Revealed Growth: A Method for Bounding the Elasticity of Demand
with an Application to Assessing the Antitrust Remedy in the Du Pont Decision

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Abstract: We propose a method for bounding the elasticity of demand that applies to growing, homogeneous-product markets which requires only minimal data—market price and quantity over a time span as short as two periods. Reminiscent of revealed-preference arguments using choices over time to bound the shape of indifference curves, we use shifts in the equilibrium over time to bound the shape of the demand curve under the assumption that growing demand curves do not cross. We apply the method to assess the effectiveness of the antitrust remedy in the 1952 Du Pont decision, ordering the incumbent manufacturers to license their patents for commercial plastics. Commentators have suggested that the incumbents may have preserved the monopoly outcome by gaming the licensing contracts. The upper bounds on demand elasticities that we compute are close to zero in many post-remedy years. Such extremely inelastic demand is inconsistent with monopoly, suggesting the remedy may have been effective. The bounds are consistent across functional forms (linear, logit) and products. Non-local information (such as the position of linear-demand intercepts) can be exploited to tighten the bounds but at considerable sacrifice of robustness.


Keywords: demand estimation · bounds · elasticity · antitrust · plastics · remedy

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I just want to say one word to you. Just one word. . . . Plastics.

(The Graduate [motion picture], 1967)

1. Introduction

In antitrust cases, whether structural remedies are effective in encouraging competition has been an ongoing concern for scholars and policymakers. Adams (1951) was an early skeptic. He highlighted a series of earlier cases, including monopolization cases, that were won by the antitrust agency but the imposed remedies failed to restore competition, the so-called pyrrhic victories of antitrust.

To understand why promising remedies might fail to achieve their goals, consider the example of the remedy in the Du Pont decision of 1952, compelling the incumbent manufacturers in the commercial plastics market to license their manufacturing patents to all applicants. As Areeda, Kaplow, and Edlin (2013) note, the remedy typically does not set the licensing terms, leaving them subject to the commercial negotiations of the parties. This leaves open the possibility for incumbent manufacturers to preserve something close to the monopoly outcome by specifying exorbitant rates. Another possibility is that the incumbents set rates that a judge or other outside party might deem “reasonable” but do not transfer adequate tacit knowledge (beyond what is written in the patents) that would allow entrants to compete on an even footing with the incumbents.

In this paper, we study the effectiveness of the antitrust remedy ordered in the aforementioned Du Pont decision. The case involved two incumbent chemical manufacturers: the U.S. firm Du Pont and the U.K. firm Imperial Chemical Industries (ICI). The two firms signed a Patents and Processes agreement in 1929, granting each company exclusive licenses for the patents and secret processes controlled by the other and dividing the global market into exclusive territories between them. The U.S. government brought suit under Section 1 of the Sherman Act for what they alleged was an illegal market division. District judge Sylvester Ryan ruled in favor of the government, ordering the defendants to cancel their exclusive-territory arrangements requiring them to license the patents behind several of their products, most significantly, two types of plastic widely used for commercial purposes: low-density polyethylene, used to make Tupperware food-storage containers, and high-density polyethylene, used to make Hula Hoops.
On the surface, the remedy appears to have had the desired procompetitive effect (Backman 1964, p. 71). Eleven manufacturers entered by the end of the decade; prices steadily declined and output rose. However, the same price declines and output increases may have arisen in a monopoly market experiencing substantial cost declines, plausibly realistic for plastics in the 1950s and 60s. The entrants may have merely been producing their share of the monopoly quantity, returning most of the rents to the incumbents.

Our study will provide formal evidence for the effectiveness of the remedy that cuts through these criticisms. Formal study is hindered by a paucity of data for this historical application, just yearly aggregate price and quantity data for polyethylene, and only in years after the remedy was ordered. We offer a new methodology that allows us to draw solid conclusions from these fairly minimal data. The methodology—which we will explain intuitively later in this introduction—provides an upper bound on the elasticity of demand in a given year assuming that demand is growing so that the demand curve in a given year is higher than in previous years and lower than in later years. While bounding methodologies that seem initially promising may turn out to produce such wide bounds as to be uninformative, the bounds we find in the polyethylene market are tightly concentrated around zero in many years, implying that demand cannot be far from perfectly inelastic. Such extremely inelastic demand is inconsistent with monopoly, suggesting the remedy may have been effective. Thus our formal evidence lends support to the use of structural remedies in monopolization cases.

Although the Du Pont decision and implications for antitrust motivate our paper, the bulk of the paper is methodological, focusing on developing the methodology for bounding the elasticity of demand using time-series information on market equilibrium prices and quantities. The methodology applies to markets that, like polyethylene, involve homogeneous products and are growing. Figure 1 can be used to explain how the method works intuitively. In the example in the figure, the researcher has price and quantity data for two years. The equilibrium point in the first year is $e_1$ and in the second is, say, $e_2'$. The researcher wants to bound the steepness of the inverse demand curve in period 1 on which $e_1$ lies. Even if the researcher is willing to assume demand is linear (at least locally, to a first approximation), with so little data, there may be little hope to say anything more than inverse demand lies somewhere between the horizontal dotted line, corresponding to an infinitely elastic demand curve, and the dotted vertical line, corresponding to an infinitely inelastic
one. But in fact we can say more. If demand is assumed to be nondecreasing over time, a curve like $D$ is ruled out because that would put the later equilibrium point $e'_2$ on a lower demand curve. The demand curve through $e_1$ must be at least as steep as the line connecting $e_1$ and $e'_2$—the line labeled $D'$—to preserve nondecreasing demand. The comparison of the two equilibrium points leads to a lower bound on the steepness of inverse demand and an upper bound on the elasticity of demand through $e_1$.

![Figure 1: Intuition for the Use of Time-Series Data to Bound Demand Elasticity](image)

The tightness of the bound is data-driven. Suppose that the observed equilibrium point is $e''_2$ in period 2 rather than $e'_2$. This leads to a tighter bound on the elasticity. The inverse demand through $e_2$ must be at least as steep as $D''$ to keep $e''_2$ from lying on a lower demand curve, violating nondecreasing demand. The funnel in which the demand curve must lie narrows from the entire shaded region to just the dark-shaded part, only leaving room for a relatively inelastic demand. If we find inverse demand this steep—and demand this inelastic—in applications, we may be able to rule out monopoly. Monopoly may be inconsistent with a steep drop from $e_1$ to a point like $e''_2$. A monopolist would never drop price for such a small increase in quantity, even if marginal costs had dropped from something positive to zero. The more likely conclusion is that it was competitive pressure that forced firms out into the inelastic part of the demand curve.

In the example in Figure 1, we have assumed that demand is linear. We develop variants of the
methodology that can narrow the bounds on the demand elasticity considerably if the researcher is willing to assume the whole demand curve is linear, from the vertical to the horizontal intercept. The variants incorporate information from the intercepts of other year’s demands to iteratively narrow the bounds on the elasticity in a given year. We redo the analysis for another widely assumed functional form, logit demand. We also generalize the method to work for any functional form the researcher chooses.

Applying our methodology to the polyethylene market after the Du Pont decision, our results in many years rule out elastic demand, and in some years only allow for extremely inelastic demand. For example, we find that absolute value of the elasticity of demand, $\varepsilon_t$, for low-density polyethylene is bounded in $[0, 0.09]$ in 1959 and $[0, 0.08]$ in 1960. These findings are robust, the same whether we assume linear or logit demand or whether we compare those years to all other years or more local comparisons to the two neighboring years. Those are the tightest bounds we find but find that $\varepsilon_t$ is bounded well away from unit elastic in all but one year, again holding for a variety of functional forms and variants. The method incorporating intercept information with the linear demand assumption powerfully narrows down the elasticity bounds, but they are so much narrower than the logit bounds that we question whether the use of nonlocal information produces trustworthy results. Perhaps projecting a given functional form too far outside of the range of the observations strains the assumption.

Our work is related to several literatures. One is the literature assessing the effectiveness of structural remedies for antitrust. We already mentioned Adams’ (1951) pioneering study. Studies by the U.S Federal Trade Commission (FTC) staff make important contributions to the literature as well. These studies include U.S. Federal Trade Commission (1999), examining the efficacy of divestiture remedies accompanying mergers approved in 1990 to 1994; Farrell, Pautler, and Vita (2009), describing a similar FTC exercise, but in this case examining ex post outcomes of hospital mergers that had been allowed by the judiciary; and U.S. Federal Trade Commission (2017), studying all 89 FTC merger orders from 2006–12. This last study judged remedies to be a success in 69 percent of cases and a qualified success in 14 percent of cases. Asker (2014) articulates the importance of remedy in merger cases.

We also contribute to the literature on patent licensing. This is a vast literature; most germane are papers on patent pools and also interesting work using historical evidence to shed light on
contemporary issues in IP.

Jean Tirole in his Nobel lecture besides surveying his theoretical contributions on intellectual property and patent pools also points out this industry and legal development. “A little known fact is that, prior to 1945, most high-tech industries of the time were run by patent pools. But the worry about cartelization through joint marketing led to a hostile decision of the US Supreme Court in 1945 and the disappearance of pools until the recent revival of interest.” ¹ Of course the chemical industry is a high tech industry of the recent past.

Lerner, Strojwas and Tirole (2007) indicate patent pools are understudied empirically. We agree. Their paper models the licensing or contractual terms that pools would offer if they had complementary patents which solves double marginalization or even worse, or a pool that involves substitute patents, which is much more likely to be anticompetitive. As they point out good pools and bad pools might be indistinguishable to a court or policymaker. The authors model and empirics focuses on whether the pool allows independent licensing of patents or also grantback requirements both of which are more likely to be complementary patents. Their results are consistent with their theoretical predictions. Their sample may include Dupont ICI but their work is complementary to ours since we focus more on remedy than the contractual terms of the pool.

In a recent paper, Watzinger, Fackler, Nagler and Schnitzer (2017) use the 1956 Bell System consent decree to examine induced innovation. Like Du pont, Bell was a monopoly albeit a regulated one. The Truman administration filed a monopolization suit against Bell in 1949, which argued that it had foreclosed entry into various related markets such as telecom equipment. In the consent decree Bell was allowed to remain vertically integrated into telecom but they had to freely license all existing licenses royalty free.

The authors find that Follow-on innovation expands as a consequence of the consent decree. They find no expansion in innovation in the telecom sector, but they do find enhanced innovation in non telecom sectors. This is a very interesting paper with great data work.

Stocking and Watkins (1946) classify the Dupont ICI patents and processes agreement as a patent pool and describe some interesting details.²

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THING “...du Pont licenses, or agrees upon request to license, ICI to use all its present and future patents, and secret “know how” for the manufacture and sale within the British Empire, of for example, dyestuffs and explosives, and ICI licenses du Pont similarly for the United States. This is a cartel agreement in fact, whatever it may be its legal status. Each of the parties has the right to use both its own and its licensee’s patents and processes only in its allotted territory.”

Reader devotes several chapters of volume II to a critical evaluation of the U.S. government’s case against ICI and dupont. Although we disagree with Reader’s evaluation, his two volume history of ICI remains a classic scholarly work. He argues that the prosecution arose out of political pressure at the Department of Justice. (p. 417). He criticizes the extra territorial reach of US antitrust law (p.431, p. 440); of Americans “trying to convert the rest of the world to the gospel of anti-trust” (p. 423). He indicates that ICI executives thought any prosecution in the US would stop with Dupont and not reach ICI. (p.419). But of course if the dupont ICI agreement were collusive in intent and effect, bringing a case against ICI does not seem a dramatic stretch of the law.

Hounshell and Smith, criticize the courts verdict for a different reason. “Judge Ryan … refused to accept the argument that ICI’s and Du Pont’s agreement resulted in the genuine exchange of scientific and technical information. But the overwhelming historical evidence demonstrates this was indeed the case.” We find the agreements division of markets geographically to be strong evidence in favor of a collusive intent.

Lampe and Moser (2013) study the first US patent pool the sewing machine pool formed in 1856. They find the creation of the pool is associated with a reduction in innovation by the substitute technology to the pool technology. And welfare consequences as they note depend on the source of this effect. Historical evidence suggests the pool affected innovation through both litigation risks but also differential licence fees for members and non members.

Does intellectual property concerns always trump antitrust, or does antitrust always trumps intellectual property concerns? Louis Kaplow in his (1984) reappraisal of the antitrust patent intersection indicated it depends. Our paper focuses on how the freeing of licenses affected product market competition but we take the patent system as given.

We also contribute to the empirical industrial organization literature developing bounds rather than point estimates on elasticities and other parameters [CITE].

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3Hounshell and Smith, p. 206.
The paper is structured as follows. The first part of the paper covers methodology. Section 2 models the situation to which the methodology will be applied. Section 3 presents methods a researcher would use assuming linear demand, Section 4 logit demand, and Section 5 general demand specifications. Then the paper moves to our application. Section 6 provides background on the Du Pont decision, Section 7 describes our data, and Section 8 presents our results. Section 9 returns to the methodology, extending it to a range of applications beyond those like the one we study, first to markets experiencing a weaker form of expansion (technically referred to as rectangular expansion) and then to markets moving in the opposite direction—to declining markets. The last section concludes. An appendix provides proofs not included in the text.

2. Model

This section lays out a general model of a growing market for a homogeneous product. Each period $t$, the interaction between producers and consumers on the market leads to an equilibrium $e_t \equiv (q_t, p_t)$, where $q_t \geq 0$ is market quantity and $p_t \geq 0$ is market price. The researcher observes the market equilibrium over some time span $t \in \{1, \ldots, T\}$. Let $E \equiv \{e_t | t = 1, \ldots, T\}$ denote the set of time-series observations of equilibrium.

The analysis can be streamlined without much loss of generality if ties between price observations or quantity observations are ruled out, accomplished by assuming $E$ is distinct according to the following definition.

**Definition.** $E$ is distinct if and only if, for all $e_t, e_{t'} \in E$ such that $t \neq t'$, we have $q_t \neq q_{t'}$ and $p_t \neq p_{t'}$.

Assuming $E$ is distinct entails little loss of generality if one supposes that two observations of, say, quantity over time are never exactly equal if measurements are carried out to arbitrary precision.

Consider each side of the market in turn starting with producers. Since the goal of our application will be to determine whether the antitrust remedy was effective in changing producer conduct, it is natural to consider producer conduct as an unknown to be determined. Thus, we will be fairly agnostic about producer behavior in the model. Producers may engage in perfect competition, in which case it may be possible to characterize their behavior with a supply curve; or they may engage in some form of imperfect competition, perhaps monopoly, whose behavior is characterized
by a supply relation deriving from a first-order condition.

Next, consider the consumer side of the market. Consumers are assumed to be price takers whose behavior is captured by the demand curve \( q = D_t(p) \) or equivalently by the inverse demand curve \( p = P_t(q) \equiv D_t^{-1}(q) \). We maintain the assumption that the law of demand holds, formally that \( D_t(p) \) is nonincreasing in \( p \). Expressed equivalently in terms of the inverse demand curve, the law of demand implies

\[
P_t(q') \geq P_t(q'') \quad \forall t \in \{1, \ldots, T\} \text{ and } \forall q'' > q' \geq 0. \tag{1}
\]

We embody the assumption of a growing market with the formal condition that \( D_t(p) \) is nondecreasing in \( t \). Expressed equivalently in terms of the inverse demand curve, a growing market implies

\[
P_t'(q) \leq P_t''(q) \quad \forall t', t'' \in \{1, \ldots, T\} \text{ such that } t'' > t' \text{ and } \forall q \geq 0. \tag{2}
\]

Not all configurations of equilibria \( E \) are consistent with growing demand. To aid the discussion of which inconsistent configurations we will rule out, it is helpful to introduce notation for subsets determined by compass directions relative to a reference equilibrium point, \( e_t \):

\[
\begin{align*}
NW(e_t) &\equiv \{(q_{t'}, p_{t'}) \in E \mid q_{t'} < q_t, p_{t'} > p_t\} \\
NE(e_t) &\equiv \{(q_{t'}, p_{t'}) \in E \mid q_{t'} > q_t, p_{t'} > p_t\} \\
SE(e_t) &\equiv \{(q_{t'}, p_{t'}) \in E \mid q_{t'} > q_t, p_{t'} < p_t\} \\
SW(e_t) &\equiv \{(q_{t'}, p_{t'}) \in E \mid q_{t'} < q_t, p_{t'} < p_t\}.
\end{align*} \tag{3}
\]

Each equilibrium point divides the nonnegative quadrant into four regions corresponding to four relative compass directions; our notation provides a label for these subregions. Figure 2 depicts these compass sets. The fact that the definitions in (3) involve strict inequalities leaves points on the dotted lines through \( e_t \) in the figure unclassified, but this is without loss of generality for distinct \( E \), which precludes equilibrium points from sharing coordinates.

Consider adding an equilibrium point observed after \( e_t \) to one of the compass sets in the figure. A downward-sloping inverse demand curve through \( e_t \) (such as the solid curve drawn in the figure) slices through regions \( NW(e_t) \) and \( SE(e_t) \). This leaves room to add the new equilibrium point so
that it lies on a higher demand curve—thus respecting assumption (2) that demand is nondecreasing over time—yet still falls in \( NW(e_t) \) or \( SE(e_t) \). The new equilibrium could also easily be added in \( NE(e_t) \) as well, placed on a higher inverse demand in that region. However, there is no way to add the new equilibrium point to \( SW(e_t) \) without having it lie on a lower demand curve. Equilibria cannot move in a south-westerly direction over time if the market is growing. In that case, we say \( E \) exhibits \textit{rectangular expansion} because equilibrium points observed after a given one \( e_t \) lie outside the rectangle determined by the \( e_t \) and the origin at opposite corners. As one can readily see in the figure, rectangular expansion is equivalent to later equilibria having at least one component (quantity or price) greater than earlier equilibria.

**Definition.** \( E \) exhibits \textit{rectangular expansion} if and only if for all \( t', t'' \in \{1, \ldots, T\} \) such that \( t' < t'' \), \( q_{t''} \geq q_{t'} \) or \( p_{t''} \geq p_{t'} \).

The analysis can be further streamlined by imposing yet more structure on \( E \). For a later equilibrium point to appear in \( NW(e_t) \), there must have been a backward shift in the supply relation. Assuming a growing market means that neither demand curves nor supply relations shift backward, one can exclude \( e_{t'} \in NW(e_t) \) for \( t' > t \). Having excluded \( e_{t'} \in SW(e_t) \) and \( e_{t'} \in NW(e_t) \), we are left with \( e_t \in NE(e_t) \cup SE(e_t) \), or in other words with later equilibria always lying to the east of earlier ones. But this is equivalent to nondecreasing quantity over time. We will call this property \textit{quantity expansion}. 

\[ p \]

\[ q_t \]

\[ q_t \]

\[ p_t \]

\[ e_t \]

\[ NW(e_t) \]

\[ NE(e_t) \]

\[ SW(e_t) \]

\[ SE(e_t) \]

**Figure 2: Sets Determined by Compass Position**
**Definition.** 

$E$ exhibits quantity expansion if and only if for all $t', t'' \in \{1, \ldots, T\}$ such that $t' < t''$, we have $q_{t''} \geq q_{t'}$. 

We will require $E$ to exhibit quantity expansion throughout most of the subsequent analysis. There is no loss from doing so in our application because all of the equilibrium points in our data that exhibit rectangular expansion exhibit quantity expansion. More generally, some applications will involve an $E$ exhibiting the weaker rectangular expansion but not the stronger quantity expansion. To cover those cases, we provide further analysis of rectangular expansion Section ???. There we show that later equilibrium points in $\text{NW}(e_t)$ do not contribute to tightening the upper bound on the demand elasticity but do contribute to tightening the lower bound.4,5

### 3. Methods for Linear Demand

Our methods for bounding demand elasticities requires the researcher to impose functional-form assumptions on demand. We begin the discussion by examining the simple case in which the sequence of inverse demand curves are all assumed to be linear: $P_t(q) = \max(0, \alpha_t - \beta_t q)$ for $t = 1, \ldots, T$. Note that the parameters $\alpha_t$ and $\beta_t$ are subscripted by $t$ and thus allowed to vary over time, in turn allowing the demand curve to shift over time.

Although inverse demand appears to involve two independent parameters, given the line must pass through the equilibrium point $e_t$, the line is pinned down to a one-parameter family. We will often focus on the vertical intercept of inverse demand, $\alpha_t$, as this key parameter. With knowledge of $\alpha_t$ and the equilibrium point $e_t = (q_t, p_t)$, one can solve for the other parameter,

$$\beta_t = \frac{\alpha_t - p_t}{q_t},$$

4In our application, the upper bound on the demand elasticity is of chief interest because that is what allows us to rule out post-remedy monopoly conduct. Thus another justification for requiring $E$ to exhibit quantity expansion in our application is that equilibrium points excluded by quantity expansion but admitted by rectangular expansion do not contribute to tightening the upper bound of interest.

5Inflation could increase nominal input prices, leading to a backward shift in the supply relation measured in nominal terms even if the supply relation shifts right in real terms. In some applications, observations that violate quantity expansion in nominal terms may satisfy it when suitably deflated. Since there is no difference between rectangular and quantity expansion even in our nominal data, we avoid the potentially controversial issue of choosing an appropriate deflator by conducting the analysis in nominal terms.
for the horizontal intercept of inverse demand,

$$\frac{\alpha_t q_t}{\alpha_t - p_t},$$

or for the absolute value of the demand elasticity,

$$\epsilon_t = \frac{p_t}{-P'(q_t)q_t} = \frac{p_t}{\alpha_t - p_t}.$$

(5) (6)

For brevity, we will drop the “absolute value” modifier and simply call $\epsilon_t$ the demand elasticity. The law of demand, assumption (1), ensures $\epsilon_t$ is nonnegative. By inverting equations (4)–(6), we can easily translate claims about one of these variables into claims about another. For example, if we are able to compute a lower bound on $\alpha_t$, this can be translated into an upper bound on $\epsilon_t$ using equation (6).

### 3.1. Comparing Equilibrium Pairs

We first present a simple, robust method for bounding the vertical intercept of inverse demand and thus the demand elasticity. This method uses only the information from pairwise comparisons of equilibrium points. Later, we will derive tighter bounds by including information from comparisons of equilibrium points with bounds on intercepts of other demand curves.

Figure 3 illustrates how the procedure works in an example with four equilibrium points. Suppose we want to bound $\alpha_3$, the vertical intercept of inverse demand through point $e_3$. We pair $e_3$ with each of the equilibrium points and see where the lines $\ell_{13}$, $\ell_{23}$, $\ell_{33}$, and $\ell_{34}$ through each pair of points intersects the vertical axis. The maximum of these intersections provides a lower bound on the vertical intercept. In the figure, the maximum of these intersections, denoted $\alpha_3^*$, is generated by the line $\ell_{34}$ through $e_3$ and $e_4$. The vertical intercept, $\alpha_3$, of the linear inverse demand through $e_3$ must be at least as high as $\alpha_3^*$. If $\alpha_3$ were any lower, the inverse demand through $e_3$ would pass above $e_4$, violating assumption (2) of nondecreasing demand over time.

Moving from the example to a general description of the procedure, let $v(e_t, e_{t'})$ be the vertical
intercept of the line through equilibrium points \( e_t = (q_t, p_t) \) and \( e'_t = (q'_t, p'_t) \). One can show

\[
v(e_t, e'_t) = \frac{p_t q'_t - p'_t q_t}{q'_t - q_t}
\]  

(7)

for \( e_t, e'_t \in E \) and \( t \neq t' \). Given that \( e_t, e'_t \in E \) and that \( E \) is admissible, we have \( q_t \neq q'_t \) for \( t \neq t' \), implying the denominator in (7) is non-zero. The definition of \( v \) can be extended to cover the case of the vertical intercept of a line through a point paired with itself:

\[
v(e_t, e_t) = p_t,
\]  

(8)

equal to the height of the horizontal line—an infinitely elastic inverse demand curve—through the equilibrium point. It is easy to verify from equations (7) and (8) that \( v \) is a symmetric function, i.e.,

\[
v(e_t, e'_t) = v(e'_t, e_t).
\]  

(9)

Define \( \alpha_t^* \) as the maximum of the vertical intercepts formed by pairing \( e_t \) with each of the equilibrium points:

\[
\alpha_t^* \equiv \max_{t' \in \{1, \ldots, T\}} v(e_t, e'_t).
\]  

(10)

The next proposition states that, under maintained assumptions, \( \alpha_t^* \) provides a lower bound on \( \alpha_t \).
Proposition 1. Consider a sequence of linear inverse demands $P_t(q) = \max(0, \alpha_t - \beta_t q)$, $t = 1, \ldots, T$, that is consistent with a distinct $E$ exhibiting quantity expansion. This sequence satisfies assumptions (1) and (2)—respectively, the law of demand and that demand is nondecreasing over time—only if $\alpha_t \geq \alpha_t^*$. 

The intuition behind the proposition is similar to that behind the example in Figure 3. A formal proof is not required here because Proposition 1 follows from subsequent results. In particular, Proposition 4 shows that $\alpha_t^{**}$, a bound proposed in the next subsection, is indeed a lower bound on $\alpha_t$. Proposition 5 then shows $\alpha_t^* \leq \alpha_t^{**}$. Combining the two results proves that $\alpha_t^*$ is a lower bound on $\alpha_t$.

An obvious shortcut can be taken in computing $\alpha_t^*$, which is most easily discussed with some additional notation. Define

$$NW^-(e_t) \equiv \{ e_{t'} \in NW(e_t) \mid t' < t \}$$
$$NE^+(e_t) \equiv \{ e_{t'} \in NE(e_t) \mid t' > t \}$$
$$SW^-(e_t) \equiv \{ e_{t'} \in SW(e_t) \mid t' < t \}$$
$$SE^+(e_t) \equiv \{ e_{t'} \in SE(e_t) \mid t' > t \}.$$ (11)

In words, $NW^-(e_t)$ is the subset of equilibrium points in $NW(e_t)$ that are later than $e_t$, and the other sets can be similarly described. The analogous definitions for four other subsets—$NW^+(e_t)$, $NE^-(e_t)$, $SW^+(e_t)$, and $SE^-(e_t)$—are deferred to later because they are empty when $E$ exhibits quantity expansion, as we now maintain. Rather than pairing $e_t$ with equilibrium points in all four of the sets defined in (11) as the definition of $\alpha_t^*$ in (10) prescribes, the sets $NE^+(e_t)$ and $SW^-(e_t)$ can be ignored. The line from points in those sets through $e_t$ slopes down, so the vertical intercept of that line lies below $p_t = v(e_t, e_t)$. As long as $e_t$ itself is included in the comparison set, the only other sets that need to be considered are $NW^-(e_t)$ and $SE^+(e_t)$.

Proposition 2. An equivalent way to compute $\alpha_t^*$ defined in (10) is

$$\alpha_t^* = \max_{e_{t'} \in NW^-(e_t) \cup SE^+(e_t) \cup \{e_t\}} v(e_t, e_{t'}).$$ (12)

Besides streamlining the calculations, the proposition emphasizes which pairwise comparisons contribute useful information to tightening the bound on $\alpha_t$. 

13
3.2. Iteratively Incorporating Intercept Information

The lower bound on $\alpha_t$ can be tightened by iteratively incorporating information about the position of intercepts of demand curves through the equilibrium points. Figure 4 illustrates how intercept information can help in a simple example with three equilibrium points. Suppose we are interested in bounding the vertical intercept $\alpha_1$ of the inverse demand curve $P_1(q)$ through point $e_1$. The top panel shows the bound derived from lines through $e_1$ and the other equilibrium points. The highest intercept is generated by the line $\ell_{13}$ through $e_1$ and $e_3$, giving us the bound $\alpha^*_1$. This is just another example the procedure discussed in the previous subsection using pairwise comparisons.

The lower panel demonstrates that this bound on $\alpha_1$ can be tightened by including intercept
information on demand curves through other points. The dotted line $\ell_{23}$ through points $e_2$ and $e_3$ has been added to the lower panel. If $P_3(q)$, the inverse demand through $e_3$, were not at least as steep as $\ell_{23}$, $P_3(q)$ would lie below $e_2$, violating the assumption that demand is nondecreasing over time. Thus, the horizontal intercept of $P_3(q)$ cannot be to the right of the horizontal intercept of $\ell_{23}$. But this means that $P_3(q)$ cuts $P_1(q)$, violating the assumption that demand is nondecreasing over time, unless $P_1(q)$ is steeper than $\ell_{13}$. In fact, $P_1(q)$ has to be at least as steep as the line from the horizontal intercept of $\ell_{23}$ through $e_1$, generating a new bound on the intercept of $P_1(q)$, denoted $\alpha_{1^{**}}$.

The position of the inverse demand curve through $e_1$ is constrained not just by the location of $e_2$ and $e_3$ but by the intercepts of the demand curves through these other points. In the example in the figure, the additional intercept information allowed us to tighten the bound on $\alpha_1$ from $\alpha_1^*$ to $\alpha_{1^{**}}$.

Moving from the example to a general description of the procedure, we first introduce an alternative formalization of the process of pairwise comparison already from equation (10). Although it generates the same bound $\alpha_t^*$ as we had before, the alternative formalization is more easily nested in the iterative procedure incorporating intercept information that will generate a new bound. To this end, let $h(e_t,e_t')$ be the horizontal intercept of the ray that originates at $e_t = (q_t, p_t)$ and passes through $e_t' = (q_t', p_t')$. We have

$$h(e_t,e_t') = \begin{cases} \frac{p_t q_t' - p_t' q_t}{p_t - p_t'} & \text{if } e_t' \in SW(e_t) \cup SE(e_t) \\ \infty & \text{else.} \end{cases}$$

(13)

Since $h$ involves a directed ray, it generates a finite value only if the ray originates at a point $e_t$ that is north of the other point $e_t'$ through which the ray passes. One can verify that the fraction in equation (13) is indeed the intersection of the ray with the horizontal axis when pointed that way. If $e_t$ equals or is south of $e_t'$, then the ray shoots off without intersecting the horizontal axis. In that case, an undefined (infinite) value is assigned to $h$.

The following equations take the extremes of collections of intercept projections.

$$V_t^{[1]} \equiv \max_{t' \leq t} v(e_t,e_t')$$

(14)
We use \( \lor \) to denote the maximum operator, as opposed to \( \land \), which we will use to denote the minimum operator. In (14), we draw lines from \( e_t \) through earlier equilibria to the vertical axis and take the maximum intercept. In (15), we project rays from \( e_t \) down through later equilibria to the horizontal axis and take the minimum. In (16), we project the minimum horizontal intercept from (15) back through \( e_t \) to get the maximum vertical intercept generated by pairing \( e_t \) with these later equilibria. Then we take the overall maximum of the intercepts generated by pairings with earlier and later equilibria. Though somewhat convoluted, it is intuitive that this process arrives at the same bound \( \alpha_t^* \) as we had in equation (10), where we just projected lines through \( e_t \) and the other equilibrium points in one direction toward the vertical axis rather than back and forth. The following proposition, proved in the appendix, formalizes this intuition.

**Proposition 3.** \( \alpha_t^* = C_t^{[1]} \).

Equations (14)–(16) provide the first iteration of the procedure. We iterate again, taking the maximum of the lower bounds on the vertical intercepts for earlier equilibrium points, the minimum of the upper bounds on the horizontal intercepts for later equilibrium points, and combining this information to ensure that the linear inverse demand through \( e_t \) does not intersect lower or higher demand curves. This leads to a new lower bound on \( \alpha_t \) given by \( \alpha_t^{* *} = C_t^{[2]} \), where

\[
H_t^{[1]} \equiv \min_{t' \geq t} h(e_t, e_{t'}) \tag{15}
\]

\[
C_t^{[1]} \equiv V_t^{[1]} \lor v(e_t, (H_t^{[1]}, 0)) \tag{16}
\]

\[
H_t^{[2]} \equiv \min_{t' \geq t} H_t^{[1]} \tag{18}
\]

\[
C_t^{[2]} \equiv V_t^{[2]} \lor v(e_t, (H_t^{[2]}, 0)) \tag{19}
\]

Despite their complicated appearance, equations (17)–(19) are just a formalization of the procedure bounding the inverse demand intercept in Figure 4. This is also the procedure we will use in the application. That is, we will augment \( \alpha_t^* \) with an additional iteration using additional intercept information to compute the new bound \( \alpha_t^{* * } \). The next proposition verifies that \( \alpha_t^{* * } \) is indeed a lower bound on \( \alpha_t \). The intuition for the proof is similar to that behind Figure 4. Formalizing the
intuition becomes quite involved, however, because the system of recursive equations needs to be unraveled; so the proof is relegated to the appendix.

Proposition 4. Consider a sequence of linear inverse demands $P_i(q) = \max(0, \alpha_i - \beta_i q)$, $t = 1, \ldots, T$, that is consistent with a distinct $E$ exhibiting quantity expansion. This sequence satisfies assumptions (1) and (2)—respectively, the law of demand and that demand is nondecreasing over time—only if $\alpha_t \geq \alpha_i^{**}$.

The next proposition states that $\alpha_i^{**}$ provides at least as tight a bound as $\alpha_i^*$.

Proposition 5. $\alpha_i^{**} \geq \alpha_i^*$.

Proof. Equations (17) and (18) imply $V_{i,2}^{[2]} \geq V_{i,1}^{[1]}$ and $H_{i,2}^{[2]} \geq H_{i,1}^{[1]}$. But then $C_{i,2}^{[2]} \geq C_{i,1}^{[1]}$ by equations (16) and (19). Thus, $\alpha_i^{**} \equiv C_{i,2}^{[2]} \geq C_{i,1}^{[1]} = \alpha_i^*$. Q.E.D.

Proposition 5 leaves open the possibility that the bounds $\alpha_i^*$ and $\alpha_i^{**}$ are equally good. However, Figure 4 provides a case in which $\alpha_1^{**} > \alpha_1^*$, proving that $\alpha_i^{**}$ is sometimes tighter than $\alpha_i^*$.

In principle, one does not have to stop at the second iteration. One could continue iterating an arbitrary number, $k$, times:

$$V_{i,k} \equiv \max_{t' \leq t} V_{i,k-1}$$

$$H_{i,k} \equiv \min_{t' \geq t} H_{i,k-1}$$

$$C_{i,k} \equiv V_{i,k} \lor v(e_t, (H_{i,k}, 0))$$

A potentially tighter bound on the vertical intercept could then be obtained by taking the process to the extreme: $C_{i,\infty} \equiv \lim_{k \to \infty} C_{i,k}$.

Though our analysis of $C_{i,\infty}$ is not finished, we have developed several conjectures. First, we conjecture that $C_{i,\infty}$ indeed is a lower bound on $\alpha_t$. Second, we conjecture that the bound provided by $C_{i,\infty}$ is tight; equivalently, the sequence of demand curves with vertical intercepts $\alpha_t = C_{i,\infty}$ can rationalize the set of observed equilibria $E$. Third, we conjecture that the lower bound is achieved in a finite number of iterations: $C_{i,\infty} = C_{i,k}$ for finite $k$. Indeed, we conjecture that two iterations may be sufficient: $C_{i,\infty} = C_{i,2}^{[2]} = \alpha_i^{**}$. Until we verify these conjectures, the tightest bound we will employ in the application is $\alpha_i^{**}$.
Substituting lower bound, \( \alpha_t^* \) on the inverse-demand intercept into equation (6) yields a corresponding upper bound on the demand elasticity: \( \epsilon_t^* = p_t / (\alpha_t^* - p_t) \). Substituting the tighter lower bound, \( \alpha_t^{**} \) yields a correspondingly tighter upper bound on the demand elasticity: \( \epsilon_t^{**} = p_t / (\alpha_t^{**} - p_t) \).

4. Method for Logit Demand

Instead of imposing linear demand, the researcher may choose to impose another popular functional form on demand, logit, microfounded by McFadden (1973), widely used in structural estimation of differentiated-product demand since Berry, Levinsohn, Pakes (1995). In the context of a homogeneous product market under study, logit demand can be specified as

\[
D_t(p) = \frac{n_t \exp(-a_t p)}{1 + \exp(-a_t p)} = \frac{n_t}{1 + \exp(a_t p)},
\]

where \( n_t \) is interpreted as a market-size parameter and \( a_t \) as a price-sensitivity parameter. For demand to be nonnegative, \( n_t \geq 0 \); for the law of demand to hold, \( a_t \geq 0 \). As in our specification of linear demand, in our specification of logit demand, the subscript \( t \) on the parameters allows them to vary over time and for demand to shift over time.

Although demand appears to involve two independent parameters, the fact that the curve must pass through the equilibrium point \( e_t \) pins it down to a one-parameter family. With knowledge of \( a_t \) and the equilibrium point \( (q_t, p_t) \), equation (23) can be solved for the market-size parameter:

\[
n_t = q_t [1 + \exp(a_t p_t)].
\]

Substituting for \( n_t \) from equation (24) into (23) yields an expression for logit demand in terms of the single unknown parameter \( a_t \) and known equilibrium point \( e_t = (q_t, p_t) \):

\[
D_t(p) = \frac{q_t [1 + \exp(a_t p_t)]}{1 + \exp(a_t p)}.
\]

We have slightly abused notation, suppressing the equilibrium point as an argument of \( D_t \), to keep the notation concise.
The analysis will focus on bounding \( a_t \). Bounds on \( a_t \) can then be translated into bounds on \( \epsilon_t \), the absolute value of the demand elasticity, via the formula

\[
\epsilon_t = \frac{-p_t D_t'(p_t)}{q_t} = \frac{-a_t p_t}{1 + \exp(-a_t p_t)}.
\] (26)

As we did initially in the linear-demand case, in the present logit-demand case, we will derive bounds from pairwise comparisons of equilibrium points. As there are no intercepts with this functional form, the is no further information to use to tighten the bound, so our analysis will stop there.

Consider the equilibrium points, \( e_t \) and \( e_{t'} \), associated with time periods \( t < t' \). For demand to be nondecreasing over time, we must have \( D_t(p) \leq D_{t'}(p) \) for all \( p \geq 0 \). In particular, the inequality must hold for \( p = p_{t'} \), implying \( D_t(p_{t'}) \leq D_{t'}(p_{t'}) = p_{t'} \). Substituting \( p_{t'} \) into equation (25) to find \( D_t(p_{t'}) \), substituting the resulting expression into the preceding inequality, and rearranging, the condition becomes

\[
\frac{1 + \exp(a_t p_{t'})}{1 + \exp(a_t p_t)} \leq \frac{q_{t'}}{q_t}.
\] (27)

Assuming the equilibrium set \( E \) is distinct and exhibits quantity expansion, \( e_{t'} \in NE^+(e_t) \cup SE^+(e_t) \) for \( t' > t \). If \( e_{t'} \in NE^+(e_t) \), it is easy to see that (27) holds for all \( a_t \geq 0 \). Thus that case does not refine the bound on \( a_t \). However, if \( e_{t'} \in SE^+(e_t) \), it can be shown that there exists a unique value of \( a_t \) solving (27) as an equality and that this value is an upper bound on the set of \( a_t \) satisfying (27): values of \( a_t \) below the threshold also satisfy (27) and values of \( a_t \) above this threshold violate (27).

One can similarly compare equilibrium points \( e_t \) and \( e_{t'} \) with \( t > t' \). The assumption that demand is nondecreasing over time can be shown to imply

\[
\frac{1 + \exp(a_t p_{t'})}{1 + \exp(a_t p_t)} \leq \frac{q_{t}}{q_{t'}}.
\] (28)

When \( t > t' \), equilibrium points \( e_{t'} \in NW^-(e_t) \) are the ones that inform the bound on \( a_t \), not those in \( SW^-(e_t) \). Just as in the previous paragraph, the solution to (28) as an equality provides an upper bound on the \( a_t \) satisfying (28), and thus an upper bound on the \( a_t \) preserving the assumption that demand is nondecreasing over time.
Each pairwise comparison of \( e_t \) to earlier and later equilibrium points provides an upper bound on \( a_t \). The lowest of these, which we will denote \( a_t^* \), provides the tightest upper bound. The following proposition combines the separate analyses for earlier and later equilibria. The proof in the appendix fills in missing details from the analysis.

**Proposition 6.** Suppose the set of equilibrium points \( E \) is distinct and exhibits quantity expansion. Let \( A(e_t, e_{t'}) \) be a solution for \( a \) in

\[
\frac{1 + \exp(a(p_t \lor p_{t'}))}{1 + \exp(a(p_t \land p_{t'}))} = \frac{q_t \lor q_{t'}}{q_t \land q_{t'}},
\]

where \( e_t, e_{t'} \in E, t \neq t' \). For \( e_{t'} \in SE^+(e_t) \) and \( e_{t'} \in NW^-(e_t) \), \( A(e_t, e_{t'}) \) exists and is unique. Defining

\[
a_t^* = \min_{e_{t'} \in SE^+(e_t) \cup NW^-(e_t)} A(e_t, e_{t'}),
\]

the sequence of logit demands defined in (25) satisfies assumptions (1) and (2)—respectively, the law of demand and that demand is nondecreasing over time—only if \( a_t \leq a_t^* \).

Equation (29) is simpler than it appears: for \( t' > t \), it reduces to (27) treated as an equality; and for \( t' < t \), it reduces to (28) treated as an equality. Given data on any pair of equilibrium points \( e_t, e_{t'} \), equation (29) is a nonlinear equation readily solved by standard software.

Proposition 6 prescribes the following procedure for computing \( a_t^* \). The researcher pairs the equilibrium point \( e_t \) in his or her data with earlier equilibria in \( NW(e_t) \) and later equilibria in \( SE(e_t) \). The equilibrium points are substituted into (29) and the resulting nonlinear equation solved for \( a \). The researcher takes the minimum of these solutions as the upper bound \( a_t^* \) on the price-sensitivity parameter. Substituting \( a_t^* \) into equation (26) yields a corresponding upper bound on the demand elasticity: \( \epsilon_t^* = -a_t^* p_t/[1 + \exp(-a_t^* p_t)] \).

**5. Method for General Functional Forms**

The method for logit demand is readily generalizable to a broad class of functional forms. Suppose the researcher specifies the form for demand \( q = D_t(p) \). This demand curve may start out as a multiple-parameter family, but assume that once it is required to pass through \( e_t \), this pins it down to a one-parameter family indexed by \( \theta_t \). Thus we have

\[
D_t(p) = D(p, \theta_t, e_t).
\]

(31)
Assume $D(p, \theta, e)$ is continuously differentiable of all orders in all arguments. Without loss of generality (as it is an arbitrary “accounting convention”), assume that increases in $\theta \in [0, \infty)$ cause inverse demand to be less steep, consistent with the behavior of the price-sensitivity parameter $a_t$ in the logit case. Translating from inverse demand to a derivative condition on demand itself, this assumption means
\[
\frac{\partial^2 D(p, \theta, e)}{\partial p \partial \theta} \leq 0. \tag{32}
\]
Finally, we will impose the following Inada-type conditions:
\[
\lim_{\theta \to 0} \frac{\partial D(p, \theta, e)}{\partial p} = 0 \tag{33}
\]
\[
\lim_{\theta \to \infty} \frac{\partial D(p, \theta, e)}{\partial p} = -\infty. \tag{34}
\]
In effect, (33) and (34) say that the domain of $\theta$ is rich enough to allow changes in $\theta$ to trace out all possible demand slopes from infinitely elastic to infinitely inelastic.

As an intermediate step in deriving bounds on the elasticity $e_t$ in period $t$, we will derive bounds on $\theta_t$. Consider some later period $t' > t$. Reflecting a growing market, we maintain the assumption that demand is nondecreasing over time. In general, this implies $D_t(p) \leq D_{t'}(p)$. Substituting from (31), we have
\[
D(p, \theta_t, e_t) \leq D(p, \theta_{t'}, e_{t'}). \tag{35}
\]
Given (35) holds for all $p \geq 0$, it must hold in particular for $p = p_{t'}$. Substituting $p_{t'}$ into (35) yields
\[
D(p_{t'}, \theta_t, e_t) \leq D(p_{t'}, \theta_{t'}, e_{t'}) = q_{t'}. \tag{36}
\]
Assuming the equilibrium set $E$ exhibits quantity expansion, then $t' > t$ implies $e_{t'} \in NE^+(e_t)$ or $e_{t'} \in SE^+(e_t)$. For $e_{t'} \in NE^+(e_t)$, condition (36) holds for all $\theta_t \geq 0$; so such points contribute nothing to bounding $\theta_t$. For $e_{t'} \in SE^+(e_t)$, assumptions (32), (33), and (34) ensure the existence of a unique solution for $\theta_t$ treating (36) as an equality. Let $\Theta(e_t, e_{t'}) > 0$ denote the solution. Further, (36) is satisfied for $\theta_t < \Theta(e_t, e_{t'})$ and violated for $\theta_t > \Theta(e_t, e_{t'})$, implying $\Theta(e_t, e_{t'})$ is an upper bound on the $\theta_t$ satisfying (36).

Similar logic applies to pairwise comparisons of $e_t$ versus earlier equilibrium points $e_{t'}, t' < t$. 21
In that case, too, the solution $\Theta(e_t, e'_t)$ provides an upper bound on the $\theta_t$ satisfying (36). Only the equilibrium points $e'_t \in NW^-(e_t)$ are informative about the bound in that case. Combining the results of this and the previous paragraph together yields the following proposition.

**Proposition 7.** Suppose the set of equilibrium points $E$ is distinct and exhibits quantity expansion. Let $\Theta(e_t, e'_t)$ be a solution for $\theta_t$ treating (36) as an equality. For $e'_t \in SE^+(e_t)$ and $e'_t \in NW^-(e_t)$, $\Theta(e_t, e'_t)$ exists and is unique. Defining

$$\theta^*_t \equiv \min_{e'_t \in SE^+(e_t) \cup NW^-(e_t)} \Theta(e_t, e'_t),$$

(37)

the sequence of logit demands defined in (31) satisfies assumptions (1) and (2)—respectively, the law of demand and that demand is nondecreasing over time—only if $\theta_t \leq \theta^*_t$.

Given the functional form chosen by the researcher and the data points $e_t$ and $e'_t = (q'_t, p'_t)$, (36)—treated as an equality—is a nonlinear equation in the single unknown parameter $\theta_t$. It is a well-behaved equation that should be solvable using any standard methods, including a simple grid search.

**6. Background on Plastics and the Du Pont Decision**

The use of plastic in consumer goods—in the products themselves and their packaging—is so widespread today that it is hard to envision the world in the 1950s and 1960s when plastic was initially developed and diffused commercially. Polyethylene was among the first commercially developed plastics. We focus on two types, low- and high-density polyethylene. As mentioned in the introduction, low-density polyethylene was famously used to make Tupperware food-storage containers (Clarke 1999, p. 2) and high-density polyethylene to make Hula Hoops (Fenichell 1996, p. 264). Polyethylene remains the highest-volume commercial plastic in terms of global sales.

In the late 1930s, the U.S. Department of Justice, under Thurman Arnold, a former Yale Law School professor and activist antitrust enforcer, sought to prosecute Du Pont, a U.S. manufacturer of polyethylene, and its U.K. co-conspirator Imperial Chemical Industries (ICI), for their Patents and Processes agreement. Signed in 1929, the agreement granted each company exclusive licenses for the patents and secret processes controlled by the other and divided the global market into exclusive territories between them. Views differ on whether the main motive for the agreement was the sharing of complementary technologies or the monopolization achieved by market di-
vision (Hounshell and Smith 1999, p. 190). The Department of Justice viewed the Patents and Processes agreement as an illegal market division, in violation of Section 1 of the Sherman Act. Franklin Roosevelt’s administration lobbied to suspend legal action during the Second World War, especially because of the military significance of the chemical industry (Reader 1975, p. 432). After the war, the case proceeded to trial. The prosecution won a liability verdict in 1951. Judge Sylvester Ryan ordered a remedy in 1952.

Judge Ryan’s order cancelled the exclusive-territory arrangements between Du Pont and ICI and required them to license patents and secret processes involved in the manufacture of several of their products to all applicants at reasonable royalty rates. The incumbents’ most significant patents covered three products: polyethylene, nylon, and neoprene. Judge Ryan did not order the compulsory licensing of neoprene. Of the remaining products, Simon Whitney, later chief economist at the U.S. Federal Trade Commission, contends that only patents for polyethylene ended up garnering “widespread interest” commercially (Whitney 1958, p. 217). Thus, we focus on the effect of Judge Ryan’s structural remedy on the polyethylene market in this paper.

Whether compulsory licensing solves the monopolization problem is the subject of debate in the antitrust literature. In their seminal legal casebook, Areeda, Kaplow, and Edlin (2013) note that the terms of such licensing typically remain subject to commercial negotiations of the parties. This leaves open the possibility that the licensor preserves the monopoly outcome by specifying exorbitant rates.\(^6\) At one point, ICI was asking for a fixed fee of $500,000 and royalties amounting to eight percent of sales revenue (Fortune, 1954). Since this royalty applied to revenue not profit, it would have added to markups, whether as much as the monopoly level it may be hard for industry outsiders to know. Expanding the incumbents’ scope to manipulate the licensing outcome in their favor, patents often do not relate all the information necessary for entrants to replicate a production process. While Judge Ryan ordered incumbents to provide accompanying manuals and training, the question remains whether these materials were even adequate to say nothing about allowing the entrants to operate on an even footing with incumbents.

The remedy appeared to have a dramatic effect on the polyethylene market (Backman 1964,\(^6\) Judge Ryan himself foresaw the potential difficulty in defining a reasonable royalty rate: “While it is true that as to [royalty rates] question might well be raised as to whether they were arrived at after arm’s length commercial negotiations, they do nevertheless furnish guide posts for future determination of the amount at which such royalties should be set. But, in any event, as to these products and all others, there is also available for judicial finding the sum a prudent licensee would pay under all existing circumstances.” (US vs. ICI, 1952, p. 227–228.)
Seven manufacturers entered in 1953–56 and four more in 1956–59. Prices steadily declined and output rose. On the face of these facts, one might be tempted to conclude that Judge Ryan’s remedy achieved its purpose of turning a monopoly into a more competitive market. However, the same price declines and output increases may have arisen in a monopoly market experiencing substantial cost declines, plausibly realistic for plastics in the 1950s and 60s. The fact of entry seems to disprove monopoly unless it is thought that the entrants are merely producing their share of the monopoly quantity, returning most of the rents to the licensor. Thus our study will try to produce evidence for effective remedy that cuts through these criticisms.

If the remedy was successful, one contributing factor may have been that the market was monopolized by independent firms, who attempted to monopolize the market via the Patents and Processes agreement, which specified exclusive territories. With the agreement cancelled by Judge Ryan’s remedy, the independent operators may have been thrown into competition to license their technologies. This may have disciplined exorbitant licensing fees and thwarted an attempt to preserve the monopoly outcome. That said, the question remains whether incumbents with a history of cooperation needed the explicit legal agreement to maintain cooperation between them.

7. Data

Our dataset consists of annual price and quantity data, aggregated across all firms the U.S. market, for two plastic products, low- and high-density polyethylene, in the period following the imposition of the remedy in the Du Pont (1952) case. Table 1 of Lieberman’s (1984) seminal paper form the foundation of our dataset, which displays the data he obtained from the annual reports, Synthetic Organic Chemicals: United States Production and Sales, issued by the U.S. Tariff Commission. The earliest date that price and quantity are reported for polyethylene is 1958, so Lieberman’s data begin then. Lieberman ends the series in 1972 because in subsequent years the OPEC crisis creates an oil shock which, together with the subsequent recession, disrupted both supply and demand in the plastics market.

Rather than taking Lieberman’s data directly, we returned to the original U.S. Tariff Commission source for two reasons. First, we noticed anomalies in the price series—identical prices to the
decimal in adjacent years—suggesting typographical errors, which required correcting. Second, he recorded production for his quantity variable. Since we are interested in estimates of demand, we went back and collected sales for our quantity variable, also reported in the original source.\(^8\) Our final dataset consists of prices (measured in nominal dollars per pound) and sales quantities (measured in million pounds) for low- and high-density polyethylene from 1958–72. The price series is an average wholesale price, computed by dividing total annual industry revenue by industry quantity sold.

Figure 5 displays the data for each of the two products in separate panels. Prices are roughly the same across the products. Quantity sold in the market for high-density polyethylene was about half that in the market for low-density polyethylene.

Each price, quantity pair can be thought of as an equilibrium point resulting from the interaction of demand and a supply relation. The figure graphs the evolution of these equilibrium points over time. The predominant pattern is for equilibrium to shift to the southeast each year. With only one exception across products and years, the set of equilibria, denoted \(E\) in our methodology section, exhibits the property that was termed “quantity expansion” in that section.\(^9\) Quantity expansion technically means that later equilibrium points lie to the east of earlier ones. Our data exhibit this for both products in all years except 1963 for low-density polyethylene. We present the analysis including that year anyway. As a robustness check will redo the analysis excluding that year and report it in appendix figures.

A strict interpretation of our model would contend that the southwest shift in the equilibrium point from 1962 to 1963 is a rejection of our assumption that demand is nondecreasing. The 1963 equilibrium point must lie on a lower demand curve than the 1962 equilibrium. Given our assumption is violated in that year, there may be other years, unknown to us, in which it is violated as well. We adopt a more liberal interpretation that 1963 is a minor exception to a pattern that holds across several products and over several decades. The latter interpretation does have consequences for our later assessment of our results, increasing our burden to demonstrate broad patterns, holding across many years, rather than drawing conclusions from a single elasticity estimate.\(^{10}\)

\(^8\) The residual between production and sales is of course change in inventory.

\(^9\) Another maintained assumption in the methodology section is that \(E\) is distinct. Our data are consistent with this assumption in that no price or quantity is observed twice over time. Some similar prices or quantities differ when measured to sufficient precision.

\(^{10}\) One strategy for addressing the anomalous 1963 observation would be to adopt a weaker assumption on the
Figure 5: Evolution of Equilibrium in the Plastics Market. Source: U.S. Tariff Commission (various years).

8. Results

This section presents results from our method for bounding $\epsilon_t$, the (absolute value of the) elasticity of demand. Our methods—when applied to these particular data—only provide upper bounds on $\epsilon_t$. The lower bound, necessitated by the law of demand, is always 0. Thus $\epsilon_t$ will always be bounded in an interval extending from 0 to the upper bound we derive.

Figure 6 presents the first set of results, which impose linear demand. Begin with the market for low-density polyethylene shown in the top panel. The grey bars represent the elasticity bounds equilibrium set $E$, only requiring it to exhibit rectangular expansion rather than quantity expansion. The southwest movement from 1962 to 1963 also violates rectangular expansion, so this change in assumptions would not help. The fact that quantity expansion is no less general than rectangular expansion in our application explains why we adopted the stronger assumption in our methodological analysis.
derived from the method only using pairwise comparisons of equilibrium points. We labeled the upper bound on the elasticity as $\epsilon_t^*$, computed from the upper bound $\alpha_t^*$ of the vertical inverse-demand intercept generated by this method.

The results are remarkable. Aside from the first year, in which the upper bound is quite high at 2.2, the upper bound on the elasticity is consistently low, never higher than 0.68, which is still quite inelastic. The upper bound can be much lower than this: in particular, 0.09 and 0.08 in 1959 and 1960, respectively. These extremely low elasticities rule out anything but almost perfectly inelastic demand in those years. This result would be inconsistent with a monopoly outcome,
which in theory leads to an equilibrium in the elastic region of demand.

Only in the first year, 1958, does the upper bound rise into the elastic region. We suspect that the high bound in 1958 results from a lack of information from previous years to bound the demand curve in that year rather than a symptom of an idiosyncratically high elasticity in that year; we have no evidence to suggest that demand for low-density polyethylene was markedly different in that compared to later years. We will see repeatedly that our methodology works best with pre- and post-year equilibrium information to produce satisfactorily narrow bounds. Throughout all of our results, there is no instance—no specification and no product—in which the bounds in the first year or years of our data are not extremely wide.

The black bars present results from the variant of the methodology that incorporates intercept information. We labeled the upper bound on the elasticity from this method as $\epsilon^{**}_t$, computed from the upper bound $\alpha^{**}_t$ on the vertical inverse-demand intercept generated by this method. Adding intercept information shrinks most of the upper bounds on the elasticity extremely close to 0 in all years apart from the first three. In the first year, adding intercept information does not change the bound at all. In the next two, the elasticity was already almost 0 even without incorporating intercept information.

If it is believed that the linearity of demand carries all the way to both axes, the conclusion from the black bars is that demand for low-density polyethylene was almost perfectly inelastic in every post-remedy year for which we have data but the first. This conclusion does seem to strain the functional-form assumption because it extrapolates outside of the range of the observed data. We will assess how robust the conclusions are when we compare to a different (logit) functional form below. That said, we did not need to resort to incorporating intercept information to bound the elasticity well within the inelastic range with the grey bars.

The lower panel presents analogous results for the other product, high-density polyethylene. This market exhibits a longer initial period in which only wide elasticity bounds are attainable. Every year after 1961, the elasticity is bounded in the inelastic region even by the methodology that ignores intercept information. The bound from this method can be quite low, 0.34 in 1963 and 0.44 and 0.43, respectively, in 1971 and 1972. Incorporating intercept information drives the bound as low as 0.15 in some years, again almost perfectly inelastic for the high-density product as we saw for the low-density product. Overall, the bounds admit higher elasticities for this product,
but with this product, too, elastic demand and monopoly is precluded in this post-remedy period.

Figure 7 presents analogous results but for the logit rather than linear functional form. Focus first on the black bars, which compare each point pairwise against all other equilibrium points as the methodology discussed in Section 4 prescribed. For low-density polyethylene shown in the upper panel, the elasticity is bounded well within the inelastic region every year but the first. The bound can be as low as 0.08, almost perfectly inelastic, in 1960. For high-density polyethylene shown in the lower panel, the bounds include elastic demand in most years, although there are some years (1962 and 1963 in particular) in which the elasticity is below 0.5.
As logit demand curves asymptote toward rather than intersecting the axes, there is no intercept information to incorporate. We use the grey bars to provide insight into the contribution of nonlocal information in a different way. The grey bars show the results from bounds from pairwise comparison of equilibrium points excluding all comparisons except for those two years or less distant from the year in question. This admittedly blows up some of the bounds. For low-density polyethylene in 1970, we the bounds include all positive reals, and the bounds are above 5 in 1971 and 1972. However, the grey bars do exhibit some degree of robustness in that many of the bounds stay roughly the same. We conclude that the tight bounds are not forced by the geometry of distant comparisons. Nor are our results polluted by a single anomalous year: the moving window behind the grey bars means that points are quickly cycled out of the comparison set.

To gauge the robustness of our results to functional form, Figure 8 juxtaposes the most comparable results from the linear and logit against each other. For the case of linear demand shown in Figure 6, the grey bars are selected, reflecting results ignoring intercept information. As logit demands lack intercepts, the most comparable of the linear results are those ignoring intercept information. For logit demand shown in Figure 7, the black bars are selected, reflecting results using all years for pairwise comparisons because this is how the comparison was done in the linear case. The selected results for the two functional forms are collected together in Figure 8 maintaining their original bar colors for clarity.

The bounds from the two functional forms closely match each other. The correlation in bar height is 0.91 in the top panel for low-density polyethylene and 0.95 in the bottom panel for high-density polyethylene. Our intuition for the correspondence of the results is that the driving force behind the bounds is often comparisons between close years, so fairly local comparisons, and different functional forms are approximately linear locally.

9. Extensions

This section briefly discusses several extensions of the methodology. First, we discuss how to handle applications in which the set of equilibria $E$ exhibits the weaker property of rectangular expansion rather than quantity expansion. Second, we discuss applying the results to declining rather than growing markets.
9.1. Rectangular Expansion

The model that we used to develop our methodology for bounding elasticities imposed a strong assumption on what it means for the market to grow. In particular, we assumed the equilibrium set \( E \) exhibits quantity expansion, that is, that \( q_t \leq q_{t'} \) for \( t < t' \). Geometrically, the assumption means that equilibrium points can only shift east over time. The economic interpretation of the assumption is that both the demand curve and the supply relation are required to increase over time. The assumption of quantity expansion streamlined our analysis by eliminating several cases.
It was without loss of generality in our application because our data satisfied the assumption (with the exception of one year for one product, which also violated alternative concepts of market growth).

Although we did not impose it, in the model section introduced the weaker requirement of rectangular expansion, entailing that either $q_t \leq q_{t'}$ or $p_t \leq p_{t'}$ for $t < t'$. In words, rectangular expansion means that if equilibrium quantity fell from one period to the next, price must have risen. This assumption admits cases excluded by quantity expansion in which the demand curve is growing but the supply relation shifts the other way. Geometrically, the weaker assumption allows the sets $NW^+(e_t)$ and $SE^-(e_t)$ to be nonempty. While this enhanced generality was not useful in our application, because those sets happened to be empty given our data, it may enhance generality in other applications.

Figure 9 illustrates an example of an equilibrium set exhibiting rectangular but not quantity expansion and shows how the analysis can be enriched in this case. Set $E$ involves three equilibrium points: equilibrium, initially at $e_1$, shifts southeast to $e_2$, consistent with quantity expansion. But then the equilibrium shifts back, northwest, to $e_3$, a shift that is inconsistent with quantity expansion but not with rectangular expansion. As before, the vertical intercept of the line $\ell_{12}$ through $e_1$ and $e_2$ provides a lower bound on the vertical intercept of the inverse demand curve through

Figure 9: Bounds from Above and Below Under Rectangular Expansion
e_2. The line ℓ_{23} through points e_2 and e_3 provides, not a lower bound, but an upper bound on the intercept of the demand curve through e_2. A higher intercept would put e_3 on a lower demand curve than e_2, violating the assumption that demand is nondecreasing over time. Thus the pairwise comparison of e_2 with e_1 and e_3 funnels the demand curve into the shaded region, which narrows down the range possibilities from infinitely elastic and inelastic demands not just in one direction but from both extremes.

The preceding example suggests a general method of bounding demand elasticities from above and below when the equilibrium set exhibits rectangular expansion. For concreteness we present the bounds under the assumption of linear demands.

**Proposition 8.** Define

\[ \underline{\alpha}_t^* \equiv \max_{e_t' \in NW^-(e_t) \cup SE^+(e_t) \cup \{e_t\}} v(e_t, e_t') \]  
\[ \bar{\alpha}_t^* \equiv \min_{e_t' \in NW^+(e_t) \cup SE^-(e_t)} v(e_t, e_t') . \]  

Consider a sequence of linear inverse demands \( P_t(q) = \max(0, \alpha_t - \beta_t q) \), \( t = 1, \ldots, T \), that is consistent with a distinct E exhibiting quantity expansion. This sequence satisfies assumptions (1) and (2)—respectively, the law of demand and that demand is nondecreasing over time—only if \( \alpha_t \in [\bar{\alpha}_t^*, \underline{\alpha}_t^*] \).

The bounds on the vertical intercepts from the proposition can be translated into bounds on the elasticity using the formula (6). Let \( \bar{\epsilon}_t^* = p_t / (\bar{\alpha}_t^* - p_t) \) and \( \underline{\epsilon}_t^* = p_t / (\underline{\alpha}_t^* - p_t) \). The reversal of the upper and lower bars in translating from intercepts to elasticities in these equations is intentional since a higher demand intercept leads to a lower elasticity. The elasticity then is bounded in the interval \( \epsilon_t \in [\bar{\epsilon}_t^*, \underline{\epsilon}_t^*] \).

We can verify that Proposition 8 is a generalization of Proposition 1 by verifying that they deliver the same results when \( E \) exhibits quantity expansion. When \( E \) exhibits quantity expansion, Proposition 2 and equation 38 can be combined to show \( \alpha_t^* = \bar{\alpha}_t^* \). Furthermore, since \( NW^+(e_t) \) and \( SE^-(e_t) \) are both empty when \( E \) exhibits quantity expansion, we have \( \bar{\alpha}_t^* = \infty \) since the minimum over an empty set is by definition infinite. Thus Proposition 8 implies \( \alpha_t \geq \alpha_t^* \)—the same as Proposition 1—when \( E \) exhibits quantity expansion.

A natural next step in the analysis of the linear-demand case would be to extend the method incorporating intercept information to the present setting in which \( E \) exhibits rectangular expansion. That extension runs into some complications. Given the questions raised about the robustness of
bounds incorporating intercept information, we do not present analysis of that method here.

Moving beyond the linear case, the analogous idea can be applied to bound the elasticities from above and below when logit demand is assumed. The nonlinear equation (29) can be solved for all \( e_t' \in NW^+(e_t) \cup SE^-(e_t) \) and the minimum taken to provide a lower bound on the price-sensitivity parameter \( a_t \), which can then be translated into a lower bound on \( \epsilon_t \). We omit formal propositions for brevity.

9.2. Declining Markets

One might suspect that analogous elasticity bounds could be provided in the opposite case of a declining rather than growing market by reversing all the signs and inequalities. In theory, this is true. In practice, several caveats need to be recognized in applying the results to that case.

First, even for markets experiencing secular declines, it may be hard to contend that the demand curve is definitively shifting back each year. Even if consumer taste is known to be shifting away from a product, population growth may offset this taste shift. Inflation may also mask a demand decline in real terms if prices are not properly deflated.

Second, even if suppliers are not investing much, the spillover of technology from other industries may lower costs in the market, shifting supply out. So the equilibrium set \( E \) may not exhibit quantity shrinkage, the analogue to quantity expansion, that we saw in the growing market. There may be very little information left in the movement of the equilibrium point to usefully bound the elasticity.

Finally, reversing all the results suggests that the main bound one would find in a declining market would be a lower bound on the elasticity. In our application, proving that the elasticity is above a threshold would not be dispositive about whether the market was monopolized post remedy or not. The tight upper bound was dispositive for us.

10. Conclusion

This paper provided a methodology for bounding the elasticity of demand that works in growing markets for homogeneous products. The method has minimal information requirements, requiring as few as two time-series observations on aggregate prices and quantities. The idea behind the
method is that the demand curve through a given equilibrium (price, quantity) pair cannot be either so steep or so flat that it passes below earlier equilibria or above later equilibria without violating the assumption that demand is nondecreasing over time. These inequality conditions place bounds on the elasticity of demand in any given year.

A potential drawback of any methodology delivering bounds rather than point estimates is that the resulting bounds may be so wide as to be uninformative. In our application to the polyethylene market in the 1958–72 period after a licensing remedy was ordered, the methodology turned out to quite informative. Even restricting attention to variants of our method that are robust to functional form Our most robust methods imposing the least demanding assumptions ) demand was virtually infinitely inelastic during virtually the whole study period. Less demanding and thus more robust methods still bounding the elasticity of demand below 0.84 in it in [0, 0.08] in 1959 and in [0, 0.09] in 1960 in the low-density polyethylene market. We conclude that it is difficult to contend that the monopoly outcome effected by the Patents and Processes agreement between Du Pont and ICI was not ended after Judge Ryan’s remedy in the Du Pont case.

The Du Pont application serves as a proof of concept for our bounding methodology, which could be applied to any growing market involving homogeneous products. We hope other researchers will find value in our methodology and are developing Stata code to facilitate its use.
Appendix: Proofs

Before presenting the proofs of propositions stated in the text, we state and prove several useful lemmas.

**Lemma 1.** The vertical intercept, \( v(e_t, (x, 0)) \), of the line through a point \((x, 0)\) on the horizontal axis and equilibrium point \(e_t\) is nondecreasing in \(x\).

**Proof.** Applying the formula in (7),

\[ v(e_t, (x, 0)) = \frac{px}{x-q_t}. \]  

(A1)

Differentiating,

\[ \frac{\partial v}{\partial x}(e_t, (x, 0)) = -\frac{p_t q_t}{(x-q_t)^2} \leq 0. \]  

(A2)

Q.E.D.

**Lemma 2.** \( \lim_{x \to \infty} v(e_t, (x, 0)) = p_t \).

**Proof.** Inspecting the expression for \( v(e_t, (x, 0)) \) in (A1), we see that the desired limit is of the \(\infty/\infty\) form. Applying l’Hôpital’s rule yields the result. Q.E.D.

**Lemma 3.** Suppose \(e_{t'} \in SE(e_t)\). Then \( v(e_t, (h(e_t, e_{t'}), 0)) = v(e_t, e_{t'}) \).

**Proof.** For \(e_{t'} \in SE(e_t)\), equation (13) gives

\[ h(e_t, e_{t'}) = \frac{p_t q_t - p_{t'} q_{t'}}{p_t - p_{t'}}. \]

Substituting this expression in the formula for \(v\) from (7),

\[ v(e_t, (h(e_t, e_{t'}), 0)) = p_t \left( \frac{p_t q_t - p_{t'} q_{t'}}{p_t - p_{t'}} \right) \left( \frac{p_t q_t - p_{t'} q_{t'}}{p_t - p_{t'}} - q_t \right) 
= \frac{p_t q_t - p_{t'} q_{t'}}{q_t - q_t} 
= v(e_t, e_{t'}). \]

The second line follows by rearranging and the last line from equation (7). Q.E.D.

**Proof of Proposition 3:** We will establish the following series of equalities:

\[ \alpha_t^n = \max_{t \in \{1, \ldots, T\}} v(e_t, e_{t'}) \]  

(A3)

\[ = \max_{t' \leq t} v(e_t, e_{t'}) \lor \max_{t' > t} v(e_t, e_{t'}) \]  

(A4)

\[ = V_t^{[1]} \lor \max_{e_{t'} \in SE(e_t)} v(e_t, e_{t'}) \]  

(A5)

\[ = V_t^{[1]} \lor \max_{e_{t'} \in SE(e_t)} v(e_t, (h(e_t, e_{t'}), 0)) \]  

(A6)

\[ = V_t^{[1]} \lor v \left( e_t, \left( \min_{e_{t'} \in SE(e_t)} h(e_t, e_{t'}), 0 \right) \right) \]  

(A7)

\[ = V_t^{[1]} \lor v \left( e_t, \left( \min_{t' \geq t} h(e_t, e_{t'}), 0 \right) \right) \]  

(A8)
Equation (A3) is the definition of $\alpha_t^*$ from (10). Equation (A4) follows from partitioning the set $\{1, \ldots, T\}$ into $\{1, \ldots, t\}$ and $\{t+1, \ldots, T\}$. In (A5), we have substituted from (14). We also used the fact that for $e_{t'} \in NE(e_t)$ as well as $e_t = e_t$, we have $v(e_t, e_{t'}) \leq p_t = v(e_t, e_t) \leq V_t^{[1]}$. Hence, without loss of generality, we can exclude $e_{t'} \in NE(e_t)$ and $e_{t'} = e_t$ from the set over which $v$ is maximized. Equation (A6) follows from Lemma 3. Equation (A7) follows because $h(e_t, e_{t'})$ is nonincreasing in the argument $h(e_t, e_{t'})$ appears in by Lemma 1, so maximizing $v$ is equivalent to minimizing that argument. Equation (A8) follows because $h(e_t, e_{t'}) = \infty$ by (13), so the minimum over $e_{t'} \in SE(e_t)$ is the same as the minimum over all $e_{t'}$ such that $t' \geq t$.

The proof follows immediately. Substituting the definition of $H_t^{[1]}$ into (A7) shows that (A7) equals $C_t^{[1]}$. Q.E.D.

**Proof of Proposition 4:** Suppose $E$ is distinct and exhibits quantity expansion. Suppose further that there exists $t \in \{1, \ldots, T\}$ such that $\alpha_t < \alpha_t^{**}$.

Since $\alpha_t^{**} \equiv C_t^{[2]}$, (19) implies that either

$$\alpha_t < V_t^{[2]}$$

or

$$\alpha_t < v(e_t, (H_t^{[2]}, 0)).$$

must hold. We will show that each of (A9) and (A10) violate one of the maintained assumptions, (1) or (2).

First suppose (A9) holds. For $t' \leq t$ we have

$$\alpha_{t'} \leq \alpha_t$$

(A11)

since $\alpha_{t'} = P_t(0)$, $\alpha_t = P_t(0)$, and $P_t(0) < P_t(0)$ by assumption (2) that demand is nondecreasing over time. In addition,

$$V_t^{[2]} = \max_{t' \leq t} V_{t'}^{[1]} = \max_{t' \leq t} \left[ \max_{t'' \leq t'} v(e_{t'}, e_{t''}) \right],$$

(A12)

where the first equality follows from (17) and the second from (14). Combining (A11), (A9), and (A12), we have

$$\alpha_{t'} < v(e_{t'}, e_{t''}) \quad \text{for some } t'' \leq t'.$$

(A13)

There are two cases to consider in analyzing (A13): $t'' = t'$ and $t'' < t'$. First suppose $t'' = t'$. Then $v(e_{t'}, e_{t''}) = v(e_{t'}, e_{t'}) = p_{t'} = \alpha_{t'} - \beta_{t'} q_{t'}$. Substituting this equality in (A13) and rearranging yields $\beta_{t'} q_{t'} < 0$, implying $\beta_{t'} < 0$ because $q_{t'} \geq 0$. But $\beta_{t'} < 0$ violates assumption (1), the law of demand.

Next suppose (A13) holds and $t'' < t'$. Substituting from (7) into (A13) yields

$$\alpha_{t'} < \frac{p_{t''} q_{t'} - p_{t'} q_{t''}}{q_{t'} - q_{t''}}.$$  

(A14)

Since $E$ is distinct and exhibits quantity expansion, $q_{t''} < q_{t'}$, implying the denominator in (A14) is positive. Cross multiplying by this positive denominator yields

$$\alpha_{t'} (q_{t'} - q_{t''}) < p_{t''} q_{t'} - p_{t'} q_{t''}.$$  

(A15)
Then
\[ p_{t''} > \alpha_{t''} - \left( \frac{\alpha_{t''} - p_{t''}}{q_{t''}} \right) q_{t''} = \alpha_{t''} - \beta_{t} q_{t''}, \]  
(A16)
where the first step follows from rearranging (A15) and the second step from (4). Equation (A16) implies \( P_{t''}(q_{t''}) = p_{t''} > P_{t'}(q_{t'}) \), violating assumption (2) that demand is nondecreasing over time.

We have thus shown (A9) violates either (1) or (2). We next turn to showing (A10) violates either (1) or (2). Now
\[ H_{t}^{[2]} = \min_{t' \geq t} H_{t}^{[1]} = \min_{t' \geq t} \left[ \min_{t'' \geq t'} h(e_{t'}, e_{t''}) \right], \]  
(A17)
where the first equality follows from (18) and the second from (15). Substituting (A17) into the right-hand side of (A10) yields
\[
v(e_{t}, (H_{t}^{[2]}, 0)) = v \left( e_{t}, \left( \min_{t' \geq t} \left[ \min_{t'' \geq t'} h(e_{t'}, e_{t''}) \right], 0 \right) \right)
= \max_{t'' \geq t' \geq t} v(e_{t}, (h(e_{t'}, e_{t''}), 0)).
\]  
(A18)

Equation (A19) follows because choosing \( t' \) and \( t'' \) to minimize an argument of \( v \) is equivalent to choosing \( t' \) and \( t'' \) to maximize \( v \) when \( v \) is nonincreasing in the argument, as we know it is from Lemma 1. Substituting (A19) into (A10), the resulting condition is equivalent to the existence of \( t', t'' \in \{1,\ldots,T\} \) with \( t \leq t' \leq t'' \) such that
\[ \alpha_{t} < v(e_{t}, (h(e_{t'}, e_{t''}), 0)). \]  
(A20)

We will show (A20) is violated if \( t' = t'' \). By (13), \( h(e_{t'}, e_{t''}) = \infty \). Lemma 2 then implies that the right-hand side of (A20) equals \( p_{t} \). But \( \alpha_{t} < p_{t} = \alpha_{t} - \beta_{t} q_{t} \) implies \( \beta < 0 \), violating assumption (1), the law of demand. Thus if (A20) is to be consistent with the maintained assumptions, it must hold for some \( t \leq t' < t'' \).

Consider the argument \( h(e_{t'}, e_{t''}) \) on the right-hand side of (A20). We will show
\[ h(e_{t'}, e_{t''}) \geq \frac{\alpha_{t'} q_{t'}}{\alpha_{t'} - p_{t'}}. \]  
(A21)
In words, the left-hand side is the horizontal intercept of the line through \( e_{t'} \) and the later equilibrium \( e_{t''} \). By (5), the right-hand side is the horizontal intercept of the inverse demand through \( e_{t'} \). If the reverse inequality holds, then \( e_{t''} \) would lie on a lower demand curve than \( e_{t'} \), violating assumption (2) that demand is nondecreasing over time.

Suppose instead
\[ h(e_{t'}, e_{t''}) < \frac{\alpha_{t'} q_{t'}}{\alpha_{t'} - p_{t'}}. \]  
(A22)
Because \( E \) exhibits quantity expansion and \( t'' > t' \), \( e_{t''} \in SE(e_{t'}) \cup NE(e_{t'}) \). But if \( e_{t''} \in NE(e_{t''}) \), then \( h(e_{t'}, e_{t''}) = \infty \) by (13), violating (A22). Hence \( e_{t''} \in SE(e_{t'}). \) Substituting the formula for \( h \) that holds in that case in (A22) yields
\[ \frac{p_{t'} q_{t''} - p_{t''} q_{t'}}{p_{t'} - p_{t''}} < \frac{\alpha_{t'} q_{t'}}{\alpha_{t'} - p_{t'}}. \]  
(A23)

38
Cross multiplying by the denominators preserves the direction of the inequality because \( e_{t''} \in SE(e_{t'}) \), implying \( p_{t'} > p_{t''} \), in turn implying the denominator on the left-hand side is positive. Cross multiplying in this way, cancelling terms, and rearranging yields
\[
p_{t''} < \alpha_t - \left( \frac{\alpha_t - p_{t'}}{q_{t'}} \right) q_{t''} = \alpha_t - \beta_{t'} q_{t''} = P_t(q_{t''}),
\]
where the first equality follows from (4). But (A24) implies \( P_{t''}(q_{t''}) = p_{t''} < P_{t'}(q_{t''}) \), violating assumption (2) that demand is nondecreasing over time. We have thus proved (A22) holds under maintained assumptions.

By Lemma 1, substituting the right-hand side of (A22) for \( h(e_{t'}, e_{t''}) \) in (A20) preserves the direction of the inequality because \( v \) is nonincreasing in that argument. Hence,
\[
\alpha_t < v \left( e_{t'}, \left( \frac{\alpha_t q_{t'}}{\alpha_t - p_{t'}}, 0 \right) \right) \quad (A25)
\]
\[
= \frac{p_t \alpha_t q_{t'}}{\alpha_t - p_{t'}} \left( \frac{\alpha_t q_{t'} - q_{t'}}{\alpha_t - p_{t'}} \right) \quad (A26)
\]
\[
= \frac{\alpha_t p_t q_{t'}}{\alpha_t q_{t'} + p_t q_{t'} - \alpha_t q_{t'}} \quad (A27)
\]
where (A26) follows from the formula (7) and (A27) follows from rearranging.

We will show assumption (2) yields the opposite inequality from (A25)–(A27). The right-hand side of (A22) is the horizontal intercept of demand curve \( P_{t''}(q) \). Evaluating \( P_{t''}(q) \) at the intercept value yields a zero price:
\[
0 = P_t \left( \frac{\alpha_t q_{t'}}{\alpha_t - p_{t'}} \right) \quad (A28)
\]
\[
\geq P_t \left( \frac{\alpha_t q_{t'}}{\alpha_t - p_{t'}} \right) \quad (A29)
\]
\[
= \alpha_t - \beta_{t'} \left( \frac{\alpha_t q_{t'}}{\alpha_t - p_{t'}} \right) \quad (A30)
\]
\[
= \alpha_t - \left( \frac{\alpha_t - p_t}{q_{t'}} \right) \left( \frac{\alpha_t q_{t'}}{\alpha_t - p_{t'}} \right) \quad (A31)
\]
where (A29) follows from assumption (2), (A30) from the equation of a linear inverse demand, and (A31) from (4). Rearranging (A31) yields
\[
\alpha_t \geq \frac{\alpha_t p_t q_{t'}}{\alpha_t q_{t'} + p_t q_{t'} - \alpha_t q_{t'}} \quad (A32)
\]
the opposite inequality from (A25)–(A27). This completes the proof that (A20) is inconsistent with the maintained assumptions. Q.E.D.

**Proof of Proposition 6:** Suppose \( E \) is distinct and exhibits quantity expansion. It remains to show \( A(e_t, e_{t''}) \) exists and is unique for \( e_{t''} \in SE^+(e_t) \) and \( e_{t'} \in NW^-(e_t) \). We will divide the analysis into two cases.
First, suppose \( e_t' \in SE^+(e_t) \). Then (29) reduces to

\[
f(a, p_t, p_t') = \frac{q_t'}{q_t},
\]

where

\[
f(a, p_t, p_t') \equiv \frac{1 + \exp(ap_t)}{1 + \exp(ap_{t'}).}
\]

Now \( f(0, p_t, p_t') = 1 < q_t'/q_t \), where the last inequality follows from \( e_t' \in SE^+(e_t) \), which implies \( q_t' > q_t \). Rewriting

\[
f(a, p_t, p_t') = \frac{\exp(-ap_t) + \exp(a(p_t - p_{t'}))}{1 + \exp(-ap_{t'})},
\]

shows \( \lim_{a \to \infty} f(a, p_t, p_t') = \infty \) since \( p_t > p_{t'} \) for \( e_t' \in SE^+(e_t) \). This infinite limit of course exceeds any finite \( q_t'/q_t \). Thus, by continuity, there exists \( a > 0 \) solving (A33).

This solution is unique since \( f(a, p_t, p_t') \) is increasing in \( a \), as can be seen by differentiating \( f \):

\[
\frac{\partial f(a, p_t, p_t')}{\partial a} = \frac{p_t \exp(ap_t) - p_t \exp(ap_{t'}) + (p_t - p_{t'}) \exp(ap_t) \exp(ap_{t'})}{[1 + \exp(ap_{t'})]^2}.
\]

This is positive since \( p_t > p_{t'} \) for \( e_t' \in SE^+(e_t) \).

The proof for the case in which \( e_t' \in NW^-(e_t) \) is similar and omitted for brevity. \( Q.E.D. \)
References


