D(\cdot) \) and \( \pi(\cdot) \) are continuously differentiable and \( D(\cdot) \) is strictly decreasing on \( \mathcal{P} \). We also make the following assumption, which, although is not crucial, simplifies our analysis.

**Assumption 2.** The monopoly profit function \( \pi(\cdot) \) is strictly concave on \( \mathcal{P} \).

There is a unique optimal monopoly price \( p^m \in (c, u'(0)) \) given by \( \pi'(p^m) = 0 \).

### 3 Preliminary analysis

#### 3.1 Buyer’s problem

Given the two firms’ offers \( \tau \in T \) and \( p \in \mathcal{P} \), the buyer’s maximization problem in (BSR) can be decomposed into two stages: in the first stage, the buyer chooses the purchase \( Q \geq 0 \) from firm 1; in the second stage, given \( Q \), the buyer chooses the
purchase $q$ from firm 2. The buyer in the second stage solves
\[ V(Q, p) \equiv \max_{q \in [0,k]} \{ u(Q + q) - pq \}, \]
and in the first stage solves
\[ \max_{Q \geq 0} \{ V(Q, p) - \tau(Q) \}. \]

Note that the second stage maximization problem has a unique maximizer, which is $\text{Proj}_{[0,k]}(D(p) - Q)$. One may call $V$ the conditional payoff or indirect utility function of the buyer. Note that $Q \leq D(p) \leq Q + k$ (so that the constraint $q \in [0,k]$ is not binding and $\text{Proj}_{[0,k]}(D(p) - Q) = D(p) - Q$ if and only if $u'(Q + k) \leq p \leq u'(Q)$.

By Milgrom-Segal Envelope Theorem, $V(Q, \cdot)$ and $V(\cdot, p)$ are absolutely continuous on any compact interval, and
\[ V_p(Q, p) = -\text{Proj}_{[0,k]}(D(p) - Q), \]
\[ V_Q(Q, p) = u'(\text{Proj}_{[Q,Q+k]}(D(p))) = \text{Proj}_{[u'(Q+k),u'(Q)]}(p). \]

We see that, $V_p$ and $V_Q$ are absolutely continuous on any compact rectangle, and $V$ is continuously differentiable. Moreover, $V$ satisfies weak increasing differences as follows:
\[ V_{qp}(Q, p) = V_{pq}(Q, p) = \begin{cases} 1 & \text{if } D(p) - k < Q < D(p) \\ 0 & \text{if } Q < D(p) - k \text{ or } Q > D(p) \end{cases}. \]

### 3.2 Non-optimality of one-bundle offering

Under our complete information setting, one may think that firm 1 can maximize its profit by offering only one bundle, characterized by a quantity $Q$ and an associated

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\(^1\)For any closed interval $X \subset \mathbb{R}$ and point $x \in \mathbb{R}$, $\text{Proj}_X(x)$ denotes the projection of $x$ on $X$. For example,
\[ \text{Proj}_{[0,1]}(x) \equiv \max\{\min\{x, 1\}, 0\}. \]
bundle price $T$ (so-called quantity forcing). After all, there can only be one purchase quantity, among all quantities offer by firm 1, that the buyer would pick in equilibrium. (Recall that we do not consider mixed strategies.) Surprisingly, this thought is wrong! Indeed, we will see in following sections that firm 1’s profit maximizing tariff must involve a continuum of quantities for the buyer to choose, although firm 1 understands that all but one would never be chosen by the buyer in equilibrium.

This subsection illustrates why quantity forcing is not optimal from firm 1’s point of view. In particular, even offering only two bundles allows firm 1 to earn more than what it would have earned if it offers only one bundle. (This subsection is mainly for providing intuitions. The analyses in the following sections do not rely on this subsection.)

For expositional simplicity, in this subsection we assume that firm 2 does not have capacity constraint (or $k \geq q^e$). Suppose for a moment that firm 1 can only offer one bundle (i.e. can only use quantity forcing). In equilibrium, firm 1 would choose the bundle quantity and the bundle price in order to maximize its profit, subject to the constraints that firm 2 and the buyer would respond optimally after firm 1’s offering.

Fix any aforementioned “one-bundle equilibrium.” Let $Q^*$ and $T^*$ denote the optimal bundle quantity and bundle price offered by firm 1 respectively. In equilibrium, the buyer must take the bundle. The resulting firm 1’s profit is

$$\Pi_1^* = T^* - cQ^*.$$  \hfill (1)

Let $p^*$ denote firm 2’s equilibrium price, so that firm 2’s output is $D(p^*) - Q^*$, and its profit is

$$\Pi_2^* = (p^* - c)(D(p^*) - Q^*).$$  \hfill (2)

Let $x^*$ denote the highest firm 2’s price at which the buyer prefers not to take firm 1’s bundle, i.e., $x^* = \sup \{p : V(0,p) \geq V(Q^*,p) - T^*\}$. It follows that $0 < x^* < p^*$ and

$$V(0,x^*) = V(Q^*,x^*) - T^*.$$  \hfill (3)
Since firm 2 must not have incentive to deviate to any other price,

\[ \Pi_2^* = \max_{p > x^*} \{(p - c)(D(p) - Q^*)\}, \]

\[ \Pi_2^* \geq (p - c)D(p) \quad \forall p \leq x^*. \]

Equivalently,

\[ \pi'(p^*) = Q^*, \quad (4) \]

\[ \Pi_2^* \geq (x^* - c)D(x^*). \quad (5) \]

(In fact, firm 1’s problem can be formulated as maximizing (1) by choosing \(Q^*, T^*, p^*, x^*, \Pi^*\) subject to constraints (2), (3), (4), and (5), where (5) must be binding. But in the following arguments, we will only need the fact that (1), (2), (3), (4), and (5) are necessary conditions of equilibrium.)

We now construct two bundles, by offering which firm 1’s profit would be strictly higher than \(\Pi_1^*\). Let \(Q_2 = Q^*\). Pick any \(Q_1 \in (0, Q^*)\). Let \(T_1(\epsilon) = V(Q_1, x^*) - V(0, x^*) - \epsilon\) and \(T_2(\epsilon) = T^* + \epsilon\). Let \(x_0(\epsilon)\) and \(x_1(\epsilon)\) be defined by

\[ V(0, x_0(\epsilon)) = V(Q_1, x_0(\epsilon)) - T_1(\epsilon), \]

\[ V(Q_1, x_1(\epsilon)) - T_1(\epsilon) = V(Q_2, x_1(\epsilon)) - T_2(\epsilon). \]

Note that (3) implies \(x_0(0) = x_1(0) = x^*\). For small \(\epsilon > 0\), we have \(0 < x_0(\epsilon) < x^* < x_1(\epsilon) < p^*\) so that, if firm 1 offers two bundles characterized by \((Q_1, T_1(\epsilon))\) and \((Q_2, T_2(\epsilon))\), the buyer would pick the large bundle \((Q_2, T_2(\epsilon))\) when firm 2’s price is above \(x_1(\epsilon)\), and would pick the small bundle \((Q_1, T_1(\epsilon))\) when firm 2’s price is between \(x_0(\epsilon)\) and \(x_1(\epsilon)\), and would not pick any bundle from firm 1 when firm 2’s price is below \(x_0(\epsilon)\). Also, when \(\epsilon > 0\) is small, (5) implies

\[ [x_0(\epsilon) - c]D(x_0(\epsilon)) < \Pi_2^*, \quad (6) \]

\[ [x_1(\epsilon) - c][D(x_1(\epsilon)) - Q_1] < \Pi_2^*. \quad (7) \]
(The left-hand side of (6) is weakly below $\Pi_2^*$ when $\epsilon = 0$, and is strictly so when $\epsilon > 0$. The left-hand side of (7) is strictly lower than $\Pi_2^*$ when $\epsilon = 0$, and this strict inequality maintains when $\epsilon$ is close to 0.)

We see that, if firm 1 offers the two bundles characterized by $(Q_1, T_1(\epsilon))$ and $(Q_2, T_2(\epsilon))$ with small $\epsilon > 0$, then firm 2 would still charge $p^*$ and the buyer would still buy the bundle with quantity $Q_2 = Q^*$. But now the price of this bundle is $T_2(\epsilon) > T^*$, so that the resulting firm 1’s profit would be strictly higher than $\Pi_1^*$.

4 Equilibrium characterization

4.1 Dominant firm’s mechanism design problem

If we were going to use the standard method, backward induction, to solve SPE, we would need to solve the three players’ sequential rationality conditions, i.e., (BSR), (RSR), and (DSR). However, those conditions are hard to solve directly. In the following, we shall transform our original problem into a mechanism design problem, which allows us to determine SPE outcomes.

The formulation of such a mechanism design problem requires definitions of conditional payoffs for the buyer and the rival firm. The buyer’s conditional payoff function we need is the $V$ function we defined in Subsection 3.1. Here we also let $\pi(Q, \cdot)$ denote firm 2’s profit function conditional on the buyer’s purchase from firm 1 being $Q$, i.e.,

$$
\pi(Q, p) \equiv (p - c) \text{Proj}_{[0,k]}(D(p) - Q) = \text{Proj}_{[0,(p-c)k]}[(\pi(p) - (p - c)Q).$$

Now we are ready to formulate a mechanism design problem that allows us to determine SPE outcomes. Observe that, every tariff $\tau \in T$ firm 1 might offer induces a continuation subgame in which firm 2 and the buyer sequentially choose their actions. When choosing $\tau$, firm 1 understands that firm 2 and the buyer would play a SPE of the continuation subgame. Given $\tau$, the buyer would optimally choose
some purchase \( Q(p) \geq 0 \) from firm 1, contingent on any possible price \( p \in \mathcal{P} \) chosen by firm 2. The payment for this purchase is \( \tau(Q(p)) \equiv T(p) \). On the other hand, given that the buyer’s optimal purchase from firm 1 is \( Q(p) \), and hence the optimal purchase from firm 2 is \( \text{Proj}_{[0,k]}(D(p) - Q(p)) \), firm 2 would optimally choose some price \( \bar{p} \in \mathcal{P} \). In the spirit of revelation principle (imagining firm 1 asks the buyer to report firm 2’s price), solving SPE for the whole game is equivalent to solving the following constrained optimization problem (OP1).

\[
\text{(OP1): Maximize } T(\bar{p}) - cQ(\bar{p})
\]

over quantity function \( Q : \mathcal{P} \to \mathbb{R}_+ \), payment function \( T : \mathcal{P} \to \mathbb{R} \) (both contingent on buyer’s report of firm 2’s price), and recommendation of firm 2’s price \( \bar{p} \in \mathcal{P} \), subject to

\[
\text{(BIC): } V(Q(p),p) - T(p) \geq V(Q(\bar{p}),\bar{p}) - T(\bar{p}) \quad \forall p, \bar{p} \in \mathcal{P}
\]

\[
\text{(BIR): } V(Q(p),p) - T(p) \geq V(0,p) \quad \forall p \in \mathcal{P}
\]

\[
\text{(RIC): } \pi(Q,p) \geq \pi(Q(p),p) \quad \forall p \in \mathcal{P}.
\]

The equivalence is formalized by the following theorem, which we prove in the Appendix.

**Theorem 1.** Take any \( Q^* : \mathcal{P} \to \mathbb{R}_+ \), \( T^* : \mathcal{P} \to \mathbb{R} \), and \( \bar{p}^* \in \mathcal{P} \). \((Q^*(\cdot),T^*(\cdot),\bar{p}^*)\) is a solution of (OP1) if and only if there is a SPE \((\tau^*,p^*,q^*)\) such that

\[
Q^*(p) = q^*_1(\tau^*,p) \quad \forall p \in \mathcal{P}, \quad (8)
\]

\[
\text{Proj}_{[0,k]}(D(p) - Q^*(p)) = q^*_2(\tau^*,p) \quad \forall p \in \mathcal{P}, \quad (9)
\]

\[
T^*(p) = \tau^*(Q^*(p)) \quad \forall p \in \mathcal{P}, \quad (10)
\]

\[
\bar{p}^* = p^*(\tau^*). \quad (11)
\]
4.2 Characterizing BIC and BIR constraints

This subsection characterizes the BIC and BIR constraints. The proofs of the three lemmas in this subsection are in the Appendix.

We first tackle the BIC constraint. Recall that $V(Q, p) \equiv \max_{q \in [0,k]} \{u(Q + q) - pq\}$.

We have seen in Subsection 3.1 that the sorting condition required to simplify (BIC) is satisfied in the weak form.

**Lemma 1.** Take any $Q : \mathcal{P} \rightarrow \mathbb{R}_+$ and $T : \mathcal{P} \rightarrow \mathbb{R}$. $Q(\cdot), T(\cdot)$ satisfy (BIC) if and only if

Weak monotonicity of $Q(\cdot)$ (BIC-1): for any $p_1, p_2 \in \mathcal{P}$ with $p_1 \leq p_2$, either $Q(p_1) \leq Q(p_2)$ or $D(p_1) \leq Q(p_2)$ or $Q(p_1) + k \leq D(p_2)$;\(^2\) and

Envelope formula (BIC-2): for any $p \in \mathcal{P}$, $T(p) - T(c) = V(Q(p), p) - V(Q(c), c) - \int_c^p Vp(Q(\tilde{p}), \tilde{p})d\tilde{p}$.

From Lemma 1, a quantity function $Q(\cdot)$ can be paired with a payment function $T(\cdot)$ to satisfy (BIC) if and only if it satisfies (BIC-1). (BIC-1) is a weakened version of the standard monotonicity condition. It is weakened because the increasing differences property of $V$ is not always strict. It says that $Q(\cdot)$ can be decreasing only in a particular way. Namely, whenever $p_1 < p_2$ and $Q(p_1) > Q(p_2)$, the rectangle $[Q(p_2), Q(p_1)] \times [p_1, p_2]$ does not intersect the region in which $D(p) - k < Q < D(p)$.

It in particular implies the following lemma.

**Lemma 2.** Take any $Q : \mathcal{P} \rightarrow \mathbb{R}_+$ that satisfies (BIC-1).

(a) $Q(p)$ is nondecreasing in $p$ on $\{p \in \mathcal{P} : 0 \leq D(p) - Q(p) \leq k\}$.

(b) $\text{Proj}_{[0,k]}(D(p) - Q(p))$ is nonincreasing in $p$ on $\mathcal{P}$.

With (BIC-2), it can be shown that $V(Q(c), c) - T(c) - V(0, c)$ is nondecreasing in $p$ and we can simplify (BIR) as follows.

**Lemma 3.** Take any $Q : \mathcal{P} \rightarrow \mathbb{R}_+$ and $T : \mathcal{P} \rightarrow \mathbb{R}$ that satisfy (BIC-2). $Q(\cdot), T(\cdot)$ satisfy (BIR) if and only if $V(Q(c), c) - T(c) \geq V(0, c)$.

\(^2\)The proof reveals that (BIC-1) is equivalent to the condition that (41) and (42) hold for all $p_1, p_2 \in \mathcal{P}$ with $p_1 \leq p_2$. In particular, given any $Q(\cdot)$ that satisfies the last condition, it is easy to construct $T(\cdot)$ such that $Q(\cdot), T(\cdot)$ satisfy (BIC-2), then the sufficiency part reveals that $Q(\cdot), T(\cdot)$ satisfy (BIC), then the necessity part reveals that $Q(\cdot)$ satisfies (BIC-1).
4.3 Equilibrium

By virtue of Theorem 1, we reduce our job of finding SPE to finding solution of (OP1). We also call any solution of (OP1) an equilibrium.

In any equilibrium, (BIR) must be binding at some \( p \in \mathcal{P} \), otherwise firm 1 can shift up \( T(\cdot) \) to raise profit. Moreover, Lemma 3 implies that, for any \( Q(\cdot), T(\cdot) \) that satisfy (BIC-2), if (BIR) is binding at some \( p \in \mathcal{P} \), it must be binding at \( p = c \) and hence

\[
\text{(BIR')}: \quad T(c) = V(Q(c), c) - V(0, c).
\]

Using (BIC-2) and (BIR') to eliminate \( T(\cdot) \), we obtain, for all \( p \in \mathcal{P} \),

\[
T(p) = V(Q(p), p) - V(0, c) - \int_c^p V_p(Q(\bar{p}), \bar{p})d\bar{p} \\
= V(Q(p), c) - V(0, c) + V(Q(p), p) - V(Q(p), c) - \int_c^p V_p(Q(\bar{p}), \bar{p})d\bar{p} \\
= \int_0^{Q(p)} V_Q(Q, c)dQ + \int_c^p [V_p(Q(p), \bar{p}) - V_p(Q(\bar{p}), \bar{p})]d\bar{p}. \tag{12}
\]

Firm 1’s profit, denoted as \( \Pi_1 \), can be written as

\[
\Pi_1 = \int_0^{Q(\bar{p})} [V_Q(Q, c) - c]dQ + \int_c^p [V_p(Q(\bar{p}), p) - V_p(Q(p), p)]dp \\
= \int_0^{Q(p)} [\text{Proj}_{[a'(Q+c), a'(Q)]}(c) - c]dQ \\
+ \int_c^p [\text{Proj}_{[0,k]}(D(p) - Q(p)) - \text{Proj}_{[0,k]}(D(p) - Q(\bar{p}))]dp. \tag{13}
\]

Figure 1 shows the area of \( \Pi_1 \) given by (13) for an example of \( Q(\cdot) \) and \( \bar{p} \). In Figure 1 area A and area B correspond to the first and the second integral in (13).
Figure 1: Firm 1’s profit $\Pi_1$ contingent on $Q(\cdot)$ and $\bar{p}$ respectively. Let $\Pi_2$ denote firm 2’s profit. (OP1) can be reduced to

\[(OP2): \text{Maximize } (13)\]

over quantity function $Q : \mathcal{P} \to \mathbb{R}^+$, firm 2’s price $\bar{p} \in \mathcal{P}$, and firm 2’s profit $\Pi_2 \geq 0$, subject to (BIC-1) and

\[(RIC): \Pi_2 \geq (p - c) \text{Proj}_{[0,k]}(D(p) - Q(p)) \quad \forall p \in \mathcal{P},\]

$$\Pi_2 = (\bar{p} - c) \text{Proj}_{[0,k]}(D(\bar{p}) - Q(\bar{p})). \quad (14)$$

The equivalence between (OP1) and (OP2) is formalized as follows.

**Lemma 4.** Take any $Q : \mathcal{P} \to \mathbb{R}^+$, $\bar{p} \in \mathcal{P}$. Let $T : \mathcal{P} \to \mathbb{R}^+$ and $\Pi_2 \geq 0$ be given by (12) and (14) respectively. Then, $(Q(\cdot), \bar{p}, \Pi_2)$ is a solution of (OP2) if and only if $(Q(\cdot), T(\cdot), \bar{p})$ is a solution of (OP1).

**Proof.** It follows from Lemma 1, Lemma 3, and the previous arguments. \(\blacksquare\)

Our strategy of solving (OP2) is as follows. We decompose (OP2) into two stages: in the first stage $\Pi_2$ is chosen; in the second stage, $Q(\cdot)$ and $\bar{p}$ are chosen contingent
on $\Pi_2$. Lemma 5 below solves the second stage for any feasible $\Pi_2 > 0$, and Lemma 6 below solves the first stage to pin down $\Pi_2$.

In order to solve the second stage given $\Pi_2$, let us take a closer look at the conditional profit $\pi(Q,p)$ of firm 2. From Assumption 2, $\pi(Q,\cdot)$ is concave on $\{p : \pi(Q,p) > 0\}$ for every $Q \geq 0$. Figure 2 shows the level curves of $\pi(p) - (p - c)Q$ (which is equal to $(p - c)(D(p) - Q)$) and Figure 3 shows firm 2’s iso-profit curves (or the level curves of $\pi(Q,p)$). If firm 2 does not have capacity constraint (or $k \geq q^e$), firm 2’s iso-profit curves are the same as the level curves of $\pi(p) - (p - c)Q$, whose slopes are $(p - c)/\pi'(p) - Q)$, as shown in Figure 2. In cases in which firm 2 has capacity constraint, the iso-profit curves are flat when $Q < D(p) - k$, and coincide the level curves of $\pi(p) - (p - c)Q$ otherwise, as shown in Figure 3.

The largest feasible $\Pi_2$ is $\pi(\max\{\rho^m, u'(k)\})$. Looking at Figures 2, 3, and 1, it can be seen that, given a $\Pi_2 \in (0, \pi(\max\{\rho^m, u'(k)\})$ and hence a firm 2’s iso-profit curve, in order to maximize $\Pi_1$ subject to (BIC-1) and (RIC), the point $(Q(\bar{p}), \bar{p})$ must be chosen to be the most rightward point on the firm 2’s iso-profit curve, and the function $Q(\cdot)$ must lie on the iso-profit curve whenever it is above the curve.
Figure 3: Firm 2’s iso-profit curves

$Q = D(p) - k$. Lemma 5 below formalizes the above claims. Figures 4 and 5 graphically show the partial solutions contingent on $\Pi_2$ for two examples. The former example assumes that firm 2 does not have capacity constraint or the capacity $k$ is large enough, while the latter example assumes $k$ is small.

**Lemma 5.** Contingent on any $\Pi_2 \in (0, \pi(\max\{p^m, u'(k)\}))$, there exist solutions $(Q(\cdot), \bar{p})$ of $(OP2)$. Any such contingent solution satisfies

$$\Pi_2 = (p - c)(D(p) - Q(p)) \quad \forall p \in [x_0, \bar{p}],$$

where $\bar{p}$ is the unique solution of

$$\max\{D(\bar{p}) - k, \pi'(\bar{p})\} = D(\bar{p}) - \frac{\Pi_2}{\bar{p} - c} \equiv \bar{Q},$$

and $x_0$ is the unique solution below $\bar{p}$ of

$$\max\{D(x_0) - k, 0\} = D(x_0) - \frac{\Pi_2}{x_0 - c} \equiv Q_0 \leq \bar{Q}.$$
Figure 4: Solution contingent on $\Pi_2$ when $k$ is large

Figure 5: Solution contingent on $\Pi_2$ when $k$ is small
One solution for $Q(\cdot)$ can be constructed as
\[
Q(p) = \begin{cases} 
D(p) - \frac{\Pi_2}{p-c} & \text{if } p \in [x_0, \bar{p}] \\
Q_0 & \text{if } c \leq p < x_0 \\
\bar{Q} & \text{if } \bar{p} < p \leq u'(0)
\end{cases},
\]
and the associated contingent maximum is
\[
\Pi_1 = \int_0^{Q_0} u'(Q + k)dQ + x_0 \cdot (\bar{Q} - Q_0) + \int_{x_0}^{\bar{p}} (\bar{Q} - Q(p))dp - c\bar{Q}.
\]

Proof. Fix any $\Pi_2 \in (0, \pi(\max\{p^m, u'(k)\}))$ and hence a firm 2’s iso-profit curve in the $Q$-$p$ space (see Figure 3). Note that $\Pi_2 > 0$ implies $\bar{p} > c$ and $D(\bar{p}) > Q(\bar{p})$. Suppose, by way of contradiction, that $(Q(\bar{p}), \bar{p})$ is not the most rightward point on the iso-profit curve. Consider the case where $(Q(\bar{p}), \bar{p})$ lies on the strictly decreasing portion of the iso-profit curve (which implies $D(\bar{p}) - Q(\bar{p}) < k$). Then, to satisfy (RIC), for small $\varepsilon > 0$, we have $Q(\bar{p} - \varepsilon) > Q(\bar{p})$. But then, from Lemma 2(a), (BIC-1) is violated. Now consider the case where $(Q(\bar{p}), \bar{p})$ lies on the nondecreasing portion of the iso-profit curve. Then, $\Pi_1$ can be raised by increasing both $Q(\bar{p})$ and $\bar{p}$ along the iso-profit curve (see Figures 3 and 1).

Therefore, $(Q(\bar{p}), \bar{p})$ must be the most rightward point on the iso-profit curve, and therefore satisfy (16) with $\bar{Q} = Q(\bar{p})$. Let $x_0$ and $Q_0$ be defined as in the lemma (i.e., defined by $x_0 \leq \bar{p}$ and the first two equalities of (17)). Then $Q_0 \leq \bar{Q}$ hold (see Figures 4 and 5). In order to maximize $\Pi_1$ subject to (BIC-1) and (RIC), $Q(\cdot)$ on $[x_0, \bar{p}]$ must lie on the iso-profit curve, otherwise $\Pi_1$ can be raised by decreasing $Q(\cdot)$ on $[x_0, \bar{p}]$. Thus, we have (15), and $Q(\cdot)$ on $[x_0, \bar{p}]$ is given by the first line of (18). From (13), the associated maximum $\Pi_1$ is given by (19). It is easy to verify that the $Q(\cdot)$ in (18) attains this maximum and satisfies constraints (BIC-1) and (RIC). In particular, $Q(\cdot)$ on $[x_0, \bar{p}]$ is increasing because any level curve of $(p-c)(D(p) - Q)$ is (horizontally) single-peaked (recall Figure 3).

To solve (OP2), it remains to pin down $\Pi_2$, which should be chosen to make the $\Pi_1$ area in Figure 4 or Figure 5 as large as possible. The corresponding first-order
condition can be simplified as (20) below. Once a solution \((Q(\cdot), \bar{p}, \Pi_2)\) of (OP2) is obtained, we can use the equivalence between (OP1) and (OP2) established in Lemma 4 to obtain a solution \((Q(\cdot), T(\cdot), \bar{p})\) of (OP1).

**Lemma 6.** (OP2) has at least one solution. For any such solution \((Q(\cdot), \bar{p}, \Pi_2)\), the \((\bar{p}, \Pi_2)\) is part of a solution \((\Pi_2, \bar{p}, x_0, \bar{Q}, Q_0)\) of

\[
\begin{align*}
\bar{p} - c &= e \cdot (x_0 - c) > 0, \\
\Pi_2 &= (\bar{p} - c)(D(\bar{p}) - \bar{Q}) = (x_0 - c)(D(x_0) - Q_0), \\
\pi'(\bar{p}) &= \bar{Q}, \\
Q_0 &= \max\{D(x_0) - k, 0\}.
\end{align*}
\]

A solution of \(Q(\cdot)\) is given by (18). For the corresponding solution \((Q(\cdot), T(\cdot), \bar{p})\) of (OP1), the \(T(\cdot)\) satisfies

\[
T(p) = u(Q_0 + k) - u(k) + \int_{x_0}^{p} \bar{p}dQ(\bar{p}) \quad \forall p \in [x_0, \bar{p}].
\]

**Proof.** Lemma 5 has characterized the optimal \((Q(\cdot), \bar{p})\) and maximum \(\Pi_1\) contingent on any \(\Pi_2 \in (0, \pi(\max\{p^m, u'(k)\}))\]. Clearly, the maximum \(\Pi_1\) contingent \(\Pi_2 = 0\) is equal to the limiting contingent maximum \(\Pi_1\) as \(\Pi_2 \downarrow 0\). After reducing the second stage (where \((Q(\cdot), \bar{p})\) is chosen), (OP2) has only one choice variable, \(\Pi_2\), and the reduced objective function is continuous in \(\Pi_2\) on \([0, \pi(\max\{p^m, u'(k)\})]\]. Thus, (OP2) has at least one solution.

If \(\Pi_2 = 0\), then the contingent maximum can be raised by increasing \(\Pi_2\). (Contemplate Figure 4.) Thus, at any optimum, \(\Pi_2 > 0\). On the other hand, if \(\Pi_2 = \pi(\max\{p^m, u'(k)\})\) or is so large that the contingent solution exhibits \(D(\bar{p}) - \bar{Q} = k\), then the contingent maximum can be raised by decreasing \(\Pi_2\). (Contemplate Figure 5 again.) Thus, at any optimum, \(\Pi_2 < \pi(\max\{p^m, u'(k)\})\) and \(D(\bar{p}) - \bar{Q} < k\). Then, it follows from (15) – (17) that \(c < x_0 < \bar{p}\) and \(Q_0 < \bar{Q}\) and (21) – (23) holds.
(12) can be rewritten as (24) and (19) can be rewritten as

\[ \Pi_1 = \int_0^{Q_0} u'(Q + k) dQ + \bar{p} \bar{Q} - x_0 Q_0 - \int_{x_0}^{\bar{p}} Q(p) dp - c \bar{Q} \]

\[ = \int_{x_0}^{\infty} \max\{D(p) - k, 0\} dp + (\bar{p} - c) \bar{Q} - \int_{x_0}^{\bar{p}} D(p) - \frac{\Pi_2}{\bar{p} - c} \] dp

\[ = \int_{x_0}^{\infty} \max\{D(p) - k, 0\} dp + TS - \int_{x_0}^{\infty} D(p) dp + \left( \ln \frac{\bar{p} - c}{x_0 - c} - 1 \right) \Pi_2 \]

\[ = TS - \int_{x_0}^{\infty} \min\{D(p), k\} dp + \left( \ln \frac{\bar{p} - c}{x_0 - c} - 1 \right) \Pi_2, \quad (25) \]

where

\[ TS \equiv u(D(\bar{p})) - cD(\bar{p}) = \int_{\bar{p}}^{\infty} D(p) dp + (\bar{p} - c)D(\bar{p}) \quad (26) \]

denote the total surplus.

The partial derivatives of (25) are

\[ \frac{\partial \Pi_1}{\partial \bar{p}} = (\bar{p} - c) D'(\bar{p}) + \frac{\Pi_2}{\bar{p} - c}, \]

\[ \frac{\partial \Pi_1}{\partial x_0} = \min\{D(x_0), k\} - \frac{\Pi_2}{x_0 - c}, \]

\[ \frac{\partial \Pi_1}{\partial \Pi_2} = \ln \frac{\bar{p} - c}{x_0 - c} - 1. \]

Note that (21) – (23) (or (16) – (17)) imply that \( \frac{\partial \Pi_1}{\partial \bar{p}} = \frac{\partial \Pi_1}{\partial x_0} = 0 \). Therefore, the total derivative of (25) with respect to \( \Pi_2 \) is

\[ \frac{d \Pi_1}{d \Pi_2} = \ln \frac{\bar{p} - c}{x_0 - c} - 1. \quad (27) \]

Therefore, the first-order condition \( \frac{d \Pi_1}{d \Pi_2} = 0 \) implies (20).\(^4\)

Finally, we can use the equivalence between solving SPE of the original game and solving (OP1) established in Theorem 1 to characterize the equilibrium outcome of

\(^4\)One can show that, if \( Q_0 > 0 \), the local second-order necessary condition is \( k + c(\bar{p} - c) \pi''(\bar{p}) \leq 0. \)
the original game. Figure 6 illustrates the features of an equilibrium tariff offered firm 1.

**Theorem 2.** There exists at least one equilibrium. In any equilibrium, \((\Pi_2, \bar{p}, x_0, \bar{Q}, Q_0)\) solves (20) – (23). Firm 2 chooses \(p = \bar{p}\), and the buyer purchases \(\bar{Q}\) units and \(D(\bar{p}) - \bar{Q} < k\) units from firm 1 and firm 2 respectively. An equilibrium tariff \(\tau(\cdot)\) offered by firm 1 can be constructed as

\[
\tau(Q) = \begin{cases} 
  u(Q_0 + k) - u(k) + \int_{Q_0}^{Q} x(\tilde{Q})d\tilde{Q} & \text{if } Q \in [Q_0, \bar{Q}] \\
  0 & \text{if } Q = 0 \\
  \infty & \text{otherwise}
\end{cases},
\]

where \(x(\cdot)\) on \([Q_0, \bar{Q}]\) is the inverse of \(Q(\cdot)\) on \([x_0, \bar{p}]\), and \(Q(\cdot)\) on \([x_0, \bar{p}]\) is given by \(D(p) - \frac{\Pi_2}{p - c}\).

**Proof.** The results are from Lemma 6 and Theorem 1. The last claim of the theorem is essentially from (24). Suppose that firm 1’s tariff \(\tau\) is given by (28). Then, firm 2’s profit would be \(\Pi_2\) if it chooses any \(p \in [x_0, \bar{p}]\). One can see from Figure 5 that, firm 2’s profit would be lower than \(\Pi_2\) if it chooses any \(p > \bar{p}\) (so that the buyer would still purchase \(\bar{Q}\) units from firm 1) or any \(p < x_0\) (so that the buyer would purchase \(Q_0\) units from firm 1).

Strictly speaking, equilibrium is never unique because \(Q(p), T(p)\) for \(p \notin [x_0, \bar{p}]\) and hence \(\tau(Q)\) for \(Q \notin [Q_0, \bar{Q}]\) are not unique. In Theorem 2, equilibrium \(\tau(Q)\) for \(Q \notin [Q_0, \bar{Q}] \cup \{0\}\) can be anything large enough. However, those values do not affect the allocation. We say the equilibrium is essentially unique if the equilibrium objects \(\Pi_1, \Pi_2, \bar{p}, x_0, \bar{Q}, Q_0\) are unique. The following proposition provides a simple sufficient condition for the uniqueness.

**Proposition 1.** The equilibrium is essentially unique if one of the following two equivalent conditions are satisfied:

\[
u'(q) - c \text{ is strictly log-concave in } q \text{ on } [0, q^*];
\]

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\[ -(p - c)D'(p) \text{ is strictly increasing in } p \text{ on } \mathcal{P}. \]  

(30)

**Proof.** From Theorem 2, the equilibrium is essentially unique if and only if the solution of the system (20) – (23) is unique. Using (22) and (23) to eliminate \( \bar{Q} \) and \( Q_0 \), the second equality of (21) can be written as

\[ -(\bar{p} - c)^2 D'(\bar{p}) = (x_0 - c) \min \{ D(x_0, k) \}. \]  

(31)

Using (20) to eliminate \( x_0 \) and dividing through by \( \bar{p} - c \), we have

\[ -(\bar{p} - c)D'(\bar{p}) = \frac{1}{e} \min \left\{ D \left( c + \frac{\bar{p} - c}{e} \right), k \right\}. \]  

(32)

(32) can be solved for \( \bar{p} \). Therefore, the equilibrium is essentially unique if and only if (32) has at most one solution.

It is easy to see that (29) and (30) are equivalent. Under condition (30), the left-hand side of (32) is strictly increasing in \( \bar{p} \), and the right-hand side is nonincreasing.
Remark 1. Consider the following conditions.
(i) $u'(q)$ is strictly log-concave in $q$ on $[0, q^e]$.
(ii) $-pD'(p)$ is strictly increasing in $p$ on $\mathcal{P}$.
(iii) $u'(\cdot)$ is concave on $[0, q^e]$.
(iv) $D(\cdot)$ is concave on $\mathcal{P}$.

It is easy to show that, given Assumption 1,

\[ \text{Assumption 2} \iff (29) \iff (30) \iff (i) \iff (ii) \iff (iii) \iff (iv). \]

Remark 2. Note that $-(p-c)D'(p) = D(p) - \pi'(p)$. Thus, a graphical interpretation of condition (29) is that, for any $k$, the curve $\pi'(p) = Q$ (or $D(p) + (p-c)D'(p) = Q$) and the curve $D(p) - k = Q$ cross at most once (as drawn in Figure 3 and Figure 5).

5 Implications of the equilibrium

5.1 Other properties of the equilibrium

In equilibrium, the total output is $D(\bar{p})$ and firm 2’s output is $D(\bar{p}) - \bar{Q} < k$. Let $TS$ denote the total surplus given by (26) and let

\[ BS \equiv u(D(\bar{p})) - \bar{p}(D(\bar{p}) - \bar{Q}) - \tau(\bar{Q}) = TS - \Pi_1 - \Pi_2 \tag{33} \]

denote the buyer’s surplus.

(20) and (25) give a simple formula to compute $\Pi_1$, i.e.,

\[ \Pi_1 = TS - \int_{x_0}^{\infty} \min\{D(p), k\}dp. \tag{34} \]

But perhaps it is more interesting to state it in the equivalent form in the following proposition, which says that what firm 2 and the buyer jointly earn in equilibrium
is equal to their joint outside option under the counterfactual situation that firm 2’s unit cost was raised to $x_0$.

**Proposition 2.** In any equilibrium,

$$\Pi_2 + BS = \int_{x_0}^{\infty} \min \{D(p), k\} dp$$

$$= u(\min \{D(x_0), k\}) - x_0 \cdot \min \{D(x_0), k\}$$

$$= u(D(x_0) - Q_0) - x_0 \cdot (D(x_0) - Q_0).$$

**Proof.** It follows from (33) and (34). ■

**Proposition 3.** In any equilibrium, firm 1’s tariff $\tau$ is strictly increasing and strictly convex on $[Q_0, \bar{Q}]$.

**Proof.** From (24) and $T(p) = \tau(Q(p))$, the first line of (28) must hold for any equilibrium $\tau$. Thus, $\tau'(\cdot) = x(\cdot)$ on $[Q_0, \bar{Q}]$. Moreover, $x(\cdot)$ is positive and strictly increasing on $[Q_0, \bar{Q}]$. ■

**Proposition 4.** There is a unique $\hat{k}$ such that $Q_0 = 0$ in equilibrium if and only if $k \geq \hat{k}$. This $\hat{k}$ satisfies $D(p^m) < \hat{k} < q^e$. The set of equilibria is independent of $k$ on $[\hat{k}, \infty]$. The following comparative statics results hold no matter how we do selection from possibly multiple equilibria for various $k$.

1. The equilibrium objects $\Pi_2, \bar{p}, x_0, \bar{p} - x_0$ (and also $D(\bar{p}) - \bar{Q}$ if we assume condition (29)) are increasing in $k$ on $(0, \hat{k}]$.

2. The equilibrium objects $\Pi_1, \bar{Q}, Q_0, TS$ are decreasing in $k$ on $(0, \hat{k}]$.

3. The equilibrium objects $\Pi_2 + BS$ and $BS$ are increasing in $k$ when $k$ is small, and are decreasing in $k$ when $k$ is close to but below $\hat{k}$.

**Proof.** Let $\hat{x}_0$ be the minimum equilibrium $x_0$ when $k = \infty$. Define $\hat{k} \equiv D(\hat{x}_0)$. From Theorem 2, $\hat{k}$ satisfies the first three claims (see Figures 4 and 5). The rest of the proof considers $k \in (0, \hat{k}]$.

Following the proof of Lemma 6, we regard $\Pi_1, \bar{p}, Q, x_0, Q_0, x(\cdot), BS, TS$ as functions of $\Pi_2$. Here we also regard them as functions of $k$. In particular, we write $\Pi_1(\Pi_2; k)$. 24
Fix \( \Pi_2 \) and let \( k \) increase on \((0, \hat{k}]\). Note that \( Q_0 > 0 \) before the increase. The \( \bar{p} \) and \( \bar{Q} \) determined by the first equality of (21) and (22) do not change. The \( x_0, Q_0, \) and \( \Pi_1 \) determined by the second equality of (21), (23), and (25) decrease (see Figure 5).

In equilibrium, \( \Pi_1 = \max_{\Pi_2} \Pi_1(\Pi_2; k) \) decreases because \( \Pi_1(\cdot; k) \) shifts down. From (27), we see that \( \partial \Pi_1(\Pi_2; k) / \partial \Pi_2 \) increases. In other words, \( \Pi_1(\cdot; k) \) satisfies strict increasing differences. Therefore, the \( \Pi_2 \) that maximizes \( \Pi_1 \) must increase. Then, from the first equality of (21) and (22), \( \bar{p} \) increases and \( \bar{Q} \) decreases (see Figure 5 again). Then \( TS \) decreases. From (20), \( x_0 \) increases. From (23), \( Q_0 \) decreases. Also, \( \bar{p} - x_0 \) increases because (20) can be written as \( \bar{p} - x_0 = (c - 1)(x_0 - c) \). To see the result for \( D(\bar{p}) - \bar{Q} \), recall Remark 2. It completes the proof of part (a) and (b).

To see the first half of part (c), note that both \( \Pi_2 \) and \( BS \) are positive and tend to zero as \( k \to 0 \).

To see the second half of part (c), first note that, from (23), we have \( \min\{D(x_0), k\} = k \) when \( k \leq \hat{k} \). From Proposition 2, \( \Pi_2 + BS = u(k) - x_0 k \) whenever \( k \leq \hat{k} \). Hence,

\[
\left. \frac{d(\Pi_2 + BS)}{dk} \right|_{k \neq \hat{k}} = u'(\hat{k}) - x_0 - \hat{k} \cdot \left. \frac{dx_0}{dk} \right|_{k \neq \hat{k}} < 0.
\]

The last inequality follows from \( u'(\hat{k}) - x_0 \leq u'(\hat{k}) - \hat{x}_0 = 0 \) and \( \left. \frac{dx_0}{dk} \right|_{k \neq \hat{k}} > 0 \). Therefore, \( \Pi_2 + BS \) is decreasing in \( k \) when \( k \) is close to but below \( \hat{k} \). This is true for \( BS \) as well since \( \Pi_2 \) is increasing in \( k \).

\[\square\]

5.2 Comparing with linear pricing

Consider a game that is similar to the one we presented in Section 2, except that firm 1 can only offer a unit price (linear pricing, or LP for short). First, firm 1 offers a unit price \( p_1 \in \mathbb{R}_+ \). Second, after observing \( p_1 \), firm 2 offers a unit price \( p_2 \in \mathbb{R}_+ \). Third, after observing \( p_1 \) and \( p_2 \), the buyer chooses the quantities \( q_1 \in \mathbb{R}_+ \) and \( q_2 \in [0, k] \) he buys from firm 1 and from firm 2. Call it the \( LP \) vs \( LP \) game, and

\[\text{If the equilibrium is essentially unique when } k = \infty, \text{ one can write } \min\{D(x_0), k\} = \min\{k, \hat{k}\} \text{ for all } k.\]

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the game presented in Section 2 the NLP vs LP game. We use superscript “LP” to denote various variables for the LP vs LP game.

**Proposition 5.** Consider the LP vs LP game. If \( k < q^e \), then there is a unique SPE outcome, in which both firms offer \( \bar{p}^{LP} \), where \( \pi'(\bar{p}^{LP}) = k \), and the buyer purchases \( q_1^{LP} = D(\bar{p}^{LP}) - k \) and \( q_2^{LP} = k \) units from firm 1 and firm 2 respectively. If \( k \geq q^e \), then there are multiple SPE outcome, in which the prevailing price can be any \( \bar{p}^{LP} \in [c, p^m] \) (either \( p_1 = p_2 = \bar{p}^{LP} \in [c, p^m] \) or \( p_1 \geq p^m = p_2 \)) and firm 1 makes no sales.

**Proof.** Straightforward and omitted. ■

**Proposition 6.** (a) \( \bar{p}^{LP}, \Pi_1^{LP} + \Pi_2^{LP}, \Pi_1^{LP} \) (and also \( q_1^{LP} \) if we assume condition (29)) are decreasing in \( k \) on \((0, q^e)\).

(b) \( TS^{LP}, q_2^{LP}, q_1^{LP} + q_2^{LP}, BS^{LP}, \Pi_2^{LP} + BS^{LP} \) are increasing in \( k \) on \((0, q^e)\).

(c) \( \Pi_2^{LP} \) is increasing in \( k \) when \( k \) is small, and is decreasing in \( k \) when \( k \) is close to but below \( q^e \).

**Proof.** Straightforward and omitted. ■

**Proposition 7.** Let \( k \in (0, q^e) \) and compare any SPE outcome of the NLP vs LP game with the unique SPE outcome of the LP vs LP game.

(a) \( D(\bar{p}) - \bar{Q} < q_2^{LP} = k, \Pi_1 > \Pi_1^{LP}, \Pi_2 + BS < \Pi_2^{LP} + BS^{LP} \).

(b) \( \bar{p} < \bar{p}^{LP}, D(\bar{p}) > D(\bar{p}^{LP}) = q_1^{LP} + q_2^{LP}, TS > TS^{LP}, \Pi_2 < \Pi_2^{LP} \) when \( k \) is small, and the opposite is true when \( k \in [\hat{k}, q^e) \). (\( \hat{k} \) is defined in Proposition 4.)

(c) \( BS < BS^{LP} \) when \( k \) is small or \( k \in [\hat{k}, q^e) \).

(d) \( \bar{Q} > q_1^{LP} = D(\bar{p}^{LP}) - k \) when \( k \) is small or close to \( q^e \).

**Proof.** Clearly, \( D(\bar{p}) - \bar{Q} < k \) and \( \Pi_1 > \Pi_1^{LP} \) hold. \( \Pi_2 + BS < \Pi_2^{LP} + BS^{LP} \) is from Proposition 2 and Proposition 5. It completes the proof of part (a).

Compare \( \pi'(\bar{p}) = \bar{Q} \) with \( \pi'(\bar{p}^{LP}) = k \) and note that \( \bar{Q} > k \) when \( k \) is small, and \( \bar{Q} < k \) when \( k \geq \hat{k} \). It proves the result for \( \bar{p}, \bar{p}^{LP} \). The results for \( D(\bar{p}), D(\bar{p}^{LP}) \) and \( TS, TS^{LP} \) follows.
Clearly, both $\Pi_2$ and $\Pi_{LP}^2$ tend to zero as $k \downarrow 0$. Since $\Pi_{LP}^2 = (\hat{p}^L - c)k$

$$\left. \frac{d \Pi_{LP}^2}{dk} \right|_{k \downarrow 0} = \hat{p}^L - c = p^m - c > 0.$$ 

Since $\Pi_2 = (\hat{p} - c)(D(\hat{p}) - \bar{Q})$ and both $\hat{p} - c$ and $D(\hat{p}) - \bar{Q}$ tend to zero as $k \downarrow 0$,

$$\left. \frac{d \Pi_2}{dk} \right|_{k \downarrow 0} = 0.$$ 

It proves the result for $\Pi_2, \Pi_{LP}^2$ when $k$ is small.

When $k \in [\hat{k}, q^e)$, (21) implies $\Pi_2 = (\hat{x}_0 - c)k$, where $\hat{x}_0$ is (as in the proof of Proposition 4) the minimum equilibrium $x_0$ when $k = \infty$. Therefore, $\Pi_{LP}^2 = (\hat{p}^L - c)k < \Pi_2$ since $\hat{p}^L < u'(k) \leq \hat{x}_0$. (In particular, as $k \nearrow q^e$, $\Pi_{LP}^2$ tends to zero but $\Pi_2$ is positive.) It proves the result for $\Pi_2, \Pi_{LP}^2$ when $k \in [\hat{k}, q^e)$. It completes the proof of part (b).

Compare $BS = TS - \Pi_1 - \Pi_2$ and $BS^{LP} = TS^{LP} - \Pi_1^{LP} - \Pi_2^{LP}$. When $k \in [\hat{k}, q^e)$, our previous results that $TS < TS^{LP}$, $\Pi_1 > \Pi_1^{LP}$, and $\Pi_2 > \Pi_2^{LP}$ together imply $BS < BS^{LP}$. (Also, it is easy to show that, as $k \nearrow q^e$, $BS^{LP}$ tends to the largest feasible total surplus $u(q^e) - cq^e$ but $BS$ is bounded away from that level.) As $k \downarrow 0$, $BS$ tends to zero (from Proposition 2) but $BS^{LP}$ is positive. Therefore, we also have $BS < BS^{LP}$ when $k$ is small. It completes the proof of part (c).

For any $k$, $\bar{Q} + k > D(\hat{p})$ (see Figure 4). It, together with part (b), implies that $\bar{Q} > D(\hat{p}^L) - k$ when $k$ is small. As $k \nearrow q^e$, $D(\hat{p}^L) - k$ tends to zero and $\bar{Q}$ tends to $q^e > 0$. It proves part (d).

\section*{5.3 A linear demand example}

This subsection considers a linear demand example. Suppose that $u(q) = q - q^2/2$ and $c \in [0, 1)$. Then $D(p) = 1 - p$, $\pi(p) = (p - c)(1 - p)$, and $\pi'(p) = 1 + c - 2p$ for all $p \in \mathcal{P} = [c, 1]$. The conditions in Proposition 1 are satisfied, so that the equilibrium
is essentially unique. The unique solution of (32) is
\[ \bar{p} = c + \frac{1}{e} \min\{k, \hat{k}\}, \]
where \( \hat{k} \) solves \( 1 - c - \frac{\hat{k}}{e^2} = \hat{k} \), or
\[ \hat{k} = \frac{e^2(1 - c)}{1 + e^2}. \]

The other endogenous objects are
\[
\begin{align*}
\bar{x}_0 &= c + \frac{1}{e^2} \min\{k, \hat{k}\}, \quad Q_0 = \frac{1 + e^2}{e^2} \max\{\hat{k} - k, 0\}, \\
\bar{Q} &= 1 - c - \frac{2}{e} \min\{k, \hat{k}\}, \quad \Pi_2 = \frac{1}{e^2} (\min\{k, \hat{k}\})^2, \\
T S &= \frac{(1 - c)^2}{2} - \frac{1}{2e^2} (\min\{k, \hat{k}\})^2, \\
\Pi_2 + BS &= (1 - c) \min\{k, \hat{k}\} - \frac{2 + e^2}{2e^2} (\min\{k, \hat{k}\})^2, \\
\Pi_1 &= TS - (\Pi_2 + BS) \\
&= \frac{(1 - c)^2}{2} - (1 - c) \min\{k, \hat{k}\} + \frac{1 + e^2}{2e^2} (\min\{k, \hat{k}\})^2 \\
&= \frac{(1 - c)^2}{2(1 + e^2)} + \frac{1 + e^2}{2e^2} (\max\{\hat{k} - k, 0\})^2, \\
BS &= (\Pi_2 + BS) - \Pi_2 \\
&= (1 - c) \min\{k, \hat{k}\} - \frac{4 + e^2}{2e^2} (\min\{k, \hat{k}\})^2.
\end{align*}
\]

As we claim generally in Proposition 4, all the above objects except \( \Pi_2 + BS \) and \( BS \) are monotone in \( k \). When \( k < \frac{e^2(1 - c)}{2 + e^2} \), \( \Pi_2 + BS \) is increasing in \( k \). When \( \frac{e^2(1 - c)}{2 + e^2} < k < \hat{k} \), \( \Pi_2 + BS \) is decreasing in \( k \). When \( k < \frac{e^2(1 - c)}{4 + e^2} \), \( BS \) is increasing in
Figure 7: Dominant firm’s equilibrium tariff schedules and the corresponding chosen purchases under assumptions $D(p) = 1 - p$, $c = 0$, and $k = 0.9$ or $0.2$, when nonlinear pricing (NLP), quantity forcing (QF), or linear pricing (LP) is feasible to the dominant firm (the LP schedule is omitted in the right panel because its scale is far below that of the NLP schedule).

$k$. When $\frac{e^2(1-c)}{4 + e^2} < k < \hat{k}$, $BS$ is decreasing in $k$. Figure 7 and Table 1 show various equilibrium objects in the linear demand example, when nonlinear pricing (NLP), quantity forcing (QF) (i.e. offering a take-it-or-leave-it quantity-payment bundle), or linear pricing (LP) is feasible to the dominant firm.

6 Concluding remarks

Recall that our model involves three kinds of asymmetries between the two firms: (1) the dominant firm is able to make nonlinear tariff schedules, while the rival firm can only choose linear pricing schemes; (2) the dominant firm commits to offering tariffs before its rival; and (3) relatively to the demand size the dominant firm has no capacity limit while its rival is capacity-constrained. Our analysis above suggests that the asymmetry in capacity is not crucial for the equilibrium adoption of nonlinear pricing by the dominant firm, but is important for the results of partial foreclosure and harming the buyer welfare.

What would happen if we relax our assumptions about the asymmetry between
the two firms by endogenizing the choices of timing and tariff options? One may consider a 4-stage extended game as follows. In Stage 0, each firm simultaneously decides whether to commit itself to use linear pricing. Any firm who makes this commitment can only offer a linear pricing scheme in later stages; and otherwise can more generally offer a nonlinear tariff schedule in later stages. In Stage 1, each firm can either offer a tariff (from the feasible set determined by its choice in stage 0), or wait until stage 2. In stage 2, any firm who chose waiting in stage 1 has to offer a tariff (again from the feasible set determined by its choice in stage 0). Lastly, in stage 3, the buyer chooses the quantities she purchases from the two firms. We can show that this extended game has a subgame perfect equilibrium with the following properties: only the rival firm commits itself to linear pricing in stage 0, the dominant firm offers a nonlinear tariff in stage 1, the rival firm waits in stage 1 and offers a linear tariff in stage 2, and their offers and the buyer’s choices are the same as those we characterized for our original 3-stage game. As a result, when both firms can choose their timing and pricing options the equilibrium outcome in the original 3-stage game remains to be part of the subgame perfect equilibrium outcome in the extended game. This demonstrates that our assumptions regarding the sequence of the moves and asymmetry in tariff options are not crucial for our main results. The asymmetry in capacity between the firms allows the unconstrained firm to take

<table>
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<th></th>
<th>$q_1$</th>
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<th>$\Pi_2$</th>
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Table 1: Equilibrium outputs ($q_1, q_2$), profits ($\Pi_1, \Pi_2$), buyer’s surplus (BS), and total surplus (TS) under assumptions $D(p) = 1 - p$, $c = 0$, and $k = 0.9$ or $0.2$, when nonlinear pricing (NLP), quantity forcing (QF), or linear pricing (LP) is feasible to the dominant firm.
advantage of a menu of tariff offers in order to restrict the choices of the constrained firm and extract surpluses from the buyer.

Appendix

The proof of Theorem 1 requires the following two lemmas.

Lemma 7. For any $Q : \mathcal{P} \rightarrow \mathbb{R}^+$, $T : \mathcal{P} \rightarrow \mathbb{R}$, and $\bar{p} \in \mathcal{P}$ that satisfy (BIC), (BIR), and (RIC), there is a $\tau \in \mathcal{T}$ and a SPE of the subgame after firm 1 offers $\tau$ such that

(i) in this SPE of the subgame, firm 2 chooses $p = \bar{p}$, and the buyer, contingent on any firm 2’s unit price $p \in \mathcal{P}$, chooses to buy $Q(p)$ and $\text{Proj}_{[0,k]}(D(p) - Q(p))$ units from firm 1 and firm 2 respectively, and

(ii) $\tau(Q(p)) = T(p)$ for all $p \in \mathcal{P}$.

Proof. Suppose that $Q : \mathcal{P} \rightarrow \mathbb{R}^+$, $T : \mathcal{P} \rightarrow \mathbb{R}$, and $\bar{p} \in \mathcal{P}$ satisfy (BIC), (BIR), and (RIC). Define

$$
\tau(Q) = \begin{cases} 
T(p) & \text{if } \exists p \in \mathcal{P} \text{ s.t. } Q(p) = \bar{Q} \\
0 & \text{if } Q = 0 \text{ and } \exists \hat{p} \in \mathcal{P} \text{ s.t. } Q(\hat{p}) = 0 \\
\infty & \text{for all other } Q \text{ in } \mathbb{R}^+ 
\end{cases}.
$$

Note that the above $\tau$ is well defined because (BIC) implies $T(p) = T(\hat{p})$ whenever $Q(p) = Q(\hat{p})$. Clearly, (ii) holds. To see that $\tau(0) \leq 0$, note that if there is no $p \in \mathcal{P}$ such that $Q(p) = 0$, then $\tau(0) = 0$. If $Q(\hat{p}) = 0$ for some $\hat{p} \in \mathcal{P}$, then $\tau(0) = T(\hat{p})$, and $T(\hat{p}) \leq 0$ from (BIR). Thus, $\tau(0) \leq 0$ in each case.

Given this $\tau$ and any $p \in \mathcal{P}$, (BIC) and (BIR) imply that a buyer’s optimal action is to buy $Q(p)$ and $\text{Proj}_{[0,k]}(D(p) - Q(p))$ units from firm 1 and firm 2 respectively. Given $\tau$ and that the buyer uses the above strategy, (RIC) implies that a firm 2’s optimal action is to choose $p = \bar{p}$. Therefore, the strategies in (i) constitute a SPE of the subgame after firm 1 offers $\tau$. It follows that $\tau$ is regular and hence $\tau \in \mathcal{T}$. ■

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Lemma 8. For any \( \tau \in \mathcal{T} \) and any SPE of the subgame after firm 1 offers \( \tau \), if \( Q : \mathcal{P} \to \mathbb{R}_+ \), \( T : \mathcal{P} \to \mathbb{R} \), and \( \bar{p} \in \mathcal{P} \) satisfy (i) and (ii) in Lemma 7, then \( Q(\cdot), T(\cdot), \bar{p} \) also satisfy (BIC), (BIR), and (RIC).

Proof. Take any \( \tau \in \mathcal{T} \) and any SPE of the subgame after firm 1 offers \( \tau \). Suppose that \( Q : \mathcal{P} \to \mathbb{R}_+ \), \( T : \mathcal{P} \to \mathbb{R} \), and \( \bar{p} \in \mathcal{P} \) satisfy (i) and (ii) in Lemma 7. Since the strategies described in (i) constitute a SPE of the subgame after firm 1 offers \( \tau \), we have (RIC) and

\[
V(Q(p), p) - \tau(Q(p)) \geq V(Q, p) - \tau(Q) \quad \forall (Q, p) \in \mathbb{R}_+ \times \mathcal{P}.
\]  

To see (BIC), take \( Q = Q(\bar{p}) \) for arbitrary \( \bar{p} \in \mathcal{P} \) in (37) and use (ii). To see (BIR), take \( Q = 0 \) in (37) and use \( \tau(0) \leq 0 \) and (ii).

Proof of Theorem 1. Necessity part. Suppose that \( (Q^*(\cdot), T^*(\cdot), \bar{p}^*) \) is a solution of (OP1). Then \( Q^*(\cdot), T^*(\cdot), \bar{p}^* \) satisfy (BIC), (BIR), and (RIC). From Lemma 7, there is a \( \tau^* \in \mathcal{T} \) (defined by (36) with \( \tau(\cdot), Q(\cdot), T(\cdot) \) replaced by \( \tau^*(\cdot), Q^*(\cdot), T^*(\cdot) \)) such that (10) holds and a SPE \( (p^*(\tau^*), q^*(\tau^*, \cdot)) \) of the subgame after firm 1 offers \( \tau^* \) is described by (8), (9), and (11).

In the subgame after firm 1 offers this \( \tau^* \), we let firm 2 and the buyer play the SPE \( (p^*(\tau^*), q^*(\tau^*, \cdot)) \), so that firm 1’s profit is \( T^*(\bar{p}^*) - cQ^*(\bar{p}^*) \). In the subgame after firm 1 offers any other \( \tau \in \mathcal{T} \setminus \{\tau^*\} \), we let firm 2 and the buyer play any SPE \( (p^*(\tau), q^*(\tau, \cdot)) \), which exists because every \( \tau \in \mathcal{T} \) is regular. By such constructions, \( p^*, q^* \) satisfy (BSR) and (RSR).

From Lemma 8, the SPE outcome of the subgame after firm 1 offers an arbitrary \( \tau \in \mathcal{T} \) must be characterized by some \( Q(\cdot), T(\cdot), \bar{p} \) that satisfy (BIC), (BIR), and (RIC), and the associated firm 1’s profit is \( T(\bar{p}) - cQ(\bar{p}) \). Since \( (Q^*(\cdot), T^*(\cdot), \bar{p}^*) \) is a solution of (OP1), firm 1 cannot make strictly higher profit than \( T^*(\bar{p}^*) - cQ^*(\bar{p}^*) \) by offering any \( \tau \in \mathcal{T} \). That is, \( (\tau^*, p^*, q^*) \) satisfies (DSR) and hence is a SPE of the whole game.

Sufficiency part. Suppose that \( (\tau^*, p^*, q^*) \) is a SPE and \( Q^*(\cdot), T^*(\cdot), \bar{p}^* \) satisfy (8), (9), (10), and (11). From Lemma 8, \( Q^*(\cdot), T^*(\cdot), \bar{p}^* \) satisfy (BIC), (BIR), and
Suppose, by way of contradiction, that \((Q^*(\cdot), T^*(\cdot), \bar{p}^*)\) is not a solution of (OP1). Then, there is some \((Q^0(\cdot), T^0(\cdot), \bar{p}^0)\) satisfying (BIC), (BIR), and (RIC), such that \(T^0(\bar{p}^0) - cQ^0(\bar{p}^0) > T^*(\bar{p}^*) - cQ^*(\bar{p}^*)\). Then, for any \(\varepsilon > 0\), we can construct some \(\tau_\varepsilon \in T\) such that the subgame after firm 1 offers \(\tau_\varepsilon\) has a unique SPE outcome, for which firm 1’s profit is at least \(T^0(\bar{p}^0) - cQ^0(\bar{p}^0) - \varepsilon\). (This can be seen from our analysis for finite bundling, which we omit here. Details are available upon request.) Hence, offering \(\tau_\varepsilon\) with small enough \(\varepsilon\) is a firm 1’s profitable deviation in the SPE \((\tau^*, p^*, q^*)\), which is a contradiction. ■

Proof of Lemma 1. Let \(U(p)\) denote \(V(Q(p), p) - T(p)\). Then (BIC) can be written as

\[U(p) - U(\bar{p}) \geq V(Q(\bar{p}), p) - V(Q(\bar{p}), \bar{p}) \quad \forall p, \bar{p} \in P,\]  
(38)

and (BIC-2) can be written as

\[U(p) - U(c) = \int_{\epsilon}^{p} V_p(Q(\bar{p}), \bar{p})d\bar{p} \quad \forall p \in P.\]  
(39)

Necessity part. Suppose that (BIC) is satisfied. Then (38) implies that, for any \(p_1, p_2 \in P\),

\[V(Q(p_1), p_2) - V(Q(p_1), p_1) \leq U(p_2) - U(p_1) \leq V(Q(p_2), p_2) - V(Q(p_2), p_1).\]  
(40)

If (BIC-1) does not hold, then there exist \(p_1, p_2 \in P\) such that \(p_1 < p_2\) and \(Q(p_1) > Q(p_2)\) and \(D(p_1) > Q(p_2)\) and \(Q(p_1) + k > D(p_2)\). But then (40) implies

\[0 \geq [V(Q(p_1), p_2) - V(Q(p_1), p_1)] - [V(Q(p_2), p_2) - V(Q(p_2), p_1)]
= \int_{p_1}^{p_2} \int_{Q(p_2)}^{Q(p_1)} V_{pQ}(Q, p)dQdp > 0,
\]

which is a contradiction. The last inequality holds because \(V_{pQ} \geq 0\) almost everywhere and, under the above hypotheses, there exists a convex combination \((\hat{Q}, \hat{p})\) between \((Q(p_2), p_1)\) and \((Q(p_1), p_2)\) such that, for all \((Q, p)\) in some open neighborhood of \((\hat{Q}, \hat{p})\), it holds that \(Q < D(p) < Q + k\) and hence \(V_{pQ}(Q, p) = 1\). Therefore, (BIC-1)
holds.

Moreover, (40) implies (39). Therefore, (BIC-2) holds.

**Sufficiency part.** Suppose that (BIC-1) and (BIC-2) hold. First, (BIC-1) implies that, for all \( p_1, p_2 \in \mathcal{P} \) with \( p_1 \leq p_2 \), we have

\[
\text{Proj}_{[0,k]}(D(p_2) - Q(p_1)) \geq \text{Proj}_{[0,k]}(D(p_2) - Q(p_2)),
\]

(41)

\[
\text{Proj}_{[0,k]}(D(p_1) - Q(p_1)) \geq \text{Proj}_{[0,k]}(D(p_1) - Q(p_2)).
\]

(42)

Indeed, \( p_1 \leq p_2 \) and (BIC-1) imply either (i) \( Q(p_1) \leq Q(p_2) \), or (ii) \( D(p_1) \leq Q(p_2) \), or (iii) \( Q(p_1) + k \leq D(p_2) \). In case (i), clearly (41) and (42) hold. In case (ii), we have \( D(p_2) \leq D(p_1) \leq Q(p_2) \) so that the right-hand sides of (41) and (42) are 0. In case (iii), we have \( Q(p_1) + k \leq D(p_2) \leq D(p_1) \) so that the left-hand sides of (41) and (42) are \( k > 0 \). Therefore, (41) and (42) hold in each case.

Recall that (BIC-2) is equivalent to (39). Therefore, for any \( p_1, p_2 \in \mathcal{P} \), no matter \( p_1 \leq p_2 \) or \( p_2 \leq p_1 \), we have

\[
U(p_2) - U(p_1) = \int_{p_1}^{p_2} V_p(Q(p), p) \, dp = -\int_{p_1}^{p_2} \text{Proj}_{[0,k]}(D(p) - Q(p)) \, dp
\]

\[
\geq -\int_{p_1}^{p_2} \text{Proj}_{[0,k]}(D(p) - Q(p_1)) \, dp = \int_{p_1}^{p_2} V_p(Q(p_1), p) \, dp
\]

\[
= V(Q(p_1), p_2) - V(Q(p_1), p_1),
\]

which proves (38) and hence (BIC).

**Proof of Lemma 2.** To see part (a), suppose that \( p_1, p_2 \in \{p \in \mathcal{P} : 0 \leq D(p) - Q(p) \leq k\} \) with \( p_1 \leq p_2 \) and \( Q(p_1) > Q(p_2) \). Then \( D(p_1) \geq Q(p_1) > Q(p_2) \geq D(p_2) - k \), violating (BIC-1). Therefore, part (a) holds.

Part (b) is implied by (41) in the proof of Lemma 1, again an implication of (BIC-1).

**Proof of Lemma 3.** We claim that \( V(Q(p), p) - T(p) - V(0, p) \) is absolutely continuous and nondecreasing in \( p \) on \( \mathcal{P} \). To see this, recall that \( V(Q, \cdot) \) is absolutely continuous...
and \( V_p(Q, p) = - \text{Proj}_{[0,k]}(D(p) - Q) \). Let \( U(p) \) denote \( V(Q(p), p) - T(p) \). (BIC-2), which is equivalent to (39), implies

\[
U(p) - V(0, p) = U(c) - V(0, c) + \int_c^p [V_p(Q(\tilde{p}), \tilde{p}) - V_p(0, \tilde{p})]d\tilde{p}.
\]

Then the claim follows from the fact that \( V_p(Q(\tilde{p}), \tilde{p}) \geq V_p(0, \tilde{p}) \) for all \( \tilde{p} \). The desired results immediately follows.

References


