The Heteroscedastic Exponomial Choice

A. Alptekinoglu and J. Semple

January 29, 2018

Abstract

We develop analytical properties of the Heteroscedastic Exponomial Choice (HEC) model to increase its appeal for building demand models in analytical and empirical research. HEC generalizes the Exponomial Choice model to accommodate choice-specific variances for the random components of utility (the error terms) in a discrete choice setting, which is analytically intractable for logit. The choice probabilities under HEC can be expressed in closed form as an exponomial, a linear function of exponential terms. The elasticity of choice probabilities and the expected consumer surplus also have closed-form, remarkably simple expressions. The HEC loglikelihood function is a concave function of the parameters of a linear utility model for a given set of error term distribution parameters (rates of the corresponding exponential distributions). Moreover, the HEC model can easily accommodate an outside option with a deterministic utility (no random error term), which is quite cumbersome to do in the logit framework. Deterministic outside option also allows choices with zero probability, which is impossible in the standard logit framework. Finally, we also derive optimal prices under monopoly (of a multi-product firm) and equilibrium prices under oligopoly (of single-product firms), and explore how the variance of error terms impact the prices. In sum, the analytical properties we develop in this paper contribute to building a case for utilizing HEC in empirical and analytical research that build demand models on a discrete choice foundation.

1. Introduction

Demand modeling is a cornerstone of theoretical research in economics (Tirole 1988) and many business disciplines, such as operations management (Talluri and van Ryzin 2004) and marketing (Lilien et al. 1992). Discrete choice theory constitutes one of the major building blocks for demand
modeling (van Ryzin 2014, Anderson et al. 1992). Within this branch of demand modeling, logit offers practically the only family of models that is analytically appealing as well as empirically versatile. Starting with the axioms of Luce (1959) and continuing with the seminal contributions of McFadden that earned him the Nobel in economics (McFadden 2001), there accumulated a vast amount of theoretical and empirical literature that use the logit framework.

In this paper we study a discrete choice model that holds promise in terms of offering an alternative to the logit framework in demand modeling: the *Heteroscedastic Exponomial Choice* (HEC) model. HEC was first proposed by Daganzo (1979, pp. 14-16). It remained largely obscure, save one major empirical test (Currim 1982), until one recent study. Alptekinoğlu and Semple (2016) use the independent and identically distributed (*iid*) error version of this model, which they dub *Exponential Choice* (EC), in pricing and assortment planning contexts. They show, among other things, that the EC loglikelihood function is concave for linear model specifications, which makes model estimation very convenient. HEC generalizes EC to accommodate choice-specific variances for the random components of utility, or, the *error* terms in a discrete choice setting (such generalization is analytically intractable for logit). In particular, we assume that the error terms follow independent but not identical exponential distributions, and that they are subtracted from the deterministic component of utility.

The purpose of this research is to make a case for utilizing HEC in empirical and analytical research that build demand models on a discrete choice foundation. To this end, we develop several analytical properties of HEC and explore its pricing implications. The HEC model’s choice probabilities can be expressed in closed form as an *exponomial*, a linear function of exponential terms. The elasticity of choice probabilities and the expected consumer surplus also have closed-form, remarkably simple expressions. The HEC loglikelihood function is a concave function of the parameters of a linear utility model for a given set of error term distribution parameters (rates of the corresponding exponential distributions). Moreover, the HEC model can easily accommodate an outside option with a deterministic utility (no random error term), which is quite cumbersome to do in a logit framework. Deterministic outside option also allows choices with zero probability, which is impossible in the logit framework. Finally, we also derive optimal prices under monopoly (of a multi-product firm) and oligopoly (of single-product firms). We also explore how the variance of error terms impact the optimal or equilibrium prices.
In sum, the analytical properties we develop in this paper contribute to building a case for utilizing HEC in empirical and analytical research that build demand models on a discrete choice foundation. We first define the HEC model in §2, present its analytical properties, and extend it to deterministic outside option. Next, in §3, we present pricing results under monopoly and oligopoly. In §2.4 we explore the maximum loglikelihood estimation of HEC. We then close the paper in §4 with a discussion of future research. (All proofs are in the appendix.)

2. The Heteroscedastic Exponential Choice Model

Consider \( n \) choices \((n \geq 2)\), let \( N = \{1, \ldots, n\} \) denote the choice set, and suppose a random consumer \( c \) derives utility

\[
U_{ic} = u_i - z_{ic}
\]

from choice \( i \in N \), where \( u_i \) is the ideal utility of choice \( i \), and \( z_{ic} \) is an exponentially distributed random variable with rate \( \lambda_i > 0 \). The random terms \( z_{ic} \) are independent (but not identical) across consumers and choices; they capture preference heterogeneity among consumers. Let \( f_i \) denote the probability density function (pdf) and \( F_i \) the cumulative distribution function (cdf) of \( z_{ic} \), i.e.,

\[
f_i(z) = \lambda_i e^{-\lambda_i z} \quad \text{and} \quad F_i(z) = 1 - e^{-\lambda_i z} \quad \text{for} \quad z \geq 0, \quad \text{and} \quad f_j(z) = F_j(z) = 0 \quad \text{for} \quad z < 0.
\]

The probability that the consumer chooses \( i \in N \) (we henceforth drop the consumer subscript \( c \) to improve readability)

\[
Q_i \equiv \mathbb{P} \{ u_i - z_i \geq u_j - z_j \ \forall j \in N, j \neq i \} = \int_0^\infty \prod_{j \neq i} (1 - F_j(u_j - u_i + z)) f_i(z) dz
\]

(1)

can be derived in closed-form using the rank-ordering of ideal utilities. Let \([j]\) denote the choice – out of the set \( N \) – with the \( j \)-th smallest ideal utility. That is, the \( n \) choices under consideration have the following rank-ordering in ideal utilities: \( u_{[1]} \leq u_{[2]} \leq \cdots \leq u_{[n]} \). Define \( L_{[j]} \equiv \sum_{k=j}^{n} \lambda_{[k]} \) and

\[
G_{[j]} \equiv \frac{\lambda_{[j]}^{[j]}}{L_{[j]}} \exp \left[ -\sum_{k=j}^{n} \lambda_{[k]}(u_{[k]} - u_{[j]}) \right] \quad \text{for} \quad j = 1, \ldots, n.
\]

(2)
for \( j = 1, \ldots, n \). Note that \( \lambda[j] \) refers to the error distribution parameter for choice \( [j] \), not the \( j \)-th smallest \( \lambda \).

**Theorem 1. (Heteroscedastic Exponential Choice Probabilities)** Out of \( n \) choices with utilities \( U_i = u_i - z_i \), where the error terms \( z_i \) follow independent exponential distributions with corresponding rate \( \lambda_i \), the probability that the consumer prefers choice \( [j] \) considering it as utility-maximizing over all choices, for some \( j \in N \), is

\[
Q[j] = G[j] - \sum_{k=1}^{j-1} \frac{\lambda[j]}{L[k+1]} G[k].
\]

(3)

These choice probabilities have the form of an exponential – a linear function of exponential terms. They were first stated without proof by Daganzo (1979, p. 15); for completeness we include the proof. Unlike in MNL (multinomial logit), heteroscedasticity is thus tractable. An additional analytical convenience HEC has over MNL and its vast number of variations is the fact that exponential terms appear in an additive form. One comparative disadvantage, however, is the fact that rank-ordering of ideal utilities matters. Despite that, we show in this paper that HEC possesses many attractive analytical properties.

In the remainder of this section we discuss the demand model that HEC implies and its various analytical properties.

**2.1 Basic Properties of HEC**

The choice probabilities in (3) should obviously sum to 1. To see this, note that the coefficients of \( G[1], \ldots, G[n-1] \) that occur in \( Q[1], \ldots, Q[n] \) sum to zero; the only term that remains after summing \( Q[j] \) is \( G[n] \), which equals 1 using (2), i.e.,

\[
\sum_{j=1}^{n} Q[j] = \sum_{j=1}^{n-1} \left( 1 - \sum_{k=j+1}^{n} \frac{\lambda[k]}{L[j+1]} \right) G[j] + G[n] = G[n] = 1
\]

The HEC choice probabilities may also be expressed in a recursive fashion. Writing out the partial sum \( Q[1] + \cdots + Q[j] = \sum_{k=1}^{j} \frac{L[j+1]}{L[k+1]} G[k] \), and multiplying both sides by \( \frac{\lambda[j+1]}{L[j+1]} \), we obtain

\[
Q[j+1] = G[j+1] - \frac{\lambda[j+1]}{L[j+1]} \left( Q[1] + \cdots + Q[j] \right)
\]

(4)
for \( j = 1, \ldots, n - 1 \). Using only (3), we have a second recursive relationship:

\[
Q_{[j]+1} = G_{[j]+1} - \frac{\lambda_{[j]+1}}{L_{[j]+1}} G_{[j]} - \frac{\lambda_{[j]+1}}{\lambda_{[j]}} \sum_{k=1}^{j-1} \frac{\lambda_{[j]}}{L_{[k]+1}} G_{[k]}
\]

\[
= G_{[j]+1} - \frac{\lambda_{[j]+1} L_{[j]} G_{[j]}}{\lambda_{[j]} L_{[j]+1}} + \frac{\lambda_{[j]+1}}{\lambda_{[j]}} Q_{[j]}
\]

(5)

A more explicit version of this – without employing the \( G \) functions – is

\[
\frac{Q_{[j]+1}}{\lambda_{[j]+1}} - \frac{Q_{[j]}}{\lambda_{[j]}} = \frac{1}{L_{[j]+1}} \left\{ \exp \left[ - \sum_{k=j+1}^{n} \lambda_{[k]} (u_{[k]} - u_{[j]+1}) \right] - \exp \left[ - \sum_{k=j}^{n} \lambda_{[k]} (u_{[k]} - u_{[j]}) \right] \right\}
\]

\[
= \frac{1}{L_{[j]+1}} \exp \left( - \sum_{k=j+1}^{n} \lambda_{[k]} u_{[k]} \right) \left[ \exp \left( L_{[j]+1} u_{[j]+1} \right) - \exp \left( L_{[j]+1} u_{[j]} \right) \right]
\]

(6)

where the right-hand-side is independent of \( \lambda_{[1]}, \ldots, \lambda_{[j]} \). It is immediate from (6) that \( \frac{Q_{[j]}}{\lambda_{[j]}} = \frac{Q_{[j]+1}}{\lambda_{[j]+1}} \iff u_{[j]} = u_{[j]+1} \). This has two important implications. First, the choice probabilities \( Q_{i} \) are continuous in the vector of \( u \)'s, because the \( Q_{i}/\lambda_{i} \) ratio is preserved at transition points where the ideal utility ranks do change. Second, when every choice has the same ideal utility \( (u_{i} = u \forall i) \), the choice probability formula (3) reduces to \( Q_{[j]} = \lambda_{[j]} / L_{[1]} \), in which case the choice probabilities are obviously free of ideal utility ranks, i.e., they can be expressed as \( Q_{i} = \lambda_{i} / (\lambda_{1} + \cdots + \lambda_{n}) \). This is also the probability that choice \( i \) has the smallest error term.

Unlike the EC model (with iid error terms), the choice probabilities may not be monotone increasing in the ideal utility parameters but they satisfy another type of monotonicity. The \( G \) functions defined in (2) must satisfy

\[
\frac{L_{[j]}}{\lambda_{[j]}} G_{[j]} \leq \frac{L_{[j]+1}}{\lambda_{[j]+1}} G_{[j]+1}
\]

(7)

Therefore, equation (5) and inequality (7) together imply

\[
\frac{Q_{[j]}}{\lambda_{[j]}} \leq \frac{Q_{[j]+1}}{\lambda_{[j]+1}}
\]

(8)

for all \( j = 1, \ldots, n - 1 \) (this can also be verified from (6) and \( u_{[j]} \leq u_{[j]+1} \)). That is, the ratio \( Q_{[j]}/\lambda_{[j]} \) is monotone increasing in \( j \).
Further, if products with higher ideal utility have error terms with smaller mean and variance, i.e., \( \lambda_{[1]} \leq \lambda_{[2]} \leq \cdots \leq \lambda_{[n]} \), then the choice probabilities themselves are monotone increasing, i.e., \( Q_{[1]} \leq Q_{[2]} \leq \cdots \leq Q_{[n]} \). This condition makes intuitive sense: Higher-quality products typically have a better reputation; it is thus quite plausible that they would have error terms with smaller mean and variance as their ideal utility would not get discounted as much by consumers. Note that \( u_{[j]} < u_{[j+1]} \) and \( \lambda_{[j]} \leq \lambda_{[j+1]} \), or \( u_{[j]} \leq u_{[j+1]} \) and \( \lambda_{[j]} < \lambda_{[j+1]} \), would imply \( Q_{[j]} < Q_{[j+1]} \).

The monotonicity of choice probabilities can thus be violated only if a product with higher ideal utility has a larger variance in its error term. Consider the two examples shown in Table 1. Consumers are presented with three products indexed in increasing order of ideal utility ([i] \( i \forall i \)). On the left-hand-side the error terms of all products are iid (\( \lambda_i = 1 \forall i \)), which results in monotone increasing choice probabilities. On the right-hand-side, product 1 is made “safer” by increasing the rate of its error term 10-fold (\( \lambda_1 = 10 \)), and product 3 is made “riskier” by doing the opposite (\( \lambda_3 = 0.1 \)). Product 3 no longer commands the highest choice probability. In fact, quite the opposite: \( Q_2 > Q_1 > Q_3 \).

Finally, we have two results on the behavior the choice probabilities as a function of ideal utilities. First, the local logconcavity of EC choice probabilities (Alptekinoğlu and Semple 2016, see Lemma 3 in Online Supplement) easily carries over to the HEC model.

Lemma 1. (Local Logconcavity) The HEC choice probabilities, \( Q_{[j]} \), are each a logconcave function of the ideal utilities provided that their rank-ordering \( u_{[1]} \leq u_{[2]} \leq \cdots \leq u_{[n]} \) is preserved.

Second, unlike the choice probabilities indexed by rank-ordered ideal utilities, the choice probabilities indexed by the choices themselves are actually smooth functions despite the complications raised by the fact that the choice probability formula (3) depends on how ideal utilities rank.

Lemma 2. (Continuity and Differentiability) The HEC choice probabilities, \( Q_i \), are continuous and differentiable in ideal utilities \( (u_1, u_2, \ldots, u_n) \), even at the transition points where ideal utility ranks change.
While these two structural properties might also be of general interest, they play an important part in our pricing (§3) and estimation (§2.4) results.

### 2.2 Elasticity of Demand in HEC

Unlike in MNL, the elasticity of choice probabilities with respect to ideal utilities is not a constant in HEC. First, taking the derivative of the $G$ function, we obtain

$$\frac{\partial G_j}{\partial u_k} = \begin{cases} -\lambda_k G_j & \forall j < k \\ L_{[j+1]} G_j & \forall j = k \\ 0 & \forall j > k \end{cases}$$

(setting $L_{[n+1]}$ to zero by convention). Now, taking the derivative of choice probabilities in (3) and writing the resulting expressions in terms of the $G$ and $Q$ functions, we obtain

$$\frac{\partial Q_j}{\partial u_k} = \begin{cases} -\lambda_k Q_j & \forall j < k \\ L_j G_j - \lambda_j Q_j & \forall j = k \\ -\lambda_j Q_k & \forall j > k \end{cases}$$

Therefore, the elasticity of the choice probability $Q_j$ is

$$E_{Q_j}^{Q_k} = \frac{\partial Q_j}{\partial u_k} \frac{u_k}{Q_j} = \begin{cases} -u_k \lambda_k & \forall j < k \\ u_j \left( L_j G_j / Q_j - \lambda_j \right) & \forall j = k \\ -u_k \lambda_j Q_k / Q_j & \forall j > k \end{cases}$$

Take three products with ideal utility ranks $j < k < l$. Comparing the elasticities of $Q_j$ and $Q_l$ with respect to the ideal utility of product $k$, we observe that $E_{Q_j}^{Q_k} \leq E_{Q_l}^{Q_k} \leq 0$, which follows from (8) and (11), and $E_{Q_j}^{Q_k} = E_{Q_l}^{Q_k}$ only if $u_k = u_{k+1} = \cdots = u_l$. Hence, lower-end products (consider $j$ with respect to $k$) would experience proportionally higher cannibalization than higher-end products (consider $l$ with respect to $k$) when the ideal utility of a mid-tier
product \([k]\) is improved, say by a price promotion. This implication of the HEC model is fully consistent with the empirical findings of Blattberg and Wisniewski (1989). In contrast, the MNL model predicts constant cross elasticities (Anderson et al. 1992, p. 44), that is, \(E_{u_j}^{Q[j]} < E_{u_k}^{Q[k]}\) can never occur under MNL.

2.3 Consumer Surplus in HEC

Setting the marginal utility of income to 1 for simplicity of exposition, the consumer surplus under HEC can be defined as \(S \equiv \max_{i \in N} \{U_i\} = \max_{i \in N} \{u_i - z_i\}\). Its expected value has a remarkably simple expression.

**Theorem 2. (Expected Consumer Surplus under HEC)** For a consumer evaluating \(n\) choices with utilities \(U_i = u_i - z_i\), where the error terms \(z_i\) follow independent exponential distributions with corresponding rate \(\lambda_i\) for \(i \in N\), the expected consumer surplus is

\[
CS \equiv E[S] = E\left[\max_{i \in N} \{U_i\}\right] = u_{[n]} - \frac{1}{\lambda_{[n]}} \cdot Q_{[n]}
\]

(12)

This is a simple and intuitive formula. For example, if all choices are identical, i.e., \(u_i = u\) and \(\lambda_i = \lambda\) for all \(i\), then the expected consumer surplus equals \(u - 1/(\lambda n)\), which is increasing in \(n\) and approaching an asymptotic limit of \(u\) as \(n \to \infty\). More generally, if all choices have the same ideal utility \((u_i = u)\) and error terms with different variances, then the expected consumer surplus is \(u - 1/L_{[1]}\). It is clear from this expression that higher variety (adding more choices with the same ideal utility \(u\)) or lower variance in the error terms (increasing \(\lambda_i\) for any existing choice \(i\)) improves consumer surplus, as both would increase \(L_{[1]}\). Also, very low variance in the error term of the choice with the highest ideal utility (very high \(\lambda_{[n]}\)) means that the expected consumer surplus is determined almost exclusively by the highest ideal utility \((u_{[n]}\)). Note that, because \(Q_{[n]} \leq 1\), the expected consumer surplus is higher than the average utility of the choice with the highest ideal utility, i.e., \(CS \geq u_{[n]} - 1/\lambda_{[n]}\).

The closed-form expected consumer surplus formula in (12) makes it easier to understand which choices are worth improving—from a consumer welfare standpoint. Taking the derivative of \(CS\)
with respect to the ideal utility of choice \([j]\), we obtain

\[
\frac{\partial CS}{\partial u_{[j]}} = Q_{[j]} \quad \forall j
\]

(13)

Therefore, improving the most popular choice (one that has the highest choice probability) improves the expected consumer surplus the most\(^1\). Recall that, under HEC, the most popular choice may not coincide with the choice that offers the highest ideal utility. So, it may be that it is better to improve a choice that is worse in its ideal utility but also less risky (has an error term with less variance). An example would be product 2 (versus product 3) in Table 1.

### 2.4 Maximum Likelihood Estimation of Ideal Utilities in HEC

Building on Lemmas 1 and 2, in this subsection we establish the concavity of the HEC loglikelihood function in model parameters. To that end, we first need to formally define the HEC loglikelihood function.

Suppose consumers are offered a set of products \(S_k\) in choice scenario \(k\). Let \(n_{i}^{k}\) represent the number of consumers who chose product \(i\) in choice scenario \(k\) (\(i \in S_k\)). Assume product \(i\)’s ideal utility in choice scenario \(k\), \(u_{i}^{k}(\gamma)\), is a linear function of its unknown parameters, represented by the vector \(\gamma\). Let \(Q_{i}^{k}\) denote the probability that a consumer chooses product \(i\) in choice scenario \(k\) (full definition remains the same as in (1)). We maintain the notational convention that \([j]\) refers to the product with the \(j\)-th lowest ideal utility; that is, \(Q_{[j]}^{k}\) denotes the probability that a consumer chooses product \([j]\) in choice scenario \(k\), which has the \(j\)-th lowest ideal utility among all \(u_{i}^{k}(\gamma)\) over \(i \in S_k\).

With consumers exposed to \(K\) choice scenarios, \(S_1, S_2, \ldots, S_K\), the loglikelihood function is

\[
\mathcal{LL}(\gamma) = \sum_{k=1}^{K} \sum_{i \in S_k} n_{i}^{k} \cdot \ln \left( Q_{i}^{k} \right) = \sum_{k=1}^{K} \sum_{|S_k|} n_{[j]}^{k} \cdot \ln \left( Q_{[j]}^{k} \right)
\]

(14)

which captures the probability of observing the data \(n_{i}^{k}\) for \(i \in S_k\) and \(k = 1, \ldots, K\). Maximum likelihood estimation of choice data with HEC, in the sense of finding \(\gamma\) that maximizes \(\mathcal{LL}(\gamma)\), obviously requires varying \(\gamma\), which will cause the rank-ordering of ideal utilities \(u_{i}^{k}(\gamma)\) to reshuffle.

\(^1\)Equation (13) identically holds for MNL also (Anderson et al. 1992, p. 45).
This dependency is expressed with $[\cdot]$ notation in (14). Lemma 2 proves very helpful in overcoming this difficulty.

**Theorem 3. (Concavity of Loglikelihood Function)** The HEC loglikelihood function $\mathcal{LL}(\gamma)$ is concave in the unknown model parameters $\gamma$, provided that the ideal utilities are linear in $\gamma$, and that the error term distribution parameters $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ are given or fixed.

The proof hinges on a key factoid: As a choice makes transitions between ideal utility ranks, say due to an optimization routine varying any of the model parameters $(\gamma)$, the choice probability and its logarithm remain differentiable even at the transition points (Lemma 2). That is, $\ln(Q^k_i)$ is a smooth function of $\gamma$. Any standard optimization package can therefore be used to maximize the HEC loglikelihood for a given set of $\lambda$'s. This fortunate feature of the HEC model may appear surprising due to the discrete nature of the ideal utility rankings. The following numerical example illustrates the result and the basic idea behind it.

**Numerical Example 1.** Consider 4 choices with ideal utilities $u_i = i$ and error term parameters $\lambda_i = 1/i$ for $i = 1, 2, 3, 4$. Varying $u_1$ from 0 to 5, we graph in Figure 1 the natural logarithm of the choice probability for choice $i$, i.e., $\ln(Q_i)$ for all $i$. Note, for example, that choice 1 has the lowest ideal utility on the interval $u_1 \in [0, 2)$, the second lowest ideal utility on $[2, 3)$, and so on. Even though its ideal utility rank changes from one segment to the next, these four segments do trace out a smooth concave curve. As mentioned earlier, the reason is that even at the transition points where a choice's ideal utility rank changes, the choice probability and its natural logarithm remain differentiable.

### 2.5 Extension to Deterministic Outside Option

The choice set in the base HEC model can include an outside option, which would need to have a random error term (with its own rate parameter $\lambda$), just like all the other choices. In this subsection we show how HEC can easily incorporate a deterministic outside option.

Consider $n$ products and an outside option. Let $N = \{1, \ldots, n\}$ denote the set of products, and $N^+ = N \cup \{0\}$ the expanded choice set that includes the outside option. Suppose product $i \in N$ yields a random utility $U_i = u_i - z_i$, exactly as defined in the beginning of §2, whereas, the outside option yields a deterministic utility of $u_0$. 

10
Consumers are looking to purchase one of the $n$ products or choose the outside option, which could represent purchasing none of the products in $N$ but rather some other product (say, a competitor’s), or it could represent opting for a default, well-known alternative (like, doing nothing) whose utility has no uncertainty. The probability that the consumer chooses product $i \in N$ is

$$Q_i = P\{u_i - z_i \geq u_0 \text{ and } u_i - z_i \geq u_j - z_j \text{ } \forall j \in N, j \neq i\}$$

$$= \begin{cases} 0 & \text{if } u_i \leq u_0 \\ \int_0^{u_i-u_0} \prod_{j \in N, j \neq i} [1 - F_j(u_j - u_i + z)] f_i(z)dz & \text{if } u_i > u_0 \end{cases} \quad (15)$$

and the probability that the consumer chooses the outside option is

$$Q_0 = P\{u_0 \geq u_j - z_j \text{ } \forall j \in N\}$$

$$= \prod_{j \in N} [1 - F_j(u_j - u_0)] \quad (16)$$

We assume without loss of generality that all $n$ products are “better” than the outside option in the sense of having a higher ideal utility, i.e., $u_i > u_0$ for all $i \in N$. If there were products with lower ideal utility than $u_0$, then their choice probability would be zero (see 15) and they could be
eliminated from consideration. This does not imply anything about how average utilities compare; there may exist products with an average utility \((u_i - 1/\lambda_i)\) that is higher or lower than \(u_0\).

As with the base HEC model, rank-ordering of ideal utilities matters in the derivation of choice probabilities. Let \([j]\) denote the product – out of the set \(N\) – with the \(j\)-th smallest ideal utility. That is, the \(n\) products under consideration have the following rank-ordering in ideal utilities: \(u_{[1]} \leq u_{[2]} \leq \cdots \leq u_{[n]}\). (Also, by assumption, \(u_0 < u_{[1]}\).) Define \(L_{[j]} \equiv \sum_{k=j}^{n} \lambda_{[k]}\) and \(G_{[j]}\) for the product in ideal-utility rank \(j \in N\) just as before in (2), and define

\[
G_{[0]} \equiv \exp \left[-\sum_{k=1}^{n} \lambda_{[k]}(u_{[k]} - u_0)\right]
\]

for the outside option. Again, note that \(\lambda_{[j]}\) refers to the error distribution parameter for product \([j]\), not the \(j\)-th smallest \(\lambda\).

**Theorem 4.** Under the HEC model with a deterministic outside option, the probability that the consumer chooses product \([j]\) is

\[
Q_{[j]} = G_{[j]} - \sum_{k=0}^{j-1} \frac{\lambda_{[j]} \cdot G_{[k]}}{L_{[k+1]}}
\]  

for \(j \in N\), and the probability that the consumer chooses the outside option is \(Q_0 = G_{[0]}\).

A few remarks are in order. First, the HEC monotonicity property still holds among the \(n\) products. That is, \(Q_{[j]} / \lambda_{[j]} \leq Q_{[j+1]} / \lambda_{[j+1]}\) for \(j = 1, \ldots, n-1\). The proof, which we omit, is very similar. Second, the outside option steals demand from each product proportional to the product’s “error rate.” In particular, consumers switching away from product \([j]\) to the outside option contribute \(\lambda_{[j]} / L_{[1]}\) fraction of the outside option demand \(G_{[0]}\) (see the summand for \(k = 0\) in (17)). Third, it is easy to see how the choice probabilities vary as the outside option improves. The first- and second-order derivatives of \(Q_{[j]}\)

\[
\frac{\partial Q_{[j]}}{\partial u_0} = -\lambda_{[j]} \cdot \exp \left[-\sum_{k=1}^{n} \lambda_{[k]}(u_{[k]} - u_0)\right] = -\lambda_{[j]}Q_0
\]

\[
\frac{\partial^2 Q_{[j]}}{\partial u_0^2} = -\lambda_{[j]}L_{[1]} \cdot \exp \left[-\sum_{k=1}^{n} \lambda_{[k]}(u_{[k]} - u_0)\right] = -\lambda_{[j]}L_{[1]}Q_0
\]

reveal that \(Q_{[j]}\) are concave decreasing in \(u_0\) (as long as the assumption \(u_0 < u_{[1]}\) remains valid).
Fourth, as $u_0$ approaches $u_{[1]}$ from below, the choice probability for the product with the smallest ideal utility

$$Q_{[1]} = G_{[1]} - \frac{\lambda_{[1]}}{L_{[1]}} \exp \left( - \sum_{k=1}^{n} \lambda_{[k]} (u_{[k]} - u_0) \right)$$

$$= \frac{\lambda_{[1]}}{L_{[1]}} \left\{ \exp \left( - \sum_{k=1}^{n} \lambda_{[k]} (u_{[k]} - u_{[1]}) \right) - \exp \left( - \sum_{k=1}^{n} \lambda_{[k]} (u_{[k]} - u_0) \right) \right\}$$

approaches zero from above.

In closing, it is fairly easy to incorporate a deterministic outside option in HEC. Whereas in MNL, it is a very cumbersome undertaking (Anderson et al. 1992, p. 235), and not very common in analytical or empirical literature. Possibly more importantly, adding a deterministic outside option in HEC allows modeling choices with zero probability, whereas MNL choice probabilities are always strictly positive.

### 3. Pricing under HEC

In this section we study pricing under a demand model built on HEC. We consider two canonical settings: a multiproduct monopoly, and an oligopoly of single-product firms. Throughout, we assume that the ideal utility of product $i \in N = \{1, \ldots, n\}$ is linear in its price $p_i$, i.e.,

$$u_i(p_i) = \alpha_i - \beta p_i.$$ 

The intercept $\alpha_i$ captures all non-price related factors and measures the *intrinsic desirability* of the product, i.e., the ideal utility of product $i$ at $p_i = 0$. The coefficient $\beta > 0$ captures the price sensitivity of consumers.

Recall our notation linking the ideal utility rank of a product to its index: $[j]$ denotes the choice with the $j$-th smallest ideal utility out of the entire choice set. Varying prices will of course result in varying ideal utility ranks. In the interest of keeping the notation simple, however, we use the same notation throughout and caution the reader when necessary.

The objective of firms in both monopoly and oligopoly settings is to maximize expected revenue. Expected profit maximization—with marginal cost $c_i$ per unit of product $i$—can be easily
accommodated. Rewriting the ideal utility as \( u_i(p_i) = (\alpha_i - \beta c_i) - \beta (p_i - c_i) \), it is easy to see that the equivalent profit maximization model would have profit margin \((p_i - c_i)\) in place of price, and \((\alpha_i - \beta c_i)\) would become the new measure of intrinsic desirability, i.e., the ideal utility of product \(i\) priced at cost \((p_i = c_i)\). Hence, we set the marginal cost of all products to zero without loss of generality.

### 3.1 Multi-product Monopoly

In this subsection we consider a multi-product monopolist’s price optimization problem. The monopolist is offering \(n\) products, \(i = 1, \ldots, n\), and there is also an outside option, indexed 0, which represents not buying from the monopolist. Let \(N = \{1, \ldots, n\}\) denote the set of products, and \(N^+ = N \cup \{0\}\) the expanded choice set that includes the outside option. Suppose product \(i \in N\) yields a random utility \(U_i = u_i(p_i) - z_i\), where \(z_i\) is exponential with rate \(\lambda_i\), and the outside option yields a random utility \(U_0 = u_0 - z_0\), where \(u_0 = \alpha_0\) and \(z_0\) is exponential with rate \(\lambda_0\).

The monopolist wants to price its products in \(N\) optimally – so as to maximize the expected revenue per customer \(R(p_1, \ldots, p_n) = \sum_{i=1}^{n} p_i Q_i\) (the market size is normalized to 1 without loss of generality). An equivalent formulation of the objective function would be \(R(p_1, \ldots, p_n) = \sum_{j=1}^{n+1} p_j Q_{[j]}\), subject to the constraint \(p_0 = 0\) without loss of generality. Even if the outside option has a price, it is fixed; therefore, its original price (times \(\beta\)) can be rolled into \(\alpha_0\) and its price reset to 0. Note that, in the latter summation, the ideal utility ranks run from 1 to \(n + 1\) because the choice set \(N^+\) has \(n + 1\) choices.

**Theorem 5.** At optimal prices, products with higher intrinsic desirability have a higher choice probability, i.e., \(Q_i \geq Q_i'\) iff \(\alpha_i \geq \alpha_i'\) for two arbitrary products \(i\) and \(i'\).

This implies a certain pricing structure under one special case.

**Corollary 1.** Relabel products in increasing order of intrinsic desirability, \(\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n\), and suppose the error distribution rates have the opposite ordering, i.e., \(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\). Then, the ideal utilities evaluated at optimal prices must be increasing, i.e., \(u_1(p_1^*) \leq u_2(p_2^*) \leq \cdots \leq u_n(p_n^*)\).
3.2 Single-product Oligopoly

In this subsection we consider the price competition among \( n \) single-product firms – assuming no outside option. The expected revenue \( R_i(u_i) \) gained by the firm offering product \( i \in N \), expressed as a function of the ideal utility \( u_i \), is

\[
R_i(u_i) = p_i \cdot Q_i = \frac{\alpha_i - u_i}{\beta} \cdot Q_i
\]

where \( u_i \leq \alpha_i \) (the market size is normalized to 1 without loss of generality). Setting the derivative of this expression to zero reveals that the equilibrium ideal utility must satisfy the first-order condition

\[
\alpha_i - u_i = \frac{Q_i}{\partial Q_i/\partial u_i}. 
\]

In order to obtain an analytical characterization of the equilibrium prices, we first study the right-hand-side ratio \( Q_i/\partial Q_i/\partial u_i \). As before, the rank-ordering of ideal utilities matters, and it depends on the price that each and every firm sets. Hence, we analyze \( Q[j]/\partial Q[j]/\partial u[j] \) for some ideal utility rank \( j \), \( 1 \leq j \leq n \), to show that it has three important properties.

**Lemma 3.** The ratio \( Q[j]/\partial Q[j]/\partial u[j] \) has the following properties.

(a) **Monotonicity and Positivity:** It is strictly positive for all \( j \), strictly increasing in \( u[j] \) for \( j \geq 2 \), and a constant for \( j = 1 \) (locally, preserving the rank-ordering of ideal utilities).

(b) **Boundary Points:** When \( u[j] = u[j-1] \) the successive ratios are related as follows:

\[
Q[j]/\partial Q[j]/\partial u[j] \left|_{u[j]=u[j-1]} \right. \leq Q[j-1]/\partial Q[j-1]/\partial u[j-1] 
\]

iff \( \lambda[j] \leq \lambda[j-1] \) (the inequality is strict iff \( \lambda[j] < \lambda[j-1] \)) for \( j \geq 2 \).

(c) **Independence from Higher-Ranked Ideal Utilities:** It is independent of the ideal utilities \( u[j+1], \ldots, u[n] \) and their rank ordering among themselves for \( j < n \).

We build on Lemma 3 to derive Nash equilibrium in prices. Note that part (c) already implies that if one finds the right sequence in which to solve the first-order conditions, right in the sense of producing ordered ideal utilities, one would find a Nash equilibrium. That is, if solving the first-order conditions in the order \( 1, 2, \ldots, n \) (say) produces optimal ideal utilities in increasing order \( (u_1^* \leq u_2^* \leq \cdots \leq u_n^*) \), then the implied prices must be Nash. Because, each firm sets their own optimal ideal utility with the full knowledge of all products with smaller ideal utilities, and the higher ideal utilities that stem from the pricing decisions of the remaining firms have no
impact on their decision. In other words, Stackelberg equilibrium with the right sequence of firms taking pricing decisions one after another, coincides with Nash equilibrium; the question is, what is the ‘right’ sequence? Before we proceed with answering this question, we first settle the issue of existence and uniqueness of Nash equilibrium.

**Lemma 4.** Given $n$ firms each offering a single product with a distinct ideal utility ($u_i$ for $i \in N$), there exists a unique price equilibrium for the HEC demand system $X_i = m \cdot Q_i$, where $m$ is the market size (which can be set to 1 without loss of generality) and $Q_i$ are the choice probabilities given in (3).

We are now ready to turn to the question of sequence (in which the first-order conditions must be solved) knowing that there can only be one sequence that works. For notational clarity, let

$$
\tau_j(x_j|x_1, \ldots, x_{j-1}) \equiv \frac{Q_j}{\partial Q_j/\partial u_j} \bigg|_{u_{[k]}=x_k, k=1, \ldots, j}
$$

for $x_1 \leq \cdots \leq x_{j-1} \leq x_j$ and $j \geq 2$, which is allowed only because the ratio for product $[j]$ depends purely on $u_{[j]}$ and all lower-ranked ideal utilities due to Lemma 3(c). In fact, writing out this ratio explicitly, we observe that the first-order condition for product $[j]$

$$
\alpha_{[j]} - u_{[j]} = \tau_j(u_{[j]|u_{[1]}, \ldots, u_{[j-1]}}) = \frac{Q_j}{L_{[j]}G_{[j]} - \lambda_{[j]}Q_{[j]}} = \frac{1}{L_{[j]}G_{[j]} - \lambda_{[j]}}
$$

where $\sigma_{j-1}(u_{[1]}, \ldots, u_{[j-1]}) = \sum_{k=1}^{j-1} \frac{\lambda_{[k]}L_{[k+1]}G_{[k+1]}}{L_{[k]}L_{[k+1]}} \cdot \exp \left( L_{[k+1]}u_{[k]} - \sum_{l=k+1}^{j-1} \lambda_{[l]}u_{[l]} \right)$, boils down to solving a single-variable implicit expression for $u_{[j]}$ (considering $u_{[1]}, \ldots, u_{[j-1]}$ as given). Note that, because $Q_{[1]} = G_{[1]}$, the ratio for product $[1]$, $\tau_1 = 1/L_{[2]}$, is a constant (it does not even depend on $u_{[1]}$), but it does depend on which product is in rank 1 via $L_{[2]}$.

Next, we analyze the first-order condition for some product $[j]$ for $j \geq 2$. Let $\hat{u}_{[j]}$ denote the
solution, i.e., the \( u_{[j]} \) that solves

\[
\alpha_{[j]} - u_{[j]} = \tau_j(u_{[j]}|\hat{u}_{[1]}, \ldots, \hat{u}_{[j-1]})
\]

(18)

for a given set of lower-ranked ideal utilities \( \hat{u}_{[1]}, \ldots, \hat{u}_{[j-1]} \) satisfying \( \hat{u}_{[1]} \leq \cdots \leq \hat{u}_{[j-1]} \). The solution for the ‘first’ product is \( \hat{u}_{[1]} = \alpha_{[1]} - 1/L_{[2]} \) (this follows directly from \( \tau_1 = 1/L_{[2]} \)).

**Lemma 5.** Consider the first-order condition for product \( [j] \) for some \( j \geq 2 \), which involves solving (18) for a given set of lower-ranked ideal utilities \( \hat{u}_{[1]}, \ldots, \hat{u}_{[j-1]} \) satisfying \( \hat{u}_{[1]} \leq \cdots \leq \hat{u}_{[j-1]} \). The solution, \( \hat{u}_{[j]} \), is unique and it satisfies the following propositions.

**Case A:** If \( \alpha_{[j-1]} + 1/\lambda_{[j-1]} \leq \alpha_{[j]} + 1/\lambda_{[j]} \) and \( \alpha_{[j-1]} \leq \alpha_{[j]} \), then \( \hat{u}_{[j]} \geq \hat{u}_{[j-1]} \), which holds with equality only if \( \alpha_{[j-1]} = \alpha_{[j]} \) and \( \lambda_{[j-1]} = \lambda_{[j]} \).

**Case B:** If \( \alpha_{[j-1]} + 1/\lambda_{[j-1]} \leq \alpha_{[j]} + 1/\lambda_{[j]} \) and \( \alpha_{[j-1]} > \alpha_{[j]} \), then \( \hat{u}_{[j]} \geq \hat{u}_{[j-1]} \) iff

\[
\alpha_{[j-1]} - \alpha_{[j]} \leq \frac{\lambda_{[j-1]} - \lambda_{[j]}}{(\frac{L_{[j-1]}G_{[j-1]}}{Q_{[j-1]}} - \lambda_{[j-1]}) (\frac{L_{[j]}G_{[j-1]}}{Q_{[j-1]}} - \lambda_{[j]})}.
\]

(19)

**Case C:** If \( \alpha_{[j-1]} + 1/\lambda_{[j-1]} > \alpha_{[j]} + 1/\lambda_{[j]} \) and \( \alpha_{[j-1]} \leq \alpha_{[j]} \), then \( \hat{u}_{[j]} \geq \hat{u}_{[j-1]} \) iff (19) holds.

**Case D:** If \( \alpha_{[j-1]} + 1/\lambda_{[j-1]} > \alpha_{[j]} + 1/\lambda_{[j]} \) and \( \alpha_{[j-1]} > \alpha_{[j]} \), then \( \hat{u}_{[j]} < \hat{u}_{[j-1]} \).

Lemma 5 shows there are four mutually exclusive and exhaustive cases that help us compare the ‘previous’ solution \( (\hat{u}_{[j-1]}) \) with the ‘next’ solution \( (\hat{u}_{[j]}) \). We are particularly interested in whether these two solutions satisfy \( \hat{u}_{[j-1]} \leq \hat{u}_{[j]} \), which, as noted before, is needed for a series of first-order condition solutions to support a Nash equilibrium. It is clear that Case A positively helps, Case D positively does not, whereas in Cases B and C it depends. The following necessary condition for equilibrium is thus immediate (we state it without proof). We use the superscript \( * \) to denote equilibrium prices or ideal utilities.

**Proposition 1.** Let \( u_1^* \leq u_2^* \leq \cdots \leq u_n^* \) be a Nash equilibrium. Then, either of the following must be true for all \( i = 2, \ldots, n \).

(i) \( \alpha_{i-1} + 1/\lambda_{i-1} \leq \alpha_i + 1/\lambda_i \), or

(ii) \( \alpha_{i-1} + 1/\lambda_{i-1} > \alpha_i + 1/\lambda_i \) and \( \alpha_{i-1} \leq \alpha_i \).

Using Case A of Lemma 5 only, we obtain the following simple procedure that traverses the
choice set only once to construct the Nash equilibrium.

**Theorem 6.** Consider the $n$-firm oligopoly. Suppose $\alpha_i + \frac{1}{\lambda_i} \leq \alpha_i' + \frac{1}{\lambda_i'}$ implies $\alpha_i \leq \alpha_i'$ for any two arbitrary products $i$ and $i'$. Then, the prices obtained by the following steps are in a Nash equilibrium:

**Step 1.** Relabel all products so that $\alpha_1 + \frac{1}{\lambda_1} \leq \alpha_2 + \frac{1}{\lambda_2} \leq \cdots \leq \alpha_n + \frac{1}{\lambda_n}$ (breaking ties arbitrarily), which implies $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$ by assumption.

**Step 2.** Solve for $u_1$ that satisfies $\alpha_1 - u_1 = \tau_1(u_1)$. (The solution is $u_1^* = \alpha_1 - 1/L_2$.)

**Step 3.** Successively moving up — for $i = 2$, $i = 3$, ..., and $i = n$ — solve for the unique $u_i$ that satisfies $\alpha_i - u_i = \tau_i(u_i|u_1^*, \ldots, u_{i-1}^*)$ and denote it by $u_i^*$. (The solution must also satisfy $u_i^* \geq u_{i-1}^*$, which holds with equality only if $\alpha_{i-1} = \alpha_i$ and $\lambda_{i-1} = \lambda_i$.)

**Step 4.** Compute the equilibrium prices: $p_i^* = (\alpha_i - u_i^*)/\beta$ for $i = 1, 2, \ldots, n$.

Theorem 6 requires a certain structure on the problem parameters: $\alpha_i + \frac{1}{\lambda_i} \leq \alpha_i' + \frac{1}{\lambda_i'}$ implies $\alpha_i \leq \alpha_i'$ for any two products $i$ and $i'$. That is, loosely speaking, larger intrinsic desirability and larger ‘error’ mean (or standard deviation) imply a larger ideal utility at equilibrium prices. Albeit not always, a larger ideal utility tends to be associated with a higher equilibrium price. For example, in the following result, a special case of Theorem 6 produces a sharper characterization of the equilibrium.

**Corollary 2.** Consider the $n$-firm oligopoly. Suppose $\alpha_i + \frac{1}{\lambda_i} \leq \alpha_i' + \frac{1}{\lambda_i'}$ implies $\alpha_i \leq \alpha_i'$ and $\lambda_i \leq \lambda_i'$ for any two arbitrary products $i$ and $i'$. Then, the linear procedure detailed in Theorem 6 produces the Nash equilibrium in prices. Moreover, the equilibrium prices and choice probabilities are monotone, i.e., relabeling all products so that $\alpha_1 + \frac{1}{\lambda_1} \leq \alpha_2 + \frac{1}{\lambda_2} \leq \cdots \leq \alpha_n + \frac{1}{\lambda_n}$ (breaking ties arbitrarily), they satisfy $p_1^* \leq p_2^* \leq \cdots \leq p_n^*$ and $Q_1^* \leq Q_2^* \leq \cdots \leq Q_n^*$. It follows that revenues in equilibrium are also monotone: $R_1^* \leq R_2^* \leq \cdots \leq R_n^*$.

Alptekinoğlu and Semple (2016) show for the EC model ($\lambda_i = \lambda$ for all $i$) that rank-ordering the products on intrinsic desirability produces the Nash equilibrium (pp. 86-87). Corollary 2 generalizes and expands this result: It generalizes to allow for heteroscedastic errors; and it expands their result by showing that equilibrium prices also have the same ordering. The result is highly intuitive in

\[18\]
that firms with a product that has higher intrinsic desirability and lower variability in the error term is able to charge more and attract more demand, thus gets rewarded more in the market.

Using Lemma 5, oligopoly equilibrium can be characterized in several more special cases.

**Corollary 3.** Consider the $n$-firm oligopoly.

(a) **Products Symmetric in Error Distribution.** Suppose $\lambda_i = \lambda$ for all $i$. Then, rank-ordering the products in increasing order of intrinsic desirability ($\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$) and solving (18) in that order produces the unique Nash equilibrium. Moreover, the equilibrium prices are monotone increasing, i.e., $p_1^* \leq p_2^* \leq \cdots \leq p_n^*$, and so are the choice probabilities ($Q_1^* \leq Q_2^* \leq \cdots \leq Q_n^*$) and revenues in equilibrium ($R_1^* \leq R_2^* \leq \cdots \leq R_n^*$).

(b) **Products Symmetric in Intrinsic Desirability.** Suppose $\alpha_i = \alpha$ for all $i$. Then, rank-ordering the products in decreasing order of error distribution parameters ($\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$) and solving (18) in that order produces the unique Nash equilibrium. Moreover, the equilibrium prices are monotone decreasing, i.e., $p_1^* \geq p_2^* \geq \cdots \geq p_n^*$, and so are the choice probabilities ($Q_1^* \geq Q_2^* \geq \cdots \geq Q_n^*$) and revenues in equilibrium ($R_1^* \geq R_2^* \geq \cdots \geq R_n^*$).

(c) **Purely Symmetric Products.** If $\lambda_i = \lambda$ and $\alpha_i = \alpha$ for all $i$, then the unique Nash equilibrium in prices is symmetric: $p_i^* = 1/\beta\lambda(n-1)$ for all $i$. The resulting market shares and revenues are $Q_i^* = 1/n$ and $R_i^* = 1/\beta\lambda n(n-1)$.

Part (a) of Corollary 3 considers the EC model (it is a special case of Corollary 2). It shows that products with higher intrinsic desirability command a higher price, attract more demand, and yield higher revenue when the error terms are iid. Part (b) shows, when the firms have products with symmetric intrinsic desirability, it is the variability of the error term that drives their success in the market: The firm that has a product with a lower-variance error term (higher $\lambda$) wins; it is able to charge a higher price and attract more demand, hence earn a higher revenue, at equilibrium. Note that higher $\lambda$ also means a lower mean for the error term; so, those firms with a higher $\lambda$ can offer consumers a higher expected utility at the same price point $p$, i.e., $E[U_i] = \alpha - \beta p - 1/\lambda_i$.

Finally, the purely symmetric case yields a very simple solution. Part (c) not only verifies the general intuition that more intense competition (higher $n$) dampens prices, but it also shows that the equilibrium price is higher for larger variance in error terms (lower $\lambda$). The intuition for the latter observation is that higher variance in error terms makes consumer choice less dependent on
price (and more on errors), which reduces the competitive pressure on the firms, allowing them to charge more. This result is consistent with what would happen under MNL (Anderson et al. 1992, p. 222). From the opposite angle, observe that the symmetric equilibrium prices and revenues are decreasing in \( \lambda \) (with lower variance in error terms). It is interesting that all firms lose out even though higher \( \lambda \) means higher expected utility \( E[U] = \alpha - \beta p^* - 1/\lambda = \alpha - [n/(n-1)]/\lambda \) for the consumers. This has an element of the Prisoner’s Dilemma: Giving out better product information seems beneficial from the unilateral perspective of each firm, but it hurts all firms in equilibrium.

Next, adopting Lemma 5 to duopoly, we obtain the following result.

**Theorem 7. (Duopoly Equilibrium in Prices)** Suppose \( n = 2 \) and \( \alpha_1 + 1/\lambda_1 \leq \alpha_2 + 1/\lambda_2 \) (breaking any tie arbitrarily). Then, the unique Nash equilibrium prices are

\[
p^*_1 = \frac{1}{\beta \lambda_2}, \quad p^*_2 = \hat{p}_2
\]

where \( \hat{p}_2 \) is the unique solution of

\[
\beta \lambda_2 \hat{p}_2 = 1 + \lambda_2 (\alpha_2 - \alpha_1) - \ln \left( \frac{\lambda_1 + \beta \lambda_1 \lambda_2 \hat{p}_2}{\lambda_1 + \lambda_2} \right)
\]

that satisfies \( 1/(\beta \lambda_1) \leq \hat{p}_2 \leq 1/(\beta \lambda_2) + (\alpha_2 - \alpha_1)/\beta \). Furthermore, the ideal utilities at equilibrium prices satisfy \( u^*_1 \leq u^*_2 \), and the resulting choice probabilities and expected revenues in equilibrium are as follows.

\[
Q^*_1 = \frac{\lambda_1}{\lambda_1 + \beta \lambda_1 \lambda_2 \hat{p}_2}, \quad R^*_1 = \frac{1}{\beta \lambda_2} \cdot \frac{\lambda_1}{\lambda_1 + \beta \lambda_1 \lambda_2 \hat{p}_2}
\]

\[
Q^*_2 = \frac{\beta \lambda_1 \lambda_2 \hat{p}_2}{\lambda_1 + \beta \lambda_1 \lambda_2 \hat{p}_2}, \quad R^*_2 = \frac{\beta \lambda_1 \lambda_2 (\hat{p}_2)^2}{\lambda_1 + \beta \lambda_1 \lambda_2 \hat{p}_2}
\]

Note that the ranking of ideal utilities at equilibrium prices matches how a particular sum \( (\alpha_i + 1/\lambda_i) \) ranks between the two choices. A lower ideal utility (at equilibrium prices) accompanies a lower sum of intrinsic desirability and standard deviation of error term. Also, comparing equilibrium revenues, we observe that firm 1 earns a higher revenue than firm 2 if and only if it can also charge a higher price at equilibrium, i.e., \( R^*_1 \geq R^*_2 \) iff \( p^*_1 \geq p^*_2 \) (or, \( \frac{1}{\beta \lambda_2} \geq \hat{p}_2 \)). A sufficient condition for firm 1 to earn a higher revenue is \( \alpha_1 \geq \alpha_2 \). That is, if it has a product with higher intrinsic desirability
(α₁ ≥ α₂) and a low-variance error term compared to competition, i.e., \( \frac{1}{\lambda_1} \leq \frac{1}{\lambda_2} - (\alpha_1 - \alpha_2) \).

Are there cases where lower variance in error term surely leads to market advantage – higher price and revenue in equilibrium? To answer this we analyze two special cases of duopoly. We show in both cases that the product with lower variance (higher \( \lambda \)) commands a higher price and earns a higher revenue at equilibrium. In other words, interpreting lower variance as better product information, the firm that gives better product information gets rewarded in the market.

**Corollary 4.** Suppose \( \alpha_1 = \alpha_2 \) and \( \lambda_1 \geq \lambda_2 \). The unique Nash equilibrium in prices is \( p_1^* = \frac{1}{(\beta \lambda_2)} \), \( p_2^* = \hat{p}_2 \), where \( \hat{p}_2 \) is the unique solution of

\[
\beta \lambda_2 \hat{p}_2 = 1 - \ln \left( \frac{\lambda_1 + \beta \lambda_1 \lambda_2 \hat{p}_2}{\lambda_1 + \lambda_2} \right)
\]

that satisfies \( 1/(\beta \lambda_1) \leq \hat{p}_2 \leq 1/(\beta \lambda_2) \). Firm 1 charges a higher price (\( p_1^* \geq p_2^* \)), attracts more demand (\( Q_1^* \geq Q_2^* \)), and earns a higher revenue (\( R_1^* \geq R_2^* \)) than firm 2.

Corollary 4 shows, when the firms have products with symmetric intrinsic desirability, it is the variance of the error term that drives their success in the market: The firm that has the product with a lower-variance error term wins; it is able to charge a higher price and earn a higher revenue at equilibrium.

**Corollary 5.** Suppose \( \alpha_1 + 1/\lambda_1 = \alpha_2 + 1/\lambda_2 \). The unique Nash equilibrium in prices is \( p_1^* = \frac{1}{(\beta \lambda_2)} \), \( p_2^* = \frac{1}{(\beta \lambda_1)} \). The equilibrium choice probabilities are \( Q_1^* = \frac{\lambda_1}{\lambda_1 + \lambda_2} \), \( Q_2^* = \frac{\lambda_2}{\lambda_1 + \lambda_2} \), and the equilibrium revenues are \( R_1^* = \frac{\lambda_1}{\beta \lambda_2 (\lambda_1 + \lambda_2)} \), \( R_2^* = \frac{\lambda_2}{\beta \lambda_1 (\lambda_1 + \lambda_2)} \). Hence, firm 1 charges a higher price (\( p_1^* \geq p_2^* \)), attracts more demand (\( Q_1^* \geq Q_2^* \)), and earns a higher revenue (\( R_1^* \geq R_2^* \)) iff \( \lambda_1 \geq \lambda_2 \).

This result is also consistent with Corollary 4 in that lower variance in error term (higher \( \lambda \)) leads to market advantage. This case can be viewed as a quasi-symmetric duopoly, because the ideal utilities at equilibrium prices are equal: \( u_1^* = \alpha_1 - 1/\lambda_2 = \alpha_2 - 1/\lambda_1 = u_2^* \). Naturally, the ratios \( Q_1^*/\lambda_1 \) and \( Q_2^*/\lambda_2 \) also match.

### 4. Concluding Remarks

The purpose of this research is to build a case for using HEC in demand modeling in theoretical and empirical research. We present a number of new analytical properties of this choice model, including
closed-form demand elasticity and consumer surplus expressions, and logconcavity of the likelihood function in utility model parameters (for a given set of error term distribution parameters).

We also study pricing under HEC in two canonical demand models, one for a monopoly offering $n$ products to consumers who also have an outside option, and the other for an oligopoly of $n$ firms each offering a single product. Higher variability in error terms generally soften the price competition, but it also is detrimental for individual firm profits.

References


Appendix: Proofs

Proof of Theorem 1

Proof. Plugging in the pdf and cdf of the error terms into (1), recognizing that they equal zero over the nonpositive domain, and adopting the notation \((x)^+ = x\) for \(x \geq 0\), \(0\) for \(x < 0\), we have

\[
Q[j] = \int_0^\infty \lambda[j] \exp \left[ -\sum_{k=1}^n \lambda[k] (u[k] - u[j] + z) \right] dz
\]

Recalling \(u[1] \leq u[2] \leq \cdots \leq u[n]\), \(Q[j]\) can be written more explicitly as

\[
Q[j] = \int_0^{u[j]-u[j-1]} \lambda[j] \exp \left[ -\sum_{k=j}^n \lambda[k] (u[k] - u[j] + z) \right] dz
\]

\[
+ \int_{u[j]-u[j-2]}^{u[j]-u[j-1]} \lambda[j] \exp \left[ -\sum_{k=j-1}^n \lambda[k] (u[k] - u[j] + z) \right] dz
\]

\[
\vdots
\]

\[
+ \int_{u[j]-u[1]}^{\infty} \lambda[j] \exp \left[ -\sum_{k=2}^n \lambda[k] (u[k] - u[j] + z) \right] dz
\]

\[
+ \int_{u[j]-u[2]}^{\infty} \lambda[j] \exp \left[ -\sum_{k=1}^n \lambda[k] (u[k] - u[j] + z) \right] dz
\]

Integrating and simplifying each term, we obtain the following, which can be expressed more compactly as in (3).

\[
Q[j] = \frac{\lambda[j]}{L[j]} \exp \left[ -\sum_{k=1}^n \lambda[k] (u[k] - u[j]) \right]
\]

\[
- \frac{\lambda[j]\lambda[j-1]}{L[j-1]L[j]} \exp \left[ -\sum_{k=j-1}^n \lambda[k] (u[k] - u[j-1]) \right]
\]

\[
- \frac{\lambda[j]\lambda[j-2]}{L[j-2]L[j-1]} \exp \left[ -\sum_{k=j-2}^n \lambda[k] (u[k] - u[j-2]) \right]
\]

\[
\vdots
\]

\[
- \frac{\lambda[j]\lambda[1]}{L[1]L[2]} \exp \left[ -\sum_{k=1}^n \lambda[k] (u[k] - u[1]) \right]
\]
Proof of Lemma 1

Proof. The proof is almost identical to the proof of a similar result for EC (Alptekinoğlu and Semple, 2016, Online Supplement, Lemma 3). First, observe that \( Q[j] \) can be factored into the following form:

\[
Q[j] = \frac{\lambda[j]}{L[j]} \cdot e^{-\sum_{k=j}^{n} \lambda[k](u[k]-u[j])} \left[ 1 - \sum_{k=1}^{j-1} \frac{L[j] \lambda[k]}{L[k]L[k+1]} \cdot e^{-\sum_{l=k}^{n} \lambda[l](u[l]-u[k])} \right].
\] (20)

Define the vector \( u \) as \( u^T = (u[1], u[2], ..., u[n]) \). Then \( Q[j] \) can be expressed in the general form

\[
Q[j] = \gamma_0 e^{-a_0^T u} \left[ 1 - \sum_{k=1}^{j-1} \gamma_k e^{-a_k^T u} \right]
\]

where \( a_0, a_1, ..., a_{j-1} \) are \( n \times 1 \) vectors that capture the coefficients of \( u \) in the exponents of equation (20), and \( \gamma_k \) are constants satisfying \( 0 < \gamma_k < 1 \). Define \( f(u) = \sum_{k=1}^{j-1} \gamma_k e^{-a_k^T u} \), and note that \( f(u) \) is a convex function of \( u \) satisfying \( 0 < f(u) < 1 \). Then we must have

\[
\ln(Q[j]) = \ln(\gamma_0) - a_0^T u + \ln(1 - f(u)).
\]

It is enough to show \( \ln(1 - f(u)) \) is concave. But this is straightforward; its Hessian is

\[
\frac{-\nabla^2 f}{(1 - f(u)) \left(1 - f(u)\right)^2}\cdot \nabla u f \cdot \nabla u^T
\]

which is negative definite because \( f(u) \) is convex. \( \square \)

Proof of Lemma 2

Proof. Continuity follows from \( Q[j]/\lambda[j] = Q[j+1]/\lambda[j+1] \) iff \( u[j] = u[j+1] \) for \( j = 1, \ldots, n - 1 \) (see §2.1). As for differentiability, suppose the ideal utilities \( u_1 \leq u_2 \leq \cdots \leq u_n \) are ordered, i.e., \([j] = j \forall j\). Differentiability is assured when this rank-ordering of ideal utilities is preserved; the
first derivatives are given in (10). The own-derivative is
\[
\frac{\partial Q_i}{\partial u_i} = \frac{\partial Q_i}{\partial u_i} = L_{i+1}G_i + \sum_{k=1}^{i-1} \frac{\lambda_k^2}{L_k+1}G_k = L_iG_i - \lambda_i Q_i
\]
for all non-transition points at which ideal utility ranks remain as presumed. Now, pick a choice \( i \in \{2, \ldots, n\} \), and consider the point at which its ideal utility crosses that of choice \( i - 1 \). The limiting value of the above partial derivative as \( u_i \) approaches \( u_{i-1} \) from above (right-limit) is
\[
\frac{\partial Q_i}{\partial u_i} \bigg|_{u_i \downarrow u_{i-1}} = \frac{\partial Q_i}{\partial u_i} \bigg|_{u_i \downarrow u_{i-1}} = \frac{\lambda_i}{\lambda_{i-1}} L_{i-1}G_{i-1} - \lambda_i^2 Q_{i-1}
\]
This follows from the fact that \( u_i = u_{i-1} \) implies \( \frac{\lambda_i}{\lambda_{i-1}} G_i = \frac{L_{i+1}}{L_{i-1}} G_{i-1} \) and \( Q_i = \frac{\lambda_i}{\lambda_{i-1}} Q_{i-1} \). The left-limit requires a bit more thought. Imagine \( u_i \) was approaching \( u_{i-1} \) from below. In that case, the ideal utility ranks of choices \( i \) and \( i - 1 \) would swap; choice \( i \) would have rank \( i - 1 \) ([\( i - 1 \) = \( i \)], whereas choice \( i - 1 \) would have rank \( i \) ([\( j \) = \( i - 1 \)], while all other choices would remain at the same rank ([\( j \) = \( j \) \( \forall j \neq i, i - 1 \)]). The left-limit would then be the limiting value of the following partial derivative:
\[
\frac{\partial Q_i}{\partial u_i} \bigg|_{u_i \uparrow u_{i-1}} = \frac{\partial Q_i}{\partial u_i} \bigg|_{u_i \uparrow u_{i-1}} = L_{i-1} \tilde{G}_{[i-1]} - \lambda_i \tilde{Q}_{[i-1]}
\]
where \( \tilde{G}_{[i-1]} \) and \( \tilde{Q}_{[i-1]} \) denote the new \( G \)- and \( Q \)-functions for choice \( i \), which is now at rank \( i - 1 \) (because \( u_i \leq u_{i-1} \)). Note that, for choice \( i \) after swapping, \( \lambda_{[i-1]} = \lambda_i \) (error terms are non-\( iid \) and go with the choice) and \( L_{[i-1]} = L_{i-1} = \lambda_{i-1} + \lambda_i + \cdots + \lambda_n \). Careful bookkeeping also reveals that \( \tilde{G}_{[i-1]} = \frac{\lambda_i}{\lambda_{i-1}} G_{i-1} \) and \( \tilde{Q}_{[i-1]} = \frac{\lambda_i}{\lambda_{i-1}} Q_{i-1} \). Thus, the left-limit (22) exactly equals the right-limit (21). A similar argument applies to the derivative as \( u_i \) approaches \( u_{i+1} \). The partial derivative \( \partial Q_i/\partial u_i \) therefore exists everywhere for all \( i \). Finally, the cross-derivatives
\[
\frac{\partial Q_i}{\partial u_i'} = \begin{cases} 
-\lambda_i Q_i & \forall i < i' \\
-\lambda_i Q_{i'} & \forall i > i'
\end{cases}
\]
also exist everywhere for all \( i \neq i' \), because \( Q_i \) and \( Q_{i'} \) are themselves continuous (Lemma 2).
Proof of Theorem 2

Proof. The cdf of consumer surplus \( S \equiv \max_{i \in N} \{ U_i \} \) can be written compactly as follows:

\[
H(s) \equiv P\left\{ \max_{i \in N} \{ u_i - z_i \} \leq s \right\} = \exp \left[ -\sum_{i=1}^{n} \lambda_i (u_i - s)^+ \right]
\]

To write \( H(s) \) more explicitly, and use it in deriving the expected value of \( S \), we need to keep track of where \( s \) falls with respect to \( u \)-values. In particular, letting \( u_0 = -\infty \),

\[
H(s) = \begin{cases} 
\exp \left[ -\sum_{j=k}^{n} \lambda_{[j]} (u_{[j]} - s) \right], & \text{if } s \in [u_{[k-1]}, u_{[k]}), \text{ for } k = 1, \ldots, n \\
1, & \text{if } s \in [u_{[n]}, +\infty) 
\end{cases}
\]

Hence, the pdf of \( S \) is

\[
h(s) \equiv H'(s) = \begin{cases} 
L_{[k]} \cdot \exp \left[ -\sum_{j=k}^{n} \lambda_{[j]} (u_{[j]} - s) \right], & \text{if } s \in [u_{[k-1]}, u_{[k]}), \text{ for } k = 1, \ldots, n \\
0, & \text{if } s \in [u_{[n]}, \infty) 
\end{cases}
\]

where \( L_{[k]} \equiv \sum_{j=k}^{n} \lambda_{[j]} \) for \( k = 1, \ldots, n \). Using \( h(\cdot) \), we write the expected value of \( S \) as

\[
E[S] = \int_{-\infty}^{\infty} s h(s) ds = \sum_{k=1}^{n} \int_{u_{[k-1]}}^{u_{[k]}} s \cdot L_{[k]} \cdot \exp \left[ -\sum_{j=k}^{n} \lambda_{[j]} (u_{[j]} - s) \right] ds
\]
Applying integration by parts to these \( n \) terms, and letting \( \Delta_k = \exp \left( - \sum_{j=k+1}^{n} \lambda_{[j]}(u_{[j]} - u_{[k]}) \right) \) for \( k = 1, \ldots, m - 1 \), we obtain

\[
E[S] = \left[ u_{[1]} \Delta_1 - 0 \right] - \frac{1}{L_{[1]}} [\Delta_1 - 0] + \\
u_{[2]} \Delta_2 - u_{[1]} \Delta_1 - \frac{1}{L_{[2]}} [\Delta_2 - \Delta_1] + \\
\ldots \\
u_{[n-1]} \Delta_{n-1} - u_{[n-2]} \Delta_{n-2} - \frac{1}{L_{[n-1]}} [\Delta_{n-1} - \Delta_{n-2}] + \\
u_{[n]} - u_{[n-1]} \Delta_{n-1} - \frac{1}{\lambda_{[n]}} [1 - \Delta_{n-1}]
\]

Cancelling the \( u_{[k]} \Delta_k \) terms and using Theorem 1, we obtain

\[
E[S] = u_{[n]} - \frac{1}{\lambda_{[n]}} \left[ 1 - \sum_{k=1}^{n-1} \frac{\lambda_{[n]} \lambda_{[k]}}{L_{[k]} L_{[k+1]}} \cdot \Delta_k \right] \\
= u_{[n]} - \frac{1}{\lambda_{[n]}} \left[ G_{[n]} - \sum_{k=1}^{n-1} \frac{\lambda_{[n]} L_{[k+1]}}{L_{[k]}} \cdot G_{[k]} \right] \\
= u_{[n]} - \frac{1}{\lambda_{[n]}} \cdot Q_{[n]}
\]

**Proof of Theorem 3**

*Proof.* Without loss of generality we focus on a single choice scenario \( S = \{1, 2, \ldots, n\} \) with \( n \) products and suppress the scenario index \( k \). We show that \( \ln (Q_i) \) is a concave function of the parameter vector \( \gamma \) for a fixed set of \( \lambda \)'s. Recall that the ideal utility ranks of products (how \( u_i(\gamma) \) rank among the products in \( S, i \in S \)) change as a function of model parameters \( \gamma \), and those ranks in turn enter into the choice probability formula \( Q_{[j]} \), hence the choice probabilities \( Q_i \).

First consider two parameter vectors, \( \gamma_0 \) and \( \gamma_1 \), and their convex combination \( \gamma(\mu) = (1 - \mu) \gamma_0 + \mu \gamma_1 \) for \( \mu \in [0, 1] \). It is enough to show that the function \( \ln (Q_i) \) is concave for \( \mu \in [0, 1] \).

Observe that if no product changes its rank on this interval, the result is clearly true; it is a direct consequence of Lemma 1 and the fact that all ideal utility functions are linear in \( \gamma \). If the ideal utility functions for two products cross (if the two products “swap ranks”) at some point, say
\( \mu = \mu' \) (a “transition point”), we know from Lemma 2 that \( \ln(Q_i) \) is still differentiable at \( \mu = \mu' \). (It is worth remembering that when ideal utility functions cross, all choice probabilities remain continuous; all choice probabilities \( Q_i \) are therefore continuous in \( \mu \).) We then use the result that if a continuous function \( f \) is concave on \([a, b]\) and concave on \([b, c]\) and differentiable at \( b \) (i.e., the one-sided derivatives agree at \( b \)), then \( f \) is concave on \([a, c]\). Applying this result to each such transition point implies the function must be concave on \([0, 1]\). Although we assume that only two ideal utility functions cross at the same time, the case where more than two cross simultaneously can be reduced to the case for two using a routine perturbation argument; taking the limit as these perturbations tend to 0 implies the result.

The total derivative of \( Q_i \) with respect to \( \mu \) therefore exists and is
\[
\frac{dQ_i}{d\mu} = \sum_{i'=1}^{n} \frac{\partial Q_i}{\partial u_{i'}} \frac{du_{i'}}{d\mu}.
\]

The latter is a continuous function of \( \mu \) at \( \mu = \mu' \) because each term in the sum is continuous (note \( u_{i'} \) is a linear function of \( \mu \)); this implies the derivative of \( \ln(Q_i) \) is continuous at \( \mu = \mu' \) too. Because \( \ln(Q_i) \) is concave on each subinterval where no rank transitions take place by Lemma 1 and differentiable at each point where a rank transition does take place by Lemma 2, the function must be concave on the entire interval \([0, 1]\). The loglikelihood function is a nonnegatively weighted combination of such functions, and this ensures its concavity.

**Proof of Theorem 4**

*Proof.* Plugging in the pdf and cdf of the error terms into (15), recognizing that they equal zero over the nonpositive domain, and adopting the notation \((x)^+ = x \) for \( x \geq 0 \), 0 for \( x < 0 \), we have

\[
Q_{[j]} = \int_{0}^{u_{[j]} - u_0} \lambda_{[j]} \exp \left[ -\sum_{k=1}^{n} \lambda_{[k]} \left( u_{[k]} - u_{[j]} + z \right)^+ \right] dz
\]
Recalling \( u_0 < u_{[1]} \leq u_{[2]} \leq \cdots \leq u_{[n]} \), \( Q[j] \) can be expressed more explicitly as

\[
Q[j] = \int_0^{u_{[j]} - u_{[j-1]}} \lambda_{[j]} \exp \left[ -\sum_{k=j}^{n} \lambda_{[k]} (u_{[k]} - u_{[j]} + z) \right] dz
\]

\[
+ \int_{u_{[j]} - u_{[j-2]}}^{u_{[j]} - u_{[j-1]}} \lambda_{[j]} \exp \left[ -\sum_{k=j-1}^{n} \lambda_{[k]} (u_{[k]} - u_{[j]} + z) \right] dz
\]

\[\vdots\]

\[
+ \int_{u_{[j]} - u_{[1]}}^{u_{[j]} - u_{[2]}} \lambda_{[j]} \exp \left[ -\sum_{k=2}^{n} \lambda_{[k]} (u_{[k]} - u_{[j]} + z) \right] dz
\]

\[
+ \int_{u_{[j]} - u_{0}}^{u_{[j]} - u_{[1]}} \lambda_{[j]} \exp \left[ -\sum_{k=1}^{n} \lambda_{[k]} (u_{[k]} - u_{[j]} + z) \right] dz
\]

Integrating and simplifying, we obtain the following expression, which can be expressed more compactly as in (17).

\[
Q[j] = \frac{\lambda_{[j]}}{L_{[j]}} \exp \left[ -\sum_{k=i}^{n} \lambda_{[k]} (u_{[k]} - u_{[j]}) \right]
\]

\[-\frac{\lambda_{[j]} \lambda_{[j-1]}}{L_{[j-1]} L_{[j]}} \exp \left[ -\sum_{k=j-1}^{n} \lambda_{[k]} (u_{[k]} - u_{[j-1]}) \right]\]

\[-\frac{\lambda_{[j]} \lambda_{[j-2]}}{L_{[j-2]} L_{[j-1]}} \exp \left[ -\sum_{k=j-2}^{n} \lambda_{[k]} (u_{[k]} - u_{[j-2]}) \right]\]

\[\vdots\]

\[-\frac{\lambda_{[j]} \lambda_{[1]}}{L_{[1]} L_{[2]}} \exp \left[ -\sum_{k=1}^{n} \lambda_{[k]} (u_{[k]} - u_{[1]}) \right]\]

\[-\frac{\lambda_{[j]}}{L_{[1]}} \exp \left[ -\sum_{k=1}^{n} \lambda_{[k]} (u_{[k]} - u_{0}) \right]\]
The choice probability for the outside option is almost immediate from (16):

\[
Q_0 = \prod_{i \in N} [1 - F_i(u_i - u_0)] = \prod_{j \in N} \exp \left[-\lambda_j (u_j - u_0)\right] = \exp \left[-\sum_{k=1}^{n} \lambda_{[k]}(u_{[k]} - u_0)\right]
\]

\[
\text{Proof of Theorem 5}
\]

\[\text{Proof.}\] Arguing by contradiction, suppose there exist products \(i\) and \(i'\) such that \(Q_i > Q_{i'}\) and yet \(\alpha_i < \alpha_{i'}\) at optimal prices. Let \(H = u_i(p_i) = \alpha_i - \beta p_i\) and \(L = u_{i'}(p_{i'}) = \alpha_{i'} - \beta p_{i'}\), which imply the prices \(p_i = \frac{\alpha_i - H}{\beta}\) and \(p_{i'} = \frac{\alpha_{i'} - L}{\beta}\). Consider the new revenue by resetting prices for \(i\) and \(i'\) as follows: \(\hat{p}_i = \frac{\alpha_i - L}{\beta}\) and \(\hat{p}_{i'} = \frac{\alpha_{i'} - H}{\beta}\). These price changes preserve the values for the ideal utility vector (as \(u_i(\hat{p}_i) = L\) and \(u_{i'}(\hat{p}_{i'}) = H\)) but interchange the choice probabilities associated with product \(i\) and product \(i'\). Because there is no change in the expected revenue contribution for any product other than \(i\) or \(i'\), the net change in total expected revenue

\[
(Q_{i'}\hat{p}_i + Q_i\hat{p}_{i'}) - (Q_ip_i + Q_{i'}p_{i'}) = (Q_i - Q_{i'}) \left(\frac{\alpha_{i'} - \alpha_i}{\beta}\right)
\]

is strictly positive, which follows from the premise. This contradicts the optimality of the original expected revenue. \(\square\)

\[
\text{Proof of Lemma 3}
\]

\[\text{Proof.}\] Recall the definition

\[
G_{[j]} \equiv \frac{\lambda_{[j]}}{L_{[j]}} \exp \left[-\sum_{k=j}^{n} \lambda_{[k]}(u_{[k]} - u_{[j]})\right]
\]

30
and how \( Q_{[j]} \) and \( \frac{\partial Q_{[j]}}{\partial u_{[j]}} \) can be expressed in terms of the \( G \) functions:

\[
Q_{[j]} = G_{[j]} - \sum_{k=1}^{j-1} \frac{\lambda_{[j]} L_{[k+1]}}{L_{[k+1]}} G_{[k]}
\]

\[
\frac{\partial Q_{[j]}}{\partial u_{[j]}} = L_{[j]} G_{[j]} - \lambda_{[j]} Q_{[j]} = L_{[j+1]} G_{[j]} + \sum_{k=1}^{j-1} \frac{\lambda_{[j]}^2}{L_{[k+1]}} G_{[k]}
\]

**Proof of Lemma 3(a):** Strict positivity follows from \( Q_{[j]} \) and \( \frac{\partial Q_{[j]}}{\partial u_{[j]}} \) above. Due to Lemma 1, \( Q_{[j]} \) is logconcave (locally, preserving the rank-ordering of ideal utilities), which implies \( \frac{\partial Q_{[j]}}{\partial u_{[j]}} / Q_{[j]} \) is monotone nonincreasing and hence the reciprocal is nondecreasing. For strict monotonicity, note that the derivative of the ratio, \( 1 - Q_{[j]} \cdot Q''_{[j]} / (Q'_{[j]})^2 \), is equivalent to \( L^2_{[j]} G_{[j]} \left[ G_{[j]} - Q_{[j]} \right] / (Q'_{[j]})^2 \), which is strictly positive except for \( j = 1 \) (because, \( G_{[j]} > Q_{[j]} \) for \( j > 1 \), and \( G_{[1]} = Q_{[1]} \)).

**Proof of Lemma 3(b):** When \( u_{[j]} = u_{[j-1]} \), we have the relationship \( \frac{L_{[j]}}{\lambda_{[j]}} G_{[j]} = \frac{L_{[j-1]}}{\lambda_{[j-1]}} G_{[j-1]} \), which implies

\[
\frac{Q_{[j]}}{\lambda_{[j]}} = \frac{Q_{[j-1]}}{\lambda_{[j-1]}}
\]

(this was shown earlier in §2.1). It also implies the following iff \( \lambda_{[j]} \leq \lambda_{[j-1]} \)

\[
\frac{\partial Q_{[j]}}{\partial u_{[j]}} = L_{[j]} G_{[j]} - \lambda_{[j]} Q_{[j]} = \frac{\lambda_{[j]} L_{[j-1]}}{\lambda_{[j-1]}} G_{[j-1]} - \frac{\lambda_{[j]}^2}{\lambda_{[j-1]}} Q_{[j-1]}
\]

\[
= \left( \frac{\lambda_{[j]}}{\lambda_{[j-1]}} \right) L_{[j-1]} G_{[j-1]} - \left( \frac{\lambda_{[j]}}{\lambda_{[j-1]}} \right)^2 \lambda_{[j-1]} Q_{[j-1]}
\]

\[
\geq \left( \frac{\lambda_{[j]}}{\lambda_{[j-1]}} \right) L_{[j-1]} G_{[j-1]} - \left( \frac{\lambda_{[j]}}{\lambda_{[j-1]}} \right) \lambda_{[j-1]} Q_{[j-1]}
\]

\[
= \left( \frac{\lambda_{[j]}}{\lambda_{[j-1]}} \right) \frac{\partial Q_{[j-1]}}{\partial u_{[j-1]}}
\]

(the condition is used in line 3). The result follows from \( \frac{Q_{[j]}}{\lambda_{[j]}} = \frac{Q_{[j-1]}}{\lambda_{[j-1]}} \) and \( \frac{1}{\lambda_{[j]}} \cdot \frac{\partial Q_{[j]}}{\partial u_{[j]}} \geq \frac{1}{\lambda_{[j-1]}} \cdot \frac{\partial Q_{[j-1]}}{\partial u_{[j-1]}} \). It holds with strict inequality iff \( \lambda_{[j]} < \lambda_{[j-1]} \).

**Proof of Lemma 3(c):** This can be confirmed by writing out the ratio and noting that every term in the numerator and the denominator depends on \( u_{[j+1]}, \ldots, u_{[n]} \) through the common factor \( \exp(-\sum_{k=j+1}^{n} \lambda_{[k]} u_{[k]}) \). In particular, this common factor cancels out from \( G_{[1]}, \ldots, G_{[j]} \) that appear both in the numerator and the denominator. Furthermore, \( \lambda_{[j+1]}, \ldots, \lambda_{[n]} \) appear only as a sum (in \( L_{[1]}, \ldots, L_{[j]}, L_{[j+1]} \) terms), therefore the ratio is also independent of how higher-ranked ideal
utilities rank among themselves.

Proof of Lemma 4

Proof. Suppose \([i] = i\) and set \(m = 1\). The first and second derivatives of the expected revenue with respect to price are

\[
\frac{\partial R_i}{\partial p_i} = Q_i - \beta p_i (L_i G_i - \lambda_i Q_i)
\]

\[
\frac{\partial^2 R_i}{\partial p_i^2} = -2\beta (L_i G_i - \lambda_i Q_i) - \beta^2 p_i [\lambda_i (L_i G_i - \lambda_i Q_i) - L_i L_{i+1} G_i]
\]

\[
\frac{\partial^2 R_i}{\partial p_i \partial p_j} = \begin{cases} 
\beta \lambda_i Q_i - \beta^2 p_i \lambda_j (L_i G_i - \lambda_i Q_i) & \text{if } i < j \\
\beta \lambda_i Q_j + \beta^2 p_i \lambda_j^2 Q_j & \text{if } j > i
\end{cases}
\]

which can be verified using (9) and (10). Evaluating \(\frac{\partial^2 R_i}{\partial p_i^2}\) at any point where the first-order-condition \((\frac{\partial R_i}{\partial p_i} = 0)\) is satisfied gives

\[
\left.\frac{\partial^2 R_i}{\partial p_i^2}\right|_{p_i=p_i^*} = -\frac{\beta}{L_i G_i - \lambda_i Q_i} \left[ (L_i G_i - \lambda_i Q_i)^2 + (L_i G_i)^2 \left( 1 - \frac{Q_i}{G_i} \right) \right]
\]

which is strictly negative. The revenue function is therefore strictly quasi-concave for all \(i \in N\), meaning, there is a unique best response denoted by \(p_i^*(p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_n)\) to any vector of prices set by other firms. If this best response function is a contraction, i.e., \(\sum_{j \neq i} |\partial p_i^*/\partial p_j| < 1\) for all \(i \in N\), then the equilibrium must also be unique (by Theorem 3.4 of Friedman, 1990, p. 84)\(^3\). Applying the implicit function theorem to the first-order-condition \((\partial R_i/\partial p_i = 0)\) and evaluating the result along the best response function, we obtain

\[
\left. \frac{\partial p_i^*}{\partial p_j} \right|_{p_i=p_i^*} = -\frac{\partial^2 R_i/\partial p_i \partial p_j}{\partial^2 R_i/\partial p_i^2} \bigg|_{p_i=p_i^*} = \begin{cases} 
0 & \text{if } i < j \\
(\lambda_i Q_j) (L_i G_i) / \left[ (L_i G_i - \lambda_i Q_i)^2 + (L_i G_i)^2 \left( 1 - \frac{Q_i}{G_i} \right) \right] & \text{if } j > i
\end{cases}
\]

\(^3\)Friedman (1990) shows it is sufficient for uniqueness that every player has (1) a compact and convex strategy space; (2) a continuous, bounded and strictly quasiconcave payoff function; and (3) a best-response function that is a contraction (Theorem 3.4, p. 84). Our proof verifies (2) and (3). We omit (1) for brevity. Although, technically, prices can be anywhere in \([0, \infty)\), firm i’s strategy space can be taken as \([0, P_i]\), which is compact (closed and bounded) and convex, for some sufficiently large constant \(P_i\).
Thus, $\sum_{j \neq i} |\partial p^*_i / \partial p_j| < 1$ if and only if

$$\frac{\lambda_i \left(\sum_{j < i} Q_j\right)}{L_i G_i} < \left(1 - \frac{\lambda_i Q_i}{L_i G_i}\right)^2 + \left(1 - \frac{Q_i}{G_i}\right)$$

This inequality holds because its left-hand-side is equal to one of the two strictly positive terms on the right-hand-side, i.e., $\frac{\lambda_i}{L_i G_i} (Q_1 + \cdots + Q_{i-1}) = 1 - \frac{Q_i}{G_i}$, which follows from (4).

In the derivation above, normalizing the market size ($m = 1$) is without loss of generality. (Any fixed market size $m$ would multiply the first and second derivatives stated in the beginning, and it eventually cancels out from $\partial p^*_i / \partial p_j$.) The assumption of $[i] = i$, however, was mainly for expositional convenience, but it also implicitly imposes a requirement (for derivatives to be valid) that all the ideal utility ranks remain the same, which of course may not be true as prices vary. Changes in ideal utility ranks do not pose a problem due to Lemma 2; the above argument holds for any ranking of ideal utilities (including the equilibrium ranking), and it follows from Lemma 2 that choice probabilities, hence firm revenues, are continuous and differentiable in ideal utilities, hence in prices.

Proof of Corollary 1

Proof. By Theorem 5, $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$ implies $Q_1 \leq Q_2 \leq \cdots \leq Q_n$ at optimal prices. Take any $i \in \{1, \ldots, n-1\}$ and suppose $u_i(p^*_i) > u_{i+1}(p^*_{i+1})$. The monotonocity argument in §2.1 would then imply $Q_i/\lambda_i > Q_{i+1}/\lambda_{i+1}$, which is in contradiction with the fact that $Q_i \leq Q_{i+1}$ and $\lambda_i \geq \lambda_{i+1}$ imply the opposite. Thus, by contradiction, it must be that $u_i(p^*_i) \leq u_{i+1}(p^*_{i+1})$. In fact, this relationship must hold for all products, implying $[i] = i$ — that the ideal utility ordering of products at optimal prices is the same as their intrinsic desirability ordering.

Proof of Lemma 5

Proof. The solution to (18) is unique, because the left-hand-side is strictly decreasing in $u[j]$ and equal to zero for $u[j] = \alpha[j]$, and the right-hand-side is strictly positive and strictly increasing in $u[j]$ due to Lemma 3(a).

Our proof strategy involves comparing $\alpha[j] - \hat{u}[j-1]$ with $\tau_j(\hat{u}[j-1]|\hat{u}[1], \ldots, \hat{u}[j-2])$. If $\alpha[j] - \hat{u}[j-1] > \tau_j(\hat{u}[j-1]|\hat{u}[1], \ldots, \hat{u}[j-1])$, then it must be that $\hat{u}[j] > \hat{u}[j-1]$, because the left-hand-side of (18) is
strictly decreasing in \( u_{[j]} \) and the right-hand-side of (18) is strictly positive and strictly increasing in \( u_{[j]} \). On the contrary, if \( \alpha_{[j]} - \hat{u}_{[j-1]} < \tau_j(\hat{u}_{[j-1]}|\hat{u}_{[1]}, \ldots, \hat{u}_{(j-1)}) \), then \( \hat{u}_{[j]} < \hat{u}_{[j-1]} \) by the same argument. Finally, if \( \alpha_{[j]} - \hat{u}_{[j-1]} = \tau_j(\hat{u}_{[j-1]}|\hat{u}_{[1]}, \ldots, \hat{u}_{(j-1)}) \), then obviously \( \hat{u}_{[j]} = \hat{u}_{[j-1]} \).

Assume \( \alpha_{[j-1]} + 1/\lambda_{[j-1]} \leq \alpha_{[j]} + 1/\lambda_{[j]} \) (Cases A and B). There are three possibilities.

**Case A1.** Suppose \( \alpha_{[j-1]} \leq \alpha_{[j]} \) and \( \lambda_{[j-1]} > \lambda_{[j]} \). Observe

\[
\alpha_{[j]} - \hat{u}_{[j-1]} \geq \alpha_{[j-1]} - \hat{u}_{[j-1]} \\
= \tau_{j-1}(\hat{u}_{[j-1]}|\hat{u}_{[1]}, \ldots, \hat{u}_{(j-2)}) \\
> \tau_j(\hat{u}_{[j-1]}|\hat{u}_{[1]}, \ldots, \hat{u}_{(j-1)})
\]

The first line is because \( \alpha_{[j-1]} \leq \alpha_{[j]} \); the second line is because \( \hat{u}_{[j-1]} \) solves its version of the first-order condition; the last line follows from Lemma 3(b) and the assumption that \( \lambda_{[j-1]} > \lambda_{[j]} \). Hence, we have \( \alpha_{[j]} - \hat{u}_{[j-1]} > \tau_j(\hat{u}_{[j-1]}|\hat{u}_{[1]}, \ldots, \hat{u}_{(j-1)}) \), which implies \( \hat{u}_{[j]} > \hat{u}_{[j-1]} \).

**Case A2.** Suppose \( \alpha_{[j-1]} \leq \alpha_{[j]} \) and \( \lambda_{[j-1]} \leq \lambda_{[j]} \). Observe

\[
\alpha_{[j]} - \hat{u}_{[j-1]} \geq \tau_j(\hat{u}_{[j-1]}|\hat{u}_{[1]}, \ldots, \hat{u}_{(j-1)}) \\
\geq \tau_{j-1}(\hat{u}_{[j-1]}|\hat{u}_{[1]}, \ldots, \hat{u}_{(j-2)}) \\
= \alpha_{[j-1]} - \hat{u}_{[j-1]}
\]

where the last line is because \( \hat{u}_{[j-1]} \) solves its version of the first-order condition, and the second line is due to Lemma 3(b) and the assumption that \( \lambda_{[j]} \geq \lambda_{[j-1]} \). The first inequality then follows from comparing the difference \( \left( \alpha_{[j]} - \hat{u}_{[j-1]} \right) - \left( \alpha_{[j-1]} - \hat{u}_{[j-1]} \right) = \alpha_{[j]} - \alpha_{[j-1]} \) with the difference \( \tau_j(\hat{u}_{[j-1]}|\hat{u}_{[1]}, \ldots, \hat{u}_{(j-1)}) - \tau_{j-1}(\hat{u}_{[j-1]}|\hat{u}_{[1]}, \ldots, \hat{u}_{(j-2)}) \), which equals

\[
\frac{Q_{[j-1]}}{L_{[j-1]}G_{[j-1]} - \lambda_{[j]}Q_{[j-1]}} - \frac{Q_{[j-1]}}{L_{[j-1]}G_{[j-1]} - \lambda_{[j-1]}Q_{[j-1]}}
\]

using the relationships derived in the proof of Lemma 3(b). It can be shown that

\[
\alpha_{[j]} - \alpha_{[j-1]} \geq \frac{1}{\lambda_{[j-1]}} - \frac{1}{\lambda_{[j]}} \geq \tau_j(\hat{u}_{[j-1]}|\hat{u}_{[1]}, \ldots, \hat{u}_{(j-1)}) - \tau_{j-1}(\hat{u}_{[j-1]}|\hat{u}_{[1]}, \ldots, \hat{u}_{(j-2)})
\]

where the first inequality follows from \( \alpha_{[j-1]} + 1/\lambda_{[j-1]} \leq \alpha_{[j]} + 1/\lambda_{[j]} \). The second inequality holds
true, because it is equivalent – using (23) – to \((\lambda_{[j]} - \lambda_{[j-1]}) L_{[j-1]} G_{[j-1]} \geq (\lambda^2_{[j]} - \lambda^2_{[j-1]}) Q_{[j-1]}\), which is always valid for \(\lambda_{[j]} \geq \lambda_{[j-1]}\). Hence, we have \(\alpha_{[j]} - \hat{u}_{[j-1]} \geq \tau_j(\hat{u}_{[j-1]}|\hat{u}_{[1]}, \ldots, \hat{u}_{[j-1]}),\) which implies \(\hat{u}_{[j]} \geq \hat{u}_{[j-1]}\), and these hold with equality only if \(\alpha_{[j-1]} = \alpha_{[j]}\) and \(\lambda_{[j-1]} = \lambda_{[j]}\).

**Case B.** Suppose \(\alpha_{[j-1]} > \alpha_{[j]}\), which implies \(\lambda_{[j-1]} > \lambda_{[j]}\). Observe

\[
\tau_{j-1}(\hat{u}_{[j-1]}|\hat{u}_{[1]}, \ldots, \hat{u}_{[j-2]}) = \alpha_{[j-1]} - \hat{u}_{[j-1]}
\]

\[
\alpha_{[j-1]} - \hat{u}_{[j-1]} > \alpha_{[j]} - \hat{u}_{[j-1]}
\]

\[
\tau_{j-1}(\hat{u}_{[j-1]}|\hat{u}_{[1]}, \ldots, \hat{u}_{[j-2]}) > \tau_j(\hat{u}_{[j-1]}|\hat{u}_{[1]}, \ldots, \hat{u}_{[j-1]})
\]

where the first line is because \(\hat{u}_{[j-1]}\) solves its version of the first-order condition, the second line follows from \(\alpha_{[j-1]} > \alpha_{[j]}\), and the third line is due to Lemma 3(b) and the assumption that \(\lambda_{[j-1]} > \lambda_{[j]}\). Thus, comparing the difference \((\alpha_{[j-1]} - \hat{u}_{[j-1]}) - (\alpha_{[j]} - \hat{u}_{[j-1]}) = \alpha_{[j-1]} - \alpha_{[j]}\) with the difference \(\tau_{j-1}(\hat{u}_{[j-1]}|\hat{u}_{[1]}, \ldots, \hat{u}_{[j-2]}) - \tau_j(\hat{u}_{[j-1]}|\hat{u}_{[1]}, \ldots, \hat{u}_{[j-1]}))\), we find that \(\alpha_{[j]} - \hat{u}_{[j-1]} \geq \tau_j(\hat{u}_{[j-1]}|\hat{u}_{[1]}, \ldots, \hat{u}_{[j-1]}))\) iff \(\alpha_{[j-1]} - \alpha_{[j]} \leq \tau_{j-1}(\hat{u}_{[j-1]}|\hat{u}_{[1]}, \ldots, \hat{u}_{[j-2]}) - \tau_j(\hat{u}_{[j-1]}|\hat{u}_{[1]}, \ldots, \hat{u}_{[j-1]}),\) which can be shown – using (23) – to be equivalent to inequality (19).

Now assume \(\alpha_{[j-1]} + 1/\lambda_{[j-1]} > \alpha_{[j]} + 1/\lambda_{[j]}\) (Cases C and D). Again, there are three possibilities.

**Case C.** Suppose \(\alpha_{[j-1]} \leq \alpha_{[j]}\), which implies \(\lambda_{[j-1]} < \lambda_{[j]}\). Observe

\[
\tau_{j-1}(\hat{u}_{[j-1]}|\hat{u}_{[1]}, \ldots, \hat{u}_{[j-2]}) = \alpha_{[j-1]} - \hat{u}_{[j-1]}
\]

\[
\alpha_{[j-1]} - \hat{u}_{[j-1]} \leq \alpha_{[j]} - \hat{u}_{[j-1]}
\]

\[
\tau_{j-1}(\hat{u}_{[j-1]}|\hat{u}_{[1]}, \ldots, \hat{u}_{[j-2]}) < \tau_j(\hat{u}_{[j-1]}|\hat{u}_{[1]}, \ldots, \hat{u}_{[j-1]})
\]

where the first line is because \(\hat{u}_{[j-1]}\) solves its version of the first-order condition, the second line follows from \(\alpha_{[j-1]} \leq \alpha_{[j]}\), and the third line is due to Lemma 3(b) and the assumption that \(\lambda_{[j-1]} < \lambda_{[j]}\). Thus, comparing the difference \((\alpha_{[j]} - \hat{u}_{[j-1]}) - (\alpha_{[j-1]} - \hat{u}_{[j-1]}) = \alpha_{[j]} - \alpha_{[j-1]}\) with the difference \(\tau_j(\hat{u}_{[j-1]}|\hat{u}_{[1]}, \ldots, \hat{u}_{[j-1]})) - \tau_{j-1}(\hat{u}_{[j-1]}|\hat{u}_{[1]}, \ldots, \hat{u}_{[j-2]})\), we find that \(\alpha_{[j]} - \hat{u}_{[j-1]} \geq \tau_j(\hat{u}_{[j-1]}|\hat{u}_{[1]}, \ldots, \hat{u}_{[j-1]}))\) iff \(\alpha_{[j]} - \alpha_{[j-1]} \geq \tau_{j-1}(\hat{u}_{[j-1]}|\hat{u}_{[1]}, \ldots, \hat{u}_{[j-1]})) - \tau_{j-1}(\hat{u}_{[j-1]}|\hat{u}_{[1]}, \ldots, \hat{u}_{[j-2]})\), which can be shown – using (23) – to be equivalent to inequality (19).
Case D1. Suppose $\alpha_{j-1} > \alpha_j$ and $\lambda_{j-1} < \lambda_j$. Observe

$$\alpha_j - \hat{u}_j < \alpha_{j-1} - \hat{u}_{j-1}$$

$$= \tau_j(\hat{u}_{j-1}|\hat{u}_1, \ldots, \hat{u}_{j-2})$$

$$< \tau_j(\hat{u}_{j-1}|\hat{u}_1, \ldots, \hat{u}_{j-1})$$

The first line is because $\alpha_{j-1} > \alpha_j$; the second line is because $\hat{u}_{j-1}$ solves its version of the first-order condition; the last line follows from Lemma 3(b) and the assumption that $\lambda_{j-1} < \lambda_j$. Hence, we have $\alpha_j - \hat{u}_j < \tau_j(\hat{u}_{j-1}|\hat{u}_1, \ldots, \hat{u}_{j-1})$, which implies $\hat{u}_j < \hat{u}_{j-1}$.

Case D2. Suppose $\alpha_{j-1} > \alpha_j$ and $\lambda_{j-1} \geq \lambda_j$. Observe

$$\alpha_{j-1} - \hat{u}_{j-1} = \tau_{j-1}(\hat{u}_{j-1}|\hat{u}_1, \ldots, \hat{u}_{j-2})$$

$$\geq \tau_j(\hat{u}_{j-1}|\hat{u}_1, \ldots, \hat{u}_{j-1})$$

$$> \alpha_j - \hat{u}_{j-1}$$

where the first line is because $\hat{u}_{j-1}$ solves its version of the first-order condition, and the second line is due to Lemma 3(b) and the assumption that $\lambda_{j-1} \geq \lambda_j$. The last inequality then follows from comparing the difference $(\alpha_{j-1} - \hat{u}_{j-1}) - (\alpha_j - \hat{u}_j) = \alpha_{j-1} - \alpha_j$ with the difference $\tau_{j-1}(\hat{u}_{j-1}|\hat{u}_1, \ldots, \hat{u}_{j-2}) - \tau_j(\hat{u}_{j-1}|\hat{u}_1, \ldots, \hat{u}_{j-1})$. It can be shown that

$$\alpha_{j-1} - \alpha_j > \frac{1}{\lambda_{j-1}} - \frac{1}{\lambda_j} \geq \tau_{j-1}(\hat{u}_{j-1}|\hat{u}_1, \ldots, \hat{u}_{j-2}) - \tau_j(\hat{u}_{j-1}|\hat{u}_1, \ldots, \hat{u}_{j-1})$$

where the first inequality follows from $\alpha_{j-1} + 1/\lambda_{j-1} > \alpha_j + 1/\lambda_j$, and the second from (23) and $\lambda_{j-1} \geq \lambda_j$. Hence, we have $\alpha_j - \hat{u}_j < \tau_j(\hat{u}_{j-1}|\hat{u}_1, \ldots, \hat{u}_{j-1})$, which implies $\hat{u}_j < \hat{u}_{j-1}$.

A summary of all the different cases regarding whether $\hat{u}_{j-1} \leq \hat{u}_j$ (“Yes”) or not are given in Table 2 for the reader’s convenience.

Proof of Theorem 6

Proof. First, note that in Step 3 the right-hand-side of the first-order condition only depends on $u_i$ and the $u$’s that have been determined so far ($u_1^*, \ldots, u_{i-1}^*$) due to Lemma 3(c). Hence,
\[\alpha_{[j-1]} + 1/\lambda_{[j-1]} \leq \alpha_{[j]} + 1/\lambda_{[j]} \]

\[\alpha_{[j-1]} \leq \alpha_{[j]} \quad \text{Yes}^{A1} \quad \alpha_{[j-1]} > \alpha_{[j]} \quad \text{Yes iff (19)}^{B} \]

\[\alpha_{[j-1]} \geq \alpha_{[j]} \quad \text{N/a} \quad \alpha_{[j-1]} \leq \alpha_{[j]} \quad \text{N/a} \]

\[\lambda_{[j-1]} \leq \lambda_{[j]} \quad \lambda_{[j-1]} > \lambda_{[j]} \]

Table 2: Is \(\hat{u}_{[j-1]} \leq \hat{u}_{[j]}\) satisfied? (Superscripts refer to cases in the proof of Lemma 5.)

the procedure can proceed sequentially and solve for one \(u\) at a time. However, for the choice probabilities and their derivatives, which make up the right-hand-side, to be well-defined, we need the whole solution to satisfy \(u_{1}^{*} \leq u_{2}^{*} \leq \cdots \leq u_{n}^{*}\).

Consider Step 3 for product \(i\), which involves solving the first-order condition for \(u_{i}\) given \(u_{1}^{*}, \ldots, u_{i-1}^{*}\) (they were determined in \(i-1\) previous iterations of Step 3 and suppose they satisfy \(u_{1}^{*} \leq \cdots \leq u_{i-1}^{*}\)). There are two possible cases that help us compare this solution to the previous one (\(u_{i-1}^{*}\)). If \(\lambda_{i-1} > \lambda_{i}\), then we are in Case A1 of Lemma 5, which shows \(u_{i}^{*} > u_{i-1}^{*}\). Else, if \(\lambda_{i-1} \leq \lambda_{i}\), then we are in Case A2 of Lemma 5, which shows \(u_{i}^{*} \geq u_{i-1}^{*}\) (holds with equality only if \(\alpha_{i-1} = \alpha_{i}\) and \(\lambda_{i-1} = \lambda_{i}\)).

Therefore, this sequential procedure naturally orders the optimal ideal utilities \(u_{1}^{*} \leq u_{2}^{*} \leq \cdots \leq u_{n}^{*}\) by construction. For the resulting prices to be a Nash equilibrium, we further claim that no firm has an incentive to deviate from their optimal ideal utility. This follows from Lemma 3(c).

The procedure starts with the product that has the lowest ideal utility and works its way up one firm at a time; when their ‘turn’ comes, each firm sets their own optimal ideal utility with the full knowledge of all products with smaller ideal utilities, and the higher ideal utilities that stem from the pricing decisions of the remaining firms have no impact on their decision. Thus, the construction does indeed produce a Nash equilibrium in ideal utilities, and therefore prices.

**Proof of Corollary 2**

**Proof.** Assume \(\alpha_{[j-1]} + 1/\lambda_{[j-1]} \leq \alpha_{[j]} + 1/\lambda_{[j]}\), \(\alpha_{[j-1]} \leq \alpha_{[j]}\), and \(\lambda_{[j-1]} \leq \lambda_{[j]}\). We are in Case A2 of the proof of Lemma 5. There we show that

\[
\begin{align*}
\alpha_{[j]} - \hat{u}_{[j-1]} & \geq \tau_{j}(\hat{u}_{[j-1]} | \hat{u}_{[1]}, \ldots, \hat{u}_{[j-1]}) \\
& \geq \tau_{j-1}(\hat{u}_{[j-1]} | \hat{u}_{[1]}, \ldots, \hat{u}_{[j-2]}) \\
& = \alpha_{[j-1]} - \hat{u}_{[j-1]}
\end{align*}
\]

37
which implies \( \hat{u}_{[j]} \geq \hat{u}_{[j-1]} \). Because \( \tau_j(u_{[j]}|\hat{u}_{[1]}, \ldots, \hat{u}_{[j-1]}) \) is monotone increasing in \( u_{[j]} \) (Lemma 3a), \( \hat{u}_{[j]} \) must satisfy \( \tau_j(\hat{u}_{[j]}|\hat{u}_{[1]}, \ldots, \hat{u}_{[j-1]}) \geq \tau_j(\hat{u}_{[j-1]}|\hat{u}_{[1]}, \ldots, \hat{u}_{[j-2]}) \). It follows that \( \alpha_{[j]} - \hat{u}_{[j]} \geq \alpha_{[j-1]} - \hat{u}_{[j-1]} \), or that the prices implied by \( \hat{u}_{[j]} \) and \( \hat{u}_{[j-1]} \) must have the same ordering, i.e., \( \hat{p}_{[j]} \equiv (\alpha_{[j]} - \hat{u}_{[j]}) / \beta \geq (\alpha_{[j-1]} - \hat{u}_{[j-1]}) / \beta \equiv \hat{p}_{[j-1]} \). Therefore, equilibrium prices and ideal utilities are monotone. The monotonicity of choice probabilities then follows from the fact that \( Q_{[j]} / \lambda_{[j]} \) is monotone increasing in \( j \) (see §2.1).

\[ \square \]

Proof of Corollary 3

Proof. We need to verify which of the six cases in the proof of Lemma 5 apply to each part.

Part (a): Relabel products such that \( \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n \). Only Case A2 survives; meaning, \( \alpha_{i-1} \leq \alpha_i \) implies \( \hat{u}_i \geq \hat{u}_{i-1} \); the other five cases can be either ruled out or they imply \( \hat{u}_i < \hat{u}_{i-1} \), which cannot support an equilibrium solution. Therefore, solving the first-order conditions for products \( 1, \ldots, n \) (in that order) produces the equilibrium ideal utilities \( u_i^* = \hat{u}_i \) that satisfy \( u_1^* \leq u_2^* \leq \cdots \leq u_n^* \). This and the assumption \( \lambda_i = \lambda \) imply \( Q_i^* \leq Q_2^* \leq \cdots \leq Q_n^* \), because the ratio \( Q_{[j]} / \lambda_{[j]} \) is monotone increasing in \( j \) (shown in §2.1). For monotonicity in prices, see the argument in the proof of Corollary 2, which shows \( p_i^* = (\alpha_i - u_i^*) / \beta \geq (\alpha_{i-1} - u_{i-1}^*) / \beta = p_{i-1}^* \).

Part (b): Relabel products such that \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \). Only Case A1 and Case A2 (restricted to \( \lambda_{i-1} = \lambda_i \)) survive; therefore, \( \lambda_{i-1} \geq \lambda_i \) implies \( \hat{u}_i \geq \hat{u}_{i-1} \), which does support an equilibrium solution. Thus, solving the first-order conditions for products \( 1, \ldots, n \) (in that order) produces the equilibrium ideal utilities \( u_i^* = \hat{u}_i \) that satisfy \( u_1^* \leq u_2^* \leq \cdots \leq u_n^* \). This and the assumption \( \alpha_i = \alpha \) imply \( p_i^* \leq p_j^* \leq \cdots \leq p_n^* \). For monotonicity in choice probabilities, first, observe that the first-order conditions (18) imply \( Q_i^* \geq Q_{i+1}^* \) iff \( (L_i G_i / \lambda_i) / [1 + 1 / (\beta \lambda_i p_i^*)] \geq (L_{i+1} G_{i+1} / \lambda_{i+1}) / [1 + 1 / (\beta \lambda_{i+1} p_{i+1}^*)] \), which is equivalent to

\[
\frac{1}{1 + \frac{1}{\beta \lambda_i p_i^*}} \geq \frac{\exp \left[ L_{i+1} (u_{i+1}^* - u_i^*) \right]}{1 + \frac{1}{\beta \lambda_{i+1} p_{i+1}^*}}
\]

\[ (24) \]
for all $i \geq 1$. Second, using the first-order condition for product $i + 1$, we obtain

$$
\alpha - u_{i+1}^* = \beta p_{i+1}^* = \left[ \frac{1}{L_{i+1} - \sigma_i \cdot \exp(-L_{i+1} u_{i+1}^*)} - \lambda_{i+1} \right]^{-1}
$$

$$
\lambda_{i+1} \left( 1 + \frac{1}{\beta \lambda_{i+1} p_{i+1}^*} \right) = \left[ \frac{1}{L_{i+1} - \sigma_i \cdot \exp(-L_{i+1} u_{i+1}^*)} \right]^{-1}
$$

which yields the following equations after replacing $\sigma_i \cdot \exp(-L_i u_i^*)$ with $\left[ \frac{1}{L_i} - \frac{1}{\beta \lambda_{i+1} p_{i+1}^*} \right]$ based on the first-order condition for product $i$.

$$
\lambda_{i+1} \left( 1 + \frac{1}{\beta \lambda_{i+1} p_{i+1}^*} \right) = \left[ \frac{1}{L_{i+1} - \sigma_i \cdot \exp(-L_i u_i^*)} \right]^{-1}
$$

$$
\exp [L_{i+1} (u_{i+1}^* - u_i^*)] = \left[ \frac{1}{L_{i+1} - \sigma_i \cdot \exp(-L_i u_i^*)} \right] \cdot \exp \left[-L_{i+1} (u_{i+1}^* - u_i^*) \right]
$$

(25)

Now, plugging (25) into (24), we conclude that $Q_i^* \geq Q_{i+1}^*$ iff

$$
\frac{1}{\lambda_i} \left( \frac{1}{\beta L_{i+1} p_i^*} - 1 \right) \leq \frac{1}{\lambda_{i+1}} \left( \frac{1}{\beta L_{i+1} p_{i+1}^*} - 1 \right)
$$

which is valid for all $i \geq 1$, because we assumed $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and showed earlier that $p_1^* \geq p_2^* \geq \cdots \geq p_n^*$.

**Part (c):** Only Case A2 survives with $\hat{u}_i = \hat{u}_{i-1}$ for all $i > 1$; hence, $u_i^* = u_1^* = \alpha - 1/L_2 = \alpha - 1/\lambda(n-1)$ for all $i$.

**Proof of Theorem 7**

First, adopting Lemma 5 to duopoly, we obtain the following result; Cases A and B of Lemma 5 collapse into one, and so do Cases C and D.

**Lemma 6.** Suppose $n = 2$ and consider the first-order condition for product [2], which involves solving $\alpha_2 - u_2^* = \tau_2(u_2^* | \hat{u}_{11})$ for $u_2^*$, where $\hat{u}_{11} = \alpha_{11} - 1/\lambda_{[2]}$. The solution, $\hat{u}_{21}$, is unique and it satisfies the following propositions.

- **Case AB:** If $\alpha_{[1]} + 1/\lambda_{[1]} \leq \alpha_{[2]} + 1/\lambda_{[2]}$, then $\hat{u}_{21} \geq \hat{u}_{11}$, which holds with equality only if

**Case CD:** If \( \alpha[1] + 1/\lambda[1] > \alpha[2] + 1/\lambda[2] \), then \( \hat{u}[2] < \hat{u}[1] \).

**Proof.** Case A1 still implies \( \hat{u}[2] > \hat{u}[1] \). In Case A2, the only difference is that, \( \alpha[1] + 1/\lambda[1] = \alpha[2] + 1/\lambda[2] \) implies \( \hat{u}[2] = \hat{u}[1] \), because

\[
\alpha[2] - \alpha[1] = \frac{1}{\lambda[1]} - \frac{1}{\lambda[2]} = \tau_2(\hat{u}[1]|\hat{u}[1]) - \tau_1(\hat{u}[1])
\]


We now prove the theorem using Lemma 6.

**Proof.** It is clear from Case CD of Lemma 6 that \( \alpha[1] + 1/\lambda[1] > \alpha[2] + 1/\lambda[2] \) fails to support an equilibrium. Case AB, however, does produce an equilibrium with \( u^*_1 = \hat{u}[1] \) and \( u^*_2 = \hat{u}[2] \), because \( \hat{u}[2] \geq \hat{u}[1] \) and neither firm wants to deviate from these ideal utilities; firm [1] would act ‘first’ and choose \( u^*_1 \) independently from what firm [2] does, and firm [2] would choose \( u^*_2 \) with the full knowledge of \( u^*_1 \). Now, relabeling the products such that \( \alpha_1 + 1/\lambda_1 \leq \alpha_2 + 1/\lambda_2 \), the price pair \( p^*_i = (\alpha_i - u^*_i)/\beta \ (i = 1, 2) \) implied by the ideal utilities \( u^*_1 = \alpha_1 - 1/\lambda_2 \) and \( u^*_2 \), which satisfies \( \alpha_2 - u^*_2 = \tau_2(u^*_2|u^*_1) \), is the unique Nash equilibrium. Furthermore, \( u^*_2 \geq u^*_1 \), which holds with equality only if \( \alpha_1 + 1/\lambda_1 = \alpha_2 + 1/\lambda_2 \).