Identification of Dynamic Games with Multiple Equilibria and Unobserved Heterogeneity with Application to Fast Food Chains In China

Yao Luo† Ping Xiao‡ Ruli Xiao§

January 29, 2018

Abstract

This paper provides sufficient conditions for non-parametrically identifying dynamic games with incomplete information, allowing for both multiple equilibria (ME) and market-level unobserved heterogeneity (UH). We assume that both ME and UH have finite support, resulting in a finite mixture structure. The model primitive then is identified sequentially, relying on the Markovian property over time induced by the markov perfect equilibrium concept. The first step identification shares the same up-to-label-swapping issue in identification of matrix eigenvalue-eigenvector decomposition in the measurement error literature. We do not impose a monotonicity condition to solve this matching-type problems. We instead exploit the fact that the eigenvalue matrices associated with different decompositions are the same to circumvent this problem. We also propose a test to distinguish between multiple equilibria and unobserved heterogeneity through comparing the corresponding payoff primitives. Finally, we apply our method to study spillover effects of fast food in China and we find evidence of unobserved heterogeneity.

JEL Classification: C14 L13

Keywords: Multiple equilibria, Unobserved heterogeneity, Discrete games, Dynamic games, Non-parametric identification

†Department of Economics, University of Toronto.
‡Department of Marketing, University of Technology Sydney.
§Department of Economics, Indiana University, 100 S. Woodlawn Ave. Bloomington, IN 47405. Email: rulixiao@iu.edu; Phone: (812) 855-3213; Fax: (480) 855-3736.

*We thank .... All errors are ours. Comments are welcome.
1 Introduction

There are two outstanding challenges in the estimation of dynamic discrete games: (1) the presence of unobserved heterogeneity (UH) at the firm or market levels; (2) the existence of multiple equilibria (ME). The existing literature either assumes that none or one of them but not both exists. Ignoring either one might result in mis-specification and inconsistent estimation of the payoff primitives. On one hand, it is challenging to distinguish between ME and UH since they generate similar mixture structures of the outcome distributions. On the other hand, it is important to distinguish between the two since they lead to different counterfactual inferences and thus have different policy implications. This paper provides sufficient conditions for non-parameterically identifying dynamic incomplete information games, which allow for ME and UH. We also apply our method to study spillover effects of fast food in China and distinguish between ME and UH empirically.

We create a latent variable to aggregate both ME and UH and propose to identify all primitives sequentially. First, we show that the cardinality of this new latent measure can be identified as the rank of a matrix constructed by the joint distribution of three periods' state variables with further specified assumptions. Second, we show that the equilibrium CCPs and state transitions can be identified via matrix decompositions, following results developed in measurement error literature but without imposing monotonicity conditions. Third, we show that payoff functions can be identified with exclusion restrictions, as in Bajari et al. (2009). Lastly, we distinguish between UH and ME based on the definition of ME and UH. Specifically, ME indicates that model primitives are the same but map with multiple equilibrium solutions, while UH indicates that model primitives vary with the value of UH. Consequently, the source of UH can be detected given that model primitives are identified non-parametrically.

This paper proposes a new perspective to tackle the prevalent label swapping problem associated with the matrix decomposition approach. The measurement error literature imposes some monotonicity conditions for unique identification. Such monotonicity conditions, however, are very restrictive in games. The presence of ME makes such conditions even more restrictive. Without a monotonicity

---


2 Similar idea has been explored in Aguirregabiria and Mira (2015) for static games.
condition, equilibrium CCPs are identified up to a permutation of the latent variable. Multiple decompositions might result in different permutations. Identification of payoff primitives requires that the permutations from multiple decompositions at least are the same. This paper exploits the identification structure so that it is feasible to identify payoff primitives. Relaxing the monotonicity assumption is important in practice, because it does not hold trivially even in single-agent discrete choice models.

It is important and complicated to address UH in identification and estimation of games. Ignoring the presence of UH is unrealistic for some empirical industrial organization applications and also problematic in explaining micro data. Not accounting for potentially UH may lead to significant biases in parameter estimates, and thus creates misunderstandings of strategic interaction between firms. Meanwhile, investigating the identification with UH is important for its widely inclusion in empirical estimations. Identification may follow the results developed in the finite mixture/measurement error literatures (Kasahara and Shimotsu (2009) and Hu and Shum (2012)). It is more difficult, however, to tackle UH in games than in discrete choice models, because of the possible coexistence of ME.

Indeed, the presence of ME is a prevalent feature in dynamic games. Identification of games with ME is not well-understood, even though the presence of ME does not necessarily preclude the identification (Jovanovic (1989)). For instance, focusing on one single market enables identification and consistent estimation of the payoff primitives since Markov Perfect Equilibrium implicitly assumes a single equilibrium is employed (Pesendorfer and Schmidt-Dengler (2008)). Relying on a long span time series, however, is often not feasible in practice sometimes due to the limited availability of data. A common estimation approach that may be used instead is to use cross market variation. For the validity of this approach, the pooled data has to be generated from the same equilibrium, which rarely has empirical evidence. Imposing such a restriction may result in the mis-specification and inconsistent estimation of payoff primitives.

To the best of our knowledge, this paper is the first to provide rigorous identification results for dynamic games of incomplete information, while incorporating UH and ME. The identification results

---

3For instance, in the empirical application in Aguirregabiria and Mira (2007), the estimation without unobserved market heterogeneity implies estimates of strategic interaction between firms (that is, competition effects) that are close to zero or even have a sign opposite to that expected under competition. While including UH in the models results in estimates that show significant and strong competition effects.
presented in this paper are of real practical importance. With fully understanding of the conditions under which the underlying data generating process can be achieved non-parametrically, one will be more comfortable about the estimation results regardless its functional assumptions. Even though some existing literature considers estimation of dynamic games allowing for UH, there is no rigorous discussion about the identification (See Aguirregabiria and Mira (2007), Bajari et al. (2007), and Arcidiacono and Miller (2011).).

This paper is closely related to Hu and Shum (2012) (HS hereafter), which provides identification for dynamic models with UH. This paper differs from HS in the following aspects. First of all, HS considers continuous UH, while we consider finite and discrete UH and also allows for ME. Secondly, HS focuses on identifying the Markov transition process, while this paper focuses on identifying the payoff primitives. Thirdly, HS relies on a monotonicity condition for unique identification, while this paper explores the identification structure to identify the equilibrium CCPs and so payoff primitives. Last, this paper proposes a test to distinguish between ME and UH, which is important for empirical studies.

This paper also contributes to the literature on identification and estimation of games with ME. See De Paula (2012) for a survey of the recent literature on the econometric analysis of games with multiplicity. See also Sweeting (2013), Ciliberto and Tamer (2009) for bound identification, Bajari et al. (2010), De Paula and Tang (2012) for identifying the sign of the strategic interaction term using ME. Otsu et al. (2015) propose several statistical tests for finite state Markov games, in order to examine whether the data can be pooled for estimation. This paper is closely related to Xiao (2014), which provides identification results for static games of incomplete information with ME. This paper is also closely related to Aguirregabiria and Mira (2015), which considers identification of static games with ME and UH.

The remainder of the paper is organized as follows. Section 2 describes the game framework and characterizes the equilibrium. Section 3 provides the nonparametric identification results without ME and UH. Section 4 provides the nonparametric identification results while incorporates ME and UH. Section 5 concludes. The Appendix contains the proofs.
2 UH vs ME in a Simple Entry Game

We use a simple entry game to illustrate the conceptual difference between UH and ME in this section. Consider a dynamic entry game with two players, i.e., 1 and 2. In each period, firm \( i \) decides whether it would enter the market if it has not or whether it would expand in the market by adding one more store if it is already in the market, i.e., \( n_i = 1 \) or \( n_i = 0 \). Before making such a decision, firms privately observe a vector of action-specific payoff shocks - the cost of entering and the cost of not entering. For illustration purpose, we assume away any market or firm-level characteristics, except the stock of the stores \( N \equiv \{N_i, N_j\} \). We also assume that the two firms are symmetric. For illustration purpose, we assume that firm \( i \)'s reduced-form payoff function is as follows:

\[
\pi_i(n_i, N; \theta) = \theta_0 + \theta_1 N_i + \theta_2 N_j - \theta_3 I(N_i = 0) - \theta_4 n_i + \epsilon_i(n_i),
\]

where \( \theta \equiv \{\theta_0, \theta_1, \theta_2, \theta_3, \theta_4\} \) collects the payoff parameter, \( N_i \) and \( N_j \) is the number of stores firm \( i \) and \( j \) owns, respectively, and \( \epsilon \) is the payoff shock. With this payoff function, firm \( i \)'s decision rule \( \delta_i \) is a mapping from the observed stocks \( N \) and the payoff shocks \( \epsilon_i \equiv \{\epsilon_i(1), \epsilon_i(0)\} \) to her entry decision \( n_i \), i.e., \( \delta_i(N, \epsilon_i) \rightarrow n_i \). Provided that the payoff shocks enter the payoff functions additively, the decision rule is a cutoff, denoted as \( \bar{\epsilon}_i \), which applies to the difference of payoff shocks of entering or not \( \epsilon_i(1) - \epsilon_i(0) \). Specifically, firm \( i \) will enter if the difference of payoff shocks is greater than this cutoff, and vice versa. That is, \( d_i(N, \epsilon_i) = 1 \) if \( \epsilon_i(1) - \epsilon_i(0) \geq \bar{\epsilon}_i(N) \); \( d_i(N, \epsilon_i) = 0 \) if \( \epsilon_i(1) - \epsilon_i(0) < \bar{\epsilon}_i(N) \). Note that this cutoff depends on the current stocks of both firms. Consequently, an equilibrium is represented by a pair of cutoffs for both firms, i.e., \( \bar{\epsilon} \equiv \{\bar{\epsilon}_1(N), \bar{\epsilon}_2(N), \forall N\} \).

The implication of ME The presence of multiple equilibria in our simple context means that for the same payoff parameter \( \theta \), there are more than one pair of cutoffs that satisfies the equilibrium conditions. Without loss of generality, we assume two equilibria, referred to as equilibrium A and B, which is equivalent to the existence of two pairs of cutoffs denoted as \( \bar{\epsilon}^A \) and \( \bar{\epsilon}^B \). These two pairs of cutoffs lead to two sets of equilibrium CCPs, denoted as \( p^A \) and \( p^B \). Suppose that our data is generated
by 100 markets, among which firms in half of the markets follow equilibrium A and in the other half use equilibrium B to make their entry decisions, respectively. We abstract away the coordination problem and assume that players always are able to coordinate on the same equilibrium in the same market. The reduced-form CCPs from the data are generated by the two equilibria with equal weights, which indicates that those reduced-form CCPs are a mixture of two equilibrium CCPs. That is,

$$\Pr(n_i = 1|N) = \frac{1}{2} p_A(n_i = 1|N) + \frac{1}{2} p_B(n_i = 1|N). \quad (1)$$

**The implication of UH** The presence of unobserved heterogeneity in our context indicates that some of the payoff parameters are different in different markets and we cannot use data to control for this difference. For instance, $\theta_0$ captures an overall consumer preference/acceptance of the western fast food in China, which is hard to use data to control for. Again for illustration purpose, we assume away the presence of multiple equilibria and assume that there are two types of markets among which consumers in one type like western fast food (type $h$) while the other type does not care much about the western food (type $l$). As a result, the payoff parameters can be represented as $\theta^h \equiv \{\theta^h_0, \theta_1, \theta_2, \theta_3, \theta_4\}$ and $\theta^l \equiv \{\theta^l_0, \theta_1, \theta_2, \theta_3, \theta_4\}$ for markets of high and low types, respectively. Since firms learn about each market’s type, their decision rules obviously will take into account the market type. Consequently, there are different cutoffs for different market types, denoted as $\bar{\epsilon}^h$ and $\bar{\epsilon}^l$. These two pairs of cutoffs lead to two sets of equilibrium CCPs, denoted as $p^h$ and $p^l$. The reduced-form CCPs from the data are generated by the two market types with equal weights, which indicates that those reduced-form CCPs are a mixture of two types. That is,

$$\Pr(n_i = 1|N) = \frac{1}{2} p^h(n_i = 1|N) + \frac{1}{2} p^l(n_i = 1|N). \quad (2)$$

It is impossible to distinguish between ME and UH only relying on the mixture structure (equations (1) or (2)). On the other hand, it is important to distinguish between ME and UH because they enter the payoff functions differently, indicating different policy implication. Specifically, UH, by definition, enters the payoff function directly, and thus results in two different payoff parameters, $\theta^h$ and $\theta^l$, where
\( \theta^h \neq \theta^l \). On the other hand, the labeling of equilibrium \( A \) or \( B \) does not enter the payoff functions. That is, using the two sets of equilibrium CCPs, \( p^A \) and \( p^B \), each enables identification of one payoff parameter, denoted as \( \theta^A \) and \( \theta^B \), respectively. The definition of ME indicates that \( \theta^A = \theta^B \). Thus, to distinguish between ME and UH, it boils down to identify the payoff functions using each set of equilibrium CCPs, which requires identification of equilibrium CCPs from the mixture structure in the data.

3 A General Dynamic Game

Consider a model of discrete-time, infinite-horizon game with \( N \) players.\(^4\) At the beginning of each period \( t \ (t \in \{0, 1, \ldots, \infty\}) \), player \( i, i \in \{1, \ldots, N\} \), simultaneously choose an action \( a_{it} \) out of a finite set \( A_i \), i.e., \( a_{it} \in A_i \equiv \{0, 1, \ldots, K\} \). Let \( a_t \) denote players’ action profile in period \( t \), i.e., \( a_t \equiv \{a_{1t}, \ldots, a_{Nt}\} \). Before making her decision, player \( i \) observes a vector of state variables \( s_t \) and a vector of action-specific private payoff shocks \( \epsilon_{it} \equiv (\epsilon_{it}(a_{it} = 0), \ldots, \epsilon_{it}(a_{it} = K)) \). Let \( \epsilon_t \) collect the private information for all players, i.e., \( \epsilon_t \equiv (\epsilon_{1t}, \ldots, \epsilon_{Nt}) \). Let \( x_t \) denote market and individual firm characteristics in period \( t \), which is discrete and finite, i.e., \( x_t \in X = \{X^1, \ldots, X^m\} \).

To characterize the equilibrium, I first introduce several assumptions which are usually imposed in the existing literature.

**Assumption 1.** (Conditional Independence) The payoff shocks \( \epsilon_{it} \) are independent across actions, players, and over time. Moreover, the payoff shocks \( \epsilon_{it} \) have a support of \( \mathbb{R}^{K+1} \).

The assumption of independence of payoff shocks is to facilitate tractability. The correlation of payoff shocks calls for a model of learning that captures the evolution of a player’s belief over opponents’ payoff shocks with knowledge of their past actions, which greatly increases the size of the state variable.

**Assumption 2.** (State Evolution) The state transition, captured by the probability with which state \( x_t \) is reached given all history information, is assumed to be only determined by the previous state \( x_{t-1} \)
and previous action profile $a_{t-1}$, i.e., $g(x_t|a_{t-1},x_{t-1},...,a_0,x_0) = g(x_t|a_{t-1},x_{t-1})$.

To simplify the model, this assumption rules out the scenario in which all past history affects the evolution of the state variable. This markov property is again widely imposed in the existing literature. Naturally, $\sum_{x'} g(x_{t+1} = x'|a_t,x_t) = 1$.

**Assumption 3.** (*Additive Separability*) The payoff for player $i$ from choosing action $a_{it}$ while her rivals choose actions $a_{-iit}$ in period $t$ is assumed to be additively separable from the shocks. That is,

$$u_i(a_t, s_t, \epsilon_{it}) = \pi_i(a_t, s_t) + \epsilon_{it}(a_{it}),$$

where $s_t$ could include the previous action as a state variable, i.e., $s_t \equiv \{x_t, a_{t-1}\}$. It is important to allow previous action affect players’ per-period payoffs in empirical studies. For example, in Sweeting (2013), the format that music stations choose to air in a given period depends on the format they aired in the previous period, due to the switching cost. Some dynamic games are to analyze firms strategic interaction regarding entry or exit, in which previous actions affect current payoffs (see also Igami and Yang (2015)). Nevertheless, this payoff function nests the payoff functions that the previous actions does not matter. That is, if previous action does not enter player’s payoff function, $s_t = x_t$.

Examples of the described games are:

1. Sweeting (2006) estimates a dynamic model of multi-station owners choose the formats of their stations for the next period. The format choice is discrete and finite.


Following Maskin and Tirole (2001), we consider pure Markovian strategies $\delta_i(s, \epsilon_i)$ in which player $i$ determines her action based on her state variable and payoff shock at the current period. We do not consider mixed strategies. The Markovian assumption implies a stationary property of the process, i.e., we can suppress time notation $t$. We first define the Markov Perfect Equilibria(MPE) in the following.
Definition 1. *(MPE)* A collection of \((\delta, \sigma) = (\delta_1, \ldots, \delta_n, \sigma_1, \ldots, \sigma_n)\) is a Markov perfect equilibrium if

1. For all \(i\), \(\delta_i\) is a best response to \(\delta_{-i}\) given the beliefs \(\sigma_i\) at all states \(s\).

2. For all \(i\), the beliefs \(\sigma_i\) is consistent with the strategy \(\delta\).

We introduce some notation before we characterize the equilibrium. Let \(\sigma_i(a|s)\) denote player \(i\)’s ex ante (before \(\epsilon_i\) is revealed) belief that action profile \(a\) will be chosen conditional on state \(s\), and \(V_i(s|\sigma_i)\) denote player \(i\)’s ex ante value function with belief \(\sigma_i\), which is the discounted sum of future payoffs for player \(i\) before the shocks are revealed and actions are taken. The ex ante value function can be expressed as:

\[
V_i(s; \sigma_i) = \sum_a \sigma_i(a|s) \left( \pi_i(a, s) + \beta \sum_{s'} V_i(s'|\sigma_i)g(s'|a, s) \right) + \sum_a E_\epsilon[\epsilon_i(a_i)|\delta_i(s, \epsilon_i) = a_i]\sigma_i(a_i|s),
\]

where \(\beta\) is the discounted factor, and \(E_\epsilon\) denotes the expectation operator with respect to player \(i\)’s shocks, which depends on the cumulative distribution function of the payoff shocks. Since the state variable is finite, we can express the above equation as a matrix equation:

\[
V_i(\sigma_i) = \sigma_i\Pi_i + D_i(\sigma_i) + \beta\sigma_iGV_i(\sigma_i)
= [I - \beta P_iG]^{-1}(\sigma_i\Pi_i + D_i(\sigma_i)),
\]

where \(V_i(\sigma_i) \equiv [V_i(s; \sigma_i)]_{s \in S}\) is the \(m_s \times 1\) dimensional vector of ex ante value function; \(P_i\) is the \(m_s \times ((1 + K)^n \cdot m_s)\) dimensional matrix consisting of player \(i\)’s belief in row \(s\), column \((a, s)\), and zeros elsewhere; \(G\) is a \(((1 + K)^n \cdot m_s) \times m_s\) vector consisting of the state transition in row \((a, s)\), vector \(s'\); \(\Pi_i\) is a \(((1 + K)^n \cdot m_s) \times 1\) vector consisting of player \(i\)’s payoffs in row \((a, s)\); \(D_i(\sigma_i)\) is a \(m_s \times 1\) vector with \(\sum a_i E_\epsilon[\epsilon_i(a_i)|\delta_i(s, \epsilon_i) = a_i]\sigma_i(a_i|s)\) in row \(s\). The above equation provides a closed form expression for player \(i\)’s ex ante value function given her beliefs.

Let \(V_i(a_i, s, \sigma_i)\) denote player \(i\)’s choice specific value function, which is the continuation net of the
payoff shocks under action $a_i$ with beliefs $\sigma_i$. $V_i(a_i, s, \sigma_i)$ can be written as:

$$V_i(a_i, s; \sigma_i) = \sum_{a_{-i}} \sigma_i(a_{-i}|s) \left( \pi_i(a, s) + \beta \sum_{s'} V_i(s'; \sigma_i) g(s'|a_i, s) \right). \quad (5)$$

A rational player $i$ should choose action $a_i$ given beliefs $\sigma_i$ if $a_i$ yields the maximal lifetime payoffs among all the alternatives. Thus,

$$\delta(s, \epsilon_i) = a_i \iff V_i(a_i, s) + \epsilon_i(a_i) > V_i(a_i', s) + \epsilon_i(a_i') \quad \forall a_i' \neq a_i$$

Define the conditional choice probability (CCP) $p_i(a_i|s)$ as the probability that player $i$ chooses action $a_i$ given state $s$ associated with player $i$’s decision rule. Thus,

$$p_i(a_i|s, \sigma_i) = \int \delta_i(s, \epsilon_i; \sigma_i = a_i) f(\epsilon_i) d\epsilon_i$$

$$= \int \delta_i(s, \epsilon_i; \sigma_i = a_i) f(\epsilon_i) d\epsilon_i$$

$$= \Psi_i(V_i(a_i = 1, s; \sigma_i) - V_i(a_i = 0, s; \sigma_i), ..., V_i(a_i = K, s; \sigma_i) - V_i(a_i = 0, s; \sigma_i)), \quad (6)$$

where $\delta_i(\cdot)$ is the indicator function. The equilibrium mapping $\Psi_i$ is determined by the distribution of the payoff shocks (Egesdal et al. (2015)). Equation (6) expresses the optimal CCPs as a function of players’ beliefs. In equilibrium, beliefs are consistent with the strategy, leading to the following equilibrium conditions:

$$p_i(a_i|s) = \Psi_i(V_i(a_i = 1, s; p) - V_i(a_i = 0, s; p), ..., V_i(a_i = K, s; p) - V_i(a_i = 0, s; p)), \quad \forall i. \quad (7)$$

An MPE in the probability space $p$ can be characterized as a solution to the following system of non-linear equations

$$\Omega(\pi, g) = \{p|p = \Psi(\pi, g, p)\}.$$ 

The existence of MPE follows from the Brouwer’s fixed point theorem. However, the equilibrium need
not to be unique. Multiple equilibria are actually a prevalent phenomenon in dynamic games.

4 Identification of the Dynamic Games

This section provides conditions with which the dynamic game is identified while allowing for both UH and ME. Note that the existing literature have explored the identification of the per-period payoff functions using the observed equilibrium CCPs from data directly. Consequently, this paper focuses on providing conditions to identify the equilibrium CCPs using results from the measurement error literature. We then can identify the per-period payoff functions following the existing literature (see Bajari et al. (2009)).

The econometrician observes firms’ action profiles \( a_{m}^{t} \) up to \( T \) periods in market \( m \) for \( m = 1, \ldots, M \), and the market and individual characteristics \( x_{m}^{t} \) in each period. Data can be summarized as

\[
\{ a_{m}^{t}, x_{m}^{t}, t = 1, \ldots, T, m = 1, \ldots, M \}.
\]

4.1 Data Generating Process

Note that potentially there may be two types of latent factors – UH and ME. Let \( \eta_{t} \) denote the time-variant market-level UH, which is assumed to be discrete and finite; that is, \( \eta_{t} \in \{ \eta^{1}, \ldots, \eta^{m_{\eta}} \} \). Note that the number of equilibria is also assumed to be finite and discrete. Denote the equilibrium and the equilibrium set as \( e^{*} \) and \( \omega(\pi, g) \), respectively, where \( \pi \) and \( g \) denote the payoff functions and the state transitions.

This paper assumes exogenous equilibrium selection process, such as by nature or some outside mechanisms.\(^5\) Specifically, the determinant of equilibrium is characterized by a probability distribution denoted as \( p^{e}(\pi, g) \equiv \{ \Pr(e^{*}), e^{*} \in \omega(\pi, g) \} \). Furthermore, we allow some equilibria to be selected with a zero probability, i.e., \( \Pr(e^{*}) = 0 \) for some \( e^{*} \). We consider an equilibrium with a positive selection probability as an active equilibrium, and denote the active equilibrium set as \( \omega^{a}(\pi, g) = \{ e^{*} : \Pr(e^{*}) > \]

\(^5\)Modeling the equilibrium selection process is too complicated and out of the scope of this paper.
0 \& e^* \in \omega(\pi, g)$. The number of the active equilibria, which is denoted as \( Q = \#\{e^*, \Pr(e^*) > 0 \& e^* \in \omega(\pi, g)\} \), may be different from the total number of equilibria.

The model primitives then can be summarized as

\[ \{\pi, g, \Omega(\pi, g), \omega(\pi, g), p^e(\pi, g)\} \].

4.2 Non-parametric Identification Results

The overall identification proceeds in three steps. First we show how to identify the cardinality of the latent measures. Next we show how to identify the equilibrium CCPs and state transition using the joint distribution of observables. Then payoff primitives can be identified the same as the scenario without ME or UH. This section mainly focuses on the identification in the first two steps.

We first lay out some assumptions for identifying the equilibrium CCPs and state transition first.

**Assumption 4.** The market observable \( x_t \) and unobservable \( \eta_t \) evolve according to the following rules:

(i). \( \Pr(\eta_t|x_{t-1}, \eta_{t-1}, a_{t-1}, \Omega_{<t-1}) = \Pr(\eta_t|\eta_{t-1}, x_{t-1}, a_{t-1}) \),

(ii). \( \Pr(x_t|\eta_t, x_{t-1}, \eta_{t-1}, a_{t-1}, \Omega_{<t-1}) = \Pr(x_t|\eta_t, x_{t-1}, a_{t-1}) \),

where \( \Omega_{<t-1} = \{x_{t-2}, \eta_{t-2}, a_{t-2}, ..., x_1, \eta_1, a_1\} \), the history up to (but not including) \( t - 1 \).

Assumption 4 (ii) indicates "limited feedback", which rules out direct feedback from the previous unobservable \( \eta_{t-1} \), on the current observable \( x_t \), but allows indirect effect of \( \eta_{t-1} \) through \( x_{t-1} \) and \( a_{t-1} \). Implicitly, this evolution process imposes a timing restriction on the game characteristics, i.e., the unobserved characteristics \( \eta_t \) being realized before the observed characteristics \( x_t \). As a result, \( x_t \) depends on \( \eta_t \) instead of \( \eta_{t-1} \). However, this limited feedback assumption is less restrictive than the assumption made in many applied settings, where the observable \( x_t \) evolves independently from the unobservable \( \eta_t \) of any periods, so that the state transition of observables can be estimated directly from the data. This assumption, however, does rule out the scenario in which the alternative timing occurs. The limited feedback assumption is trivial when the unobserved market type does not vary overtime. This assumption is similar to Assumption 1 in HS. Let \( w_t \) denote the observables in period \( t \),
Lemma 1. Given that Assumption 4 is satisfied, observables and unobservables satisfy the following properties, and the joint distribution of observables satisfy the following representations in a given market.

(i). \( \{ w_t, \eta_t \} \equiv \{ a_t, x_t, \eta_t \} \) follows a stationary first-order Markov process,

(ii). \[ \text{Pr}(w_{t+2}, w_{t+1}, w_t) = \sum_{\eta_{t+1}} \text{Pr}(w_{t+2}| w_{t+1}, \eta_{t+1}) \text{Pr}(w_{t+1}| \eta_{t+1}) \text{Pr}(\eta_{t+1}), \]

(iii). \[ \text{Pr}(w_{t+3}, w_{t+2}, w_{t+1}, w_t) = \sum_{\eta_{t+2}} \text{Pr}(w_{t+3}| w_{t+2}, \eta_{t+2}) \text{Pr}(w_{t+2}| w_{t+1}, \eta_{t+2}) \text{Pr}(w_{t+1}| \eta_{t+2}) \text{Pr}(w_t, \eta_{t+2}). \]

Proof. See Appendix.

Lemma 1 holds for any individual markets over time. Consequently, one can consistently estimate the payoff primitives using a long time series for one individual market without worrying about multiple equilibria. However, the existence of a long time series for even one market is very demanding and limits the scope of the analysis using dynamic games. In most if not all empirical applications, identification and estimation relies on variation across market by pooling data from different markets together. With pooling data, estimation is consistent if the data is generated by a single equilibrium. With the presence of ME, the reduced-form outcome distributions computed from the data represent a mixture of outcome distributions associated with different equilibria. Moreover, estimation becomes more complicated with the existence of UH.

To incorporate both latent variables, we create an overall latent measure, denote as \( \tau = \tau(\eta, e^*) \), where \( \tau(\cdot) \) aggregates the UH (\( \eta \)) and the equilibrium (\( e^* \)). Since both \( \eta \) and \( e^* \) are discrete and finite, so does \( \tau \), and denote \( \tau \)'s cardinality as \( m_{\tau} \). Intuitively, \( \tau \) provides overall information on both latent variables. For instance, suppose \( \eta_{t+1} \) represents the unobserved market demand \{high, low\}, and there
are two equilibria \(\{1, 2\}\). One example of the overall latent factor \(\tau_{t+1}\) is:

\[
\tau_{t+1} = \begin{cases} 
1, & \text{if the market employs equilibrium 1, and the current consumer demand is } \text{high}, \\
2, & \text{if the market employs equilibrium 1, and the current consumer demand is } \text{low}, \\
3, & \text{if the market employs equilibrium 2, and the current consumer demand is } \text{high}, \\
4, & \text{if the market employs equilibrium 2, and the current consumer demand is } \text{low}. 
\end{cases}
\]

Note that the cardinality of the aggregate latent factor \(\tau\) is unknown, while the UH is assumed to be continuous in HS. We first show how to identify the cardinality of \(\tau\). Using lemma 1, we connect the joint distribution of three period’s observables with the components of unknowns in the following equation:

\[
\Pr(w_{t+2}, w_{t+1}, w_t) = \sum_{\tau_{t+1}} \Pr(w_{t+2}|w_{t+1}, \tau_{t+1}) \Pr(w_{t+1}, w_t|\tau_{t+1}) \Pr(\tau_{t+1}). \tag{8}
\]

Note that the state variables two periods apart are independent from each other if the state variable in the intermediate period is fixed when there is not unobserved factor. Intuitively, the correlation between \(w_{t+2}\) and \(w_t\) comes from \(w_{t+1}\) due to the markov property of the game. Consequently, the correlation of \(w_{t+2}\) and \(w_t\) besides \(w_{t+1}\) indicates the presence of some unobserved factors. Moreover, the more correlation between \(w_{t+2}\) and \(w_t\) with \(w_{t+1}\) being control for suggests more variation of the unobserved factors. We show that the cardinality (variation) of the latent measure can be bounded by the rank of the overall matrix constructed by the joint distribution of \(\Pr(w_{t+2}, w_{t+1}, w_t)\) fixing \(w_{t+1}\). Specifically, we rewrite equation 8 into a matrix form:

\[
F_{w_{t+2}, w_{t+1}, w_t} = A_{w_{t+2}|w_{t+1}, \tau_{t+1}} D_{\tau_{t+1}} B_{w_{t+1}, w_t|\tau_{t+1}}, \tag{9}
\]
where

\[
F_{w_{t+2},\bar{w}_{t+1},w_t} \equiv \begin{bmatrix} \Pr \left( w_{t+2} = w^k, \bar{w}_{t+1} = w^j \right) \end{bmatrix}_{k,j},
\]

\[
A_{w_{t+2}|\bar{w}_{t+1},\tau_{t+1}} \equiv \begin{bmatrix} \Pr \left( w_{t+2} = w^k|\bar{w}_{t+1}, \tau_{t+1} = \tau^q \right) \end{bmatrix}_{k,q},
\]

\[
B_{\bar{w}_{t+1},w_t|\tau_{t+1}} \equiv \begin{bmatrix} \Pr \left( \bar{w}_{t+1}, w_t = w^k|\tau_{t+1} = \tau^q \right) \end{bmatrix}_{q,k},
\]

\[
D_{\tau_{t+1}} \equiv \text{diag} \begin{bmatrix} \Pr(\tau_{t+1} = 1) & \ldots & \Pr(\tau_{t+1} = m_{\tau}) \end{bmatrix}.
\]

\(F_{w_{t+2},\bar{w}_{t+1},w_t}\), with dimensions of \(m_w \times m_w\), aggregates the overall correlation of \(w_{t+2}\) and \(w_t\) with \(w_{t+1}\) being fixed. \(A_{w_{t+2}|\bar{w}_{t+1},\tau_{t+1}}\) of dimensions \(m_w \times m_\tau\) provides the variation of \(w_{t+2}\) with \(w_{t+1}\) being fixed. \(B_{\bar{w}_{t+1},w_t|\tau_{t+1}}\) of dimensions \(m_\tau \times m_w\) provides the variation of \(w_t\) with \(w_{t+1}\) being fixed. \(D_{\tau_{t+1}}\) is a diagonal matrix of dimensions \(m_\tau \times m_\tau\) consisting of the marginal distribution \(\Pr(\tau_{t+1})\) as the diagonal elements.

This matrix expression provides information on the cardinality of the aggregate latent variable, which we summarize in the following lemma.\(^6\)

**Lemma 2.** The rank of the observed matrix \(F_{w_{t+2},\bar{w}_{t+1},w_t}\) serves as a lower bound for the cardinality of the aggregated latent variable \(\tau_t\), i.e., \(m_\tau \geq \text{Rank}(F_{w_{t+2},\bar{w}_{t+1},w_t})\). Furthermore, the cardinality \(m_\tau\) is identified, in particular, \(m_\tau = \text{Rank}(F_{w_{t+2},\bar{w}_{t+1},w_t})\), when the following conditions are satisfied:

1. \(m_w > m_\tau\); 2. both matrices \(A_{w_{t+2}|\bar{w}_{t+1},\tau_{t+1}}\) and \(B_{\bar{w}_{t+1},w_t|\tau_{t+1}}\) have full column rank.

**Proof** See Appendix. \(\blacksquare\)

The first condition indicates that the measurement’s cardinality needs to be greater than that of the latent variable. Secondly, the full rank condition indicates the requirement of enough variations in the equilibrium CCPs. This full rank conditions are counterparts of the invertibility assumption in HS, which are equivalent to completeness conditions employed in the nonparametric IV literature. Note that the full rank condition is required for only one value of \(w_{t+1}\). In practice, we can check the rank of the matrix associated with different values of \(w_{t+1}\), and the cardinality can be determined by the

\(^6\)The identification argument is similar to that of Kasahara and Shimotsu (2014).
biggest rank.

Having identified the cardinality of the aggregate latent variable, I now proceed to identify the law of transition for both the observables and unobservables. The identification again relies on the joint distribution of the observables, but requires four periods of data. In this case, the joint distribution of the observables becomes:

$$\Pr(w_{t+3}, w_{t+2}, w_{t+1}, w_t) = \sum_{\tau_{t+2}} \Pr(w_{t+3}|w_{t+2}, \tau_{t+2}) \Pr(w_{t+2}|w_{t+1}, \tau_{t+2}) \Pr(w_{t+1}, w_t, \tau_{t+2}). \quad (10)$$

To use the invertibility of the matrix, we need to partition the state space to create square invertible matrices. Specifically, let $z_t$ represent a potential partition, i.e., $z_t = z(w_t)$, which maps some of the $w_t$ into same values such that the corresponding matrix $F_{z_{t+2}, \bar{w}_{t+1}, z_t}$ is full rank. Note that such a partition always exists, which is sufficient for identification purpose, as in Xiao (2014). Multiple valid partitions result in over-identification. Given a valid partition and fixing $w_{t+2}$ and $w_{t+1}$, we can rewrite equation (10) into the following matrix expression

$$F_{z_{t+3}, \bar{w}_{t+2}, \bar{w}_{t+1}, z_t} = A_{z_{t+3}|\bar{w}_{t+2}, \tau_{t+2}} D_{\bar{w}_{t+2}|\bar{w}_{t+1}, \tau_{t+2}} B_{\bar{w}_{t+1}, z_t, \tau_{t+2}}. \quad (11)$$

Matrices $F_{z_{t+3}, \bar{w}_{t+2}, \bar{w}_{t+1}, z_t}$, $A_{z_{t+3}|\bar{w}_{t+2}, \tau_{t+2}}$, $D_{\bar{w}_{t+2}|\bar{w}_{t+1}, \tau_{t+2}}$, and $B_{\bar{w}_{t+1}, z_t, \tau_{t+2}}$ have dimensions of $m_\tau \times m_\tau$, and the definition of these matrices are similar as above. $\Pr(w_{t+3}|w_{t+2}, \tau_{t+2})$ is identified by evaluating the above joint distribution by varying the fixed pair $(w_{t+2}, w_{t+1})$. That is, four different pairs of $(w_{t+2}, w_{t+1}), (\bar{w}_{t+2}, w_{t+1}), (w_{t+2}, \bar{w}_{t+1}), (\bar{w}_{t+2}, \bar{w}_{t+1})$ generate four matrix equations, which share some common components. Thus, we can form an eigenvalue-eigenvector representation between the observed and unknown matrices in the following:

$$F_{z_{t+3}, \bar{w}_{t+2}, \bar{w}_{t+1}, z_t} F_{z_{t+3}, \bar{w}_{t+2}, \bar{w}_{t+1}, z_t}^{-1} F_{z_{t+3}, \bar{w}_{t+2}, \bar{w}_{t+1}, z_t}^{-1} F_{z_{t+3}, \bar{w}_{t+2}, \bar{w}_{t+1}, z_t}^{-1} F_{z_{t+3}, \bar{w}_{t+2}, \bar{w}_{t+1}, z_t}^{-1} A_{z_{t+3}|\bar{w}_{t+2}, \tau_{t+2}}, \quad (12)$$

where the diagonal matrix $D_{\bar{w}_{t+2}, \bar{w}_{t+1}, \bar{w}_{t+1}, \tau_{t+2}} \equiv \text{diag} \left( \frac{\Pr(w_{t+2}|w_{t+1}, \tau_{t+2}=k) \Pr(w_{t+2}|\bar{w}_{t+1}, \tau_{t+2}=k)}{\Pr(w_{t+2}|\bar{w}_{t+1}, \tau_{t+2}=k) \Pr(w_{t+2}|\bar{w}_{t+1}, \tau_{t+2}=k)} \right)$. 

16
Equation (12) implies that the matrices on both sides of the equation are similar to each other. Note that the matrices on the left-hand side can be computed directly from the data while the matrices on the right-hand side are our identification target. Using the decomposition results, the matrices $A_{z_{t+3}|w_{t+2},\tau_{t+2}}$ and $D_{w_{t+2},w_{t+1},\bar{w}_{t+1}|\tau_{t+2}}$ can be identified as the eigenvectors and eigenvalues of the matrix on the right-hand side, respectively. Decomposition is not unique without the eigenvalues being distinctive, stated in the following conditions:

**Assumption 5.** For a pair of states in period $t+2$ ($\{w_{t+2}, \bar{w}_{t+2}\}$), there exists a pair of states in period $t+1$ ($\{w_{t+1}, \bar{w}_{t+1}\}$), such that

1. *(Positive)* $\Pr(\bar{w}_{t+2}|w_{t+1}, \tau_{t+2})\Pr(w_{t+2}|\bar{w}_{t+1}, \tau_{t+2}) > 0$ for all $\tau_{t+2}$;
2. *(Distinctive)* Eigenvalues are distinctive,

$\frac{\Pr(w_{t+2}|w_{t+1}, \tau_{t+2}, \tau_{t} = \tau_i)\Pr(\bar{w}_{t+2}|\bar{w}_{t+1}, \tau_{t} = \tau_i)}{\Pr(\bar{w}_{t+2}|\bar{w}_{t+1}, \tau_{t} = \tau_i)\Pr(w_{t+2}|w_{t+1}, \tau_{t} = \tau_i)} \neq \frac{\Pr(w_{t+2}|w_{t+1}, \tau_{t+2}, \tau_{t} = \tau_j)\Pr(\bar{w}_{t+2}|\bar{w}_{t+1}, \tau_{t} = \tau_j)}{\Pr(\bar{w}_{t+2}|\bar{w}_{t+1}, \tau_{t} = \tau_j)\Pr(w_{t+2}|w_{t+1}, \tau_{t} = \tau_j)} \forall i \neq j.$

Condition (1) indicates that for any state combination in period $t+2$, there exists a state combination in period $t+1$, such that the transition between these two states is with a positive probability for any type. This condition is not restrictive as it seems. Remember that the goal is to identify the markov law of motions. If for some states, this condition fails, i.e., $\Pr(\bar{w}_{t+2}|w_{t+1}, \tau_{t+2}) = 0$, this implies that the transition between these states are zero too, i.e., $\Pr(\bar{w}_{t+2}, \tau_{t+2}|w_{t+1}, \tau_{t+1}) = 0$. Law of motions is identified trivially.

Condition (2) indicates that the eigenvalues from the representation linking the observed and unobserved matrices are distinctive. This condition is empirically testable because the matrix for the eigen-decomposition can be computed from the data.

**Proposition 1.** *(Identification of $\Pr(w_{t+3}|w_{t+2}, \tau_{t+2})$):* Given that Assumptions 4-5, and the conditions in Lemma 2 are satisfied, the following claims hold using joint distributions of four periods of data.

1. *(Permutation)* $A_{w_{t+3}|w_{t+2},\tau_{t+2}}$ is identified up to a permutation of its columns ($\tau_{t+2}$), $\forall w_{t+2}$.
2. *(The Same permutation)* The identified $A_{w_{t+3}|w_{t+2},\tau_{t+2}}$ can be structured to have the same permutation for all $w_{t+2}$.  

17
**Proof**  See Appendix

Even with a unique decomposition, the identification is not unique in the sense that we still cannot pin down the order of the eigenvectors. The order of the eigenvalues or eigenvectors can be any permutation without further assumptions. That is, eigen-decomposition enables identification of $A_{w_{t+3}|w_{t+2},\tau_{t+2}}I_{w_{t+2}}$ as the eigenvector matrix, where $I_{w_{t+2}}$ is an elementary matrix for interchanging columns of $A_{w_{t+3}|w_{t+2},\tau_{t+2}}$. More importantly, the permutation matrix $I_{w_{t+2}}$ could vary with $w_{t+2}$. This is a prevalent feature in eigenvalue-eigenvector decomposition. A monotonicity assumption, as imposed in HS, is necessary for fixing the order of the eigenvalues and eigenvectors from eigenvalue-eigenvector decomposition in equation (12). Without a similar monotonicity condition, the eigenvector matrix $A_{w_{t+3}|w_{t+2},\tau_{t+2}}$ cannot be identified uniquely.

The second claim of Proposition 1 allows us to circumvent the impact of the permutation on identification of the payoff primitives without imposing a monotonicity condition to fix the order of the eigenvectors. This is different from HS, and the novel aspect of the identification. Relaxing the monotonicity assumption is of practice importance and expands the applicable scope of the identification method in the measurement error literature. The monotonicity assumption is restrictive because $\tau_{t+2}$ combines information from both market-level UH and ME. Moreover, the monotonicity assumption is not necessarily valid, even when the data is generated by a single equilibrium. We use the following simple example to illustrate the importance of fixing the order of the latent measure for identifying the payoff primitives. Suppose $w_t$ and $\tau_t$ are binary, i.e., $w_t \in \{w^1, w^2\}$ and $\tau_t \in \{\tau^1, \tau^2\}$. Fixing $w_{t+2} = w^1$, the eigenvector matrix identified from eigen-decomposing the left-hand side matrix of equation (12) could be

$$A_{w_{t+3}|w_{t+2}=w^1,\tau_{t+2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$ or $$A_{w_{t+3}|w_{t+2}=w^1,\tau_{t+2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Similarly, fixing $w_{t+2} = w^2$, the eigenvector matrix identified from eigen-decomposing the left-hand side matrix of equation (12) could be
Claim 2 makes sure that the permutations are the same for $w^1$ and $w^2$. That is, eigenvector matrices from both decompositions either are
\[
A_{w_{t+3}|w_{t+2}=w^2,\tau_{t+2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad A_{w_{t+3}|w_{t+2}=w^2,\tau_{t+2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]
or
\[
A_{w_{t+3}|w_{t+2}=w^1,\tau_{t+2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad A_{w_{t+3}|w_{t+2}=w^2,\tau_{t+2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]
The first and second column correspond to \{\tau^1, \tau^2\} and \{\tau^2, \tau^1\} in the first and second cases, respectively $w_{t+2} \in \{w^1, w^2\}$. Essentially, claim 2 allows us to keep the order of the latent factors the same across $w_{t+2}$ but unknown, while the monotonicity condition dictates an order by imposing monotonicity, which is empirically untestable. We further show that even the order is unknown, but we can identify the payoff primitives again with order unknown. Without claim 2, which suggests a constructive approach to structure the permutation to be the same for different values of $w_{t+2}$, payoff primitives cannot be identified.

In what follows, we summarize that all relevant components can be identified up to the same order permutation.

**Lemma 3. (Markov Law of Motion)** Given that Assumptions 4-5, and the conditions in Lemma 2 are satisfied, for any combination of \{\( w_{t+3}, w_{t+2} \)\}, the Markov law of motion $A_{w_{t+3},\tau_{t+3}|w_{t+2},\tau_{t+2}}$ can be identified up to permutations of both columns and rows with four periods of data.

**Proof** See Appendix
and rows. The law of motion then can be decomposed as follows:

\[
Pr(w_{t+1}, \tau_{t+1}|w_t, \tau_t) = Pr(a_{t+1}, x_{t+1}, \tau_{t+1}|a_t, x_t, \tau_t) = \frac{Pr(a_{t+1}|x_{t+1}, \tau_{t+1}, a_t) Pr(x_{t+1}|\tau_t, x_t, a_t) Pr(\tau_{t+1}|x_t, a_t, \tau_t)}{\text{equilibrium CCPs \ transition of observables \ transition of unobservables}}
\]

(13)

As a result, the equilibrium CCPs and state transitions can be identified up to the same permutation.

**Lemma 4.** Given that Assumptions 4-5, and the conditions in Lemma 2 are satisfied, the equilibrium CCPs \(Pr(a_{t+1}|x_{t+1}, \tau_{t+1}, a_t)\), and the state transition for observables \(Pr(x_{t+1}|\tau_t, x_t, a_t)\) and unobservables \(Pr(\tau_{t+1}|x_t, a_t, \tau_t)\) can be identified up to a permutation with four periods of data.

**Proof**  See Appendix

With equilibrium CCPs and the state transition identified, payoff primitives can be identified following the existing literature (Bajari et al. (2009)) by treating the type as observed. To non-parametrically identify the payoff functions, the following two assumptions are necessary, which are usually imposed in the existing literature.

**Assumption 6.** \((\text{Normalization})\) For all \(i\), all \(a_{-i}\) and all \(s\), \(\pi_i(a_i = 0, a_{-i}, s) = 0\).

This assumption sets the mean utility from a particular choice as equal to zero, which is similar to the outside good condition in the discrete choice model. This assumption is a conventional assumption imposed in the existing literature.

**Assumption 7.** \((\text{Exclusion Restriction})\) For each player \(i\), the state variable can be partitioned into two parts denoted as \(s = \{s_i, s_{-i}\}\), so that only \(s_i\) enters player \(i\)’s payoff, i.e. \(\pi_i(a_i = k, a_{-i}, s) \equiv \pi_i(a_i = k, a_{-i}, s_i)\).

An example of exclusion restrictions is a covariate, which shifts the profitability of one firm but can be excluded from the profits of all other firms. For example, firm-specific cost shifters are commonly used in empirical work. For example, Jia (2008) and Holmes (2011) demonstrate that distance from firm
headquarters or distribution centers is a cost shifter for big box retailers such as Walmart. Identification of payoff primitives is the same as that without any latent factors. We summarize the identification results in the following Lemma.

**Lemma 5. (Identification of Payoff Primitives)** Given that Assumptions 1-7 and the conditions in lemma 2 are satisfied, type-specific payoff primitives can be non-parametrically identified up to a permutation of the latent variable with four periods of data.

Initial condition $\Pr(w_t, \tau_t)$ plays an important role in simulating the game to do estimation while this information is impossible to obtain from the data. However, following lemma states that the initial condition can also be uniquely recovered as a byproduct from the main identification.

**Lemma 6. (Initial Condition):** Given that Assumptions 4-5, and the conditions in lemma 3 are satisfied, the initial density distribution $\Pr(w_t, \tau_t)$ can be identified up to a permutation of $\tau_t$ from four periods of data.

**Proof** See Appendix

A byproduct of the identification of the initial distribution $\Pr(w_t, \tau_t)$ is the identification of the marginal distribution of the unobservables. The equilibrium selection, therefore, is identified once the unobserved types and ME are identified.

### 4.3 Distinguishing between ME and UH

The model is not fully identified without distinguishing between ME and the UH. In empirical studies, UH refers to some state variables affect players’ payoffs but unobserved to the econometrician. However, another possible source of UH could be that some factors affect the state transition but not the payoff primitives. Consequently, to distinguish ME and UH, we have to consider both potential sources.
4.3.1 Tests based on Non-parametric payoff functions

Note that we can identify both payoff functions and the transition probabilities for observables and unobservables non-parametrically. The essential difference between ME and UH lies in their definitions. Specifically, ME implies that there are more than one solution to the equilibrium conditions. This is equivalent to the same payoff functions map to more than one set of equilibrium CCPs. In contrast, UH by definition indicates that its value affects players’ payoff or the transition probabilities. To illustrate the difference, suppose $\tau = l$ and $\tau = h$ are two equilibria,

$$p^l = \Psi(\pi, g, p^l)$$
$$p^h = \Psi(\pi, g, p^h)$$

The payoff functions $\pi$ and transition probabilities $g$ are the same, but $p^l$ and $p^h$ are two solutions to the equilibrium conditions. As a result, if payoff and transitions can be identified, the payoff and transition probabilities are the same from CCPs $p^l$ and $p^h$.

In contrast, suppose $\tau = l$ and $\tau = h$ are two unobserved types, the equilibrium conditions become

$$p^l = \Psi(\pi^l, g^l, p^l)$$
$$p^h = \Psi(\pi^h, g^h, p^h),$$

which implies that $p^l$ is the equilibrium CCPs associated with type l’s payoff $\pi^l$ and transition $g^l$, while $p^h$ is the equilibrium CCPs associated with type l’s payoff $\pi^h$ and transition $g^h$. Consequently, given that all the sufficient conditions are satisfied for identification, equilibrium CCPs $p^l$ maps to payoffs $\pi^l$ and transition $g^l$, while equilibrium CCPs $p^h$ maps to payoffs $\pi^h$ and transition $g^h$. The payoffs or/and the transitions are different from each other.

As a result, we can conduct the following test to distinguish between ME and UH.

$$H_0 : \quad \pi^l = \pi^h \quad \& \quad g^l = g^h$$
against the alternative

\[ H_1 : \pi^l \neq \pi^h \lor \text{or} \; g^l \neq g^h \]

If we reject the null hypothesis, \( \tau^l \) and \( \tau^h \) are different unobserved heterogeneity. We can further test where exactly the UH come from through conducting the above test separately for payoffs and the transition probabilities. Specifically, to test if payoff functions are the source of UH,

\[ H_0^\pi : \pi^l = \pi^h \]

against the alternative

\[ H_1^\pi : \pi^l \neq \pi^h. \]

Rejecting the null indicates that payoff functions are the source of the UH.

On the other hand, if the source of UH lies in the transition probabilities of exogenous variables instead of the payoff functions, forming a test based on the type-specific transition probability enables to detect the source of UH. Specifically,

\[ H_0^g : g^l = g^h \]

against the alternative

\[ H_1^g : g^l \neq g^h \]

As a byproduct of the comparison of payoffs, the cardinality of the UH can be identified as the number of groups. Additionally, the number of equilibria can be identified as the number of components in each group. Moreover, the equilibrium selection mechanism can be identified through the marginal distribution of \( \tau \), which is a conditional distribution within each group. In another word, all aspects of the game can be identified non-parametrically.
Theorem 2. (Identification of Dynamic Games with Incomplete Information) Given that Assumptions 1-7 and the conditions in lemma 2 are satisfied, the cardinality and initial marginal distribution of the UH, the number of equilibria, the equilibrium selection, players’ equilibrium CCPs, state transitions of observables and unobservables, and payoff primitives are non-parametrically identified in dynamic games with four periods of data.

5 Empirical Application

This section applies the proposed identification method to study the strategic interaction regarding entry in Chinese markets between Kentucky Fried Chicken (KFC) and Mcdonald(McD).

5.1 Data and Industry

5.2 Entry Competition between KFC and MCD

In each period \( t = 1, 2, ..., \infty \), two fast food chains, Kentucky Fried Chicken and Mcdonald’s, \( i \in \{KFC, McD\} \), decide simultaneously whether enter district \( m \) and/or how many new stores to open if it has already been operating in that district. Let \( n_{imt} \in \mathcal{N} \equiv \{0, 1, 2, ..., K\} \) and \( N_{imt} \) denote the number of stores newly added and the outlet stock in market \( m \) in period \( t \), respectively. For ease of notation, we skip the market and time indicator, when there is no ambiguity. The market structure is characterized by all firms’ outlet stocks, \( N = \{N_i\}_i \). Together with market structure, demand and cost shifters denoted as \( z \) also affect firms’ per period payoff. Assume that the market state variable evolve following a first-order markov process. Let \( x \) denote payoff-relevant market states, i.e., \( x \equiv (N, z) \).

The game takes place as follows. In each period, both firms observe market state information \( x_t \). Meanwhile, both firms obtain private payoff shocks which vary with the number of stores they decide to adjust. Firms then determine the number of stores they would like to add into market \( m \). After firms’ decision, the state variable is updated. Following the exiting literature (Assumption 2), we assume that the state evolves according to a first order Markov process.
The payoff shocks enter firm’s per period payoff function additively (Assumption 3) as follows,

$\Delta_i(n_i, x, \epsilon_i) = \pi_i(n_i, x) + \epsilon_i(n_i).$  \hspace{1cm} (14)

Firm $i$ chooses $n_{it}$ to maximize the expected net present value of the stream of profits $\sum_s \beta^{t+s} \Delta_{it+s},$ where $\beta$ is the discount factor. Even though we show that payoff primitives can be identified non-parametrically, we consider estimating the payoff primitives parametrically given the data limitation. We describe the parameterization of the payoff function in the following. In particular, we assume a reduced-form per-period payoff,

$$\pi_i(n_i, x) = \theta_{i,0}(\tau) + \theta_{i,N_1}N_i + \theta_{i,N_i^2}N_i^2 + \theta_{i,N_j}N_j + \theta_{i,N_j^2}N_j^2 + \theta_{i,GDP}GDP - \theta_{i,e}I(N_i = 0)I(n_i > 0) - \theta_{i,n}n_i - \theta_{i,n^2}n_i^2. \hspace{1cm} (15)$$

where $x \equiv \{\tau, N, GDP\},$ and $\theta_{i,0}(\tau)$ captures the effect of the unobserved factor $\tau$ on the chain’s overall payoff. We allow the unobserved factor affects different chains differently. Note that we cannot distinguished where the unobserved factor comes from. It could be either from demand side or supply side.

In addition, we impose the following assumptions regarding the unobserved common market characteristic $\tau$.

**Assumption 8.** The unobserved common market characteristic satisfies the following conditions:

(1). It has a discrete and finite support $\{\tau^1, \tau^2, ..., \tau^l\},$ where $l$ is the cardinality of unobserved factor. Note that $l$ could be 1.

(2). It is time-invariant and is independent of observables (Assumption 4 satisfied trivially).

(3). It does not enter the conditional transition probability, $g(s'|n, x, \tau) = g(s'|n, x).$ (Assuming the heterogeneity is in the payoff functions.)

Allowing for this potential unobserved factor enters into chain’s payoff function is important in the following aspects.
As the existing literature does (Assumption 1), we assume that the payoff shocks are independent across actions, players and over time. If we further assume that payoff shocks follow a type one extreme value distribution, the CCPs have the following closed-form expression:

\[ p_i(n_t|x) = \frac{\exp(V_i(n_t,x))}{\sum_{n_t'} \exp(V_i(n_t',x))} \equiv l_i(n_t|x; \pi_i, p, g), \]

where \( V_i(n_t,x) \) is the choice specific value function, and \( p \) and \( g \) collect the equilibrium CCPs and state transition, respectively.

5.3 Estimation

We estimate the dynamic competition model combining several estimation methods from the literature. In particular, first we estimate the cardinality of the latent factor through estimating the rank of the matrix constructed by the joint distribution of three periods data. The rank of a general matrix is estimated via a sequential test. Second, we estimate the equilibrium CCPs per each value of the latent factor and the profit functions combining the EM algorithm as in Arcidiacono and Miller (2011) and the Nest as in Sweeting (2013). We then distinguish between UH and ME through testing of the associated payoff parameters.

5.3.1 The Number of Latent Type

Note that the rank of the joint distribution matrix \( F_{w_{t+2}, \bar{w}_{t+1}, w_t} \) is an estimator for the cardinality of the latent measure, where \( w_t = \{n_t, N_t, z_t\} \). We can use a simple frequency estimator for the population matrix and denoted as \( \hat{F}_{\bar{w}_{t+2}, \bar{w}_{t+1}, w_t} \). We estimate the rank of matrix \( F_{w_{t+2}, \bar{w}_{t+1}, w_t} = \bar{w}, w_t \) for all possible values of \( \bar{w} \) and use the biggest rank as the estimator of the cardinality of the latent measure.

We estimate the rank of matrix \( F_{w_{t+2}, \bar{w}_{t+1}, w_t} \) using a sequence of tests,\(^7\) as in Robin and Smith (2000). Specifically, the hypotheses are constructed as: \( H_0^r : \text{Rank}(F_{w_{t+2}, \bar{w}_{t+1}, w_t}) = r \) against the alternatives \( H_1^r : \text{Rank}(F_{w_{t+2}, \bar{w}_{t+1}, w_t}) > r \) with \( r = 1, 2, ..., |w_t| - 1 \). This test is based on a characteristic

---

\(^7\)See also Kleibergen and Paap (2006) and Camba-Mendez and Kapetanios (2009) for a review.
root (CR) statistics of the matrix quadratic form, denoted as $\mathcal{CR}_r$. The sequence of tests starts with a null hypothesis of the rank of matrix $F_{\bar{w}_{t+2},\bar{w}_{t+1},w_t}$ being 1. If the null hypothesis is rejected, $r$ is augmented by one and the test is repeated. When I fail to reject the null that the rank equals $r$ for the first time, the rank of $F_{\bar{w}_{t+2},\bar{w}_{t+1},w_t}$ is estimated as $r$.

To guarantee weakly consistency of the rank estimator, we adjust the asymptotic size $\alpha_r$ of the CR test at each stage $r$ to depend on the sample size $M$ with a certain rate. The revised critical region at stage $r$ is given by $\{\mathcal{CR}_r \geq c^r_{1-\alpha_rM}\}$ with the critical value $c^r_{1-\alpha_rM}$ along with an asymptotic size $\alpha_rM$ under the null $H^*_r : \text{Rank}(F_{\bar{w}_{t+2},\bar{w}_{t+1},w_t}) = r, r = 1, 2, ..., |w_t| - 1$. The estimator for the number of equilibria $Q$ then is defined as

$$\hat{r} \equiv \min_{r \in \{1, ..., |w_t| - 1\}} \{r : \mathcal{CR}_r \geq c^i_{1-\alpha_iM}, i = 1, ..., r - 1, \mathcal{CR}_r < c^r_{1-\alpha_rM}\}.$$ 

Note that the test is feasible only when the true rank is strictly less than the cardinality of $w_t$, denoted as $|w_t|$. If we fail to reject the null till $|w_t| - 1$, the matrix is full rank, i.e., $\text{Rank}(F_{\bar{w}_{t+2},\bar{w}_{t+1},w_t}) = |w_t|$. In this case, it is likely that the number of equilibria is greater than $|w_t|$. Thus, we cannot identify the number of equilibria exactly but only its lower bound.

Before the estimation, I process the data as follows. The state variable $x_{mt} = \{N_{mt}, z_{mt}\}$, where $N_{mt} = \{N_{imt}\}_{i=1}^2$ consists of the number of stores of KFC and McD, and $z_{mt}$ consists of the per capita GPD. We discretize the state variable gdp per capital in a way where each bin has a reasonable size of observations but not too coarse so it is still informative. Specifically, we discretize the gdp per capita into four categories based on its quantile. We consider two scenarios for handling the number of stores.

1. the number of stores added in each period and the stock of stores is fewer than 2 and 6 in most of observations (95% percentile), respectively, so we limit the set of entry choose and the stock of stores to $n_{imt} = \{0, 1, 2+\}$ and $N_{imt} = \{0, 1, 2, 3, 4, 5, 6+\}$, respectively.

2. the number of stores added in each period and the stock of stores is fewer than 1 and 2 in most of observations (90% percentile), respectively, so we limit the set of entry choose and the stock of stores to $n_{imt} = \{0, 1+\}$ and $N_{imt} = \{0, 1, 2+\}$, respectively.
For both data processing methods, we estimate the rank to be either 1 or 2 with different values of $w_{t+1}$. For scenario 1, the proportion of the rank estimated as 1 and 2 is 92% and 8%, respectively. For scenario 2, the proportion of the rank estimated as 1 and 2 is 91% and 9%, respectively.

5.3.2 Equilibrium Strategies and Payoff functions - EM Algorithm

Even though the payoff functions can be identified non-parametrically, we parameterize it and denote the structural parameters as $\theta^*_i$, which is allowed to be type-specific. Let $\theta_i \equiv \{\theta^*_i\}$. We assume that the state transition matrix can be estimated directly from the data, and denote it as $\hat{g}$. The equilibrium conditions with latent factor can be expressed as:

$$p_i(n_i|x, \tau) = l_i(n_i|x; \theta_i, \tau, p, g).$$

Following the EM algorithm developed in Arcidiacono and Miller (2011), estimation involves iterating on four steps. We need to first set initial values for $\theta_1$, $\phi^1$ and $p^1$. The $j$th iteration follows:

**Step 1** We first update the equilibrium CCP using the structure of the model as:

$$p_i^{j+1}(n_i|x, \tau) = l_i(n_i|x; \theta_i^j, \tau, p^j, \hat{g}) \quad \forall i. \quad (16)$$

**Step 2** We next update the conditional probability that market $m$ belongs to type $\tau$ using the updated CCPs $p_i^{j+1}(n_i|x, \tau)$ through

$$\phi_{m, \tau}^{j+1} = \frac{\phi^j \Pr(n_{mT}, x_{mT}, ..., n_{m0}, x_{m0}|\tau)}{\sum_{\tau} \phi^j \Pr(n_{mT}, x_{mT}, ..., n_{m0}, x_{m0}|\tau)}$$

$$= \frac{\phi^j \Pi_{t=0}^T \Pi_i p_i^{j+1}(n_{imt}|x_{mt}, \tau)}{\sum_{\tau} \phi^j \Pi_{t=0}^T \Pi_i p_i^{j+1}(n_{imt}|x_{mt}, \tau)}, \quad (17)$$

where $\phi_\tau$ is the unconditional probability that a market belongs to type $\tau$, $\Pr(n_{imt}|x_{mt}, \tau)$ is the equilibrium CCPs that firm $i$ opens $n_{imt}$ number of new stores given state $x_{mt}$ in type-$\tau$ market, and $\Pr(x_{m0})$ is the initial distribution of $x_{m0}$, which is type independent by assumption.
Step 3  We then use the market-specific type probability \( q^j_{m, \tau} \) to compute the population type probability according to:

\[
\varphi^{j+1}_\tau = \frac{1}{M} \sum_m q^{j+1}_{m, \tau},
\]

where \( M \) is the number of markets.

Step 4  we update the structural parameters \( \{\theta^{j+1}_i\}_i \) from

\[
\theta^{j+1}_i = \arg\max_\theta \sum_m \sum_\tau \sum_t q^{j+1}_{m, \tau} \ln[l_i(n_{imt}|x_{mt}; \theta_i, \tau, p^j_\tau, \hat{g})] \quad \forall i.
\]

We constantly update the four components through equations (16) - (19) until they converge.

Initial values for the EM Algorithm  We first estimate the CCPs \( p_i(n_i|x) \) using sieve series expansions as follows. Let \( \{q_l(s), l = 1, 2, \ldots\} \) as the known basis functions and \( k(M) \) the number of basis functions to be used with a sample size of \( M \). \( k(M) \) satisfies the following properties, \( k(T) \to \infty, k(M)/M \to 0 \). Denote the \( 1 \times k(M) \) vector of basis functions as \( q^{k(M)}(x) = (q_1(x), \ldots, q_{k(M)}(x)) \) and its collection into a matrix as \( Q_M = ((q^{k(M)}(x_1))', \ldots, (q^{k(M)}(x_M))')' \). The sieve estimator for the CCPs can be represented as

\[
\hat{p}_i(n_i = k|x) = q^{k(M)}(x)(Q'_M Q_M)^{-1} \sum I(n_{imt} = k)q^{k(M)}(x_{mt})',
\]

which is the standard formula for a linear probability model where the regressors are the sieve functions \( q^{k(M)}(x) \).

From the above rank estimation, the cardinality of the latent factor is 2. We initialize the iterations for the EM Algorithm by a few different ways of setting \( p_i(n_i = n|x, \tau), \varphi(\tau) \). 1. we use the values from the data without unobserved factor and treat them the same. ie., \( p_i(n_i = n|x, \tau) = p_i(n_i = n|x) \), and \( \varphi(\tau) = 1/2. \) 2. we disturb the initial values from (1) randomly. 3. using randomized initial values.
5.4 Findings

6 Conclusion

This paper presents a methodology for non-parametrically identifying finite action games with incomplete information allowing for the presence of ME and UH. Specifically, the cardinality of the overall latent factors can be identified non-parametrically. The law of motion and the equilibrium CCPs which are latent factor variant can also be uniquely recovered. Once the CCPs and transition functions have been identified, the payoffs can be non-parametrically identified with exclusion restrictions. Disentangling equilibria and UH can be obtained by testing the hypothesis that payoffs from different levels of latent variables are the same or different.
Appendix

Following are proofs of lemmas and propositions presented in the paper.

A Proofs

Proof of Lemma 1 We first prove (i): \( \{w_t, \eta_t\} \) follows a stationary first-order Markov chain. The distribution of state variables in period \( t \) conditioning on all of the history can be written as

\[
\Pr(w_t, \eta_t | w_{t-1}, \eta_{t-1}, \Omega_{<t-1})
= \Pr(a_t, x_t, \eta_t | a_{t-1}, x_{t-1}, \eta_{t-1}, \Omega_{<t-1})
= \Pr(a_t | x_t, \eta_t, a_{t-1}, \Omega_{<t-1}) \Pr(x_t | \eta_t, x_{t-1}, a_{t-1}, \Omega_{<t-1}) \Pr(\eta_t | x_{t-1}, \eta_{t-1}, a_{t-1}, \Omega_{<t-1})
= \Pr(a_t | x_t, \eta_t, x_{t-1}, a_{t-1}) \Pr(\eta_t | x_{t-1}, \eta_{t-1}) \Pr(a_{t-1})
= \Pr(w_t, \eta_t | w_{t-1}, \eta_{t-1}).
\]

The third equality holds because of Assumption 4 and the Markov perfect equilibrium assumption. With this Markovian property, we can express the joint distribution of observables in three periods for a given market as follows.

\[
\Pr(w_{t+2}, w_{t+1}, w_t) = \sum_{\eta_{t+1}} \Pr(w_{t+2}, w_{t+1}, \eta_{t+1}, w_t)
= \sum_{\eta_{t+1}} \Pr(w_{t+2} | w_{t+1}, \eta_{t+1}) \Pr(w_{t+1}, \eta_{t+1}) \Pr(\eta_{t+1}).
\] (A.1)
Moreover, with the limited feedback assumption on observed state variable and Markov perfect equilibrium assumption, we can this limited feedback property carries over to \( w_t \).

\[
\Pr(w_{t+2}|\eta_{t+2}, w_{t+1}, \eta_{t+1}) = \Pr(a_{t+2}, x_{t+2}|\eta_{t+2}, a_{t+1}, x_{t+1}, \eta_{t+1}) \\
= \Pr(a_{t+2}|x_{t+2}, \eta_{t+2}, a_{t+1}, x_{t+1}, \eta_{t+1}) \Pr(x_{t+2}|\eta_{t+2}, a_{t+1}, x_{t+1}, \eta_{t+1}) \\
= \Pr(a_{t+2}|x_{t+2}, \eta_{t+2}, a_{t+1}) \Pr(x_{t+2}|\eta_{t+2}, a_{t+1}, x_{t+1}) \\
= \Pr(a_{t+2}, x_{t+2}|\eta_{t+2}, a_{t+1}, x_{t+1}) \\
= \Pr(w_{t+2}|\eta_{t+2}, w_{t+1}). \quad (A.2)
\]

In a given market, the joint distribution of four periods of observables can be represented as follows

\[
\Pr(w_{t+3}, w_{t+2}, w_{t+1}, w_t) \\
= \sum_{\eta_{t+2}, \eta_{t+1}} \Pr(w_{t+3}, w_{t+2}, \eta_{t+2}, w_{t+1}, \eta_{t+1}, w_t) \\
= \sum_{\eta_{t+2}, \eta_{t+1}} \Pr(w_{t+3}|w_{t+2}, \eta_{t+2}) \Pr(w_{t+2}|\eta_{t+2}, w_{t+1}, \eta_{t+1}) \Pr(\eta_{t+2}|w_{t+1}, \eta_{t+1}) \Pr(w_{t+1}, \eta_{t+1}, w_t) \\
= \sum_{\eta_{t+2}, \eta_{t+1}} \Pr(w_{t+3}|w_{t+2}, \eta_{t+2}) \Pr(w_{t+2}|\eta_{t+2}, w_{t+1}) \Pr(\eta_{t+2}|w_{t+1}, \eta_{t+1}) \Pr(w_{t+1}, \eta_{t+1}, w_t) \\
= \sum_{\eta_{t+2}, \eta_{t+1}} \Pr(w_{t+3}|w_{t+2}, \eta_{t+2}) \Pr(w_{t+2}|\eta_{t+2}, w_{t+1}) \Pr(\eta_{t+2}, w_{t+1}, \eta_{t+1}, w_t) \\
= \sum_{\eta_{t+2}} \Pr(w_{t+3}|w_{t+2}, \eta_{t+2}) \Pr(w_{t+2}|\eta_{t+2}, w_{t+1}) \sum_{\eta_{t+1}} \Pr(\eta_{t+2}, w_{t+1}, \eta_{t+1}, w_t) \\
= \sum_{\eta_{t+2}} \Pr(w_{t+3}|w_{t+2}, \eta_{t+2}) \Pr(w_{t+2}|\eta_{t+2}, w_{t+1}) \Pr(\eta_{t+2}, w_{t+1}, w_t) \quad (A.3)
\]

The second equality is due to the first order Markov property. The fourth equality is due to the limited feedback property on observables.
Proof of equation 8

\[
\Pr(w_{t+2}, w_{t+1}, w_t) = \sum_{e^* \in \omega} \Pr(w_{t+2}, w_{t+1}, w_t | e^*) \Pr(e^*) \\
= \sum_{e^* \in \omega} \Pr(w_{t+2}, w_{t+1}, w_t | e^*) \\
= \sum_{e^* \in \omega} \sum_{\eta_{t+1}} \Pr(w_{t+2}, \eta_{t+1}, e^*) \Pr(w_{t+1}, \eta_{t+1}, e^*) \Pr(\eta_{t+1}, e^*) \\
= \sum_{\tau_{t+1}} \Pr(w_{t+2}, \tau_{t+1}) \Pr(w_{t+1}, \tau_{t+1}) \Pr(\tau_{t+1}).
\]

Proof of Lemma 2  Based on the MPE assumption and Assumption 4, we have the following joint distribution

\[
F_{w_{t+2}, \bar{w}_{t+1}, w_t} = A_{w_{t+2}|\bar{w}_{t+1}, \tau_{t+1}} D_{\tau_{t+1}} A_{\bar{w}_{t+1}, w_t|\tau_{t+1}}. \quad (A.4)
\]

Given the assumptions that \(m_w > m_\tau\) and both matrices \(A_{w_{t+2}|\bar{w}_{t+1}, \tau_{t+1}}\) and \(A_{\bar{w}_{t+1}, w_t|\tau_{t+1}}\) are full rank, according to the following inequality regarding the rank of matrix \(F_{w_{t+2}, \bar{w}_{t+1}, w_t}\)

\[
\text{Rank}(A_{w_{t+2}|\bar{w}_{t+1}, \tau_{t+1}}) + \text{Rank}(A_{\bar{w}_{t+1}, w_t|\tau_{t+1}}) - m_\tau \\
\leq F_{w_{t+2}, \bar{w}_{t+1}, w_t} \leq \min\{\text{Rank}(A_{w_{t+2}|\bar{w}_{t+1}, \tau_{t+1}}), \text{Rank}(A_{\bar{w}_{t+1}, w_t|\tau_{t+1}})\}.
\]

we conclude that \(\text{Rank}(F_{w_{t+2}, \bar{w}_{t+1}, w_t}) = m_\tau\).

Proof of Proposition 1  we first show that for any value of \(w_{t+2}\), \(\Pr(z_{t+3}|w_{t+2}, \tau_{t+2})\) can be identified up to ordering (claim 1). Next we move to show that the permutation can be constructed to the same for multiple compositions (claim 2).

Claim 1  Fixing \(w_{t+2}\) and \(w_{t+1}\), matrix \(F_{z_{t+3}, w_{t+2}, w_{t+1}, z_t}\) defined as below, is invertible. As a result, matrices \(A_{w_{t+3}|w_{t+2}, \tau_{t+2}}\) and \(B_{w_{t+3}, w_{t+1}, \tau_{t+2}}\) are also invertible. Evaluating the joint distribution of four
periods of data at the four pairs of points \((w_{t+2}, w_{t+1}), (\bar{w}_{t+2}, w_{t+1}), (w_{t+2}, \bar{w}_{t+1}), (\bar{w}_{t+2}, \bar{w}_{t+1})\), each pair of equations will share one matrix in common. Specifically,

\[
(w_{t+2}, w_{t+1}) : F_{z_{t+3}, w_{t+2}, w_{t+1}, z_t} = A_{z_{t+3}|w_{t+2}, \tau_{t+2}} D_{w_{t+2}|w_{t+1}, \tau_{t+2}} B_{w_{t+1}, z_t, \tau_{t+2}} \tag{A.5}
\]

\[
(\bar{w}_{t+2}, w_{t+1}) : F_{z_{t+3}, \bar{w}_{t+2}, w_{t+1}, z_t} = A_{z_{t+3}|\bar{w}_{t+2}, \tau_{t+2}} D_{\bar{w}_{t+2}|w_{t+1}, \tau_{t+2}} B_{w_{t+1}, z_t, \tau_{t+2}} \tag{A.6}
\]

\[
(w_{t+2}, \bar{w}_{t+1}) : F_{z_{t+3}, w_{t+2}, \bar{w}_{t+1}, z_t} = A_{z_{t+3}|w_{t+2}, \tau_{t+2}} D_{w_{t+2}|\bar{w}_{t+1}, \tau_{t+2}} B_{\bar{w}_{t+1}, z_t, \tau_{t+2}} \tag{A.7}
\]

\[
(\bar{w}_{t+2}, \bar{w}_{t+1}) : F_{z_{t+3}, \bar{w}_{t+2}, \bar{w}_{t+1}, z_t} = A_{z_{t+3}|\bar{w}_{t+2}, \tau_{t+2}} D_{\bar{w}_{t+2}|\bar{w}_{t+1}, \tau_{t+2}} B_{\bar{w}_{t+1}, z_t, \tau_{t+2}} \tag{A.8}
\]

Matrices \(A_{z_{t+3}|w_{t+2}, \tau_{t+2}}\) and \(B_{w_{t+1}, z_t, \tau_{t+2}}\) are invertible by construction. Assume that \(Pr(w_{t+2}|w_{t+1}, \tau_{t+2})\) is positive for every combination of \(w_{t+2}\) and \(w_{t+1}\); then matrix \(D_{w_{t+2}|w_{t+1}, \tau_{t+2}}\) is also invertible. Consequently, we can post-multiply the inverse of equation A.6 to equation A.5, to obtain

\[
Y = F_{z_{t+3}, w_{t+2}, w_{t+1}, z_t} F^{-1}_{z_{t+3}, \bar{w}_{t+2}, w_{t+1}, z_t} = A_{z_{t+3}|w_{t+2}, \tau_{t+2}} D_{w_{t+2}|w_{t+1}, \tau_{t+2}} D^{-1}_{\bar{w}_{t+2}|w_{t+1}, \tau_{t+2}} A^{-1}_{z_{t+3}|\bar{w}_{t+2}, \tau_{t+2}} \tag{A.9}
\]

Similarly,

\[
Z = F_{z_{t+3}, \bar{w}_{t+2}, w_{t+1}, z_t} F^{-1}_{z_{t+3}, \bar{w}_{t+2}, \bar{w}_{t+1}, z_t} = A_{z_{t+3}|\bar{w}_{t+2}, \tau_{t+2}} D_{\bar{w}_{t+2}|w_{t+1}, \tau_{t+2}} D^{-1}_{\bar{w}_{t+2}|\bar{w}_{t+1}, \tau_{t+2}} A^{-1}_{z_{t+3}|\bar{w}_{t+2}, \tau_{t+2}} \tag{A.10}
\]

Consequently, we postmultiply Eq. A.9 by Eq. A.10, leading to

\[
YZ = A_{z_{t+3}|w_{t+2}, \tau_{t+2}} \left( D_{w_{t+2}|w_{t+1}, \tau_{t+2}} D^{-1}_{\bar{w}_{t+2}|w_{t+1}, \tau_{t+2}} D_{\bar{w}_{t+2}|\bar{w}_{t+1}, \tau_{t+2}} D^{-1}_{w_{t+2}|\bar{w}_{t+1}, \tau_{t+2}} \right) A^{-1}_{z_{t+3}|w_{t+2}, \tau_{t+2}}
\]

where

\[
D_{w_{t+2}|w_{t+1}, \tau_{t+2}} D^{-1}_{\bar{w}_{t+2}|w_{t+1}, \tau_{t+2}} D_{\bar{w}_{t+2}|\bar{w}_{t+1}, \tau_{t+2}} D^{-1}_{w_{t+2}|\bar{w}_{t+1}, \tau_{t+2}} = \frac{Pr(w_{t+2}|w_{t+1}, \tau_{t+2}) Pr(\bar{w}_{t+2}|\bar{w}_{t+1}, \tau_{t+2})}{Pr(w_{t+2}|w_{t+1}, \tau_{t+2}) Pr(\bar{w}_{t+2}|\bar{w}_{t+1}, \tau_{t+2})},
\]

The matrix on the left-hand side of equation A.11 can be directly computed from the data, while the matrices on the right-hand size are of particular interest. Moreover, this representation indi-
cates that the joint distribution of observables on the left-hand side of the equation includes the same eigenvalue-eigenvector decomposition of the unknown matrix on the right-hand side. (Hu (2008)) Consequently, $D_{w_{t+2},\bar{w}_{t+2},w_{t+1},\bar{w}_{t+1}|\tau_{t+2}}$ can be identified as eigenvalues up to permutations of columns, and $A_{w_{t+3}|w_{t+2},\tau_{t+2}}$ can be identified as eigenvectors up to scale and permutations of columns. Since each column in the matrix $A_{z_{t+3}|w_{t+2},\tau_{t+2}}$ represents an entire distribution, the column sum should equal to 1, basing on which normalization can be performed. The decomposition leads to a matrix $A_{w_{t+3}|w_{t+2},\tau_{t+2}} I_{w_{t+2}}$, where $I_{w_{t+2}}$ is an elementary matrix generated by interchanging columns of the identify matrix. Note that we have to conduct the decomposition for every possible values of $w_{t+2}$, which may result in different permutations of the columns. That is, the $I_{w_{t+2}}$ matrix varies with $w_{t+2}$.

Given that $\Pr(z_{t+3}|w_{t+2}, \tau_{t+2})$ is identified, we show that $\Pr(w_{t+3}|w_{t+2}, \tau_{t+2})$ is also identified and subjects to a permutation for every value of $w_{t+2}$. Fixing $w_{t+2}$, equation (A.4) leads to

$$F_{z_{t+3},w_{t+2},z_{t+1}} = A_{z_{t+3}|w_{t+2},\tau_{t+2}} A_{w_{t+2},\tau_{t+2},z_{t+1}},$$

$$F_{w_{t+3},w_{t+2},z_{t+1}} = A_{w_{t+3}|w_{t+2},\tau_{t+2}} A_{w_{t+2},\tau_{t+2},z_{t+1}}.$$  

Thus, matrix $A_{w_{t+3}|w_{t+2},\tau_{t+2}}$ can be identified up to a permutation of columns through the following equation

$$A_{w_{t+3}|w_{t+2},\tau_{t+2}} I_{w_{t+2}} = F_{w_{t+3},w_{t+2},\tau_{t+2}} F_{z_{t+3},w_{t+2},z_{t+1}}^{-1} A_{z_{t+3}|w_{t+2},\tau_{t+2}} I_{w_{t+2}}.$$  

Claim 2 We now move to show that the permutation matrix $I_{w_{t+2}}$ can be constructed to be the same different values of $w_{t+2}$. Matrix $A_{w_{t+3}|w_{t+2},\tau_{t+2}}$ is identified through evaluating the joint distribution of four periods of data at four pairs of points $(w_{t+2}, w_{t+1}), (\bar{w}_{t+2}, w_{t+1}), (w_{t+2}, \bar{w}_{t+1}), (\bar{w}_{t+2}, \bar{w}_{t+1})$ and as the eigenvectors of the matrix on the left-hand side based on Equation A.11.

Similarly, postmultiplying Eq. A.10 by Eq. A.9 leads to

$$ZY = A_{z_{t+3}|\bar{w}_{t+2},\tau_{t+2}} \left(D_{\bar{w}_{t+2},\bar{w}_{t+1},\tau_{t+2}} D_{w_{t+2},\bar{w}_{t+1},\tau_{t+2}}^{-1} D_{w_{t+2},\bar{w}_{t+1},\tau_{t+2}}^{-1} D_{\bar{w}_{t+2},\bar{w}_{t+1},\tau_{t+2}}^{-1} A_{w_{t+3}|\bar{w}_{t+2},\tau_{t+2}}^{-1}\right),$$  

where

$$A_{w_{t+3}|\bar{w}_{t+2},\tau_{t+2}} = A_{w_{t+3}|w_{t+2},\tau_{t+2}}.$$  

(A.12)
\[D_{\bar{w}_{t+2}, w_{t+2}, \bar{w}_{t+1}, w_{t+1} \mid \tau_{t+2}} = \frac{Pr(w_{t+2} \mid w_{t+1}, \tau_{t+2}) Pr(\bar{w}_{t+2} \mid \bar{w}_{t+1}, \tau_{t+2})}{Pr(\bar{w}_{t+2} \mid w_{t+1}, \tau_{t+2}) Pr(w_{t+2} \mid \bar{w}_{t+1}, \tau_{t+2})}.\]

Importantly, the diagonal matrix for the right hand-side of equations A.11 and A.12 are the same. This feature enables us to make sure the permutation matrix \(I_{w_{t+2}}\) and \(I_{\bar{w}_{t+2}}\) are the same. Specifically, from Equations A.11 and A.12, we identify

\[(A_{w_{t+3}} w_{t+2}, \tau_{t+2} I_{w_{t+2}}, I_{w_{t+2}} D_{w_{t+2}} I_{w_{t+2}}^{-1}),\]

and

\[(A_{\bar{w}_{t+3}} \bar{w}_{t+2}, \tau_{t+2} I_{\bar{w}_{t+2}}, I_{\bar{w}_{t+2}} D_{\bar{w}_{t+2}} I_{\bar{w}_{t+2}}^{-1}),\]

respectively, where the permutation might vary with \(w_{t+2}\). Note that the eigenvalue matrices are the same by construction, i.e., \(D_{w_{t+2}} = D_{\bar{w}_{t+2}}\). If we force the eigenvalue matrices from the decomposition to be the same \((I_{w_{t+2}} D_{w_{t+2}} I_{w_{t+2}}^{-1} = I_{\bar{w}_{t+2}} D_{\bar{w}_{t+2}} I_{\bar{w}_{t+2}}^{-1})\), the permutations in the two cases are the same \((I_{w_{t+2}} = I_{\bar{w}_{t+2}})\). Consequently, exploring the identification structure, the permutations of the latent variable can be preserved for different values of observables \(w_{t+2}\).

To illustrate the intuition, assume that the latent factor \(\tau \in \{H, L\}\) is time invariant. Specifically, to identify \(Pr(z_{t+3} \mid w_{t+2} = 0, \tau)\), we do eigen-decomposition with respect to the observed matrix \(YZ\), leading to two possible results

\[
YZ = \begin{bmatrix}
Pr(z_{t+3} \mid 0, \tau = L) & Pr(z_{t+3} \mid 0, \tau = H)
\end{bmatrix}
\begin{pmatrix}
f(\tau = L) & 0 \\
0 & f(\tau = H)
\end{pmatrix}
egin{pmatrix}
Pr(z_{t+3} \mid 0, \tau = L)^T \\
Pr(z_{t+3} \mid 0, \tau = H)^T
\end{pmatrix},
\]

or

\[
YZ = \begin{bmatrix}
Pr(z_{t+3} \mid 0, \tau = H) & Pr(z_{t+3} \mid 0, \tau = L)
\end{bmatrix}
\begin{pmatrix}
f(\tau = H) & 0 \\
0 & f(\tau = L)
\end{pmatrix}
egin{pmatrix}
Pr(z_{t+3} \mid 0, \tau = H)^T \\
Pr(z_{t+3} \mid 0, \tau = L)^T
\end{pmatrix}.
\]
Similarly, decomposition with respect to the observed matrix $ZY$ leads to

$$ZY = \begin{bmatrix} \Pr(z_{t+3}|1, \tau = L) & \Pr(z_{t+3}|1, \tau = H) \end{bmatrix} \begin{pmatrix} f(\tau = L) & 0 \\ 0 & f(\tau = H) \end{pmatrix} \begin{pmatrix} \Pr(z_{t+3}|1, \tau = L)^T \\ \Pr(z_{t+3}|1, \tau = H)^T \end{pmatrix},$$

or

$$ZY = \begin{bmatrix} \Pr(z_{t+3}|1, \tau = H) & \Pr(z_{t+3}|1, \tau = L) \end{bmatrix} \begin{pmatrix} f(\tau = H) & 0 \\ 0 & f(\tau = L) \end{pmatrix} \begin{pmatrix} \Pr(z_{t+3}|1, \tau = H)^T \\ \Pr(z_{t+3}|1, \tau = L)^T \end{pmatrix}.$$  

Without information from the eigenvalue matrix, there are four possible matches after the decomposition such as \{[L H][L H]\}, \{[L H][H L]\}, \{[H L][LH]\}, and \{[H L][H L]\}, among which matches \{[L H][L H]\} and \{[H L][L H]\} are consistent, while \{[L H][H L]\} and \{[H L][LH]\} are inconsistent. However, the diagonal matrices from the two compositions should be the same when there are consistent matches in both cases. As a result, we can rule out the inconsistent matches case of \{[L H][H L]\}, \{[H L][LH]\}. Yet, without further assumptions, we are not able to recover the order of the unobserved latent factor; thus, we still have the relabeling issue, but it does not affect the identification of the payoff primitives.

Furthermore, for other values of $w_{t+2} = \bar{w}_{t+2}$, one can uses a similar logic in exploring the four pairs of $(w_{t+2}, w_{t+1}), (w_{t+2}, \bar{w}_{t+1}), (w_{t+2}, \bar{w}_{t+1}), (\bar{w}_{t+2}, \bar{w}_{t+1})$. □

**Proof of Lemma 3: (Identification of the Law of Motion)** Again, with four periods of data, the joint distribution of observables can be factorized as the components that we want to identify in the following equations:

$$\Pr(z_{t+3}, w_{t+2}, w_{t+1}, z_t) = \sum_{\tau_{t+2}} \Pr(z_{t+3}|w_{t+2}, \tau_{t+2}) \Pr(w_{t+2}, \tau_{t+2}, w_{t+1}, z_t), \quad (A.13)$$

$$\Pr(w_{t+2}, \tau_{t+2}, w_{t+1}, z_t) = \sum_{\tau_{t+1}} \Pr(w_{t+2}, \tau_{t+2}, w_{t+1}, \tau_{t+1}, z_t)$$

$$\quad = \sum_{\tau_{t+1}} \Pr(w_{t+2}, \tau_{t+2}|w_{t+1}, \tau_{t+1}) \Pr(w_{t+1}, \tau_{t+1}, z_t). \quad (A.14)$$

Fixing $w_{t+2} = \bar{w}_{t+2}$ and $w_{t+1} = \bar{w}_{t+1}$, and rewrite above equations into a matrix format similar to that
defined at the outset:

\[ F_{z_{t+3}, \bar{w}_{t+2}, \bar{w}_{t+1}, z_t} = A_{z_{t+3}|\bar{w}_{t+2}, \tau_{t+2}} B_{\bar{w}_{t+2}, \tau_{t+2}, \bar{w}_{t+1}, z_t}, \quad (A.15) \]

\[ B_{\bar{w}_{t+2}, \tau_{t+2}, \bar{w}_{t+1}, z_t} = A_{\bar{w}_{t+2}, \tau_{t+2}|\bar{w}_{t+1}, \tau_{t+1}} A_{\bar{w}_{t+1}, \tau_{t+1}, z_t}. \quad (A.16) \]

Consequently, we have

\[ F_{z_{t+3}, \bar{w}_{t+2}, \bar{w}_{t+1}, z_t} = A_{z_{t+3}|\bar{w}_{t+2}, \tau_{t+2}} A_{\bar{w}_{t+2}, \tau_{t+2}|\bar{w}_{t+1}, \tau_{t+1}} A_{\bar{w}_{t+1}, \tau_{t+1}, z_t}. \quad (A.17) \]

We show in lemma 1 that \( A_{z_{t+3}|\bar{w}_{t+2}, \tau_{t+2}} \) is identified up to a permutation of its columns; that is, we can only know \( A_{z_{t+3}|\bar{w}_{t+2}, \tau_{t+2}} I_{w_{t+2}} \). Moreover, based on Lemma 2, the permutation matrix \( I_{w_{t+2}} \) is invariant of \( W_{t+2} \). Additionally, the left-hand side matrix can be computed from the data.

Identification of the law of motion depends on the identification of matrix \( A_{\bar{w}_{t+1}, \tau_{t+1}, z_t} \), because both \( A_{\bar{w}_{t+1}, \tau_{t+1}, z_t} \) and \( A_{z_{t+3}|\bar{w}_{t+2}, \tau_{t+2}} \) are invertible by construction. We next show that \( A_{\bar{w}_{t+1}, \tau_{t+1}, z_t} \) can be identified again up to a permutation through the following equation:

\[ \Pr(z_{t+2}, w_{t+1}, z_t) = \sum_{\tau_{t+1}} \Pr(z_{t+2}|w_{t+1}, \tau_{t+1}) \Pr(w_{t+1}, \tau_{t+1}, z_t). \quad (A.18) \]

Using a similar logic, fixing \( w_{t+1} = \bar{w}_{t+1} \), the above equation’s matrix counterpart is as follows:

\[ F_{z_{t+3}, \bar{w}_{t+1}, z_t} = A_{z_{t+3}|\bar{w}_{t+1}, \tau_{t+1}} A_{\bar{w}_{t+1}, \tau_{t+1}, z_t}. \quad (A.19) \]

\( A_{w_{t+2}|\bar{w}_{t+1}, \tau_{t+1}} \) is identified up to a permutation of \( I_{w_{t+2}} \) by the stationary assumption and is invertible. That is, we can identify \( A_{w_{t+2}|\bar{w}_{t+1}, \tau_{t+1}} I_{w_{t+2}} \). Consequently, matrix \( A_{\bar{w}_{t+1}, \tau_{t+1}, z_t} \) is identified up to a permutation of rows. That is, \( I_{w_{t+2}}^{-1} A_{\bar{w}_{t+1}, \tau_{t+1}, z_t} \) is identified. Substituting the identified matrix back to equation A.17, we have

\[ F_{z_{t+3}, \bar{w}_{t+2}, \bar{w}_{t+1}, z_t} = A_{z_{t+3}|\bar{w}_{t+2}, \tau_{t+2}} I_{w_{t+2}}^{-1} A_{\bar{w}_{t+2}, \tau_{t+2}|\bar{w}_{t+1}, \tau_{t+1}} I_{w_{t+2}}^{-1} I_{w_{t+2}}^{-1} A_{\bar{w}_{t+1}, \tau_{t+1}, z_t}. \quad (A.20) \]
Consequently, the law of motion is identified up to a permutation of both rows and columns:

\[ I_{w_{t+2}}^{-1} A_{\bar{w}_{t+2}, \tau_{t+2}} \bar{w}_{t+1} \tau_{t+1} I_{w_{t+2}}. \]

\[ \blacksquare \]

**Proof of Lemma 6: (Identification of the Initial Condition)** Given that we have already identified the transition matrix \( Pr(w_{t+1}|w_t, \tau_t) \), the following equation provides identification of the initial distribution

\[ \text{Pr}(z_{t+1}, w_t) = \sum_{\tau_t} \text{Pr}(z_{t+1}|w_t, \tau_t) \text{Pr}(w_t, \tau_t). \] (A.21)

Fix \( w_t = \bar{w}_t \), and rewrite above equation in matrix format

\[ V_{z_{t+1}, \bar{w}_t} = A_{z_{t+1}|\bar{w}_t, \tau_t} V_{\bar{w}_t, \tau_t}. \] (A.22)

Since \( A_{z_{t+1}|\bar{w}_t, \tau_t} \) is invertible and identified up to a permutation of columns, \( V_{\bar{w}_t, \tau_t} \) is identified up to a permutation of rows for all \( w_t \).

\[ \blacksquare \]

**Proof of Lemma 4: (Identification of the Policy Functions and Transition Functions)** With the identification of the Law of Motion, we can easily identify the transition functions for unobservables \( \tau_{t+1} \) through the following equations:

\[ \text{Pr}(\tau_{t+1}|x_t, \tau_t, a_t) = \sum_{x_{t+1}, a_{t+1}} \text{Pr}(x_{t+1}, \tau_{t+1}, a_{t+1}|x_t, \tau_t, a_t) = \sum_{w_{t+1}} \text{Pr}(w_{t+1}, \tau_{t+1}|w_t, \tau_t), \] (A.23)

Note that the policy function \( \text{Pr}(\tau_{t+1}|x_t, \tau_t, a_t) \) is identified up to a permutation of \( \tau_{t+1} \) and \( \tau_t \).

The transition function for the observed state variable can be identified up to a permutation of \( \tau_{t+1} \) through the following equation:

\[ \text{Pr}(x_{t+1}|\tau_{t+1}, x_t, a_t) = \frac{\text{Pr}(x_{t+1}, \tau_{t+1}|x_t, a_t, \tau_t)}{\text{Pr}(\tau_{t+1}|x_t, \tau_t, a_t)} = \frac{\text{Pr}(w_{t+1}|w_t, \tau_t)}{\text{Pr}(\tau_{t+1}|x_t, \tau_t, a_t)}. \] (A.24)
The policy function \( \Pr(a_{t+1}|x_{t+1}, \tau_{t+1}, a_t) \) can be identified up to a permutation of \( \tau_{t+1} \) through the following equation:

\[
\Pr(a_{t+1}|x_{t+1}, \tau_{t+1}, a_t) = \frac{\Pr(a_{t+1}, x_{t+1}, \tau_{t+1}|a_t, x_t, \tau_t)}{\sum_{a_{t+1}} \Pr(a_{t+1}, x_{t+1}, \tau_{t+1}|a_t, x_t, \tau_t)}.
\]

(A.25)

Proof of equation (17)

\[
g_{m,t}^{j+1} = \frac{\varphi_j \Pr(n_{mT}, x_{mT}, ..., n_{m0}, x_{m0}|\tau)}{\sum_{\tau} \varphi_j \Pr(n_{mT}, x_{mT}, ..., n_{m0}, x_{m0}|\tau)}
\]

\[
= \frac{\varphi_j \Pi_{t=0}^T \Pr(n_{mt}|x_{mt}, \tau) \Pi_{t=1}^T \Pr(x_{mt}|x_{mt-1}, n_{mt-1}) \Pr(x_{m0})}{\sum_{\tau} \varphi_j \Pi_{t=0}^T \Pr(n_{mt}|x_{mt}, \tau) \Pi_{t=1}^T \Pr(x_{mt}|x_{mt-1}, n_{mt-1}) \Pr(x_{m0})}
\]

\[
= \frac{\varphi_j \Pi_{t=0}^T \Pi_i p_i(n_{mt}|x_{mt}, \tau)}{\sum_{\tau} \varphi_j \Pi_{t=0}^T \Pi_i p_i(n_{mt}|x_{mt}, \tau)}
\]

\[
= \frac{\varphi_j \Pi_{t=0}^T \Pi_i l_i(n_{mt}|x_{mt}; \theta_i, \tilde{\theta}_i, p_i \hat{\theta}_i, \hat{\theta}_i \hat{g})}{\sum_{\tau} \varphi_j \Pi_{t=0}^T \Pi_i l_i(n_{mt}|x_{mt}; \theta_i, \tilde{\theta}_i, p_i \hat{\theta}_i, \hat{\theta}_i \hat{g})}
\]

\[
= \frac{\varphi_j \Pi_{t=0}^T \Pi_i p_{i+1}^{j+1}(n_{mt}|x_{mt}, \tau)}{\sum_{\tau} \varphi_j \Pi_{t=0}^T \Pi_i p_{i+1}^{j+1}(n_{mt}|x_{mt}, \tau)}
\]

References


